MÉCANIQUE CÉLESTE.

Nathl Bowditch

MÉCANIQUE CÉLESTE.

BY THE

MARQUIS DE LA PLACE,

PEER OF FRANCE; GRAND CROSS OF THE LEGION OF HONOR; MEMBER OF THE FRENCH ACADEMY, OF THE ACADEMY OF SCIENCES OF PARIS, OF THE BOARD OF LONGITUDE OF FRANCE, OF THE ROYAL SOCIETIES OF LONDON AND GÖTTINGEN, OF THE ACADEMIES OF SCIENCES OF RUSSIA, DENMARR, SWEDEN, PRUSSIA, HOLLAND, AND ITALY; MEMBER OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES; ETC.

TRANSLATED, WITH A COMMENTARY,

BY

NATHANIEL BOWDITCH, LL. D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON, EDINBURGH, AND DUBLIN; OF THE ASTRONOMICAL SOCIETY OF LONDON, OF THE PHILOSOPHICAL SOCIETY HELD AT PHILADELPHIA, OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES; CORRESPONDING MEMBER OF THE ROYAL SOCIETIES OF BERLIN, PALERMO; ETC.

VOLUME IV.

WITH A MEMOIR OF THE TRANSLATOR,

BY HIS SON,

NATHANIEL INGERSOLL BOWDITCH.

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M DCCC XXXIX.

MEMOIR.

Mary Bowditch

THIS

TRANSLATION AND COMMENTARY

ARE DEDICATED, BY THE AUTHOR,

To the Memory of his Wife,

MARY BOWDITCH;

WHO DEVOTED HERSELF TO HER DOMESTIC AVOCATIONS WITH GREAT JUDGMENT, UNCEASING KINDNESS, AND A ZEAL WHICH COULD NOT BE SURPASSED;

TAKING UPON HERSELF THE WHOLE CARE OF HER FAMILY,

AND THUS PROCURING FOR HIM THE LEISURE HOURS TO PREPARE THE WORK;

AND SECURING TO HIM,

BY HER PRUDENT MANAGEMENT,

THE MEANS FOR ITS PUBLICATION IN ITS PRESENT FORM,

WHICH SHE FULLY APPROVED;

AND

WITHOUT HER APPROBATION THE WORK WOULD NOT HAVE BEEN UNDERTAKEN.

MEMOIR.

The children of the late Dr. Bowditch feel assured that every reader of this Translation and Commentary will be desirous to know something of the life and character of him who planned and executed so vast a work, and of her to whose memory it is dedicated. We have been blessed in our parents far beyond the lot of others. Such a father and such a mother are but rarely given by Heaven to any one. Both now sleep in the grave; and our kindred dust will soon be mingled with theirs in that last resting-place. But after the lapse of many years, this work, devoted to the elucidation of one of the most abstruse and profound subjects of human investigation, will still endure, a memorial of the genius of its author. Upon this monument we would inscribe our filial tribute — a record of the parents' virtues, of the children's gratitude and affection.

Nathaniel Bowditch was born at Salem, in Massachusetts, March 26, 1773, being the fourth of seven children of Habakkuk Bowditch, by his wife Mary, who was the daughter of Nathaniel Ingersoll. His ancestors had always resided in that place from its earliest settlement, having been, for the four last generations, ship-masters. Tradition has handed down the fact, that three

brothers, Jonathan, Joel, and William Bowditch, emigrated to this country from England, and, as is believed, from the city of Exeter, or its immediate vicinity. William became an inhabitant of Salem in 1639. His humble situation in life may be inferred from the title of "Goodman," by which he is mentioned, as distinguished from the more dignified appellation of "Mr." He was living in 1649, in which year he had a grant of thirty acres of land from the town. The time of his death is unknown.

He left an only child, of the same name, who was born in 1640, and died in 1681. He was collector of the port of Salem under the Colonial government, and owned a warehouse and other real estate, and several small vessels, but died insolvent. He likewise left an only child, also named William, who was born in September, 1663, and died May 28, 1728, in the sixty-fifth year of his age. He was actively engaged as a ship-master for many years, and was well known as an enterprising merchant. A dangerous rock in the channel of the harbor, of which the original Indian name was "Tenapoo," still bears the name of "Bowditch's Ledge," which was given to it in consequence of a vessel called the "Essex Galley," under his command, having struck upon it about the year 1700. He was for many years one of the selectmen for managing the affairs of his native town, and served also, during several sessions, as a representative in the General Court of the Province. He married, August 30, 1688, Mary, the daughter of Thomas Gardner, Esq., a wealthy merchant. She died in 1724, four years before her husband. He left an estate valued at between four and five thousand pounds. The grave-stones of both husband and wife are still to be seen in the burial-ground at Salem,

though the inscriptions are partially effaced. There were eleven children from this marriage, the seventh of whom was Ebenezer, who was born April 26, 1703, and died February 2, 1768, also in the sixty-fifth year of his age. He was the only son of this numerous household who left male descendants, and thus became the common ancestor of all who now bear the family name in Salem. He married, August 15, 1728, Mary, the daughter of the Hon. John Turner, one of the most distinguished citizens of Salem, long a member of the Provincial Council, and well known in the local history of that time. The annalist of Salem says of him, "His deserts were equal to his honors."*

Ebenezer Bowditch preserved through life an irreproachable character, and possessed in a high degree the confidence and respect of the community. He pursued his father's occupation, but, as it seems, without much success, since he left but little property at his death. His wife survived him till May 1, 1785, living in reduced circumstances, and being dependent upon her young grandson, the subject of this memoir, for many little attentions, by which her declining years were rendered more comfortable. They had six children, of whom the fifth was Habakkuk, born in 1738. He also was in early life a ship-master, and, as was the custom of that day, learned the trade of a cooper, a practical acquaintance with which was then deemed an important, though subordinate qualification for the discharge of the appropriate duties of his situation. At the commencement of the revolutionary war, he met with misfortunes in business,

* Annals of Salem, from its first Settlement; by Joseph B. Felt; p. 423.

by which his circumstances became very much reduced; and he was so disheartened, that he had not the energy to attempt to retrieve them. He subsequently worked at the trade of a cooper, which he had originally learned from the motive above stated. But he was able in this manner to earn only a scanty subsistence for his wife and children. Some idea may be formed of his poverty, from the circumstance that, for many successive years, he received fifteen or twenty dollars annually from the charity funds of the Salem Marine Society, of which he was a member, deriving from this sum, small as it was, quite an essential aid toward the support of his family. He was a man very remarkable for strong natural good sense, but had enjoyed no advantages of education. He was an attentive observer and an intelligent judge of men and events as they passed before him. He was extremely conversant with the Scriptures, and entertained liberal and enlightened views on the subjects of religion and revelation. He possessed a cheerful temper and an amiable disposition. But having yielded to the reverses which he had encountered, and which would but have stimulated others to greater efforts, he outlived those prospects of usefulness and happiness, which, upon his entrance into life, seemed to be within his reach. Upon one occasion, his son, alluding to the latter years of his father's life as having been less happy than his earlier ones, expressed the hope that he might not himself "live too long." Habakkuk Bowditch was married July 23, 1765, and died July 18th, 1798, at the age of sixty years.

Mary, the wife of Habakkuk Bowditch, had died December 16, 1783. She was an excellent woman, discharging all the duties

of life with exemplary fidelity. By her death Dr. Bowditch was deprived of his earliest and best friend, at the age of ten years. But there can be no doubt that she had lived long enough to exercise a most salutary influence over her son's mind, and that it is in a great degree to the teachings and example of this pious and affectionate mother, that we may trace the inflexible integrity, the unwavering love of truth, and the high moral principle, for which he was through life distinguished. From her he learned his first lesson as to the value of truth. While a child, playing behind his mother's chair, he had, unobserved by her, unrolled a ball of yarn, from which she was knitting, and involved it in inextricable confusion. When she discovered the mischief, and addressed him with some severity of manner, he denied having done it. She at once entered into a very serious conversation with him, and while she told him that the original matter of offence was but trifling, she explained to him so fully the meanness and criminality of falsehood, and urged him with so much earnestness never again to be guilty of it, that this lesson of his infancy became indelibly impressed upon his heart. He was always a favorite child. She was interested in the early development of his character and talents, and, it is said, was sometimes obliged even to restrain and check his fondness for study, as being excessive. She has been heard to say that he would be "something decided." To use his own expression, "she idolized him;" and he never spoke of her but in terms of the most respectful and affectionate remembrance. Her excellent influence extended to all her children. They grew up together, as one observed of them, "a loving household," remarkable

as a family for their excellent moral character and their strong mutual attachment.

Dr. Bowditch survived all his brothers and sisters for nearly thirty years. The eldest, Mary, born March 27, 1766, took upon herself, after her mother's death, the whole charge of the family, and almost supplied, towards the younger children, the place of that excellent parent whom they had lost. She married, April 20, 1791, Mr. David Martin, a ship-master, who died a few years afterwards. She herself died December 2, 1808, at the age of forty-two years, leaving to her brother's care an only child, who afterwards always resided in his family, repaying his kindness by the attentions of a daughter, and being to us as an elder sister. As such, she unites with us upon the present occasion.

The second child, Habakkuk, was born May 2, 1768, and was drowned in Boston about forty years ago. It is not known that he manifested any peculiar taste for the study of mathematics.

The third child, Elizabeth, was born May 16, 1771, and died December 9, 1791, in consequence of a fall down a flight of steps, having lingered in great agony a short time after the accident. Dr. Bowditch often mentioned, with much emotion, the circumstances of the death of this sister, who, being but two years older than himself, had always been the object of his peculiar regard and affection. In the midst of life and health, and with a countenance radiant with smiles and joy, she was about to leave her friends for a few moments, when a single misstep removed her from them forever.

The fifth child, William, was born May 5, 1776. He embraced a sea-faring life, and, while absent upon one of his voyages, died at Trinidad, in the autumn of 1799, at the age of twenty-three years. He was quite as remarkable for his mathematical talents as his elder brother, and, had he lived, might perhaps have been equally distinguished for the successful cultivation of this branch of science. In the first edition of the Practical Navigator he is mentioned as the author of one of the notes to Table XIV. Dr. Bowditch delighted to speak of the purity, and almost holiness of character of his brother William; and another, who knew well his early virtues, has said of him, "He was sanctified from his birth."

The sixth child, Samuel, was born September 13, 1778, and died April 5, 1794, at the age of sixteen years. He also possessed great quickness at mathematical calculations. But he pursued his studies with a waywardness and eccentricity which would probably have prevented his acquirements from being as great as might have resulted from a more regular and systematic cultivation of his naturally excellent talents.

The seventh child, Lois, was born March 29, 1781, married her cousin, Mr. Joseph Bowditch, September 28, 1806, and died without children, July 29, 1809, aged twenty-eight years. The eldest and youngest daughters only of this family were married, and they were also both on their death-beds at one time and place, prostrated by the same fatal disease, consumption.* Dr. Bowditch,

* One of them, at this time, presented to the eldest child of Dr. Bowditch a little silver

in his last illness, said that it had always been a source of pleasure to him to remember that these sisters, when dying, had each told him that he had been through life a good brother to them; "but," added he, "it gives me greater pleasure now than ever before." He also said that "they died with the calmness of two Stoics." He once mentioned that, in settling the estate of his deceased sister, the Judge of Probate thought that he had discovered a mistake in the fact that the estate had not been represented insolvent, although more money had been paid away than had been received. The matter was explained, however, by the statement of the administrator, that he should have felt himself disgraced by leaving undischarged the few small debts incurred by his sister, chiefly during her last illness, while he possessed any means of his own with which to pay them.

Such were the parents and such the brothers and sisters of Dr. Bowditch; and amid the domestic influences which have been described were the years of his infancy and childhood passed. Many amusing and interesting incidents of this period of his life might be mentioned. It was one to which he himself always recurred with pleasure, as having been very happy, notwithstanding its many privations. If he was obliged, from motives of economy, to wear the thin garments of summer when the near approach of winter made them less comfortable, he would reply to the laugh of his schoolmates or playfellows by charging them with effeminacy for preferring warmer clothing. If, as was often the case, he sat

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medal, bearing upon it their names, and the inscription, "Virtue and Religion lead to Happiness." Such had been the result of her own experience.

down to a dinner consisting chiefly of potatoes, he felt that a mealy potato was as good fare as he desired. He humorously described one occasion, upon which, when sent to buy a warm loaf of bread for breakfast, he found himself unable, on his way home, to resist the temptation of gradually eating out the soft part, so that, upon his arrival, the upper and under crusts had come in contact. Possessing health and activity of body, he engaged at one moment with earnestness and ardor in all the amusements of boyhood, and in the next returned with increased pleasure to his studies. Yielding sometimes to the impulse of the moment, as in the instance last cited, he committed trifling indiscretions, but nothing mean or vicious was ever developed in his character. Blessed with a joyous and happy temper, he contentedly accommodated himself to the necessities of his situation. The son of a poor mechanic, with no expectations from family or friends, he had within him an energy of purpose by which he was finally to surmount all obstacles.

While he was yet in his infancy, his father removed with the family from Salem to the adjoining village of Danvers, and resided there several years. The house which he occupied is still standing. It is a humble cottage. The main building, as seen in front, exhibits but one door and one window. It was here that his mother first showed him the slight crescent of the new moon, and the fuller orb of the harvest moon, and perhaps first awakened in his mind a curiosity to know more of the nature and laws of the planetary system. He here received instruction from a school-mistress, whose aged relatives still live in the immediate vicinity, and by whom it is distinctly remembered that he was

"a likely, clever, thoughtful boy;" that "his instructress took mightily to him;" that "he was the best scholar she ever had;" that "he learnt amazing fast, for his mind was fully given to it;" and that "he did not seem like other children; he seemed better."

Upon the return of the family to Salem, he was sent, at the age of seven years and three months, to the best school in the town, kept by a Mr. Watson. The character of this "seminary of learning" may perhaps be better realized from the following circumstances than from any more general or elaborate description of it. There was but one dictionary in the school; and a gentleman, who was a fellow-pupil with Dr. Bowditch, never saw one while he was there. Each day, the scholars were called upon to spell aloud, all together, in chorus, the word *honorificabilitudinity;** spelling and pronouncing the first syllable, then the two first, three first, &c., which process, applied to the whole word, of course occupied several minutes. He early showed a great fondness for mathematics; but, on account of his extreme youth, his master, it is said, refused to permit him to enter on that branch of study until he had obtained and produced from his father a special request to that effect. He, upon one occasion, solved a problem in arithmetic, which the instructer thought must be above his comprehension, and therefore charged him with having procured the assistance of some older scholar, giving him a severe reprimand for the attempt to deceive him. The timely interference and explanations, indeed, of his

* This word, meaning *honor*, may be found in Bailey's English Dictionary; and Shakspeare uses *honorificabilitudinitatibus*, in Love's Labor Lost, Act 5, Scene 1.

eldest brother, saved him from any actual chastisement; but this indignity and act of injustice Dr. Bowditch never forgot.

But even the slight elementary instruction which he might have obtained at this school, he was obliged to forego altogether at the age of ten years and two months, when he was taken by his father into his cooper's shop, that he might by his labor assist in the support of the family. After remaining here a short time, he entered as a clerk or apprentice into the ship-chandlery shop of Messrs. Ropes and Hodges, when he was about twelve years of age. In this shop he remained till his employers retired from business, at which time, as early as 1790, he entered the similar shop of Mr. Samuel Curwin Ward, where he remained until he sailed on his first voyage, in 1795. Here, when not engaged in serving customers, he spent his time in reading, and particularly in the study of mathematics, for which he then felt a confirmed and decided taste. Upon one occasion, a visiter entered the shop, and, looking at the two clerks, one of whom was asleep behind the counter, and the other diligently occupied with his slate and pencil, smiled and said, "Hogarth's apprentices!" Another visiter observed that, if he kept on ciphering so, he had not any doubt that, in time, he would become *an almanac maker!* And in fact, in the year 1788, he computed an almanac for the year 1790, which is still preserved in his library, being one of the most curious, if not most valuable, manuscripts in that collection. It is also stated that he occasionally tried his dexterity at philosophical experiments; one instance mentioned being that, while in the shop of Ropes and Hodges, he constructed quite a curious barometer. There is now in his library, also, among

other similar articles, a very neat wooden sun-dial, which he made in the year 1792.

These pursuits were, however, only the amusement of his leisure hours. He never allowed them to interfere with the discharge of his duty towards his employers. Upon one occasion, indeed, a customer called and purchased a pair of hinges, at a time when the young clerk was deeply engaged in solving some problem in mathematics, which he thought he would finish before charging the delivery of them upon the books, and when the problem was solved, he forgot the matter altogether. In a few days, the customer called again to pay for them, when Mr. Hodges himself was in the shop. The books were examined, and gave no account of this purchase. The clerk, upon being applied to, at once recollected the circumstance, and the reason of his own forgetfulness. From that day, he made it an invariable rule to finish every matter of business which he began, before undertaking any thing else. Upon his recommendation, given quite late in life, one of his sons adopted as a motto for a seal, "End what you begin." He has himself more than once said, that he never forgot the hinges.

Having once heard, in 1787, from his brother William, a vague account of a method of working out problems by *letters*, instead of *figures*, he succeeded in borrowing the book which contained it, and was so much interested and excited by his first glance at algebra, that he could not get the least sleep during the whole of the next night. An old British sailor, residing at Salem on half pay, and who ended his days as an inmate of the Greenwich

Hospital, taught him the elements of navigation; and when they last met, as he was about to embark for Europe, he patted him on the head, saying, "My boy, you have a taste for these things: keep on studying, and you will be a great man yet:" — an approval which greatly stimulated and encouraged him. He rose each day at the earliest dawn, and devoted his morning hours to study. He has often been heard to say, that the time which he thus gained from sleep, gave him, substantially, all his mathematics. He passed the long winter evenings, too, by the kitchen fireside of his employer, Mr. Hodges, — which his diffidence, as well as the security it offered him from interruption, led him to prefer to the parlor, — quietly engaged in his favorite pursuit; and occasionally, it is said, also rocking, at the same time, the infant's cradle, at the request of the attendant, who wished to be doing something else.

It happened that Mr. Hodges and another gentleman owned together in moieties a very irregularly-shaped field in Salem, and wished to divide it. Accordingly, the young apprentice undertook to make the proposed survey and division, and completed the task with the most minute accuracy. The co-tenant, however, refused to abide by this survey, since he thought that, as it was made by one who was in the employment of Mr. Hodges, it was probable that there had been an unfair bias in his favor. A regular surveyor was then employed, and Dr. Bowditch, who was very indignant at the suspicions entertained in regard to his own result, said that he could not help feeling a malicious pleasure when he found that the gentleman alluded to received for his half part several square feet less than he was entitled to. In 1794, he was employed by the town to assist

Captain John Gibaut in making a survey of Salem, which labor he accordingly performed; and the exact area of the town, as ascertained by this survey, was computed by him.

Being very fond of books, and having no guide in the selection of them, his reading in early life was of the most miscellaneous character. Thus he read through the whole of Chambers's Encyclopædia, in four folio volumes, without omitting an article; and, as his memory, except as to persons and names, was wonderfully retentive, he in this manner acquired a fund of the most varied information. His intimate friends have often been surprised at finding him quite conversant with subjects apparently the most foreign from his favorite studies; and one of the most profound scholars among them observed, that he could hardly form an adequate estimate of the extent of his general attainments. He was an ardent admirer of Shakspeare, whose most beautiful passages were treasured up in his memory from earliest youth. His familiarity with the Old and New Testament was very great, surpassing that of many professed theologians. The family Bible, which he first read, is still preserved, having in it a curious map of the wanderings of the Israelites, and various engravings calculated to awaken that interest in the young reader, which every subsequent perusal in manhood and old age has a tendency to strengthen and confirm. Through the kindness of his friends, and especially of Dr. Prince and Dr. Bentley, both Unitarian clergymen of Salem, the former of whom was distinguished for his fondness for natural philosophy, he obtained the use of books which would otherwise have been unattainable by him. It happened that, in his youth, the extensive scientific library of the late Dr.

Richard Kirwan was captured in the British Channel by a privateer fitted out from Beverly, the next town to Salem. The enlightened and liberal owners of the vessel permitted the library thus captured to be sold at a very low rate to an association of gentlemen in Salem, and it became the basis of the present Salem Athenæum.* From this extremely valuable library, which was a better one than then existed in any part of the United States, except at Philadelphia, he obtained leave freely to take out books, and to consult and study them at pleasure. Among its treasures were the Transactions of the Royal Society of London. All the mathematical papers in these Transactions, and many scientific works, were wholly or partially transcribed by him, and are still preserved in his library, contained in more than twenty folio and quarto common-place books and other volumes. And this immense labor he was obliged to undergo chiefly from his inability to purchase the books in question, — which he wished to have by him permanently, for the purpose of convenient reference, — and partly, perhaps, from his desire to impress their contents more strongly upon his memory than could be done by a mere perusal. The title-page of one of these volumes states that it contains, with the next volume, "A complete Collection of all the Mathematical Papers in the Philosophical Transactions; Extracts from various Encyclopædias, from the Memoirs of the Paris

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* It is an interesting fact, that, many years afterwards, "an offer of remuneration was made to Dr. Kirwan, who respectfully declined it, expressing his satisfaction that his valuable library had found so useful a destination." — *An Eulogy on the Life and Character of Nathaniel Bowditch, LL. D., F. R. S., delivered at the Request of the Corporation of the City of Salem, May* 24, 1838; *by Daniel Appleton White;* p. 43.

Academy; a complete Copy of Emerson's Mechanics; a Copy of Hamilton's Conics; Extracts from Gravesande's and Martyn's Philosophical Treatise, from Bernoulli, &c. &c." He always read with close attention, and endeavored to ascertain the exact meaning of every word about which he was doubtful. This led him, in after life, to collect around him dictionaries, which he constantly consulted. He had more than one hundred in his library.

He began to learn Latin, January 4, 1790, without an instructer, that he might read Newton's Principia, which he had before attempted to understand by means only of his knowledge of mathematical subjects and the various equations and diagrams which it contained. He now read a copy of Euclid, which had been given him by his brother-in-law, Mr. Martin, and which was once the property of Dr. Mather Byles, a clergyman of Boston, distinguished for his humor and eccentricity. The book still retains his original pencil marks, recording the meaning of the simplest Latin words — "*tamen*, nevertheless; *rursus*, again;" &c. He had previously read Euclid in English, and the letters "Q. E. D.," which had remained an impenetrable mystery to him on his previous attempt to read the Principia, were now explained by the "*quod erat demonstrandum*" which he here discovered; but he was for a long time perplexed by the words "*mutatis mutandis*," and unable to conjecture what particular change they indicated. He had received from Dr. Bentley a copy of the Principia, which had formerly been presented by that gentleman to a young friend, who kindly consented to relinquish his prior claims; and this work he at last mastered, as he had done Euclid

before. The Hon. Nathan Reed, then an apothecary in Salem, afterwards a member of Congress, being himself fond of scientific pursuits, was attracted by this love of science manifested by Dr. Bowditch, and formed an intimate acquaintance with him. In his shop was an assistant who was a schoolmate and friend of Dr. Bowditch, and here their Sunday evenings were often passed together. Mr. Reed states as a fact, that Dr. Bowditch, while in Mr. Ward's employment, actually translated Newton's Principia into English. No such translation is, however, now to be found among his papers; though translations of parts of it, indeed, are contained in the manuscript volumes before mentioned. In a similar manner, and from the same motive, he acquired the elements of the French language; to perfect himself in which, he took lessons during sixteen months, from a foreigner then in Salem, whom, in return, he instructed in English. At first, he declined learning the pronunciation, as a matter which could not be of any use to him; but at last, the foreigner was so shocked at hearing him read the words *parlez vous*, &c., as if these had been English, that he almost insisted upon instructing him in the true pronunciation, telling him that it might be of importance in the business of life. And in fact, he had scarcely learned it, before his first voyage was decided upon, and to a French port, where he was thus enabled to act as a successful interpreter.

Excepting a few lessons which he took in book-keeping from Mr. Michael Walsh, it is believed that he received no other regular instruction after leaving school. But it has been stated that Drs. Bentley and Prince rarely passed his employer's shop, without stopping to converse with him; and thus, perhaps,

by the interest he had awakened in their minds, he had secured to himself the gratuitous and invaluable assistance of the two ablest instructers whom the town then contained. The world, indeed, was his school, and Nature herself his best instructer. She offers her lessons to all, though many overlook or disregard her teachings. But his was one of those powerful intellects which only at intervals appear among men: it was stimulated and aroused to action by that sternest though best of monitors, necessity; and it mastered every thing within its reach. Dr. Bowditch never considered that the obstacles in his path had the slightest tendency to retard his progress. On the contrary, he felt that they afforded him a foothold by which that progress was rendered more sure and steady. Much as he valued all the "means and appliances" of learning, — and he did value them beyond all price, — he thought it a great disadvantage to any one to be born and educated in the midst of ease and luxury, even though surrounded with every facility for mental cultivation; since, to such a one, the needful stimulus or inducement to use the means within his reach would be almost surely wanting. He often mentioned with approbation, as containing much truth, the remark of a distinguished French mathematician to a young pupil, whose ready and intelligent answers had awakened his interest, and who, in reply to the question of his instructer, had told him his parentage and situation in life, — "Ah! I am sorry. You are too rich. You must give up mathematics." One remarkable exception, indeed, to this rule, Dr. Bowditch readily admitted in the instance of him whose genius reflected as bright a lustre on the noble house of Cavendish as had been received from it.*

* "En sorte qu'il n'y a nulle témérité à présager qu'il fera rejailler sur sa maison autant de lustre qu'il en a reçu d'elle." — *Cuvier's Eulogy on Cavendish, before the Institute of France.*

In the manuscript volumes before mentioned are often contained the precise dates at which he was studying and recording the mathematical papers there collected, and occasionally they contain mottoes or sentiments upon other subjects. Thus the title-page of one of them, under the date of December 13, 1794, has the well-known quotation, "Nullius addictus jurare in verba magistri." A minute analysis, indeed, of these volumes, in a more extended biography, might perhaps enable the reader to trace, step by step, the mental progress of Dr. Bowditch. It is only necessary, however, here to state, that he who, at the age of twenty-one years, had read the immortal work of Newton, was, even then, unsurpassed, and probably unequalled, in mathematical attainments by any one in the commonwealth. Those habits were then formed which were to render him as eminent among men of business, as, by his talents and acquirements, he was to become eminent among men of science. And his character, also, then exhibited all those beautiful and harmonious elements which it ever afterwards retained. That deep religious principle, which sustained and cheered him in the last hours of his life, had guided his boyhood, and was now the familiar and inseparable companion of his mature years; and already were displayed those various social and personal virtues, which were to render him a moral exemplar to the community in which he lived.

Dr. Bowditch began life with the same pursuits which his ancestors had followed for so many generations. Between the years 1795 and 1804, he made five voyages, — performing the first in the capacity of clerk, and the three next in that of supercargo, —

all under the command of Captain Henry Prince, of Salem. On his fifth and last voyage, he acted as both master and supercargo. He sailed upon the first of these voyages, January 11, 1795, in the ship Henry, bound to the Isle of Bourbon, and was absent exactly one year. His three next voyages were in the ship Astrea, which sailed, in 1796, for Lisbon, Madeira, and Manilla, and arrived at Salem in May, 1797; and again in August, 1798, sailed for Cadiz, thence to the Mediterranean, loaded at Alicant, and arrived at Salem in April, 1799; and in July, 1799, sailed from Boston to Batavia and Manilla, and returned in September, 1800; — and his fifth voyage was in the Putnam, which sailed from Beverly, November 21, 1802, bound for Sumatra, and arrived at Salem December 25, 1803.

He has related that, upon the first of these voyages, he carried out, as an adventure, a small box of shoes, which article proved on his arrival at the Isle of Bourbon to be in great demand. He sold them for about three times the first cost, and having made an advantageous investment of the proceeds, he returned home quite elated, and feeling that the fickle goddess had smiled upon him more propitiously than she ever had done upon any mortal before.

Of his second voyage, Captain Prince relates, that one day, when dining at the table of the American consul at Madeira, "his supercargo laid down his knife and fork, and, after squeezing the tips of his fingers for two minutes," gave to the lady of the house an answer to an intricate question which she had proposed; to the

great astonishment of her clerk, who, after a long calculation, had succeeded in solving it, and "who exclaimed that he did not believe there was another man on the island who could have done it in two hours."

During his third voyage, on the passage from Cadiz to Alicant, they were chased by a French privateer; but, being well armed and manned, they determined on resistance. The duty assigned to Dr. Bowditch was that of handing up the powder upon deck. And in the midst of the preparations, the captain looked into the cabin, where he was no less surprised than amused at finding his supercargo quietly seated by his keg of powder, and busily occupied, as usual, with his slate and pencil. He said to him, "I suppose you could now make your will," to which he smilingly assented. He did in fact give to his eldest son his instructions in regard to his last will, with the like calmness and composure, when there was not only an apparent danger, but an absolute certainty, of the near approach of death.

Upon his arrival at Manilla, during his fourth voyage, the captain, being asked how he contrived to find his way, in the face of a north-east monsoon, by mere dead-reckoning, replied, "that he had a crew of twelve men, every one of whom could take and work a lunar observation as well, for all practical purposes, as Sir Isaac Newton himself, were he alive." During this conversation, Dr. Bowditch sat "as modest as a maid, saying not a word, but holding his slate pencil in his mouth;" while another person remarked, that "there was more knowledge

of navigation on board that ship than there ever was in all the vessels that have floated in Manilla Bay." *

In his last voyage, Dr. Bowditch arrived off the coast in mid-winter, and in the height of a violent north-east snow-storm. He had been unable to get an observation for a day or two, and felt very anxious and uneasy at the dangerous situation of the vessel. At the close of the afternoon of December 25, he came on deck, and took the whole management of the ship into his own hands. Feeling very confident where the vessel was, he kept his eyes directed towards the light on Baker's Island, at the entrance of Salem harbor. Fortunately, in the interval between two gusts of wind, the fall of snow became less dense than before, and he thus obtained a glimpse of the light of which he was in search. It was seen by but one other person, and in the next instant all was again impenetrable darkness. Confirmed, however, in his previous convictions, he now kept on the same course, entered the harbor, and finally anchored in safety.† He immediately went on shore, and the owners were very much alarmed at his sudden appearance on such a tempestuous night, and at first could hardly be persuaded that he had not been wrecked. And cordial indeed was the welcome

* An interesting incidental notice of Dr. Bowditch, in the case of a black cook who could work lunar observations, may be found in Zach's *Correspondance Astronomique*, Vol. IV. p. 62.

† Upon this occasion, he had given his orders with the same decision and preciseness as if he saw all the objects around, and thus inspired the sailors with the confidence which he felt himself. One of them, who was twenty years older than his captain, exclaimed, "Our old man goes ahead as if it was noon-day!"

which he received from one who had been listening to the warfare of the elements with all the solicitude of a sailor's wife.

In his transactions with custom-house officers upon the continent of Europe, he found that they almost universally required a fee, not less for the performance of duty than for a violation of it; and several amusing instances might be mentioned as illustrating his own experience in this matter. Indeed, all his subsequent observation convinced him that there is hardly a labor or duty in life that is not rendered more light and easy by gratuitous compensation; and therefore it was always his rule, not only during these voyages, but through life, to make it *for the interest* of those about him to be upon the alert in attending to his wishes, or complying with his requests; though never did he attempt by this means to persuade any one to what he considered, in the slightest degree, a violation of duty, or breach of trust.

During these voyages, he perfected himself in the French language, and acquired a knowledge of the Italian, Portuguese, and Spanish, especially of the latter language. Thus he read through the whole of the voluminous Spanish History of Mariana, during one of these voyages; and many interesting facts there stated respecting Cardinal Ximenes and the Great Captain, &c., he distinctly remembered in his recent perusal of Prescott's Ferdinand and Isabella, — the last work which he read before his death. An interpreter, with whom he was transacting business, and whose piety consisted in the external observances of a good Catholic, cautioned him against reading so many books, lest some of them should bring him into the hands of the

Inquisition. It is worthy of remark, that it was to this acquisition of the Spanish language, and the consequent opportunity afforded him of conferring an obligation upon an active merchant in Salem, by gratuitously translating for him a Spanish protest, that Dr. Bowditch always attributed his appointment to the situation which he, a few years afterwards, obtained against a powerful competitor, and for which he was much indebted to the influence and friendship of the merchant alluded to. In view of this circumstance, and that before mentioned respecting his knowledge of French pronunciation, with other incidents of a similar character, he used to say that nothing which he ever learned came amiss. It may here be mentioned that, as late in life as the age of forty-five, he learned the German language thoroughly,* and acquired, at about the same time, a slight knowledge of Dutch. A manuscript in his library contains probably ten thousand German words and English meanings, which he had transcribed that he might better remember them. He delighted to trace analogies between different languages, and especially to discover resemblances of foreign words to those of his mother tongue, of which many striking examples were detected and mentioned by him — the *Handschuh* of the German, meaning *glove;* and the verse in the Dutch New Testament in which the stoning of

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* In a letter of Dr. Bowditch to Baron Zach, dated November 22, 1822, published in Zach's *Correspondance Astronomique*, Vol. X. p. 224, he states that he had, three years previously, purchased several of the most important works of the German mathematicians, and among others Zach's *Monatliche Correspondenz*, in twenty-eight volumes, — and adds, "With this work I began to learn German, and have been amply rewarded for the labor." His own experience led him to say, that this language could be acquired, in a degree sufficient for reading all mathematical works, by studying two hours each day for four months.

Stephen is described, and where it is added that the apostles made "eenen grooten rowe over hem," &c. The serious attempt to prove that *jour* was derived from *dies*, he thought not so absurd as he might have done, had not a Spanish boy who once shipped with him, having the Christian name of *Benito*, been in the next voyage entered upon the books by the good American cognomen of *Ben* or *Benjamin Eaton*. He was often amused at discovering in the dictionary of some foreign language, a definition expressing more clearly than elegantly the precise signification of a word.* He also acquired some knowledge of Greek, but how early in life is not known. He always began to learn a language by taking the New Testament and dictionary, and attempting immediately to translate. Thus he left in his library the New Testament in more than twenty-five different dialects or languages.

But the long intervals of leisure which a sailor's life afforded, he chiefly devoted to his favorite study, pursuing with unremitting zeal those researches in which he had already made such progress, notwithstanding the interruptions and embarrassments of his earlier days. Here, with only the sea around him, and the sky above him, protected alike from all the intruding cares and engrossing pleasures of life, he especially delighted to hold converse with the master-spirits who had attempted to explain the mysteries of the visible universe, and the laws by which the great energies of nature are guided and controlled. M. Lacroix mentioned to one of the sons of Dr. Bowditch, that from him he had received several corrections and notices of errata in his works,

* See Ebers's German Dictionary, *passim*.

which our navigator had discovered during these long India voyages. And in the ship in which he sailed were witnessed not merely the labors and vigils of the solitary student, but the teachings of the kind and generous instructer, anxious and eager to impart to others the knowledge which he had himself acquired. "He loved study himself," says Captain Prince, "and he loved to see others study. He was always fond of teaching others. He would do any thing if any one would show a disposition to learn. Hence," he adds, "all was harmony on board; all had a zeal for study; all were ambitious to learn." On one occasion, two sailors were zealously disputing, in the hearing of the captain and supercargo, respecting sines and cosines. The result of his teaching, in enabling the whole crew of twelve men to work a lunar observation, has been before stated. Every one of those twelve sailors subsequently attained, at least, the rank of first or second officer of a ship. It was a circumstance highly in favor of a seaman, that he had sailed with Dr. Bowditch, and was often sufficient to secure his promotion. Connected with much testimony of this sort, is that of the uniform affability and kindness of manner displayed by Dr. Bowditch in his intercourse with all on board, which were especially calculated to increase the self-respect of the sailor, and inspire him with a due sense of his own powers, and of the importance of his occupation. In a letter from an officer in the United States navy, who sailed twice in the Astrea with Dr. Bowditch, at first as a cabin boy, and who died a few months after the friend of whom he speaks, the writer states some of the above particulars respecting Dr. Bowditch, and adds that "his kindness and attention to the poor sea-sick

cabin boy are to this hour uppermost in my memory, and will be so when his logarithms and lunar observations are remembered no more."*

It is unnecessary to state, that Dr. Bowditch discharged his duty toward his employers with the utmost fidelity and exactness. His voyages were conducted with uniform skill and success, and to their entire satisfaction. It is said by Captain Prince, that Dr. Bowditch, though he had such a thorough knowledge of navigation, knew but little of what is called *seamanship;* that he never went to see a launch in his life, &c. It is without doubt true, that the mere detail of seamanship was always irksome to him. He has often told his children that, upon common occasions, he left the management of the ship to his first officer; but upon any emergency, he was not only ready and desirous, but, as is believed, perfectly competent, to perform all the duties which could, on such occasions, be required of an experienced and practical seaman.

The following is the account of his habits when at sea, given by one who was his companion during several voyages. "His practice was to rise at a very early hour in the morning, and pursue his studies till breakfast, immediately after which he walked rapidly for about half an hour, and then went below to his studies till half past eleven o'clock, when he returned and walked till the hour at which he commenced his meridian observations. Then came the dinner, after which he was

* Charles F. Waldo, Esq., died August 31, 1838.

engaged in his studies till five o'clock; then he walked till tea time, and after tea was at his studies till nine in the evening. From this hour till half past ten, he appeared to have banished all thoughts of study, and, while walking at his usual quick pace, he would converse in the most lively manner, giving us useful information, intermixed with amusing anecdotes and an occasional hearty laugh. He thus made the time delightful to the officers who walked with him. Whenever the heavenly bodies were in distance to get the longitude, night or day, he was sure to make his observations once, and frequently twice, in every twenty-four hours, always preferring to make them by the moon and stars, as less fatiguing to his eyes. He was often seen on deck at other times, walking, apparently in deep thought, when it was well understood by all on board that he was not to be disturbed, as we supposed he was solving some difficult problem; and when he darted below, the conclusion was that he had got the idea. If he were in the fore part of the ship when the idea came to him, he would actually *run* to the cabin, and his countenance would give the expression that he had found a prize."* Another correspondent states that sometimes, when he wished to pursue his studies without disturbing those in the cabin by introducing a candle or lamp, he has seen him standing in the companion-way with his slate and pencil, working out some problem, at eleven o'clock at night, by the aid only of the binacle lamp.†

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* Judge White's Eulogy, p. 27.

† Another companion of his voyages says, "He never manifested any moral failings whatever, but was always remarkable for his strict principles of conduct, and for the utmost

Such was Dr. Bowditch's seafaring life, — not wasted in ennui or idle reveries, but every moment of it devoted to the improvement alike of his own mind and character, and those of every individual in the little world around him. Already, too, as might have been expected, he was beginning to win the honors of science; and domestic life, from which the sailor is almost wholly debarred, was preparing for him its sweetest home.

From our venerable University at Cambridge he received the highest encouragement to pursue the career upon which he had entered. In July, 1802, when his ship, the Astrea, was wind-bound in Boston, he went to hear the performances at the annual commencement of the College; and among the honorary degrees conferred, he thought he heard his own name announced as Master of Arts; but it was not until congratulated by a townsman and friend, that he became satisfied that his senses had not deceived him. He always spoke of this as one of the proudest days of his life; and amid all the subsequent proofs which he received of the respect and esteem of his fellow-citizens, and the distinctions conferred upon him from foreign countries, he recurred to this with the greatest pleasure. It is, indeed, made the subject of express mention in his will. It was

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purity of mind and character; detesting any thing of an opposite nature, even in word. His feelings, indeed, were quick, and sometimes, though rarely, he was thought to give a too quick utterance to them; but the excitement passed off in a moment." Another says, "I have known Dr. B. intimately for more than fifty years, and I know no faults. This may seem strange; for most of your great men, when you look at them closely, have something to bring them down; but he had nothing. I suppose all Europe would not have tempted him to swerve a hair's breadth from what he thought right." — *Judge White's Eulogy*, p. 56.

gratefully repaid by the services of a long life. For the last twelve years, he was one of the select body of seven individuals intrusted with the immediate management and control of the College, having for many years before been a member of the more numerous body of Overseers, who have the general and more remote supervision of its affairs. Upon his decease, his associates in the Corporation of Harvard College state, "that he so acquired the confidence of his contemporaries, as to be regarded as the pillar and the pride of every society of which he was an active member, the effects of which never failed to be seen and acknowledged by its prosperity and success;" and they proceed to admit the benefit which that institution "has derived from the extraordinary endowments he possessed, and by which, in the exercise of his characteristic zeal, intelligence, and faithfulness, he ever sustained and advanced all its interests."

On the 28th day of May, 1799, he was chosen a member of the American Academy of Arts and Sciences. Some of the most valuable and interesting papers in its Transactions were the subsequent contributions of his pen; and the presidency of this society, to which he was elected in May, 1829, in the place of John Quincy Adams, late President of the United States, is one of the highest honors which Science offers to her votaries upon this side of the Atlantic. A letter received from the officers of the Academy bears a like honorable testimony to the merits and services of their deceased associate and President: — "It is the common fate of mankind to die, and be forgotten. It is the privilege of the just and good to be associated in the

remembrance with tender and grateful recollections. It is the destiny of minds gifted above the common lot, and acting beyond the common sphere, to involve in general regret the communities that have known their worth. It is thus that, on the present occasion, our sincere and general regret is necessarily mingled with the sadness of domestic affliction."

On March 25, 1798, Dr. Bowditch married Elizabeth, daughter of Francis Boardman, Esq., who is said to have been a lady of remarkable intelligence, and worthy of his choice. After a few months spent in her society, he went upon his third voyage, and upon his return found his home desolate. She whom he first selected as his companion, was not to be the mother of his children. His wife had died, October 18, 1798, aged only eighteen years. Dr. Bowditch felt that an alliance so abruptly terminated, did not justly entitle him to retain to his own use the property of which he thereby became legally possessed; and accordingly, he surrendered to the relatives of his late wife, every thing which he had thus acquired, including even certain small articles of plate, &c., which, but for the general character of the motive which influenced him, he would have gladly retained. Upon his second daughter and youngest child, he, many years afterwards, bestowed the name which had been borne by the wife of his youth.

On October 28, 1800, he married his cousin Mary, the only daughter of his uncle, Jonathan Ingersoll, Esq. She was then scarcely nineteen years of age, having been born December 4, 1781. Her father had been an active ship-master, and

was then living upon his estate in Danvers, from which, more than twenty-five years ago, he removed to Windsor, Vermont, where he still cultivates a farm upon the beautiful banks of the Connecticut, finding in the cares and labors of husbandry, at the advanced age of eighty-eight years, a pleasure greater than he ever experienced amid the more stirring scenes of his youth. Long may his honorable and peaceful life be preserved! This marriage, which lasted more than thirty-three years, may be regarded as the most happy circumstance of Dr. Bowditch's life. With personal attractions of no common order, domestic in her habits, of the most lively and cheerful disposition, with affections which age never chilled, governed ever by the strictest religious principle, the wife and mother was devotedly attached to her husband and children, sympathizing in the pursuits of the former, and guiding and directing those of the latter, making home the scene of the purest and most delightful influences and recollections, and associating with her presence in life and her memory in death, the idea of a being whose every act and thought were blameless. The stranger was attracted by her winning smile and affable manners. She made her house the agreeable resort of friends and visiters. Many sons and daughters of sorrow acknowledged in her that active benevolence and liberal aid which discovered and supplied their wants, or that kindly sympathy which soothed where it could not relieve. But there was one who was her heart's idol, whom she reverenced almost as a being of a higher order than herself, regarding as worthless every thing else, in comparison with his approving smile. He, indeed, had reason always to rejoice, that a benignant Providence had made her the sharer and the guardian

of his home and his happiness. That bosom, where his head had reposed in life, with undoubting faith and trust, was the fittest pillow upon which it could be placed for its final rest in death!

The most important result of this period of Dr. Bowditch's life, was the publication of *The New American Practical Navigator*, a manual in which were imbodied a scientific explanation of the principles of navigation, and also the practical application of these principles in the simplest and most effective manner.* Dr. Bowditch had prepared for publication two editions of the treatise of John Hamilton Moore, with notes and corrections, and in preparing a third revised edition of that work, he corrected so very many errors, that, in 1802, he was induced to publish it under his own name. From that time to the present, it has been exclusively used by every ship-master who has sailed from this country, and its tables and rules have been adopted in the works used in England and elsewhere. Into the original work, and the eight succeeding editions, many improvements, of great practical utility, have from time to time been introduced. Thus in the last edition, published in 1837, "the body of tables has been increased from thirty-three to

* This work is mentioned by Zach, in his *Correspondance Astron.*, Vol. VI. p. 206, A. D. 1822, who gives the entire title-page of the third edition, printed at Newburyport; and in Vol. VII. p. 167, is an example taken from it. In Vol. X. p. 234, A. D. 1824, we find the title-page of the fifth edition, printed in 1821. And many other notices occur in the same journal, of astronomical methods given by Dr. Bowditch in this work, which, as they were also the subjects of particular communications made by him to the American Academy, will be noticed hereafter.

fifty-six, some being entirely new, and others essentially improved or corrected."* The successive additions thus made by the author, have prevented the competition of any other work. More than eight thousand errors were corrected by Dr. Bowditch in his first edition of it under his own name; and when it is considered that one of these was no less than the very criminal inattention of setting down the year 1800 as a leap year, in the tables of the sun's declination,† thereby making a mistake in some of the numbers of twenty-three miles, and causing the actual destruction of several ships, and the imminent danger of others, some idea may be formed of the great service thus rendered by him to the cause of nautical science.‡ The amount of labor requisite for insuring accuracy in the tables, by actually going through all the calculations necessary to a complete examination of them, was immense almost beyond conception. The following striking contrast is presented by the modest Preface of the American editor, and the boasting language of the original compiler. The one says, that "the author had once flattered himself that the tables of this collection which did not depend on observations, would be absolutely correct; but

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* Eulogy on Nathaniel Bowditch, LL. D., President of the American Academy of Arts and Sciences; including an Analysis of his scientific Publications; delivered before the Academy, May 29, 1838; by John Pickering, Corresponding Secretary of the Academy; p. 13.

† See Preface to the last edition of the work.

‡ For many interesting details respecting this work, see "A Discourse on the Life and Character of the Hon. Nathaniel Bowditch, LL. D., F. R. S., delivered in the Church on Church Green, [Boston,] March 25, 1838; by Alexander Young;" pp. 34 to 39;—and Mr. Pickering's Eulogy, p. 10, &c., and Notes, pp. 84 to 87.

in the course of his calculations, he has accidentally discovered several errors in two of the most correct works of the kind extant, viz., Taylor's and Hutton's Logarithms, notwithstanding the great care taken by those able mathematicians in examining and correcting them. He therefore does not absolutely assert that these tables are entirely correct, but feels conscious that no pains have been spared to make them so." The other says, that "he sells no *sea-books*, charts, or instruments, but such as may be depended on; consequently he excludes all those *old, inaccurate, and erroneous publications*, the depending upon which *has often proved fatal to shipping and seamen*." * The following is the summary elsewhere given of this work: It "has been pronounced by competent judges to be, in point of practical utility, second to no work of man ever published. This apparently extravagant estimate of its importance, appears but just, when we consider the countless millions of treasures and of human lives which it has conducted, and will conduct, in safety through the perils of the ocean. But it is not only the best guide of the mariner in traversing the ocean; it is also his best instructer and companion every where, containing within itself a complete scientific library, for his study and improvement in his profession. Such a work was as worthy of the author's mind, as it is illustrative of his character; — unostentatious, yet profoundly scientific and thoroughly practical, with an effective power and influence of incalculable value." † So, also, the London Athenæum says of this work, "It goes, both in American and British craft, over every sea of the globe, and is probably the best work of the sort ever published."

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* Mr. Pickering's Eulogy, p. 11. † Judge White's Eulogy, p. 29.

Dr. Bowditch, however, did not himself consider this work as one which would much advance his scientific reputation. It was, in his view, only a "practical manual."* But it was the work by which, almost exclusively, he was, for a long time, known in this country, and it laid the basis of a wide-spread popularity, such as few, if any, works upon scientific subjects have ever gained for their authors. Several years ago, he was much amused by the following incident. Two young men came into the shop of his bookseller to purchase a copy of the Navigator. Upon being shown one bearing on its title-page the number of the edition, and purporting to have been revised and corrected by the author, one said to the other, "That is all a mere cheat; the old fellow must have been dead years ago!" They were astonished, and perhaps a little embarrassed, at being introduced to an active, sprightly gentleman, in full health and good spirits, as the author of this work, which they had known from their earliest entrance upon a sailor's life. It was in honor, especially, of the memory of him who had written the Practical Navigator, that,

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* Dr. Bowditch, in his letter to Baron Zach, *Corr. Ast.* Vol. X. p. 225, says, "You will see that I have studiously avoided all *scientific parade*, and have published the work according to the method of instruction used in our country, where we prefer, in these matters, practice to theory." Thus, owing to the errors incident to all nautical observations, he deemed it useless to aim, in the nautical tables, at the most minute degree of exactness, but only at that measure of it which was requisite for practical purposes; so that these tables might be used with the greatest possible promptness, and might, at the same time, lead to the greatest accuracy of result which was in fact really attainable. And he, in this letter, states that he is delighted to find that Zach, in a previous publication, concurs in this opinion. Upon this the editor remarks, p. 244, "mais c'est à nous de nous féliciter de nous trouver d'accord avec un navigateur d'une si grande expérience; la nôtre n'est qu'une induction."

when the news of his death was received at Cronstadt, all the American shipping, and many of the English and Russian vessels, hoisted their flags at half-mast in that naval depôt of the Czars, — a tribute of respect which had been previously paid in the ports of Baltimore, Boston, and Salem. From the same motive, the badge of mourning was adopted by the members of the Naval School of the United States, as for the loss of a valued friend and instructer.

Immediately upon the close of his seafaring life, Dr. Bowditch was elected President of the Essex Fire and Marine Company, which situation he held till his removal to Boston, in 1823. Here, also, he displayed his usual good judgment and discretion, and his usual success attended him. It was, indeed, an office for which he was eminently qualified by his whole previous life. After paying to the stockholders an average annual dividend of ten or twelve per cent. for the whole twenty years of his presidency, he left the institution with a large surplus of profits earned. In this situation, where he was necessarily brought more in contact with men of business than ever before, his easy and affable manners soon made him generally known; and the intrinsic excellences of his character made him no less generally beloved and respected.

During the years 1805, 1806, and 1807, he employed himself in making a survey of the harbors of Salem, Marblehead, Beverly, and Manchester; and the result of his labors was a chart of remarkable beauty and exactness, upon which all the old and familiar landmarks of the pilots, though not known by him to be

such, were so accurately placed in their true distances and bearings, as to excite among them the greatest surprise.

His principal occasional labors, during his residence in Salem, consisted of twenty-three contributions to several volumes of the Transactions of the American Academy of Arts and Sciences, of which the following is an accurate list: —

VOLUME SECOND. PART SECOND.

Published in 1800.

1. *New Method of working a Lunar Observation.*

The object of this method was to establish a uniform rule for the application of corrections, so that there should be no variation of cases resulting from the distance and altitude of the observed bodies. Dr. Bowditch says of this method, in a note, that "it was written several years ago, and before the publication of the Transactions of the Royal Society for 1797, in which is inserted a method somewhat similar, invented by Mr. Mendoza y Rios. An appendix to the New Practical Navigator has lately been published, in which the corrections are all additive, and the work is shorter." It is particularly noticed and commended in the *Connoissance des Tems*, (1808,) then published under the direction of M. Delambre.*

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* Zach (*Corr. Astron.*, Vol. VI. p. 553, A. D. 1822) says, "M. *Bowditch* dans son *New american practical navigator* a aussi donné pour la réduction des distances lunaires une nouvelle méthode abrégée, avec des tables, qui mérite d'être plus connue ; aucun auteur européen n'en a encore parlé ; il vient de la perfectionner dans sa quatrième édition stéréotype publiée à New York en août 1817. Nous la recommandons à l'attention des professeurs et auteurs des traités de navigation." In Vol. X. p. 321, A. D. 1824, he says, "La méthode de M. *Bowditch* a l'avantage sur toutes les autres méthodes d'approximation, que toutes les corrections sont toujours additives, et qu'on n'a jamais besoin de faire attention à des cas

VOLUME THIRD. PART FIRST.

Published in 1809.

2. *Observations on the Comet of* 1807. [pp. 1—18.]

3. *Observations on the total Eclipse of the Sun*, *June* 16, 1806, *made at Salem.* [pp. 18—23.]

In a note to this communication, Dr. Bowditch makes, as is believed, the first public mention of an error in Laplace's *Mécanique Céleste*, in the estimate of the oblateness of the earth, as calculated from the observed length of pendulums; showing that Laplace's result ought to have been, upon his own principles, $\frac{1}{315}$ instead of $\frac{1}{336}$.

4. *Addition to the Memoir on the Solar Eclipse of June* 16, 1806. [pp. 23—33.]

5. *Application of Napier's Rules for solving the Cases of Right-angled Spheric Trigonometry to several Cases of Oblique-angled Spheric Trigonometry.* [pp. 33—38.]

This communication so alters Napier's rules, as to make them include most of the cases of oblique-angled spheric trigonometry, and is marked by the same neatness, elegance, and simplicity, which characterized his first communication. These rules are now familiarly known in the text-books of Harvard College as "Bowditch's Rules."

VOLUME THIRD. PART SECOND.

Published in 1815.

6. *An Estimate of the Height*, *Direction*, *Velocity*, *and*

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particuliers; les règles sont générales;" and proceeds to give a detailed account of it.—See also note to article 15.

Magnitude of the Meteor that exploded over Weston, in Connecticut, December 14, 1807. [pp. 213—237.]

This communication is of a very interesting character, and it rests upon numerous observations collected with great labor and assiduity. Dr. Bowditch considers the meteor in question to have had a course about eighteen miles above the earth, a velocity of more than three miles a second, and a probable cubic bulk of six millions of tons — which others have estimated to be the contents of the pyramid of Cheops.*

7. *On the Eclipse of the Sun of September* 17, 1811, *with the Longitudes of several Places in this Country, deduced from all the Observations of the Eclipses of the Sun, and Transits of Mercury and Venus, that have been published in the Transactions of the Royal Societies of Paris and London, the Philosophical Society held at Philadelphia, and the American Academy of Arts and Sciences.*† [pp. 255—305.]

8. *Elements of the Orbit of the Comet of* 1811. [pp. 313—326.]

In this, as in his second communication, he arrived at his results after almost incredible labor, rendered necessary by the want of the

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* The *Zeitschrift für Astronomie*, Vol. I. p. 37, A. D. 1816, gives the results arrived at in this communication, and calls it "einer interressanten Arbeit."

† The *Zeitschrift für Astronomie*, Vol. I. p. 90, A. D. 1816, mentions the observations of the eclipses of the sun, June 16, 1806, and September 17, 1811, as contained in these volumes, &c., and states that "Bowditch hat den grössern Theil davon zu Längenbestimmungen benutzt und zugleich dabey, für eine Menge amerikanischer Orte, Hülfsgrössen zur leichtern Berechnung des Nonagesimus gegeben;" and Zach, in his *Corr. Astron.* Vol. X. p. 494, A. D. 1824, has a table of the longitudes and latitudes of places determined by astronomical observations calculated by Dr. Bowditch.

improved methods of the present day.* The original volume, containing his calculations in the case of this latter comet, now preserved in his library, contains one hundred and forty-four pages of close figures, probably exceeding one million in number, though the result of this vast labor forms but a communication of twelve pages.†

9. *An Estimate of the Height of the White Hills in New Hampshire.* [pp. 326—328.]

10. *On the Variation of the Magnetic Needle.* [pp. 337—344.]

This communication, in like manner, which is of quite an interesting character, and of considerable practical importance, was the result of five thousand one hundred and twenty-five observations, during a period of four years.

11. *On the Motion of a Pendulum suspended from two Points.* [pp. 413—437.]

This communication is also one of interest and value; and the little wooden stand, from which a leaden ball was suspended, still exists, to remind us of the zeal and assiduity with which Dr. Bowditch watched the various curves and lines which the ball described.‡

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* See Dr. Bowditch's letter (Zach, *Corr. Astron.* Vol. X. p. 228) before referred to, where this fact is stated. The editor, in p. 248, gives the elements of the orbits of the comets calculated by Dr. Bowditch wholly from American observations.

† Mr. Encke, in speaking to a friend of Dr. Bowditch, at Berlin, in 1836, said that he had known him from the time when this paper appeared; and that he had never seen an American since, without asking him what he could tell him about its author; — and the *Zeitschrift für Astronomie*, Vol. I. p. 41, gives an account of this communication "von dem amerikanischen Astronomen Bowditch."

‡ This subject is mentioned in his letter to Baron Zach, before alluded to, (*Corr. Astron.* Vol. X. p. 227.) The editor, in his note, p. 246, says the remarkable variety of the

12. *A Demonstration of the Rule for finding the Place of a Meteor, in the second Problem, page* 218 *of this Volume.* [pp. 437—439.]

VOLUME FOURTH. PART FIRST.

Published in 1818.

13. *On a Mistake which exists in the Solar Tables of Mayer, Lalande, and Zach.** [pp. 2, 3.]

14. *On the Calculation of the Oblateness of the Earth, by Means of the observed Lengths of a Pendulum in different Latitudes, according to the Method given by*

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motions of a pendulum thus suspended, and the very curious experiments of Professor Dean, who explains, in this mode, the apparent motion of the earth as seen from the moon, engaged Dr. Bowditch in the examination of the theory of these motions. The result has been, he adds, "une recherche très intéressante." "Comme ce mémoire mérite d'être mieux connu, et qu'il ne l'est pas généralement, vu la difficulté de se procurer des livres américains, nous en donnerons la traduction dans un de nos cahiers."

* Dr. Bowditch states, that "The attraction of Jupiter produces an equation in the expression of the Sun's distance from the Earth, and a Table is given for its computation, by Mayer, in 1770," &c.; "and ever since this Table was first published, which is about fifty years, an error of six signs has always existed in the argument by which the correction is found; so that, when the equation is really *subtractive*, it will frequently be found by the Table to be *additive*, and the contrary." — "In De Lambre's Solar Tables, published in 1806, the form of the table is wholly altered, the method of entry by a double argument being used; and by thus taking a different path, the error is avoided, without noticing that it really does exist in the other works."

Baron Zach, in his *Monatliche Correspondenz*, Vol. VIII. p. 449, A. D. 1803, says that Bowditch, an American astronomer, has called his attention to this mistake; and, after admitting its importance, frankly adds, "Allen Astronomen, welche sich mit Verfertigung der Sonnen-Tafeln beschäftigt haben, einen *La Caille, Tob. Mayer, La Lande, De Lambre* und *mir* ist dieser Fehler entgangen."

Laplace, in the Second Volume of his "Mécanique Céleste;" with Remarks on other Parts of the same Work relating to the Figure of the Earth. [pp. 3—24.]

The object of this communication is to correct certain errors in the article "Earth" in Rees's Cyclopædia, to the end that currency should not be given to inaccurate ideas on the subject, by that popular work.

15. *Method of correcting the apparent Distance of the Moon from the Sun, or a Star, for the Effects of Parallax and Refraction.* [pp. 24—31.]

This is but the rule given in the Practical Navigator, making all the corrections in question additive. It is another instance of the simplicity at which he always aimed in his rules and formulas.*

16. *On the Method of computing the Dip of the Magnetic Needle in different Latitudes, according to the Theory of Mr. Biot.* [pp. 31—36.]

17. *Remarks on the Methods of correcting the Elements of the Orbit of a Comet, in Newton's "Principia" and in Laplace's "Mécanique Céleste."* [pp. 36—48.]

This communication proves that two equations in the Principia, the accuracy of which several commentators upon that work had

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* In Zach's *Monatl. Corres.*, Vol. XVII. p. 411, A. D. 1808, this method is mentioned as being in the Appendix to the New American Practical Navigator, printed at Newburyport, 1804; and the editor says, "Der Verfasser ist ein Americaner, Bowditch, und Delambre hat es der Mühe werth gehalten, eine umständliche Darstellung dieses Verfahrens zu geben." Then follows a somewhat minute account of the method. — See note to article 1.

attempted to prove, and as to which no doubts had yet been expressed or insinuated, always made the corrections in question "double of what they ought to be," and restricts the method of Laplace as appropriate only where the number of observations is small.

18. *Remarks on the usual Demonstration of the Permanency of the Solar System, with Respect to the Eccentricities and Inclinations of the Orbits of the Planets.* [pp. 48—51.]

19. *Remarks on Dr. Stewart's Formula for computing the Motion of the Moon's Apsides, as given in the Supplement to the Encyclopædia Britannica.* [pp. 51—61.]

This is a very curious and interesting communication. A method which, notwithstanding doubts had been expressed respecting it, had been sanctioned as accurate by Dr. Hutton, by Lalande, and Playfair, — the latter of whom even considered its accuracy to have been demonstrated, — is in this memoir proved to have been true only in the particular case supposed; and it is shown that, as a general method, it wholly fails.

VOLUME FOURTH. PART SECOND.

Published in 1820.

20. *On the Meteor which passed over Wilmington, in the State of Delaware, November* 21, 1819. [pp. 3—14.]

21. *Occultation of Spica by the Moon, observed at Salem.* [p. 14.]

22. *On a Mistake which exists in the Calculation of Mr. Poisson relative to the Distribution of the Electrical Matter upon the Surfaces of two Globes, in Vol. XII.*

of the "Mémoires de la classe des sciences mathématiques et physiques de l'Institut Impérial de France." [pp. 15—17.]

23. *Elements of the Comet of* 1819.* [pp. 17—19.]

Besides the above contributions to the Memoirs of the American Academy, Dr. Bowditch was the writer of several other articles, among which may be mentioned the following: —

1. *Notice of the Comet of* 1807. Published in the Monthly Anthology for December, 1807, Vol. IV. [pp. 653, 654.]

2. *Review of a "Report of the Committee (of Congress,) to whom was referred, on the 25th of January,* 1810, *the Memorial of William Lambert, accompanied with sundry Papers relating to the Establishment of a First Meridian for the United States, at the permanent Seat of their Government."* Published in the Monthly Anthology for October, 1810, Vol. IX. [pp. 245—266.]

   This article occupies twenty-one pages, and proves very conclusively the great advantages of continuing to estimate the longitude from Greenwich, which Mr. Lambert considered "a sort of degrading and unnecessary dependence on a foreign nation," and an "encumbrance unworthy of the freedom and sovereignty of the American people." This Memorial the reviewer shows to be "a compilation, with needless repetitions and palpable mistakes, evincing a great want of knowledge in the principles of the

* For a statement of Dr. Bowditch's communications to the Memoirs of the Academy, and an abstract of their contents, from which several of our remarks in the text are condensed, see Mr. Pickering's Eulogy, pp. 17—31.

calculations;" and that, "both as respects its object and execution, it was wholly undeserving the patronage of the National Legislature."

3. *Defence of the Review of Mr. Lambert's Memorial.* Published in the Monthly Anthology for January, 1811, Vol. X. [pp. 40—49.]

Mr. Lambert having made an angry reply, charging his reviewer with "twistical cunning," "ingenious quibbling," "zeal for the honor of the British nation, and the convenience of British mariners," and challenging him "to examine his computation of the longitude of the Capitol at Washington from Greenwich, *and to point out a mistake that can be made palpable*," — Dr. Bowditch, in this reply, considers these charges of Mr. Lambert as beneath his notice, but accepts his challenge, and proves *that there is an error in every one of the six examples he has given.*

These two papers were fatal to the proposed project; and, fortunately for the interests of science, Greenwich continues to be the first meridian of all who speak the English language.

4. *Review of "A Treatise on the most easy and convenient Method of computing the Path of a Comet, from several Observations; by William Olbers, M. D.; Weimar,* 1797;" *— and of "Theoria Motus Corporum Cœlestium in Sectionibus Conicis Solem ambientium;" by Charles Frederick Gauss; Hamburg,* 1809. Published in the North American Review for April, 1820, Vol. X.* [pp. 260—272.]

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* A copy of this article Dr. Bowditch sent to Baron Zach, with the letter before referred to, *marking a part of it as written by Mr. Everett, the editor.* Zach publishes

This article gives an account of several German astronomers and their most noted periodical publications. Thus it contains a notice of Dr. Olbers — "the Columbus of the planetary world" — and of Gauss, the authors of the two works reviewed; an account of Bode's *Astronomisches Jahrbuch*, Zach's *Monatliche Correspondenz*, and the *Zeitschrift für Astronomie*. It states the fact that, "out of *thirteen* primary planets and satellites, discovered since the year 1781, we are indebted to persons born in Germany for *twelve*; and that, in the determination of the orbits of these new bodies, they have done more than all the other astronomers in the world."

5. ***Review of* "*A remarkable Astronomical Discovery, and Observations of the Comet of July*, 1819; *by Dr. Olbers of Bremen; published in Bode's Astronomisches Jahrbuch for* 1822;"** [and of two other articles in the same work, for the years 1822 and 1823, on the same subject, by Professor Encke of the Ducal Observatory at Seeberg, near Gotha.] Published in the North American Review for January, 1822, Vol. XIV. [pp. 26—34.]

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extracts from it in his notes upon this letter, Vol. X. p. 231, and says, "It will be interesting to the reader to learn how men of science in America render justice to those of Germany, while they reproach their brethren beyond the water for the little attention which they have bestowed upon our productions." Dr. Bowditch mentions in this review an interesting paper which Mr. Ivory had published in the Transactions of the Royal Society of London, 1814, giving a method of his own for computing the orbit of a comet, which, "upon examination, turns out to be nothing more than that which Dr. Olbers had published in his work above seventeen years before, although this coincidence must have been wholly unknown to Mr. Ivory, and to the other members of the Royal Society. We consider this as a striking instance of the little attention paid in Great Britain to works of mathematical science printed in Germany." The passage added by Mr. Everett was merely that which states a like neglect of German *literature*.

A copy of this article on Encke's comet Dr. Bowditch also sent with the letter before mentioned, to Baron Zach, who, in his notes, states that Dr. Bowditch "has here collected all that has been said and done respecting this famous comet." In the concluding paragraph of this article, the reviewer expresses his regret that, "while Great Britain alone can boast of more than thirty public and private observatories of considerable note, we have not, in the whole United States, one that deserves the name." He also speaks of the duties imposed on the importation of mathematical instruments and scientific works, as *fines* and *penalties*, which had been justly called "a bounty upon ignorance," &c. This whole paragraph is extracted by Zach, (Vol. X. p. 245,) and he says, "Voici de quelle manière un bon républicain exhale son chagrin en public; c'est au moins *quelque chose*," &c.

6. ***Letter to Baron Zach, dated November*** **22, 1822;** ***with a Postscript, dated December*** **20, 1823. Published in his Correspondance Astronomique for the year 1824,* Vol. X.** [pp. 223—230.]

7. ***Review, entitled "Remarks on several Papers published in former Volumes of this Journal;"*** **[the first being remarks on "A New Algebraical Series, by Professor Wallace, of Columbia, S. C.;"] published in Silliman's Journal for 1824, Vol. VIII.** [pp. 131—139;] **— and** ***Remarks on Mr.***

* This letter has been already more than once referred to, and contains many interesting facts. The editor's comments upon it occupy twenty pages. With this letter Dr. Bowditch had sent, besides his articles mentioned in the two last preceding items, a copy of the fifth edition of the Practical Navigator. The editor says of him, "C'est le premier, et jusqu'à-présent le seul grand géomètre en Amérique."

Wallace's Reply; published in the same Journal for 1825, Vol. IX. [pp. 293—304.]

The reviewer expresses his surprise that any offence should have been given by the mere statement of the *historical fact* that this "new series" was but the usual development of the binomial theorem, and the same which had been given by Euler fifty years before.*

8. *Review of "Fundamenta Astronomiæ," by Frederick William Bessel;* 1818; — *of the Tables of the Moon, by M. Burckhardt;* 1812; — *of the New Tables of Jupiter and*

* Professor Wallace, in his Reply, states that he did not claim the series as *new,* and appeals to a reference which he had made in his original article to Mr. Stainville, &c., and, not knowing who his opponent was, says that he does not, like his reviewer, refer his readers "to the *Complement des Elémens d'Algèbre, however useful as a school-book,*" &c. He also states, "that the results which Euler has given do not include a single case of a transcendent function, and were only given as examples of the applications of the simplest case of the binomial theorem," &c. Dr. Bowditch, in his rejoinder, mentions the vague terms in which Mr. Stainville had been originally referred to, and says, "It now appears that Mr. Stainville gave it as new for the *first* time in 1818, and Professor Wallace for the *second* time in 1824, Euler's having been published in 1775:" and again; "It is believed that most persons, after reading what Professor W. has written, would suppose he claimed some, if not a very large portion, for his own. But the real fact is, that *none* of it is *his.* The whole of the first seven pages, and a large portion of the two remaining pages, of Professor W.'s first communication, are merely literal translations from Stainville and Gergonne; and what is not copied from them is quite unimportant." He also says, "It is a fact, *notwithstanding the positive declaration of Professor W. to the contrary,* that Euler's demonstration is not restricted to this very simple case, but is general for all values of the exponent, whether integer, fractional negative, or surd; and it is characterized by Lacroix as being elegant and rigorous." This review will be found quite amusing and *piquant.* It is, like the articles on Mr. Lambert's Memorial, both as to matter and style, a fair specimen of Dr. Bowditch's powers as a controversial writer.

of Saturn, by M. Bouvard; 1808; — *of the Tables of the Satellites of Jupiter, &c., by M. Delambre;* 1817; — *of the Tables of Venus, of Mars, and of Mercury, by B. de Lindenau;* 1810, 1811, *and* 1813; — *and of the Memoir on the Figure of the Earth, by M. de Laplace;* 1817 *and* 1818.* Published in the North American Review for April, 1825, Vol. XX. [pp. 309—367.] †

This brief but most comprehensive article upon modern astronomy will be found to possess an uncommon degree of interest. It consists of a series of biographical sketches, in which are described all who have been remarkable for the successful cultivation of physical science in modern times, bringing into view their actual and relative services and merits, and awarding to each the degree of approval to which he was entitled; — the writer now dwelling with enthusiasm upon his favorite Lagrange, now bestowing a more qualified and guarded approbation, or a positive censure, upon others inferior in powers and attainments to that distinguished mathematician, or opposite to him in character.‡ It comprises,

* The titles of the particular works reviewed, are here given in an abridged form.

† In the Notes to Mr. Pickering's Eulogy, p. 95, a list is given, without comment, of six of the above eight articles, the fifth and seventh not being noticed. All these occasional publications of Dr. Bowditch, excepting the letter to Baron Zach, were collected by him in two volumes, now in his library.

‡ Thus he says, "Upon the decease of Euler, Lagrange remained undisputedly the greatest mathematician then living," &c.; while of Dr. Bradley's successor he says, "Dr. Bliss was wholly unworthy of the office of astronomer royal. The account of his life by La Lande is comprised in less than a dozen words — 'Bliss était astronome royal; il mourut en 1765.'" This article is the one, of all Dr. Bowditch's occasional publications, which exhibits in the clearest light his peculiar talents and acquirements. Evidently the work of one possessing a knowledge of the actual state of mathematical science, in its various departments,

especially, a very full account of Dr. Bradley's observations, and of Bessel's services in reducing them; of the best makers of mathematical instruments — Graham, Bird, Ramsden, Troughton, Jones, Reichenbach, Frauenhofer, Herschel, &c.; of the successive astronomers royal at Greenwich, and of the other chief European observers; and, lastly, "it gives an account of the labors of those mathematicians who have improved the science of astronomy by their calculations of the effects of the mutual attractions of the heavenly bodies."

Dr. Bowditch was also, for many years, a contributor to the Annalist and Mathematical Diary, solving every question there proposed, in his usual style of simple elegance. He also wrote or corrected various articles in the American edition of Rees's Cyclopædia. And all these various publications were the employment merely of those leisure hours which were left to him after all the calls of active business, and all the claims of social and domestic life, had been most fully answered; and more than this, and notwithstanding all these duties and engagements, and all the occasional scientific labors which have been mentioned, such was his wonderful economy of time, that, within the same period, he also completed what has justly been characterized as the gigantic undertaking of making the Translation and Commentary now before the reader, — a work upon which, almost exclusively, will rest his fame as a man of science.*

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as extensive and minute as was possessed by any individual then living, — it is, throughout, a record of the most sound and impartial criticism. Any biography of him, which has not this review in an appendix, must be incomplete.

* Baron Zach, in his *Correspondance Astronomique*, Vol. X. p. 234, A. D. 1824, says, "Nous finirons cette note par apprendre à nos lecteurs ce que nous a révélé le professeur

Upon recurring to the Translator's Preface, in the first volume, it will be found there stated that "the notes were written at the time of reading the volumes, as they were successively published. The translation was made between the years 1814 [misprinted 1815] and 1817, at which time the four first volumes, with the several appendices and notes, were ready for publication." The fifth volume, published by La Place twenty years after the others, was never translated by Dr. Bowditch, though he wrote many important notes upon it.* It was his intention, however, had he lived, to translate the volume. Death has defeated forever that intention. The work which he had so nearly completed, no one lives to finish as he would have finished it; but, like the beautiful painting from which was taken the engraving prefixed to this memoir, and which never received the final touch of the dying artist, it is the more interesting from the circumstances under which it was left incomplete.

Everett, que M. *Bowditch* a traduit en anglais toute la *Mécanique Céleste* de M. *La Place*, avec un ample commentaire, mais qu'on n'a pu encore le persuader de publier cet ouvrage qui ne pourrait que lui faire un honneur infini, ainsi qu'à son pays, mais nous soupçonnons qu'il attend pour cela l'ouvrage de MM. *Plana* et *Carlini*, qui est sur le métier, et qui ne tardera pas à paraître." A similar public announcement of this fact had been made in the North American Review for April, 1820, Vol. X. p. 272; and the editor says that Dr Bowditch "has not, however, yet been prevailed upon to do honor to himself and to his country, by the publication of so great and arduous a work."

* A day or two only before his death, he received from Europe a translation, executed by a young lady whom he had never seen, but who was soon to become his daughter, embracing in seventy manuscript pages the first part of the fifth volume; — a suitable offering of filial duty to one who never lived to thank her in person for her kindness, but who left for her at his decease an affectionate letter, written exactly a week before his death.

As, in the course of publication, it became necessary to incorporate into the notes much additional matter, owing to the subsequent progress of mathematical science, they were all, in a great measure, rewritten; and thus, perhaps, the present four volumes will be found to contain almost every thing of importance in the whole five volumes of the original work, excepting what relates to the earth's temperature and the velocity of sound.* Still, it was Dr. Bowditch's intention to introduce into the fifth volume more original matter than into either of the preceding ones, making it, as it were, the general depository alike of all the results of his extensive theoretical investigations, and of the practical experience of a long life. It was, especially, a source of regret to him, that he could not prepare the Index to the work, which he felt assured, from his intimate knowledge of its contents, and of the relative importance of the different matters of which it treats, he was more competent to prepare than any one else. That duty, we believe, however, will at a future time be ably performed by a friend, (Benjamin Peirce, Esq., Professor of Mathematics in Harvard University,) whose revision of the entire work, when in the process of publication, and vigilance in detecting typographical errors, Dr. Bowditch always valued as an additional means of insuring its accuracy.

It would not be our desire, were we competent to the task, to offer any criticism upon the present work. It will itself speak to every reader. A few remarks, however, upon the motives, views, and objects of the translator may not be inappropriate.

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* Mr. Pickering's Eulogy, p. 54.

In the first place, then, his great design was to supply those steps in the author's demonstrations, which were not discoverable without much study and research, and which had rendered the original work so abstruse and difficult, as to lead a writer in the Edinburgh Review to say there were not twelve individuals in Great Britain who could read it with any facility.* Dr. Bowditch himself was accustomed to remark, "Whenever I meet in La Place with the words 'Thus it plainly appears,' I am sure that hours, and perhaps days, of hard study will alone enable me to discover *how* it plainly appears." So important did he consider the object which he thus had in view, that every letter which he received, proving to his satisfaction the fact of some young man's having read his Translation and Commentary, afforded him much more pleasure than the favorable mention of it in popular journals, or even than the flattering approbation bestowed by competent judges; since, while the one would be but an opinion, the other would be *a proof*, that the great end of his labors had been accomplished. He received several such letters. M. Lacroix wrote to him that he had recommended the work to a young professor at Lausanne. There can, indeed, be no doubt

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* "We will venture to say, that the number of those in this island who can read that work with any tolerable facility, is small indeed. If we reckon two or three in London and the military schools in its vicinity, the same number at each of the English Universities, and perhaps four in Scotland, we shall hardly exceed a dozen; and yet we are fully persuaded that our reckoning is beyond the truth." — *Edinburgh Review*, 1808, Vol. XI. p. 281.

In America, two, and perhaps three persons, besides Dr. Bowditch, were able to read the original work critically; but a competent judge has doubted whether the whole of it had been so read even by one.

that it has been truly said by a late foreign review,* respecting this Translation and Commentary, "a work which existed in mere abstraction before, has been made as accessible, to all public and popular purposes, as its essential nature would permit;" and by another review,† "the notes to each page leave no step in the text, of moment, unsupplied, and hardly any material difficulty of conception or reasoning unelucidated."‡ Mr. Babbage, in a letter to the translator, August 5, 1832, says, "It is a proud circumstance for America, that she has preceded her parent country in such an undertaking; and we in England must be content that our language is made the vehicle of the sublimest portion of human knowledge, and be grateful to you for rendering it more accessible."

A second great object of the translator was, to continue the original work to the present time, so that it should place in possession of the reader the many recent improvements and discoveries in mathematical science. That the most eminent living mathematicians consider this end to have been attained, clearly appears by the following extracts from letters addressed by them to Dr. Bowditch, and now before us: — M. Lacroix says, July 5, 1836, "I am more and more astonished at your continued perseverance in a task so laborious and

* London Athenæum, 1838.

† London Quarterly Review, Vol. XLVII. p. 558.

‡ An English professor of mathematics, who was at Rome in the winter of 1836–1837, told a friend of Dr. Bowditch, that he was indebted to these notes for his knowledge of La Place; and that, though he did not expect to be very long absent from England, he had ordered the next volume to be sent after him to Italy, if it should appear before his return home.

extensive. I perceive that you do not confine yourself to the mere text of your author, and to the elucidations which it requires; but you subjoin the parallel passages and subsequent remarks of those geometers who have treated of the same subjects; so that your work will embrace the actual state of science at the time of its publication." — M. Legendre, in a letter dated at Paris, July 2, 1832, says, "Your work is not merely a translation with a commentary; I regard it as a new edition, augmented and improved, and such a one as might have come from the hands of the author himself, if he had consulted his true interest, that is, if he had been solicitously studious of being clear," &c. — Mr. Bessel, also, in a letter dated at Königsberg, February 18, 1836, writes, "Through your labors on the Mechanism of the Heavens, La Place's work is brought down to our own time, as you add to it the studies of geometricians since its first appearance. You yourself enrich this science by your own additions, for which especial obligations are due to you." — M. Puissant (in a letter dated May 31, 1835, and addressed to D. B. Warden, Esq., through whose agency Dr. Bowditch had transmitted to him a copy of this work) observes, "The numerous additions which accompany the text, and which, in their turn, deserve to be translated into French, are the more important, as they clear away the difficulties which the subject frequently presents, and moreover include whatever Dr. Bowditch and other geometers have added to the theory of the motions of the heavenly bodies."

A third object of the translator was one which, though wholly subordinate to the others, he still thought of considerable

importance. La Place had so modified, by the action of his own mind, the various productions of other men of genius, that, while he stated the results of their labors, or adopted their improvements, he did not remember, or at least did not think it necessary to admit, the source to which he was, in each particular instance, indebted. His work told the great truths of science, but omitted to state by whom those truths had been first discovered and announced. But it must be remembered that it was concise in all its processes and expressions; and he probably felt that every reader, whose genius could follow him into the depth of his abstruse speculations, must necessarily have previously read the same works from which he had himself derived assistance; and that, familiar as they must be to the reader already, it would be superfluous, by any acknowledgment or quotation, to direct attention to them. Be this as it may, the fact is certain that, in the original work, credit is frequently not given to the eminent mathematicians of ancient and modern times, by whose labors those of its author were rendered less difficult or more effective. But Dr. Bowditch thought it due to the cause of literary justice, that, in every such instance, the omission in the original work should be supplied. Several of the communications which he received, mention his course in this respect with high approbation, and express the regret of the writers that La Place should himself have thought and acted otherwise. Dr. Bowditch was well aware of that natural self-love, by which every one is gratified at finding his labors approved by others; and he could especially realize how great must have been the pleasure felt at being quoted by La Place for some important process or discovery which had contributed to the completeness of the *Mécanique Céleste*. He had communicated to the public, and

to La Place himself, a notice of an error in the original work, which was corrected in a subsequent edition, but without even a private acknowledgment. That it was not *publicly* noticed by the author, was of course, for the reason above stated, no cause of any especial complaint. And Dr. Bowditch well knew by personal experience, upon more than one occasion, that it was very possible his own letter to La Place, or La Place's reply, might have miscarried.* It has also been suggested, that La Place was extremely averse to the act of writing a letter at any time, however strong or urgent a motive existed for so doing. This was certainly the case with Dr. Bowditch. Often, upon the receipt of an epistle or note, he has taken his hat, called personally upon the writer, and given him a verbal answer.†

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* Eight copies of the first volume of this Translation and Commentary, sent by him as presents to the most distinguished institutions and astronomers of Germany, wholly failed to reach their place of destination; and several copies of the first volume of Struve's Observations, transmitted by the author as presents from the Imperial Observatory at Dorpat in Russia to the Royal Society of London, in 1821, found their way, with the original letter which accompanied them, to a bookstore in Boston, in 1824, where Dr. Bowditch accidentally saw and purchased them; retaining one of which for his own use, he transmitted all the others as at first directed. He received no reply whatever, and presumed that the same evil destiny had again followed them. But some years afterwards, he found in Schumacher's *Astronomische Nachrichten*, Vol. IV. p. 398, a letter to the editor, from Francis Baily, Esq., of London, dated January 19, 1826, giving an account of their curious wanderings; of the agency of "Mr. Bowditch, the celebrated American astronomer," in the matter; and of their safe arrival at last, "after a long and circuitous voyage of five years, from Dorpat to Boston, and from Boston to London.

† When his third son went to Europe to pursue his medical studies, he gave him no letters of introduction, but, as a substitute for them, certain copies of a newly-published volume of this work to deliver as presents.

We should much regret that any of the preceding remarks should be construed as in the least degree attributing to La Place an intentional or unfair appropriation to himself of the fruits of the labor of others. We allude only to an error of judgment, which, though easily accounted for upon the above suppositions, is much to be regretted, as having occasionally exposed him to this more serious imputation. Any such omission of the author, as far as Dr. Bowditch was himself concerned, was, long before his decease, wholly effaced from his memory by the kindness shown to one of his sons by the widow of La Place, who transmitted, by his hands, as a present to his father, a bust of her late husband, which has ever since been one of the ornaments of his library, and which, by a provision of his will, is eventually to be deposited, with the manuscript of this work, in the Library of Harvard College, there to remain an interesting joint memorial of the author and the commentator.

Such were the three chief objects which it was the design of this Commentary to accomplish; and the general merits of the work have been acknowledged, in language no less strong than that already quoted, by Professor Airy, Francis Baily, Esq., the late Bishop of Cloyne, and other astronomers of Great Britain, as well as by those of France, Germany, and Italy. Thus, Sir John Herschel, in a letter to the translator, dated March 8, 1830, says, "It is very gratifying to me to commence a scientific intercourse, which I have long desired, with the congratulations which the accomplishment of so great a work naturally calls for; and I trust that its reception by the public will be such (of which, indeed, there can be little doubt) as to encourage you to proceed to the

publication of the succeeding volumes, and that you will be favored with health, strength, and leisure, to enable you to complete the whole of this gigantic task in the masterly manner in which you have commenced it. It is a work, indeed, of which your nation may well be proud, as demonstrating that the spirit of energy and enterprise which forms the distinguishing feature of its character, is carried into the regions of science; and every expectation of future success may be justified from such beginnings." — There was also one delicate attention which he received from a female hand. Mrs. Somerville sent a copy of her translation of a part of this work to him who was so happily and successfully engaged in the same labors. This volume, invested by him in a rich and beautiful binding, still attests the pleasure which he derived from it as a tribute of respect to his genius from one of the most gifted women of the age. — M. Lacroix, in a letter of April 5, 1830, writes, "Besides doing honor to the able, patient, and conscientious geometer, who has undertaken this great labor, your work, by the beauty of its typographical execution, does honor to the country where it is published. It is perhaps the most beautiful book which has appeared upon mathematics. The calculations in it possess the greatest neatness; and the figures which you have inserted in the body of the work itself unite the greatest elegance with convenience. An undertaking so remarkable entitles you to the gratitude of those who are desirous of studying to the bottom the theory of the system of the world, which rests upon transcendental mechanics; and it makes us wish for the speedy publication of the remaining volumes." — So also Mr. Encke of Berlin, in a letter dated May 5, 1836, speaks of it as a work "which, by the depth

of the researches with which it is accompanied, will insure to you a distinguished place among the astronomers who have employed themselves on the difficult branch of physical astronomy." — Mr. Cacciatore, conductor of the Royal Observatory at Palermo, in a letter dated May 1, 1836, mentions it as having "excited the enthusiasm of all who took an interest in the subject of it;" and in his treatise on Goniometry he remarks, "The profoundness and clearness which are conspicuous in that work, demonstrate that it was only by the aid of such powers of analysis that a commentary could be written upon the immortal work of La Place, and that La Place cannot be read with advantage unless it is accompanied with the notes of Bowditch. Italy must have a translation of it." *

The translation of this work was, as has been stated, completed as early as 1817; but so limited was Dr. Bowditch's income, that it hardly sufficed to meet the expenses of a growing family, upon all of whom he was desirous to confer the advantages of the best education which the country afforded; and all that was not needed for this purpose was expended in collecting around him the choicest scientific works of ancient and modern times. The American Academy, with that kindness and liberality which ever marked their intercourse with their late President, and which have characterized their proceedings since his decease, offered to publish the work at their own expense. He was also solicited to publish it by subscription. But his natural and praiseworthy independence of spirit induced him unhesitatingly to decline these gratifying

* See Notes to Mr. Pickering's Eulogy, pp. 96—100

proposals. He was aware, from the character of the work, that it would find but few readers, and he did not wish any one to feel compelled, or to be induced to subscribe for it, lest he should have it in his power to say, "I patronized Mr. Bowditch by buying his book, which I cannot read." He was thus obliged to wait even longer than the time prescribed by the poet, "nonumque prematur in annum," — until, under more favorable circumstances, he was enabled to commence the publication at his own expense.

But, though this work was not yet published, his fame as a mathematician had become fully established, and several of the scientific institutions of this country and of Europe conferred upon him their highest honors. The following are the foreign societies of which he was admitted a member, and the date of the several diplomas: — The Edinburgh Royal Society, January 26, 1818; the Royal Society of London, March 12, 1818; Royal Irish Academy, March 16, 1819; Royal Astronomical Society of London, April 13, 1832; Royal Academy of Palermo, March 12, 1835; British Association, June 29, 1835; Royal Academy of Berlin, March, 1836. It is worthy of remark, that France, the labors of whose greatest author have been by him rendered of so much more practical value and extensive usefulness, should alone have withheld from him the like honors.* In this country, he was elected a member of the American Philosophical Society held at Philadelphia, April 21, 1809; of the Connecticut Academy of Arts and Sciences, October 26, 1813; and of the Literary and

* It is indeed mentioned in Mr. Pickering's Eulogy, p. 101, that, but for his death, he would probably soon have been elected a member of the Royal Institute of France.

Philosophical Society of New York, January 17, 1815; &c. &c.; and at the annual commencement in 1816, he received from Harvard College the honorary degree of Doctor of Laws.

Dr. Bowditch always felt a deep interest in the various literary, scientific, and charitable institutions of his native town. It was chiefly through his instrumentality that, in 1810, a union was effected between the Philosophical Library, before referred to, and the Social Library, so called, which was the origin of the Salem Athenæum; and from that period, during his whole residence in Salem, he continued to be one of its most active and influential trustees. Having felt, in early life, the importance of a ready access to books, he labored to promote a most free and extensive circulation through the community, of the works in this institution, and to make its advantages as easy of attainment as possible, by every deserving individual. The gratitude inspired in his breast by the recollection of his own obligations of this nature, when he was but a poor apprentice, ended only with his life. Recurring to this subject in his will, he says, "These inestimable advantages have made me deeply a debtor to the Salem Athenæum;" and in return he bequeathed to it the sum of one thousand dollars. In accepting this bequest, the trustees admit most fully that "the early benefits which he thus gratefully remembered in his will," he had before repaid by his services and donations; and they add that they "see in this last act the unconscious and disinterested devotedness with which he, through a life of activity and business, fostered all the interests of learning and education."

Equally strong and lasting was his gratitude towards another

excellent institution, the Salem Marine Society. Composed exclusively, as its name denotes, of such as had made the sea the scene of their enterprising labors, Dr. Bowditch and his father had been both successively enrolled among its members. The kind and timely aid, to which, during several years of his infancy and childhood, he had been indebted for some of the absolute necessaries of life, he mentioned with emotion to his children, during his last illness, and deemed that he but paid a debt to this institution when he bequeathed to it a like sum of one thousand dollars, in aid of its charitable objects and purposes. His associates in this society were peculiarly competent judges of the value of his labors and services; and we doubt if our language could be made to present a more simple and beautiful expression of gratitude and regard than is imbodied in the following extracts from resolutions adopted by them upon the occasion of his decease: — "In his death a public, a national, a human benefactor has departed. Not this community, nor our country only, but the whole world, has reason to do honor to his memory. When the voice of Eulogy shall be still, when the tear of Sorrow shall cease to flow, no monument will be needed to keep alive his memory among men; but as long as ships shall sail, the needle point to the north, and the stars go through their wonted courses in the heavens, the name of Dr. Bowditch will be revered as of one who helped his fellow-men in a time of need, who was and is a guide to them over the pathless ocean, and of one who forwarded the great interests of mankind."

Each stranger who visits the hospitable city of Salem, is desirous to see the Museum of the Salem East India Marine

Society, or, as it is familiarly called, the Salem Museum. It is readily and gratuitously opened to his inspection. As he enters its spacious hall, his attention is arrested by a full length portrait of its late President. There the Commentator on the *Mécanique Céleste* seems still to preside in person over a favorite scene of his labors, inviting the attention of the visiter to what he has himself, in his will, described as "a museum of a very rare and peculiar character, collected from distant countries, and affording a proof alike of the enterprise, taste, and liberality of such of the citizens of Salem as have followed a seafaring life." The members of this society are such only as have sailed, in the capacity of masters or supercargoes, beyond the Cape of Good Hope or Cape Horn; and besides the obtaining of curiosities from these distant regions, an object of much greater practical importance, the collection of facts and observations in aid of nautical science, has always been zealously promoted by this society, and is believed to have been suggested by Dr. Bowditch himself. A blank book is furnished to each member, uniformly prepared for recording these facts and observations during each voyage; and, upon the return of the vessel, it is deposited with the society. It is then examined by a committee, who select and record in other volumes, having a convenient index for reference, all that they consider important; and the result is a mass of nautical information, such as, probably, exists nowhere else in the world, and which Dr. Bowditch found of great service in preparing for the press the various editions of the Practical Navigator. He was for many years Inspector of its Journals, before he became its President, and, in both of these relations to the society, highly promoted its progress and

success. Owning Salem for our birthplace, we feel proud of this institution; and we know that the bequest to it by its former President of the like sum of one thousand dollars, was made from an actual sense of obligations conferred, as in the case of the two other institutions which he thus remembered.

Besides these duties and engagements of a public nature, Dr. Bowditch became, in 1818, and was at his death, trustee for managing an estate of nearly half a million of dollars, which had been left by a merchant of Salem; and it may truly be said that there seemed to be no end to the various little services and good offices which he constantly delighted to render, and for which he was always sure to find the requisite leisure. His fondness for imparting as well as acquiring knowledge, was still manifested; as an instance of which it may be mentioned that he instructed several young ladies of Salem in French.

In his political opinions, he was a decided Federalist, and during the late war between this country and Great Britain, he took great interest in the absorbing and important events of the time. It has been stated that, when this war was first declared, he was, for two or three days, wholly unable to attend to his usual engagements of business or study. At the end of this time, however, he addressed his wife with "This will never do;" and, summoning a resolution to hope for the best, as the evil could not be avoided, he returned with alacrity to his ordinary course of life; and nothing more was ever heard from him about the war, except that he often expressed the ardent hope of obtaining a speedy and an honorable peace, and used all those

exertions which he thought the crisis required to accomplish so desirable an end.* It is believed that, later in life, and in view of its incidental and remote results, he regarded this war, if not as a necessary vindication of the national honor, at least as far less disastrous in its consequences than he had anticipated.

An instance may be mentioned of a fearless and independent discharge of duty, upon an occasion involving quite an exciting political topic. The legislature had passed a law in which a comma was inserted contrary, probably, to the true intention of the law-makers; but the mistake (if it were one) existed in the original draft, and in all the printed copies of the statute. Certain acts had been done by the Federalists, under authority of the law as actually promulgated, which their political opponents thought indictable offences, and which would have been so, if the comma had been transposed. A term of court was held in Salem, and Dr. Bowditch was returned upon the grand jury, and his associates elected him foreman. He had made himself thoroughly acquainted with this particular case, and had with him a legal opinion drawn up by one whose knowledge of law commanded universal respect. The prosecuting officer of the government attended before the jury, and, after stating that it was incumbent

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* On one occasion, he was distributing votes at the ballot box, upon a very inclement day, by the side of a political opponent, whose efforts just counteracted his own. Each was troubled with a severe cold, and Dr. Bowditch proposed that they should both go home, as he was satisfied that their absence would leave the final result the same as if they both continued their labors. His opponent smiled, and objected to the proposition, that it would indicate a lukewarmness in a good cause. This reason immediately operated upon Dr. Bowditch to withdraw his proposition, and the equal struggle was resumed.

upon him to give them any legal information which might be needed in the course of their duties, added that complaints would be laid before them of violations of this statute, which he accordingly proceeded to explain. Dr. Bowditch said, "Sir, I doubt the accuracy of your explanation. Have you got the statute with you?" The legal adviser said he had, and produced it, and read it with a wrong emphasis, as if the comma were otherwise inserted. Dr. Bowditch indignantly interrupted him: "Please, sir, to show me the book:" and on looking at it, he added, "Why did you so read it as, by your emphasis, to give us a false impression of its meaning?" The reply was, "There is no doubt the comma is inserted where it now is, only by mistake." Dr. Bowditch said "It is your duty, sir, to tell us what the law *is* as you find it, not to tell how you think it ought to be improved or altered. We have no further occasion for your services at present; when we wish them, we will send for you." His associates on the jury, though nearly equally divided in political sentiments, were highly gratified by his characteristic promptness and energy, and refused to find any bills of indictment for the supposed violations of law, and unanimously passed a very full vote of thanks to him for the fairness and independence with which he had presided over their deliberations.

For the last twenty years of his life, he retired altogether from the exciting scenes of political strife to what he called his "peaceful mathematics;" though he still continued to entertain and express decided opinions upon public men and measures, and to act upon these convictions. Dr. Bowditch was never fond of public life. He never held a seat in the House of Representatives of his native

state, and was never a speaker in the assemblies of his fellow-citizens. He was, however, elected to the honorable office of one of the Executive Council of Massachusetts, which he held during the years 1815 and 1816, being, during one of those years, at the council board under the administration of Governor Strong, for whose dignified manners, commanding talents, and exalted character, he entertained the highest respect; and this sentiment, it is believed, was cordially reciprocated on the part of the chief magistrate. At this board, upon more than one trying occasion, he gave his vote and exerted his influence in support of the law, and refused to screen from its penalties the murderer and other criminals who had deliberately violated its provisions without any palliating circumstances; notwithstanding the strong and urgent appeals in their behalf, made by many excellent and benevolent citizens, among whom were some of his own personal friends. He considered that a capricious exercise of even the prerogative of mercy, would, in effect, convert a government of law into a government of men.

Dr. Bowditch's father had originally worshipped at the Episcopal church in Salem, but became a member of Dr. Bentley's society while his son was quite young. Upon his second marriage, Dr. Bowditch removed to a different part of the town, and, for this and other reasons, became a member of the society under the pastoral care of his friend Dr. Prince, and always continued so during his residence in Salem. Within the walls of its ancient church was the first simple rite of the Christian religion administered to all his children. A coolness on the part of Dr. Bentley, originating in this removal from his society, resulted,

from political causes during the war, in an entire estrangement, which was always a source of regret to Dr. Bowditch, who made the first advances toward a reconciliation, by a direct call at his house with a friend who desired an introduction. The visiters were received with the utmost cordiality, and the intercourse, thus happily renewed, was never afterwards interrupted; and the family still retain in their possession memorials both of the early and the late friendship of Dr. Bentley. It was indeed particularly delightful to Dr. Bowditch to find that the restoration of peace to the country, brought with it a renewal of that social intercourse which political dissensions had wholly interrupted. He often mentioned the visit of Mr. Monroe, the President of the United States, to the town of Salem, in 1817, as an occasion never to be forgotten, because it was the first upon which, after a separation of many years, were again brought together within the same circle so many of his earliest and most valued friends.

Many and very flattering and advantageous proposals were made to Dr. Bowditch, from time to time, to induce him to leave Salem; but his attachment to his native place proved stronger than any temptation to which he was thus exposed. In 1806, he was elected Hollis Professor of Mathematics in Harvard University.* In 1818, President Jefferson desired him to accept

* By a singular coincidence, it happened that, at the annual commencement of that year, he was seated between two strangers, one of whom, reaching forward, observed to the other the fact of his nomination to this office, and asked whether it would probably be accepted; to which the other replied, that he rather thought not, since Mr. Bowditch would probably be afraid of "singing small on classic ground." But with the *classics* of his own science Dr. Bowditch was sure that he was more conversant than any one there, and his ability to

the like professorship in his University at Charlottesville in Virginia; and in his letter containing this request he says, "We are satisfied we can get from no country a professor of higher qualifications than yourself for our mathematical department." In the same year, he was also urgently requested to take charge of an insurance office in Boston. In 1820, Mr. Calhoun, the Secretary of War of the United States, requested him to consent to a nomination to the vacant professorship of mathematics at West Point, and says, "I am anxious to avail myself of the first mathematical talents and acquirements to fill the vacancy."

In 1823, he received an invitation to take upon himself the charge of an institution in Boston, — similar to that which he then managed, — jointly with another, recently incorporated by the name of the Massachusetts Hospital Life Insurance Company, for which latter institution his services were considered almost indispensable. The salary offered at first was exactly three times that which he then enjoyed. After mature consultation with his friends, and after bestowing upon the proposal his own most careful deliberation, it was also decidedly declined. But those who made it would admit of no refusal. A new offer was forthwith made of a still more liberal compensation, ($5000;) and as he felt it to be more than an equivalent for any services which he could render, — and as any further refusal on his part, which should have led to the offer of a still higher salary, he would

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teach to others what he knew himself, he had often abundantly tested. But he declined the appointment, solely from an unwillingness to break away from all the pleasant associations connected with Salem.

have regarded as a mere extortion, — he felt that, in listening to the proposal, he was now obeying a call of duty, and he accordingly, though with great reluctance, determined to quit the town where, as he says in his will, "he had passed so pleasantly the first fifty years of his life." He could, indeed, hardly determine to make the sacrifice in question; and, even when it was determined upon, a vague hope and anticipation were long cherished both by himself and his wife, that eventually they should return and end their days amid the scenes of their childhood. Until his death he continued to take the same lively interest as ever in the affairs of that city.

He left his early home attended with as cordial and sincere expressions of respectful and affectionate regret as could possibly have honored his departure. A public dinner was given to him upon that occasion, which will long be remembered by those present as a scene of the most interesting character; while the recorded account of the festival will ever attest that it was a tribute paid to "science supported by genius, guided by benevolence, and attended by all the virtues;" — to their "distinguished citizen, the first of his countrymen in the walks of science, and second to no man on earth for purity and honor;" — to "their respected guest, who reflected upon his country the brightest honors of science, and diffused in social life the warmest influences of benevolence." And the wish was expressed, that he might "enjoy a happiness as pure as his fame, and constant as the activity of his virtues;" and it was declared that, "as the monarchy of France had done homage to her La Place, so would the republic of America not be ungrateful to her Bowditch."

It may here be mentioned, as an instance of Dr. Bowditch's diffidence and aversion to all public display, that he previously obtained from the president of the day a promise not to call upon him to address, however briefly, his assembled friends ; — an incident which probably never before occurred upon a like occasion. It is also an interesting circumstance, that that gentleman (Hon. Benjamin Pickman) was the same individual to whom he was indirectly indebted in early life for his copy of Newton's Principia.*

This was the last occasion upon which Dr. Bowditch was personally to receive from his native city any public expression of those sentiments which continued, however, to be uniformly cherished and manifested towards him by its citizens to the close of his life. But honorable indeed to his memory were the proceedings of Salem consequent upon his decease, and gratefully will his children ever cherish the remembrance of them. The resolutions then adopted describe him to have been "a townsman of singular simplicity, integrity, purity, and benevolence of character ; attaining from humble life, by his intellectual and moral energy, the highest honors of science, and the respect

* Dr. Bowditch had, many years before, been a member of an engine club in Salem, — a voluntary association of gentlemen for securing each other's property from the ravages of fire. At the occasional meetings of this club, of a social character, he stipulated that he should never be called upon for toasts or sentiments, unless he could be allowed to get them written and delivered by proxy. This was agreed to, and a friend, of "infinite humor," prepared accordingly a number of them, of a most appropriate character, which were from time to time produced, and highly applauded, and the more warmly from the above circumstance respecting their origin, which, though ostensibly concealed with suitable gravity, was yet known to all the members.

and gratitude of the community as a public benefactor;" and "earnestly commend to the admiration and imitation of all, and especially of the young men of his native place, the noble example of active and patient industry, unconquerable perseverance, unbending uprightness and faithfulness in all the relations of life, and ardent love and constant pursuit of knowledge and truth, which were the foundations of a character of such honorable distinction and rare usefulness;" and declare that "the people of Salem have ever retained a deep interest in his happiness and fame since he reluctantly left his native place for a sphere of more extended usefulness;" and that they "now receive and acknowledge with grateful sensibility the evidence of his generous remembrance of his first home in the last days of his life, contained in his liberal bequests to three of the most useful and important institutions of the city." *

Thus, in 1823, Dr. Bowditch removed to Boston, with his wife and a family of six children, — four sons and two daughters, — the eldest of whom had not completed his professional studies, and the youngest of whom was but an infant. The remains of one interesting child, who died in 1820, at the age of ten years, and those also of an infant boy, repose in the burial-grounds of Salem. All the anticipations and motives which

* Pursuant to another resolution, a public Eulogy was pronounced upon the deceased by Hon. Daniel Appleton White, which, listened to at the time with the deepest interest, will in its published form remain as true, beautiful, and discriminating a delineation of character, as might have been expected from one who himself possesses a high order of talent, who was long an intimate personal friend of the deceased, and whose thoughts are always clothed in a classic elegance of style.

determined Dr. Bowditch to this removal, were fully realized and justified by its ultimate results. In Boston he found many of his early friends, who had preceded him in this removal, and was in a few years followed by others. Strangers extended to him the hand of friendship, and gradually became endeared to him. He saw his three eldest sons engaged in the professions or pursuits which their tastes had led them to select, under circumstances more advantageous than his former place of residence would have afforded. His own increased income allowed him to enrich still more his valuable library; and he found himself surrounded by sources of the purest and highest enjoyments. In his will he speaks of Boston as "the home of his adoption, where as a stranger he met with welcome, and had continued to receive constantly increasing proofs of kindness and regard."

The affairs of the Commercial Insurance Company were successfully conducted by him till the increasing labor of his office as Actuary of the Life Insurance Company, induced the latter institution to offer him the same salary which had been previously paid by both together, and which was subsequently still further increased to six thousand dollars. He now relinquished the charge of the other corporation, whose charter was surrendered, and its concerns prosperously closed. And it was without any regret that Dr. Bowditch bade farewell to the cares and anxieties attendant upon marine insurance, where occasionally an unforeseen accident intervenes to destroy the fairest prospect of success. The company of which he was President had met with two losses, of thirty thousand dollars each, within one week; and though

one was a case of piracy, of which none lived to tell the tale, and the other a case of a tempest and shipwreck, and in each instance the vessels lost were of the first class, so that no error of judgment could be attributed to him, still the immediate influences of these disasters were disheartening; and he felt that, with the multitude, success, and that alone, is wisdom, and that, in the majority of cases, their verdict is a just one. He himself, indeed, always was of opinion that continued ill-luck indicated incapacity. On one occasion, when he had refused to underwrite upon a vessel commanded by Mr. A . . . ., because "he was unlucky," the captain called upon him to complain of his imputing to him as a fault what was but a misfortune; and, after trying for some time to evade a direct reply, Dr. Bowditch at last said, "If you do not know that, when you got your vessel on shore on Cape Cod, in a moon-light night, with a fair wind, you forfeited your reputation as an intelligent and careful ship-master, I must now tell you so; and THIS IS WHAT I MEAN BY BEING UNLUCKY."

It was with pleasure, therefore, that he now turned his undivided attention to the management of the institution which was truly "the child of his affections." The act incorporating this company with a capital of a half a million of dollars, conferred powers of effecting insurance upon lives and granting annuities; and Dr. Bowditch, before he had even removed to Boston with his family, expressed so decidedly the opinion that the business would not be a source of profit with these limited powers, that, at his suggestion, an additional act was obtained, recognizing the right of the company to take money in trust to manage for individuals. His judgment proved perfectly correct upon both

points: while the former branch of business has been very trifling in its results, the amount of property already received in trust exceeds five millions of dollars, and the charge deducted for its management is the chief, almost the only, source of the profits of the company. He calculated interest tables, for the common year and leap year, specially designed for the use of this corporation, involving a great amount of labor; and a few copies were privately printed. These tables have saved the constant employment of at least one clerk. The continually increasing degree of public confidence and general popularity which this institution has enjoyed, has been chiefly attributable to the financial skill, sound judgment, strict integrity, and watchful vigilance, with which he devoted himself to its administration, and the fearless and decided manner in which he always checked, prevented, and guarded against, every possible abuse. He considered the institution as being morally the guardian of the property intrusted to it belonging to widows, minors, and others, and was careful that they should fully understand the contracts made by them, or on their behalf, and that those contracts, when made, should be observed strictly according to their true intent and meaning. Displaying the utmost courtesy, and the most liberal spirit of accommodation towards other institutions and individuals who dealt with the company, he had always in view, in its widest sense, the permanent and ultimate good of the institution over which he presided, and never compromised its interests or rights. Disarming all jealousy upon the part of the legislature, by the open and frank communications which he made to its committees, he gradually overcame much of that prejudice which a republican form of government naturally tends to foster

against all large moneyed institutions. Identified almost with himself, the public, no less than the stockholders and depositors, reposed in it a degree of trust, which has probably never been exceeded by the most extensive and well-earned popularity of any similar institution. In the settlement of estates of deceased persons in the Probate Office for the county, the records often speak of it as "the Bowditch Office."

Hardly a day passed which did not exhibit in full view all his most peculiar and methodical habits of business, and many of the most valuable and important of the distinguishing traits of his character. Instances without number might be cited. One of the wealthiest citizens of Boston, himself a member of the Board of Control of the company, wished, upon a Saturday, to deposit ten thousand dollars to be managed in trust. His balance in the bank, however, was less than that sum by three hundred dollars, and he offered to the actuary his check for that part, to be good on the next Monday. Dr. Bowditch said, "I cannot, sir, receive any check payable at a future day as cash. It is a rule of the office, which you yourself assisted in making, that I shall never part with the money of the institution, or make any engagement in its name, without an actual payment, or sufficient collateral security received in return. It is my duty to enforce this rule against the most powerful and influential, as well as the most humble, individual who deals with the institution." The gentleman was at first not a little astonished at such a novelty as the refusal to trust him for three hundred dollars for one day. Dr. Bowditch resumed, — "I am happy, sir, that it has become necessary to enforce this rule in an extreme

case. Having been once applied to yourself, no one else can ever object to a compliance with it. And it is in itself an excellent regulation." A moment afterwards, finding that his own private balance in the bank was more than that sum, he offered to take the gentleman's check himself, giving to the company his own check payable that day; which was done accordingly.

Upon another occasion, a person called to take away a policy for which he had contracted. Dr. Bowditch asked him the time of making it and the amount; then turned in a moment to two books in succession, went into the vault in which was contained the property of the company, and looking over a small file of papers in one corner, came out again, and said, "You have got it, sir." — "No, sir, I have not." — "I am certain you have." — "Nothing but your being so certain that I have, makes me doubt at all that I have not got it." — "I am ready to take my oath in court, if necessary, that it has been delivered to you." — "O, then, you remember, I suppose, placing it in my hands." — "No, sir, I have no particular recollection about the matter at all. But when a policy is once recorded in *that* book (pointing to a volume before him,) and has received the examination both of myself and the secretary, the original policy is always put by me in that corner of the safe. It is the rule of this office, that nobody shall deliver out an original paper but myself. I have the key of that safe; your paper is not there. *Therefore*, if I were called upon in court, I could take my oath that you have received it." The lost paper was of course found.

A female had deposited with the office all her property, in

strict trust for her own life, the sum being sufficient to secure her an income of about six dollars a week. She subsequently became insane, and a guardian was appointed, who took her into his own family to reside. He complained every year that the income was not enough to pay the necessary expenses of taking care of her, and said that he must have part of the principal. Dr. Bowditch told him it was impossible; that the company never would consent to any violation or modification of the original contract which the lady had made when in possession of her reason; and added, "You can have her placed in any private institution for the insane, at a much less weekly expense than you yourself charge." Finally, one day, when he called for the annual income which was payable, he refused to receive it unless he could obtain also part of the principal; and added that if the company would not pay it voluntarily, he should commence a suit to compel them to do so. Dr. Bowditch, fired with indignation, said, "The moment a writ is served upon the company for such an object, I will institute a complaint against you as an unfaithful guardian, and get you removed from your trust." From this time, the income was amply sufficient to meet all the wishes of the guardian.

A gentleman wished to obtain a loan upon mortgage. On examination, it appeared that the former owner of the estate had, *before his purchase of it*, devised all his property, of every kind, to the lady he was about to marry, and, several years afterwards, died without children, leaving her his widow; and that she had conveyed the estate in question to the applicant, with warranty. Notwithstanding the clear intent of the testator, this particular

estate legally belonged not to his widow, but to his brothers and sisters as his heirs at law. The loan must therefore be declined. But the equity of the case was so strong, that, upon the applicant's giving such further security as was required, in addition to that afforded by the improvements which he had made upon the estate, the loan was at last agreed to; and the secret of the defect of title thus discovered, was long preserved inviolate. It happened, by a singular coincidence, that the widow had died, and that her property, including the proceeds of this very estate, had been placed with the company in trust for a daughter, who, with a large family, was dependent upon the income which it afforded. The gentleman, ascertaining this fact, and being impatient of waiting for the expiration of about a year, when his title to the land would be rendered perfect by the statute of limitations, actually disclosed the defect to those legally entitled to the estate, feeling sure that, if they recovered it from him, he should be able to obtain his indemnity from the property thus placed in trust with the company. The heirs at law, as soon as they became apprized of their rights, brought a suit to enforce the legal claim, which had originated about thirty-nine years before. When Dr. Bowditch learned these circumstances, and found that the person in question, rather than wait silently a few months longer, had been willing to give effect to this unjust claim, and thus indirectly to deprive of her last resource this female and her family, he said to him, "You have involved yourself in one suit, and must lose it; and never will I voluntarily part with one dollar of the widow's money intrusted to me, to make good a loss which you have thus brought upon yourself.

You shall first have still another suit, against all the weight and influence which this company can command." The gentleman died during the pendency of the original suit, by which event the action was ended; and the period of limitation having been previously completed, a new suit could not be instituted. Nothing but the loss of life could have prevented his losing the cause.

A gentleman called to deposit a small sum of money in behalf of a young lady, his ward, to remain till she was of age. It was readily received. Before he retired, another gentleman entered, who happened to be a very particular friend of the actuary. He said, "Will you receive twenty or thirty thousand dollars in trust for me?" — "No, I cannot receive it from you." — "Why not from me, as well as any one else?" — "Because you can take care of the money yourself. Whenever, as at present is the case, there is so much money in possession of the company, uninvested, that it will not be a decided advantage for them to take any more, I receive it only from such as cannot take care of it themselves. For such cases especially was the company designed. It is a sort of Savings' Bank, except that it is on a larger scale than usual."

He also considered it very important that no money should be received in trust from foreigners or residents out of New England; both as a means of preventing ill-will of any kind, and that the whole affairs of the company might be more strictly local, and therefore more safe, than they could be if its dealings were more widely extended. And thus it once happened, in a severe financial crisis, when it would, in his own opinion, have

been advantageous to the community, in the particular case, to have dispensed with the rule, that he yet looked to ultimate consequences, and refused a deposit of one or more hundred thousand dollars, which a resident in Nova Scotia wished to place with the institution.

His intercourse with the three individuals associated as immediate officers of the institution under him, was uniformly of the most affectionate character. Requiring at all times great promptness and accuracy from them in the discharge of their appropriate duties, his kindness of heart won from them all, the same attachment which they would have felt towards a parent. These officers will never forget that he summoned them as witnesses to his will, telling them that before he died he wished to see them once more together, adding, " This is, probably, the last time that I shall have that pleasure." It was the last time. One of them (the secretary) had been his colleague from the foundation of the office; and there had existed between them a daily intercourse of the most friendly character, without the slightest interruption, during fifteen years. To him Dr. Bowditch expressed, on his death-bed, the earnest wish that he would in no event desert the institution. Having been addressed by the deceased as a son, he, as such, was one of the four individuals who, besides ourselves, attended his remains to the tomb. He will, we know, pardon the relation of the following anecdote: — Dr. Bowditch had one day gone out of the office for a few moments, and, on his return, found that he had accidentally left open the trunk containing all the convertible property of the company. The secretary might have had access to it. Without any

remark at the time, he took out the trunk, and the schedule of the property which ought to have been there, and carefully examined each item. He told us at the time, and the secretary of the company himself afterwards, that, though he would have unhesitatingly left his own property uncounted, and have felt that there was not the slightest risk from the exposure, he could not answer it to his conscience, as the responsible guardian of the property of others, knowingly to subject it even to a possibility of loss.

An instance may be mentioned of his exact and equal justice, where a member of his own household was made its subject. Several years ago, it was the duty of the individual then the messenger of the office, to receive the interest paid upon notes and mortgages, and hand it immediately to the actuary, that the proper endorsements might be made; and if, after business hours, persons called to make such payments, and were willing to leave the money with the messenger, taking his word that the proper endorsements should be made the next day, this officer was in the habit of accommodating them by so receiving it. Yielding to temptation, he spent a small sum thus received, ($120,) intending to replace it by his salary, which would be due in a few days. Dr. Bowditch's eldest son, who then was, and still is, solicitor of the company, was called upon to write certain letters to persons supposed to be delinquent in payment of interest. When he was preparing to do so, the messenger of the company called, and with many tears confessed his wrongful appropriation of the money, and begged that, at least for the sake of his wife and children, it might be concealed till the next day, when he had

always intended to replace the sum; adding the most solemn assurances that it should then certainly be done. The son consented, though reluctantly, to conceal from the father an act which he was induced to believe had been committed without any deliberate intent to defraud. On the next morning, the salary was paid, and almost immediately afterwards, instead of the promised application of it, was paid over by the messenger to an urgent creditor, who threatened him with the utmost severity of the law. Upon ascertaining this fact, the original offence was without delay disclosed by the solicitor of the company to its actuary. Dr. Bowditch's reply was, "Had it been your own money, you would have been at liberty to listen to the dictates of compassion and humanity; but as an officer of this institution, you have committed, though unintentionally, a great fault, which I can with difficulty overlook. You must give me your own check for the whole amount of the deficit, since by a timely exposure the company could have withheld the salary which has just been paid. This being done, all further action I leave to the directors." The check was then given; and this important though painful lesson of duty was cheerfully learned at the time, and has been as gratefully remembered since as the most kind and affectionate instructions which a parent's love ever communicated. Before this incident, and, if possible, still more scrupulously since that time, Dr. Bowditch determined that no one should remain in any situation attached to the office, who was laboring under pecuniary embarrassments. To see the note of one of its officers offered upon change, would with him, at any time, have been a conclusive reason for his instant dismissal. He knew intimately the weakness of human nature; that

honesty and integrity may in a moment be lost by those fatal entanglements; and he regarded the prayer for delivery from temptation as one of vital importance. In his own conduct, he practised upon the same rule. He never endorsed or became surety for any of his children, or made any engagements by which he might become liable to forfeit his independence.

In adopting the forms for various blanks, and the books for the various accounts of the company, Dr. Bowditch introduced, at the first establishment of the institution, such perfect simplicity of method and arrangement, that scarce any subsequent change has been found from experience to be necessary; the books having different columns ruled, and the matters stated in print at the top of each, which are to be recorded in it, so that a glance suffices to decide what would otherwise require perhaps a long search. Dr. Bowditch was very rapid and exact in all his calculations, such as computing interest, &c., and each one's business was in succession finished with the utmost despatch, so that it was wonderful how much he was able to accomplish. He always bestowed his own final revision upon every contract made by the company, and every note or mortgage or other security made to or taken by it;* and frequently his minute and careful scrutiny

* When a mortgage is paid off, the law makes it the duty of the lender to go to the public office where the same is recorded, and to acknowledge satisfaction in the margin of the record; for making which entry the officer is entitled to a small fee, which is payable by the borrower. Dr. Bowditch, on such occasions, always took this fee, and, wrapping round it a piece of paper, on which were minuted such particulars of the mortgage as would identify it, and prevent him from discharging the wrong one, "This money," said he, "will answer as ballast for the paper, and prevent that from getting out of my pocket, and the affair out of my memory."

would detect some clerical error, which had escaped all who had preceded him. He was equally exact and particular in his mode of transacting all the other business of the company. Every day, at two o'clock, he balanced the cash account before he closed the office, that he might leave nothing unfinished. Only the day before his death, having a week previously found himself too feeble to make an endorsement upon a promissory note of half the principal, and to look over and execute a deed of release of half the mortgaged premises, he sent to the secretary to bring him the papers again, saying, "You know I never like to leave any thing unfinished." He made the endorsement, and executed the release in question only forty-seven hours before he died. He would never listen to two speakers, or attempt to attend to two matters at once. "One thing at a time," was his rule. It brought order out of chaos; all the elements of confusion vanished at its magic influence. It was certainly the most efficient, and probably the only rule, that could have been devised for finishing all the various and complicated transactions which each successive day brought with it. Often, when engaged in making an entry, if, upon looking up, he saw a friend, he would exclaim, "In one moment!" and then proceed and deliberately finish the matter before him; after which he would say, "Now I am free, and will talk with you." He had his La Place habitually by his side, and in the occasional intervals of leisure from the calls of business or friendship, he constantly recurred with delight to the teachings of this his favorite author.

Dr. Bowditch enjoyed most heartily any laughable incident which occurred, and often, by his amusing comments or anecdotes,

awakened a like hilarity in others. Thus, upon one occasion, a person who called to buy a life annuity moved so feebly, and made so many grimaces and contortions, and groaned so dolefully, lamenting his ill health, and the short time he had to live, that it was very evident he was acting a part, with a view to make as good a bargain as possible. Dr. Bowditch enjoyed the affair highly, and, after the applicant had retired, he was describing the incident to a friend with so much comic effect, "suiting the action to the word, and the word to the action," that he even surpassed his original; and the two officers of an insurance company in the room immediately beneath his own, came running up stairs with some anxiety to know the cause of such sounds of distress and such piteous ejaculations.

It was indeed wonderful with what facility Dr. Bowditch could in an instant divert his attention from any subject to another of the most opposite character; at one moment engaged in the every-day detail of the business of his office, at the next abstracted from all around him by the most elevated investigations of science; and then, again, displaying either the utmost cordiality of friendship, or almost the wild hilarity of childhood, and apparently finding from each change an equal degree of relaxation.

Dr. Bowditch's disposition to afford every possible facility and accommodation to annuitants, depositors, and stockholders, was manifested upon all occasions. He habitually kept the office open during the afternoon of the day preceding that for the general payment of interest or dividends, of which he sent a private notice to those individuals who had the largest number

of different sums to receive, stating his readiness then to pay them; which arrangement saved both them and all others who might have applied at the same time with them on the following day, from the disagreeable but unavoidable delay to which they must then have been subjected. This not only gratified the individuals in question, but was indeed, indirectly, an equal accommodation to every one else, besides that it insured greater accuracy than if the entries were made in a hurry, with many standing around waiting impatiently for their own turn.

He was also desirous that females who had annuities or deposits in trust, should come in person to the office to receive their payments, as he wished them to see and judge for themselves as to the management of their property, and that he might himself give them any explanations and information which they desired; and the moment a lady entered, she took immediate precedence of every one else, and the claims of some of the most considerable depositors have often been thus postponed to those of a poor widow who had intrusted to the institution her little all. No female annuitant, indeed, ever left the presence of Dr. Bowditch without having been delighted with his courteous and polite reception, and with the ready, frank, and kind manner in which her inquiries had been answered or her wishes attended to. The courtesy thus shown to female annuitants was extended to females applying for loans upon mortgage — but in rather a peculiar way, viz. *the uniform refusal of their request.* And it was certainly with some ingenuity that this rule of the company was supported exclusively by reasons based upon gallantry towards the sex. He said to them, "It is impossible to accede to your

request, because, should any delinquency occur, the company could never be so rude or harsh as to institute a suit against a woman, or to take forcible possession of her estate. Therefore we never lend except to a *man*, with whom we can immediately resort to all the strict measures of the law, in case it becomes necessary." So plausible a reason was always satisfactory.

Prompted by a similar motive of politeness was another of his private rules. Aware of a difficulty which he through life experienced in remembering names, and that the self-love of applicants at the office would be hurt at the necessity of informing him who they were, he was in the habit of referring every one whose familiar features thus perplexed him, to another officer of the institution, to get the number of the policy or mortgage respecting which question had been made. The clerk understood this request, and began by asking *the name*, which was a less mortifying question from him, than it would have been from the principal of the office. He then handed the name and number to the actuary on paper.

The most difficult duty to be performed by the actuary of this company, and at the same time one of almost daily recurrence, was that of refusing applications for loans of money which he thought it not safe for the institution to grant. It often required great firmness and decision. Powerful influences, direct and indirect, were often resorted to in order to obtain a favorable answer. But it is emphatically true that Dr. Bowditch understood the art of saying "No;" and while he decidedly and peremptorily declined an offer as inadmissible, so that no time should be wasted

in profitless discussion, it was always his endeavor to do it with as much courtesy of manner as possible. He was well aware, however, that this was the most thankless part of the actuary's duties; that though a manly, independent, and decided course, would certainly secure the respect and approbation of the majority, and promote the interests of the institution, it must also necessarily give offence in individual cases. Such cases did occur. There never lived the man whom Dr. Bowditch feared to address in what he considered the language of truth, and he often spoke with a plainness and directness to which his hearers had not been accustomed.

It was always a painful duty, however, to be compelled to disappoint applicants by the refusal of their requests, though it was one which, as has been stated, Dr. Bowditch never hesitated to perform. Sometimes, indeed, he declined requests, which he subsequently thought might, with some slight modification, have been admissible; and in such cases he was always ready and willing to recede from his first position. An instance of this kind occurred a few weeks before his death. The proposal made by a friend was declined, as not coming within the rules of the office. The applicant had no idea that Dr. Bowditch at that time was laboring under a serious disease, and manifested some surprise and irritation at this unfavorable answer. With a slight change in the terms of the proposal, by which the original objection was removed, it was, in a day or two afterwards, acceded to. It was soon known that Dr. Bowditch was alarmingly ill. No one was more earnest and constant in his inquiries respecting his health, than the gentleman alluded to. The day but one before

his death, Dr. Bowditch made some remarks to his eldest son, desiring him to communicate them in his name to that gentleman. A letter was accordingly written, in which, after stating his uniform sentiments of esteem and respect during a long intercourse, and alluding to a common descent from the same remote ancestor, (John Turner,) as having strengthened by relationship the feelings of good-will which a knowledge of character had first produced, — Dr. Bowditch proceeds to say that if any incident has ever occurred between them of a less friendly description, he has never let the sun go down upon his remembrance of it, and hopes that it has been equally forgotten by his friend. A reply was received, in which the writer says, "I have ever been inclined to reverence the silver hairs of an honest man. Associated with the consideration that they are connected with great public services, inflexible independence of thought and action, and a very high order of intelligence, duty, not less than inclination, commands our respect." A copy of the original communication, sent by his son, as he had desired, and the reply to it, were read to Dr. Bowditch only twenty-three hours before his death. They were the last to which he ever was a listener. He died, as it were, in the very act of forgetting and forgiving, and asking a like forgetfulness and forgiveness of, all the incidents connected with one occasion upon which he feared that, as actuary of the Life Insurance Company, he had perhaps unnecessarily said "No." *

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* Of a similar character is the anecdote of his once asking the pardon of some young men in the Salem Athenæum for having upon a certain occasion spoken, as he thought, somewhat too quickly to them. — *Judge White's Eulogy,* p. 57.

During the late disastrous period, when every bank in the United States was compelled to suspend specie payments, Dr. Bowditch conducted the affairs of the Company with such caution, that — though this was the largest moneyed institution in New England, having a capital equal to that of ten common banks, and though its dealings were necessarily extended throughout the community — the actual loss sustained by the reckless management of other institutions, and by the numerous bankruptcies which destroyed all commercial confidence, was less than that of any one bank in the city, and was more than balanced by the reserved profits resulting from the success of a financial measure which he had previously suggested and executed.

Such was Dr. Bowditch, the Actuary of the Massachusetts Hospital Life Insurance Company. He had qualifications rarely, if ever, found united in one individual, and they had here their happiest and fullest exercise, and accomplished a most useful result. Not inferior to his fame throughout the scientific world as the author of this work, will be the reputation which he has left among all connected with that institution, as one who, (in the language of the Board of Control in one of their resolutions adopted after his decease,) "by the clearness and simplicity of the regulations he devised and adopted, and the intelligence, fidelity, and inflexible resolution with which they were adhered to and executed, has preëminently contributed to the present stability and prosperity of the institution." In another of these resolutions they describe him as "one who lived long enough to perform all the duties of a long life, although not permitted to attain old age; who has left to his family a bright example,

and a name that will be known and honored throughout the world so long as virtue and science shall be held in reverence."

It is gratifying to us to reflect that the institution whose continued prosperity was almost the last earthly wish of Dr. Bowditch, has been intrusted to the hands of one whom, of all others, he would have himself selected to be his successor ; under whose auspices we doubt not that it will long possess, as heretofore, the unlimited confidence of its friends and of the community.

Dr. Bowditch was, from 1826 to the end of 1833, a Trustee of the Boston Athenæum. At the time of his appointment, it was in a situation far from prosperous. One whose name has ever stood foremost upon the list of public benefactors in this city, generously offered to the institution eight thousand dollars, if a like sum in addition could be obtained. Dr. Bowditch, with the assistance of a friend equally zealous in the cause, undertook the task of procuring the performance of this condition. They first waited upon a nephew of the original donor, who, upon the circumstances being stated to him, immediately said, "I will follow the example of my uncle, and give the same sum, provided you can get from others sixteen thousand dollars." This brilliant success in the outset, in reality, as he perceived by the condition thus annexed, doubled his future labors. But he saw in it only an opportunity of urging more strongly upon others a like munificence, as the withholding of each small sum might endanger the loss of the whole promised bounty. His efforts, therefore, were unremitting. With that persuasive eloquence which is always inspired by disinterested zeal in a good cause, and which

few could resist from his lips, he appealed so forcibly to those he addressed, that he obtained much more than the requisite sum. One great object to which the funds thus gained for the institution were applied, was that of perfecting its collection of works of science; and here his labors were no less useful than they had been before. He had previously accomplished one measure, far more important in his view than any other, and without which he felt that any future labors would be of but little advantage, — namely, that of permitting subscribers to take books from the library for the use of themselves and their families. The benefit of a like arrangement he had long experienced while connected with the Athenæum in Salem. This was at first vehemently opposed by some of the most intelligent of his associates, who apprehended from this plan evil consequences, which have been proved by experience not to result from it. There can be no doubt, indeed, that the final attainment of this his favorite object has been of great benefit to the citizens of Boston.

These services were remembered so gratefully by this institution, that, on his decease, its trustees felt themselves called upon publicly to declare their nature and extent, in order that the community might duly appreciate its obligations to him. Death had removed the necessity of that silence which was more grateful to the modesty of the living, than would have been even that just and appropriate eulogy, which, after alluding to his particular services above mentioned, proceeds thus: — "But Dr. Bowditch has far higher claims to notice; he stood at the head of the scientific men of this country, and no man living has contributed more to his country's reputation.

His fame is of the most durable kind, resting on the union of the highest genius with the most practical talent, and the application of both to the good of his fellow-men. Every American ship crosses the ocean more safely for his labors, and the most eminent mathematicians of Europe have acknowledged him their equal in the highest walks of their science. His last great work ranks with the noblest productions of our age." — "But it is not merely the benefactor of this institution, and the illustrious mathematician, whose labors have given safety to commerce and reputation to his country, whom we lament. It is one whose whole life was directed to good ends; who combined the greatest energy with the kindest feelings; who was the friend of every good man and every good undertaking; the enemy of oppression, the patron of merit, the warm-hearted champion of truth and virtue. It is the companion, whose simple manners and amiable disposition put every one at ease in his presence, notwithstanding the respect which his genius inspired; and who could turn, apparently without effort, from the profoundest investigations, to take his part, with the light-heartedness of a child, in the mirth of the social circle. His heart was as tender as his intellect was powerful. His family found him as affectionate as he was wise; he was equally their delight and their pride. They could have no richer inheritance than his character; and nothing but such a character could afford them consolation for such a loss." And for this consolation they refer us, in their concluding resolution, to "the contemplation of a life so gloriously spent, and which has left such enduring monuments of excellence in every department, whether of science or of practical utility, to which it has been devoted."

A marble bust of Dr. Bowditch, executed some years since by John Frazee, of New York, was presented to this institution by the gentlemen at whose request it was taken. Though it accurately represents the features, the artist has not succeeded in arresting that bright and cheerful expression of the deceased, which his children will ever most delight to recall.

Under his auspices as President of the American Academy, one volume of its Transactions * has been published. He also procured an important modification of the terms upon which a donation had been made to that body by Count Rumford; so that, from being wholly worthless, it has been rendered available for the general objects of the society. He obtained the hall over the Life Insurance Company's office for the use of this institution, where its excellent collection of books was neatly and elegantly arranged under his direction. His youngest son, who had succeeded in making a fine catalogue of his own library, he had requested at his leisure hours to prepare one likewise for this; which labor has been nearly completed since his decease. An artist has, at the request of the Academy, recently executed a marble bust of their late president, in whose death they lament the loss of "their distinguished associate and head, whose name

* This is called Vol. I. of a New Series, to avoid the necessity of sending to foreign members of the Academy copies of all the earlier volumes; as he considered that Vol. I. of the Old Series contains some mathematical papers of so inferior a character, as to indicate a low state of that science in this country at the time of its publication. It may be observed that, by this arrangement, he was obliged also to consign to the like obscurity all his own communications, as they are recorded in the subsequent volumes of the Old Series.

has for many years conferred honor upon their institution, and whose communications are among the most valuable contents of the volumes of their Memoirs;" — "of a friend and fellow-citizen, whose services were of the highest value in the active walks of life, whose entire influence was given to the cause of good principles, whose life was a uniform exhibition of the loftiest virtues, and who, with a firmness and energy which nothing could shake or subdue, devoted himself to the most arduous and important duties, and made the profoundest researches of science subservient to the practical business of life." *

Dr. Bowditch was accustomed to say, after his appointment to a seat in the Corporation of Harvard College, that his two high holidays were those occasioned by the literary exercises and festivities of the annual Commencements of that institution. On these days he might always be seen listening with interest and attention to the various performances.

Though not himself a practical mechanic, there was no class in the community whom he more valued and respected. Many intelligent mechanics will remember the familiar and friendly manner in which Dr. Bowditch has often joined them when

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* The Eulogy pronounced, pursuant to another resolution of the Academy, by John Pickering, Esq., one of Dr. Bowditch's earliest and most intimate friends, has been before referred to. It presents to the reader, with the utmost fidelity and accuracy, and with great thoroughness of research, an analysis and estimate of the scientific labors and services of the deceased, to which we with pleasure acknowledge our own obligations in preparing the present memoir.

walking, and continued to walk with them arm in arm. Living in a republic, he respected in others, and aspired himself to no aristocracy, but that of character and talents — that which results from useful and honorable labors either of the hands or of the head. No sight ever afforded him such pleasure as that of the working classes of the city, upon one day in seven, dressed in their Sunday clothes, and forgetting the laborious occupations of the week, enjoying with their wives and children the pure air and beautiful scenery of the Boston Common.

The Boston Mechanics' Institution considered him justly entitled to the honor of being elected its President, even though his manual dexterity in any particular craft might be doubtful. He received this appointment January 12, 1827, and resigned the office April 27, 1829.* A valuable apparatus was purchased by subscription, which he promoted by his influence and example; and lectures upon the steam engine, and other similar important subjects, were delivered with much success. And it is believed that all the lectures now delivered so generally before various institutions in Boston, upon almost every evening in the week, and by which so much valuable information is diffused through the community, find almost their first precedent in this country in the course given by the mechanics of Boston, with Dr. Bowditch at their head.

By the same body of men Dr. Bowditch was placed on the select list of honorary members of the Massachusetts Charitable

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* He was chosen *first* honorary member of this society, May 15, 1829.

Mechanic Association, (February 4, 1828;) while those whom he through life had most benefited were proud to own him as one of their profession, and elected him a member of the Boston Marine Society, (March 2, 1830.) The latter society, upon his death, say, "'He hath wrought a good work, and rests from his labors.' His intuitive mind sought and amassed knowledge, to impart it to the world in more easy and comprehensive forms. His life and example, in all their phases, present more to admire and approve than we may hope to see imitated and achieved by another individual." Both these societies, receiving unitedly in his will the same affectionate and honorable mention, unitedly listened to the Discourse which the Rev. Alexander Young had previously delivered before his parishioners,* in the church where the deceased had worshipped.

Thus various and important were the public relations which Dr. Bowditch sustained in the community around him, and thus

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* The first in the series of the publications of this class in point of time, it details very fully the incidents of Dr. Bowditch's life, and especially those illustrating his personal and social habits and character. Some slight errors in this Discourse, chiefly respecting the time when, and the circumstances under which Dr. Bowditch gained particular acquirements, a subsequent investigation has enabled others to correct; but the substantial accuracy and fidelity of those *moral* delineations, which it was his peculiar province and design to present, will ever remain unimpeached. On the afternoon of the day when it was delivered in Boston, it was repeated at the church of the late Dr. Prince in Salem. Of its delivery upon this latter occasion, it is recorded that "in that compacted audience, there were several present who had witnessed the whole career of Dr. Bowditch, from earliest childhood to the lofty summit of his usefulness and fame; and among others, Captain Henry Prince, under whose command he had performed his four first voyages;" — and it is added that "it was a striking evidence of the sincere and deep sympathy of the audience in the commendations bestowed by the preacher upon the

uniform and unqualified is the testimony given respecting the value of his diversified labors and services. Beautifully has one of his eulogists* said, "The world has been the wiser and the happier that he has lived in it." And his death was announced in one of the public journals,† with but the following brief comment: "A star has fallen." Connected with so many different classes of society, he seemed, as he walked through the streets, constantly to meet an acquaintance or a friend. None were so high that they did not feel themselves honored by his notice, and none so humble as to be beneath it. The little child on its way to school was often arrested by some kind inquiry from one who had been prepossessed by its sweet or intelligent countenance, and detained by a dialogue which ended in a kiss, or some other act of endearment; and virtuous age ever received his reverence. The Chief Magistrate of the Commonwealth, with friendly sympathy, was present at those simple funeral rites by which we felt that the deceased would be most appropriately honored; and one in humble life, who was to us a stranger, asked permission to take a last look at his lifeless remains, "because he had known Dr. Bowditch and *loved* him."

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character and merits of the departed, that, after the discourse was finished, though large numbers of them had been standing for three hours, they continued in the church to listen to the dirge commencing with the unrivalled lines, —

'Unveil thy bosom, faithful tomb;
Take this new treasure to thy trust,
And give these sacred relics room
To slumber in thy silent dust.'"

* Mr. Young.

† National Intelligencer, March 21, 1838.

But for his residence in Boston, Dr. Bowditch would never have possessed the means of publishing this work at his own expense ; and notwithstanding all his daily duties and occasional labors above described, during his residence here, and the performance of multiplied good offices to individuals as before, and by which, in the aggregate, almost as much service was rendered to society as by his more public efforts, — he also found the leisure which he needed for this the last great undertaking of his life.* The estimated cost of publishing the five volumes exceeded twelve thousand dollars, and was equal to one third of all his property at that time. To this undertaking, involving so much expense and labor, he was strongly urged by his wife, who assured him of her willingness to make any sacrifice which

* The first volume was published in the year 1829, the second in 1832, and the third in 1834. These three volumes were from the press of Isaac R. Butts. The fourth volume, from page 684, has been stereotyped at the Boston Type and Stereotype Foundry. If Dr. Bowditch had lived, he would probably have stereotyped all the others. Connected with the publication of this work, may be mentioned two anecdotes exhibiting a trait of character in the translator which has been before alluded to. Robert W. Macnair, whose name is affixed to the second volume as a compositor, was one with whose accuracy, neatness, and assiduity, Dr. Bowditch was always much pleased ; and he was gratified to find that he felt such a pride in the appearance of the volume, as the result, in part, of his own manual dexterity, that he wished the fact of his agency in preparing it to be thus always known. Dr. Bowditch, on learning his death, (which took place after a short illness, February 27, 1833,) expressed to his widow his sympathy for her loss, and, notwithstanding her husband had been employed and paid by another, gave her at parting the sum of fifty dollars, as an acknowledgment of the zeal and fidelity which had been shown in his service. And during his own last illness, he told his children that with the printers and publishers of this work he wished them "to deal liberally." And when, after his death, this circumstance was communicated to them, they said, "It is like him ; he always acted so towards us while living."

it might render necessary.* She knew that, in the event of his death, he had made her his sole legatee. But, like himself, she valued money only as a means of attaining desirable ends. And to what more noble or worthy purpose could it possibly be applied? She did not live to see the publication completed, though she found the reward of her advice, in those high terms of commendation with which each successive volume was mentioned by the most eminent scientific men of the age, in this and in foreign countries, and in the constantly increasing fame and reputation which were thus gained by her husband. The letters of this description, already, in part, laid before the reader, the wife listened to as to the sweetest music, for they contained the praises of one dearer to her than herself. And so deep was Dr. Bowditch's conviction that, but for her disinterested advice and urgent solicitation, the publication would never have been commenced, that he prepared a dedication of the work to her memory. This document, in his own hand-writing, manifesting, as it does throughout, a deep feeling of affection, he had always preserved; and, during his last illness, he gave to it the sanction of a love stronger even than death, by enjoining on his children as his last wish, that they should prefix it to this the posthumous volume of his work, and thus pay a public tribute of respect to the memory of their mother. Willingly do we perform this sacred duty. It is indeed fitting that they who in life were inseparably connected, no less by the bonds of an earthly marriage, than by the more intimate union of mind and heart, should forever remain associated in the memory

* The publication was indeed decided upon in a family conclave, in which there was no dissenting voice.

of every reader of this work, as those to whom he is jointly and alike indebted for any pleasure or profit which he may have derived from its perusal.

As early as 1826, Dr. Bowditch first perceived in his wife the symptoms of that fatal disease which had deprived him of two sisters, and which, after the lapse of eight years, was to remove from him a still nearer friend. At intervals during this period, his wife enjoyed her usual health, and her accustomed cheerfulness never deserted her. She gradually became more and more feeble. Aware of her situation, and resigned to it, no one except her confidential medical adviser heard from her lips those convictions of her approaching end, which she knew, if expressed to them, would send sadness to the hearts of her husband and children. They were not, however, deceived. When, a few days before her death, she was borne in a chair by two of her sons into that library where she ever delighted to sit, it was only her pale countenance and debilitated frame which told us — and they did, alas! tell us but too truly — that soon one seat would be vacant, and one voice silent, in that assembled household. The blow, however, fell suddenly and heavily at last. We were awakened at midnight, and told that the fatal hour had come. To him who first reached her apartment she extended her hand, and, giving to his a gentle pressure, — a proof of consciousness and of love, — she murmured a few words so feebly that they did not reach his ear, but they were distinctly heard by her attendant: "My dear, you have come to bid me farewell." She died also in the presence of her eldest children. The unbroken slumbers of the youngest left them, for a few hours longer, happily unconscious of their

loss; and one was destined to learn the event in a distant land, who therefore had not a personal knowledge of those consoling circumstances which a brother's pen could at best but inadequately describe. It was truly, as the historian of America * has said, when speaking of a similar death-bed, "too serene for sorrow, too beautiful for fear." The wish which she had often expressed had been granted. She died before her husband, April 17, 1834, and was followed to the tomb only by those few whose home had ever been gladdened by her presence.†

Dr. Bowditch bore this heavy calamity as a Philosopher and a Christian. The early morning witnessed the funeral obsequies which he attended; and that forenoon saw the Actuary of the Life Insurance Company engaged in his usual routine of business, and at intervals examining the proof-sheets of this work, upon whose progress he was never more to look with a pleasure heightened by her participation; and the kindly ministrations of time were aided by this cheerful discharge of duty, and by this devoted pursuit of science, till he was himself summoned to receive the glorious rewards of eternity. To the stranger he appeared as he had ever done before. To his friends and family his character displayed a strength and grandeur never until then fully appreciated. Most deeply, however, did Dr.

* George Bancroft. See his History of the United States, Vol. I. p. 388.

† Trinity Church, in Boston, was rebuilt after Dr. Bowditch removed to this city; and he became proprietor of one of the new tombs constructed beneath it. On the day of his wife's funeral, he executed an instrument transferring this tomb to his four sons in trust, as the future burial-place of himself and his descendants. That trust has already been fulfilled towards himself, and also towards a granddaughter, born subsequently to his death.

Bowditch feel this loss; and sometimes, particularly during his own last illness, he alluded to it with much sensibility. His countenance, after her death, exhibited, more frequently than before, a degree of thoughtfulness sometimes amounting almost to sadness. Indeed, he frequently stated to his children, though the fact may not have been apparent to the public, or even to his friends, that though life had still many charms for him, it had lost forever what he had always regarded as its brightest attraction. And we felt that this event had devolved upon us additional duties of filial tenderness and regard towards him who had been so severely bereft.

The various other incidents of Dr. Bowditch's life, during his residence in Boston, which led to the display of his peculiar talents and virtues, were few of them so conspicuous and remarkable as to be especially deserving of selection, though scarce one can be mentioned which would not add greater clearness to the reader's previous impressions. He one day fearlessly seized a carman who was cruelly beating a horse, and obliged him to desist by the mere alarm which his vehement and indignant manner inspired, though in bodily strength wholly his inferior. When Lafayette visited this country, Dr. Bowditch found himself, he hardly knew how, in the street near his chariot wheels; and amid the acclamations of the multitude, he too waved his hat and joined his voice to the praises of a virtuous and honorable life, which were then spontaneously rising from countless numbers of grateful citizens.

When a Roman Catholic school, in the adjoining town of

Charlestown, occupied by defenceless females,* was attacked at night, and its frightened inmates dispersed by the imprecations and torches of a band of deluded fanatics, he felt indeed that the fair fame of the state had received a deep, if not an indelible stain, and that the same town which is memorable as the scene of the first of freedom's battles in modern times, would also exhibit a monument of the most ruthless violation of private rights. He most openly expressed his abhorrence of this act, and calling upon the Bishop, whose church and residence almost adjoined his own, said to him, "Though our forms of worship are the most opposite and widely separated of all the creeds by which the Christian church has ever been divided, upon this ground I make common cause with you. This act has awakened me from a pleasant dream of security, and shown to me that the fanaticism of one class of this our orderly community, if it had the power, would not want the will, to attack with fire and sword all those whose peculiar modes of faith or religious institutions should happen to excite suspicion or incur hatred." And he at the same time gave him a small sum toward the immediate relief of those whom the flames had deprived of the necessaries of life. In recollection, doubtless, of this incident, the bells of the Catholic church were prevented, by orders from the Bishop, from being rung during Dr. Bowditch's last illness, — although it was at the season of Lent, — "that the last days of a good man might not be disturbed."

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* The immediate cause of this outrage was the supposed confinement of a female against her will; and this belief was chiefly occasioned by the popular prejudice against Catholics, convents, and nunneries.

Such, indeed, was his respect for the law of the land, that, when he had but a few days to live, he expressed the determination to make the effort to see the Governor of the Commonwealth, should he again call to inquire respecting his health, that he might assure him of the pleasure he felt at a recent act, by which he considered the law to have been suitably vindicated; namely, the disbanding of certain military companies, for an open violation of discipline on a day of public parade. The interview accordingly took place.

The existence of domestic slavery in the Southern States of the Union is a subject of so much importance, and its discussion has been the source of so much excitement in the community, that it may perhaps be proper briefly to state Dr. Bowditch's views in regard to it. Considering slavery to be one of the greatest of moral evils, his whole principles and sympathies were on the side of the oppressed. He scorned the selfish and timid considerations by which many were led to refrain from or to check the free discussion of its character and tendency. It was, however, a subject upon which he thought and acted for himself. The blacks he regarded as a race of men naturally less intelligent than the whites; and he believed that their present servile condition had so degraded them, that an immediate emancipation, extorted from the slaveholders, while it would find the slaves ill fitted for self-government, would also prevent the experiment from having that fair chance of success, which would be afforded by a cordial coöperation of their former masters. He would gladly have seen a national debt, even of immense magnitude, voluntarily incurred for the purpose of accomplishing this object, and at the same time

indemnifying the slaveholders, and thus securing to the slaves their aid and good-will. Indeed, under the original compact made between the several states, he did not think that the *moral* right existed in the free states to attempt to compel the emancipation of the slaves without making such compensation. And though Congress has exclusive jurisdiction within the District of Columbia, he did not think that, even there, the measure of the immediate abolition of slavery should be introduced, without first obtaining the consent of those states by which the District was ceded to the general government. He had the greatest horror at the thoughts of the proposed annexation of Texas to this Union, and was delighted with Dr. Channing's pamphlet, as he would also have been with Mr. Adams's speech in Congress upon this subject. He was in like manner utterly hostile to the admission of any new slave-holding state into the Confederacy. He often said that he never wished to shake hands with, or even to see, a northern man who, surrounded by free institutions at home, had voted for any extension of the evils of slavery. Such a person he deemed rightly characterized as one of the " white slaves of the north." Though he did not himself approve of all the views and measures of those who advocated the immediate abolition of slavery, he admitted that no great moral or religious revolution had ever been accomplished except through the agency of a few enthusiastic and excited spirits, whose apparently excessive and over-zealous efforts at last aroused *the many* to a sound, moderate, and successful reformation of abuses. Such he hoped would be the issue of the like efforts in the present instance. He considered the movement begun which would sooner or later prove fatal to this institution.

Upon one occasion, Dr. Bowditch was introduced by a friend to a stranger, who had heard much of his reputation, and was obsequious and almost servile in his manner of addressing him. Dr. Bowditch replied with stateliness and reserve. After the interview was ended, the stranger said, "If there does not go an aristocrat, there never was one;" to which remark the friend replied, "He an aristocrat! I care not how many such we have among us. The truth is, you treated him as one, and he despised you for your cringing manners, and want of a proper self-respect."

At the time when Dr. Bowditch was preparing to leave forever the home of his ancestors, almost his last act had been to repair with pious reverence the dilapidated monument beside which he had seen his grandmother's remains deposited,* and beneath which reposed the ashes of all her relatives of many former generations — the tomb of John Turner. In September, 1835, the board whose peculiar province it was to *take care* of the burial-grounds of Salem, finding several tombs out of repair, advertised them for sale, and unceremoniously ejected the remains of some who, in their day, had been Salem's greatest benefactors. The act was at first the result of a want of due consideration in two or three

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* Dr. Bowditch often mentioned that his grandmother, on her death-bed, refused to be buried in this tomb, saying that, many years before, at the funeral of one of the family, a mourner took up her father's skull, and holding it before her, said, "This is the skull of an Indian warrior." She seemed to have a prophetic dread of the possibility of the outrage subsequently to be committed, and preferred that her remains should be consigned to the safer custody of her parent earth.

individuals, members of this board; but finding themselves actually committed by it so far that they could not retract, they induced their associates officially to adopt and defend the measure. The tomb in question was thus violated. Dr. Bowditch was indignant at an act which was alike revolting to his private feelings, at variance with every dictate of humanity and civilization, and which, if acquiesced in, would be a permanent public disgrace to the city which he loved. He headed an address to that board, and subsequently one to the selectmen. The public press was loud in its denunciations of the act. The board reconsidered their decision. None indeed, apparently, at last regretted it more sincerely than themselves. Dr. Bowditch said, on this occasion, that had the act been rendered necessary for the promotion of any public object, he would have cheerfully surrendered his own private wishes to the interests of the community. Accordingly, when, a short time afterwards, the city authorities of Boston wished to lay out an avenue or public walk through one of the burial-grounds, — and had met with such sincere and vehement opposition from two or three individuals, whose relatives were there buried, that a useful public measure was in danger of being abandoned, — Dr. Bowditch waited on those gentlemen, and, sympathizing as he did most fully in all their feelings, yet wholly succeeded in conquering the repugnance, which he satisfied them ought to yield to other and higher considerations.

Dr. Bowditch was in person under the common size. His hair, originally of a light color, was entirely gray at the age of twenty-one years, and gradually became of a silvery whiteness. His high forehead, bright and penetrating eye, open and intelligent

countenance, are, we think, accurately shown in the annexed engraving; though the changes which, with the rapidity of lightning, passed across those expressive features, as they in turn exhibited the feelings of benevolence, or the most intense thoughtfulness, — at one moment radiant with smiles, and at another dark with virtuous indignation, — can never be realized but by such as have themselves seen and studied there the outward manifestation of all that was most excellent and beautiful in his character. His, indeed, was a face never to be forgotten. Intellect there altogether predominated over sense.

He always possessed great bodily activity, and late in life he might often be seen running along or across the streets with as much quickness as in youth. In his daily walks, indeed, he seemed constantly eager to outstrip all his competitors. He was very methodical in his habits of exercise, seldom walking less than five or six miles each day. He fully appreciated the importance of this practice to a person of sedentary pursuits. Throughout the summer, he was in the habit of driving with a horse and gig eight or ten miles in the afternoon; and during one or more seasons, he mounted his horse and rode before breakfast.*

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* He always drove with great rapidity. A friend, who was riding at a very moderate rate, was once passed on the road by him, and when they next met said, "You whisked by me like the tail of a comet." At another time, a person called upon one of Dr. Bowditch's sons, and, after a few remarks upon the furious mode in which some *young* men were in the habit of driving, demanded of him compensation for a slight injury which had been thus occasioned, as he believed, by him. The supposed *youthful* offender proved to be Dr. Bowditch himself, by whom, however, the blame of the accident was laid wholly upon the

It has been strikingly said of him that "*he was a live man!*"* All his processes of body and of mind, all his thoughts, all his actions, were full of life. When any thing pleased him, he would rub his face with his hands, or rub his hands together, with an expression of the most free and unrestrained delight; and when any thing displeased him, and he felt excited enough to determine to speak, he always, as he said, found himself upon his feet, without knowing how he got there; and except in a standing position, his tongue never became effectually loosed. On such occasions, his vehement and earnest manner was most impressive in its effect upon the beholders, and it truly appalled the individual against whose unjustifiable opinions or conduct his censures were directed.†

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other party. He once attended Commencement at Cambridge with quite a spirited horse, and in the evening started to return to Salem. His horse, however, seemed very unwilling to move, and almost insisted upon turning into the yard of a clergyman's house on the road. Dr. Bowditch resorted to the argument of the whip, and at last reached Salem, after a drive at the rate of about three or four miles an hour. On the contrary, a country clergyman, who had also attended Commencement, was very much alarmed at the rapidity with which his horse carried him home, and at his impetuous and almost ungovernable movements. The double mystery was easily explained; and when the clergyman received back his own animal, he said, "I am delighted, Dr. Bowditch, that my poor beast fell into such good hands. If the mistake had happened, as I was afraid it had, with some gay young collegian, my horse would have been terribly beaten." Dr. Bowditch said that his conscience smote him as he listened, and thought how little cause there was for this self-congratulation.

* See anecdote in Judge White's Eulogy, p. 57.

† It has been observed of Dr. Bowditch, that, "though no 'rude and boisterous captain of the sea,' there may have been occasions when a happier combination would have been produced, had the same measure of the *fortiter in re* been blended with more of the *suaviter in modo*." (*North American Review, January*, 1839.) We do not deny that such instances sometimes, though they rarely occurred. Dr. Bowditch, in a conversation with his eldest son

Dr. Bowditch generally enjoyed excellent health, the result, beyond doubt, of his regular and temperate habits. At the age of thirty-five years, however, his life was considered in danger from the disease of which, at that precise time, (1808,) his two sisters were dying. He, like them, was attacked with the alarming symptom of bleeding from the lungs. Upon this occasion, his friend Thomas W. Ward, Esq., relinquished all his own engagements, and devoted himself to the invalid during a journey of several weeks. As they were leaving an inn in a town about twenty miles from Salem, the landlord beckoned to Mr. Ward, and asked him where his friend lived, and, on being told, advised their return, in the apprehension that the latter could not even live to reach the next stage in their intended route. By the invigorating effect, however, of the exercise thus taken in the open air, his disorder was checked, and his health completely reestablished. Until this time, he had never tasted wine. It was then prescribed as a medicine. When a young man, — but at what precise age is not known, — he had agreed to sit up with a friend who was ill, and, being unwilling that so much precious

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upon this subject, once said, "There is a gentleman in this city, (naming him,) who possesses such courtly manners, that he can utter a bitter sarcasm, or express profound contempt, in the most mild and conciliatory language. Such, however, is not my case. If I am obliged to measure my words, or even to think the least about them, I lose the substance of what I intended to say. When I feel that I cannot remain silent, I speak — and in such terms that *no one can mistake my meaning*. But, my speech being ended, the whole affair is over. I pour out, indeed, the contents of my vial of wrath, but I then let it be seen that it is left empty." And though it is certain that his was not that guarded demeanor, which, upon every occasion of life, prevents the utterance of a word which it may be desirable to recall, it is also certain that this was a source of more regret to himself than of pain to others.

time should be idly spent, he passed the whole night in mathematical computations. He was much alarmed, the next morning, to find his vision obstructed by little motes or specks passing before him in grotesque variety and constant succession. It was ascertained that he had taxed his eyes beyond their powers, and it was two years before he was able again to use them freely.

Dr. Bowditch removed to Boston a few months before the necessary arrangements could be made for his family to join him in that city. Hardly had he been there two days, when, under the influence of a disorder to which he had never till then been subject, he fell senseless in the street. It happened that the hospitable mansion of the same friend, to whom, as just stated, he had before been so much indebted, was now freely offered him as a temporary home. Only once again did this vertigo cause him to fall in a similar manner; and then great indeed was the consternation excited in his family as they perceived a crowd approaching bearing his apparently lifeless body, while, from a wound in his head, blood was flowing profusely. A tendency to this species of attack, however, always continued. But, ascertaining that it was brought on by exercise immediately after eating, and that it was always carried off by sitting down and resting a few minutes, he avoided its exciting cause, and thus never experienced any subsequent ill effect from it. It was rarely, however, after this, that he walked alone. And often, when attended by one of his sons, has he stopped to look in at the window of some shop which they were passing, or even walked in, and asked for a seat, because he felt the sure indications of approaching danger. He well knew the delicate organization

of human life, upon which depend alike all the functions of the body and of the mind, and he often expressed his surprise that what seemed so fragile should yet be able to resist so much. One of his favorite quotations, indeed, was that of the beautiful lines by Watts —

> "Strange, that a harp of thousand strings
> Should keep in tune so long!"

And it should ever be remembered that the publication of this work was commenced after he knew that he could no more expect the robust and fearless health of his youthful days.

Dr. Bowditch always continued his habits of early rising. He greeted the rays of the morning sun of summer as they first entered his library, and by hours of study anticipated its tardy beams in the winter. Often has he been heard to exclaim, as it then first met his eye,

> "See, from ocean rising, bright flames the orb of day!"

with as much enthusiasm as at that period of his life when he had made the same glorious luminary his guide over the trackless deep.

The following may not be uninteresting as a strictly accurate description of each day of his life, for all the period of his residence in Boston:* — He had formerly been in the habit of walking before breakfast, but during this period he breakfasted immediately after rising, in the winter by candle light, and always before the rest of his family. He then applied himself to

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* See the similar account, drawn up by us at the request of Judge White, and printed in the notes to his Eulogy, p. 70, &c.

mathematics, gaining from two and a half to three hours' study; after which he walked about a mile and a half, attended by one of his sons, and commenced the business of the day at his office a few minutes after nine o'clock. There also, as has been stated, mathematics was the occupation of all the moments left at his own disposal. He frequently walked home in the forenoon for a few minutes, as he found his eyes strengthened and refreshed by being at intervals in the open air for a short time.* Every day at two o'clock the office was closed, and he then walked as before, being usually accompanied by a friend who still lives to find in the recollection of this daily intercourse one of the most pleasing reminiscences of the past. He dined at a quarter before three, P. M. After dinner he indulged in a short "siesta," which lasted from fifteen minutes to an hour, sometimes even longer. He always awoke bright, and prepared to recommence his studies, which he pursued for about an hour and a half to two hours. He always, near the close of the afternoon, went to his office again, though it was not open for the transaction of business, to see if any thing needed his attention or explanation; and in the latter months of the year, he was frequently detained there a considerable time. He then walked a third time, usually with one of his sons, and returned to tea. At all his meals, his diet was perfectly simple. His health was indeed, latterly, wholly dependent upon the observance of a very exact and particular regimen. During the evening he continued his studies, and from

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* It was from this motive that he performed his ablutions as regularly and frequently as the most pious Mussulman. A basin of cold water was as habitually resorted to by him upon entering or leaving his house, as his books were at other times.

time to time joined in conversation with his family, or threw aside his books to devote himself to his visiters and friends. It has been well remarked, that "you never saw the mathematician, unless *you inquired for him*,"* as mathematics was a topic which he never obtruded upon any one. He had other and most abundant resources of knowledge, with which he could instruct or amuse. He always expected the members of his household to be at home by ten o'clock. The house was then closed, and he usually retired between ten and eleven. There is no doubt that, taking the whole year together, he got as much as six, and perhaps eight hours a day, for his mathematics, besides the time devoted to his business and other pursuits.†

Dr. Bowditch was never fond of reading works upon logic, or even upon moral philosophy, or any abstract speculations upon the nature and powers of the soul. He felt his mind perplexed rather than enlightened by most treatises of this sort. They produced, he said, upon him the effect described by Milton, as produced upon those who

> "reasoned high
> Of providence, foreknowledge, will, and fate,
> Fixed fate, free-will, foreknowledge absolute;
> And *found no end, in wandering mazes lost.*"

He *felt* that he was a free and an accountable agent, and he did

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* Christian Review, September, 1838.

† Astronomy even entered somewhat into his management of his family. Thus his children, for the first few years of their lives, on going into the library in the morning, if they had behaved well during the preceding day, received three dots on the arm from his pen, which he called "*the Belt of Orion.*"

not care to analyze very nicely the source of this feeling. He also considered the time spent in reading most works of imagination unprofitably employed. He preferred history and biography. Boswell's Johnson delighted him. Raynal's History of the Indies he read in early life with great interest, and he never forgot its facts or its peculiarities of style. The Éloges of Cuvier he regarded as master-pieces. There was, indeed, hardly a striking anecdote of any of the eminent men of the present and of former times, which he did not seem to have gathered up in the course of his miscellaneous reading; and his excellent memory placed them constantly at his disposal. He mentioned with approbation the remark, "Why read any thing which you cannot quote?" Not that he was himself ever in the least degree pedantic, or ostentatious; but only because he valued fact far before fiction.

Of late years, certainly, his reading was almost exclusively confined to mathematics. He owned the works of Scott, which he highly valued for their true delineations of nature, and for their freedom from the immorality which characterizes the pages of some of the earlier novelists; but he rarely indulged himself in the recreation of reading even *his* works of fiction. He reserved them, as he said, till the thermometer stood at 90°, and he read them when he did not feel the energy to devote himself to abstruse studies. His recollection of the characters and incidents of these novels was remarkable. He would dwell with delight on Jeanie Deans, and often recall some of the amusing and characteristic scenes of the Antiquary. The earlier volumes of Lockhart's

Life of Scott had been republished in this country before he died, and he had read them with avidity and delight. There were many traits in the character of Scott, as there described, which, as we think, greatly resembled his own; and those later volumes, which carry the reader with a saddened interest to the closing scene of his life, and which Dr. Bowditch never saw himself, spoke to us, who had been thus recently bereaved of a parent in every respect equally entitled to our love, with a peculiar pathos. We had seen the same lofty virtues displayed through many years, which invested the poet's death-bed with its high moral interest; and we actually beheld the euthanasia which, though mentioned by Scott, we fear he was himself hardly able fully to realize; and many of the precise expressions which had fallen from the dying lips of the one, had been also used by the other.—The works of Byron, on the contrary, Dr. Bowditch never admitted into his library; and many years ago he owned a small French work, in four volumes, which had been presented to him during one of his voyages, but which was not a book of very exemplary morality. It had engravings which attracted the notice of one of his sons, when he had begun to study French. Soon after, the books disappeared: Dr. Bowditch had burned them, though he had kept them many years on account of the donor, and the really beautiful execution of the work. He subscribed to very many periodicals, and by glancing his eye over them cursorily, he seemed to find out what articles were worth a careful perusal, and made himself master of whatever was important.

He used playfully to denominate as "the poet's corner" that

part of his library where were to be found Shakspeare, Pope, Milton, &c.;* and it is quite a remarkable fact, that upon the inside of the two leather covers, in which he kept the proof-sheets of this work while in the process of publication, and which were therefore constantly before him, he had entered in his own handwriting extracts from Burns's "Cotter's Saturday Night," and the following stanza of Hafiz, the Persian poet, as given by Sir William Jones: —

"On parents' knees, a naked, new-born child,
Weeping thou sat'st, whilst all around thee smiled.
So live, that, sinking in thy last, long sleep,
Calm thou mayst smile, whilst all around thee weep."

So likewise there and in his Newton's Principia, we find copied by him verses of Voltaire and other French writers in honor of that illustrious author.† Among the poets of America, Bryant was his favorite. He has often said that he thought "The Old Man's Funeral" was one of the most beautiful poetical pieces in the English language. Never can it be hereafter perused by us without recalling one of the most interesting and touching scenes at the close of his own life. Dr. Bowditch often delighted to

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* He, a few years ago, expressed his satisfaction at having been tempted to read Milton again, by the beauty of a new Boston edition of that author. — *Mr. Young's Eulogy*, p. 81.

† Upon these covers he had also written the mottoes, "Ohne Hast, ohne Rast," (*Goethe,*) and "Ne tentes aut perfice;" with extracts from Virgil, Ovid, Lucretius, Halley, Cumberland, Bolingbroke, and Charles Lamb. Among the lines quoted from Voltaire are the following upon "The Academicians who measured the Earth in Lapland:" —

"Vous avez recherché dans ces lieux pleins d'ennui,
Ceque Newton connut sans sorter de chez lui."

quote the stanzas "On two Swallows that flew into a Church during divine Service," commencing,

"Gay, guiltless pair,
What seek ye from the fields of heaven?
Ye have no need of prayer,
Ye have no sins to be forgiven."

The author* was his friend, and he deemed that Boston might well be proud to own him as her son.

Dr. Bowditch was fond of music, and when young played a good deal on the flute; but he soon abandoned it altogether, as leading to an unprofitable employment of time, and the formation of bad habits. For the same reasons, he through life altogether abstained from the use of tobacco in any of its forms, and never played at any game of cards. Chess he also avoided, as not affording any relaxation of body or mind, and as leading to no useful or practical object. Dr. Bowditch was rarely induced to pass an evening at the theatre. Fictitious representations of life, either under tragic or comic aspects, always left upon his mind a feeling of dissatisfaction with its realities. But when he did go to hear some popular actor, his laugh was more loud and cordial, or the starting tear betrayed itself more readily, than if this had been an excitement to which he was more habituated. Much as he was gratified by the sight of innocent hilarity, he did not feel at home in the ball-room or crowded assembly. He seldom, it might almost be said never, went into general society, but nothing contributed more to his happiness than a familiar intercourse with his friends.

* Charles Sprague, Esq.

Dr. Bowditch was very quick in his judgments of character, and having formed his opinion, he was slow to change it. A moral failing once noticed in any one, he always associated with the idea of that individual; and a character which once attracted his respect and love he ever continued to regard with interest, apparently overlooking the slighter blemishes which a more intimate acquaintance may have disclosed. He had a few particular friends, in whose society he especially delighted. Thus while he lived at Salem, and also during his residence in Boston, there were three or four individuals with whom he associated more than with all his other friends and acquaintance together. They were the companions of his daily walks, and at their houses almost exclusively he made his evening visits.

Dr. Bowditch showed a like constancy and perseverance in any course of life, or in the prosecution of any measure which he had undertaken. Deciding only after due deliberation, he acted without the slightest hesitancy or vacillation of purpose. He believed fully in the scripture, "Unstable as water, thou shalt not excel." He has often reproved the use of the expression "I can't do it;" saying, "Never undertake any thing but with the feeling that you *can* and *will* do it. With that feeling success is certain; and without it failure is unavoidable."

Dr. Bowditch's intercourse with his family was entirely free and unreserved. No feeling of restraint was ever inspired by his presence. Among his children, he was himself a child. One occasion is remembered, when, after partaking with them in some

frolic, he laughed at his own want of dignity, and proceeded humorously to contrast the scene around him with a description of the formal observances and requirements of past times. A model for the imitation of all parents, he avoided every thing calculated to interrupt the mutual confidence and familiarity which existed between him and his family. Though readily granting any reasonable favor, he was never weakly indulgent. Inculcating by precept and example the most valuable lessons of life, affection ever prompted and directed his admonitions, and a sound judgment always controlled the impulses of affection. The censure of an instructer uniformly brought with it the weight of a father's displeasure; since Dr. Bowditch never weakened the authority which he had thus delegated to another, by expressing a doubt whether, in any particular instance, it had been judiciously exercised. He devoted much of his own time (though not so much of late years as formerly) to the instruction of his children, particularly the elder ones; his chief endeavor being to awaken in them a taste for mathematics. He persuaded one of his sons to learn French when very young, by the stimulus of a small compensation for the translation of a certain number of pages. The result satisfied him, however, that this was inexpedient. The best works in the language were read before they could be duly appreciated, and they could never afterwards be read with the interest of a first perusal. His experience, also, led him to acquiesce in a child's pursuit of any study, though comparatively useless in itself, if voluntarily undertaken, and prosecuted with ardor; as he believed that it might be attended with incidental advantageous results, and that it would certainly assist in forming a habit of industry.

If a predisposition were manifested for any occupation in life, the father candidly stated his own opinion, and enforced his views by such arguments as occurred to him, but left the final choice of his child free. In one instance of this kind, he, by his advice, induced the adoption of a profession other than that for which a slight preference had been at first felt; while in another case, he readily yielded at last his own wishes to the strong predilection which one of his sons manifested for a seafaring life; judging wisely in both cases. He often spoke of the feeling of independence resulting from the consciousness that one is able to maintain himself by his own exertions, saying that "A man whose *capital* is in his head is free from all anxiety about investments, and has a much more certain income than any one else." He early impressed upon his sons the necessity which they would be under of earning their own livelihood, and he regarded it as a most fortunate necessity. One of his eulogists says, "He would not, as we happen to know, have accepted the offer of a fortune for one of his sons, at the risk of any unpropitious influence upon his opening mind and character." *

As his children grew up, they became his companions. His most intimate friends were those who day by day met around his own fireside. To them his most secret thoughts were disclosed, except only in those cases where silence was a duty which he owed to others. Each of his children may well apply to him (as was indeed done by one of them who communicated in a letter to a younger brother the information of his dangerous

* Judge White's Eulogy, p. 50.

illness) the beautiful language in which Marcia speaks of Cato: —

"Though stern and awful to the foes of Rome,
He is all goodness, Lucia — always mild,
Compassionate, and gentle to his friends;
Filled with domestic tenderness, — the best,
The kindest father! I have ever found him
Easy and good, and bounteous to my wishes."

We feel assured that only one who had often seen Dr. Bowditch by his own fireside, could have penned the resolutions received after his decease from the Faculty of a neighboring university, (Yale College,) which state that they "respectfully and feelingly sympathize with the children of the illustrious deceased, whose memory, justly dear to the country which he honored, is cherished still more affectionately by those who were so happy as to call him their *father*."*

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* We have thought that the reader might be interested in the following remarks of Rev. N. L. Frothingham, D. D., being an incidental notice of the death of Dr. Bowditch, in a discourse delivered at the First Church in Boston, on Sunday, March 25, 1838: — "We need not wait for the consummations of a future world, to see that the righteous spirit is more than a match for death. It wins the victory even now. The eyes of the public have been turned, within a few days, to a remarkable instance of this; and they will long remain fixed upon so serene and noble a spectacle. A great man has been struck down among us. A good man has gone his way from us. His was a mind eminent among the loftiest, and as benignant as it was strong. His renown, that travelled over the world, was the least portion of his deserts. His unaffected goodness was as noble as his genius. His character was as striking as his fame. Who, that ever saw him, forgot him? There was a divine stamp set upon his clear, high brow. A healthy vigor looked out of his cheerful but thoughtful eyes. He was in the midst of the abstractest science, and in the midst of the world's busiest interests, at the same time, — not absorbed by the one, not disturbed by the other, seeing calmly through

We have endeavored, by these various details, to lay before the reader such facts and circumstances as would in some degree

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both. Strangers might well turn as he passed, to ask who he was; and his most intimate friends would feel that they themselves knew of him but the half.

"I can hardly bear to hear him described chiefly as an Astronomer or a Mathematician, — though among the most illustrious that have lived, — he was so honestly, heartily, bravely, entirely, a man. There was something in him brighter than talent, and deeper even than that profound knowledge which led the way with a modest silence where there were few intellects that could so much as attend him. It was the light and depth of a true soul. While he demonstrated the subtlest problems, and scaled starry heights, he displayed the simplest, the most practical, the most engaging worth. It was an instruction to behold him. All the affections of youthful life beamed from his face. His feeling was as keen as his intelligence. To be with him was a wholesome delight; for his was the energy and the very inspiration of good sense, — a free, natural, unseducible spirit, playful and sublime. He was full of humanity. And in using that word, I do not understand it in the technical sense in which it is commonly taken, being applied often to the weakly charitable, and assumed often, as if it were exclusively their own, by visionary schemers and itinerant philanthropists. But I mean that he was rich in the elements and endowments that best distinguish our nature; wise beyond books; benevolent without theory, or feebleness, or parade; active, affectionate, manful; pursuing his way without fear or favor; poised upon himself, and seeming to be lifted by a calm philosophy above all the groveling interests, and fanciful systems, and transient fashions, and heated delusions of the world.

"Alas, that such a one should be withdrawn in the midst of his labors and glory! But that 'alas' he left for others to say. For himself, it was neither expressed nor felt. He left life as cheerfully as he had traversed it. There was no difference between his last days and those which had gone before them, but that they were still more admirable. He had thought as a philosopher. He showed now the most precious fruits of his thought. He submitted and suffered like a Christian disciple. He expired like a saint. Such a 'euthanasia,' as he himself called it, with nothing but peace and hope in it, exhibits the full power of Christian principle. It ought not to be confined to the knowledge of a few, and cannot be. It will spread as far as his name, and do good, as his studies had done before." — *MS.*

enable him to form his own judgment respecting the most striking peculiarities which marked the habits and character of Dr. Bowditch. We thought it not advisable to attempt an elaborate analysis of what we felt ourselves incompetent fully to measure and comprehend. In his general manners he was affable and courteous, social in his feelings, and in all the domestic relations most kind and tender. Crowned with the honors of science, he retained the modesty and simplicity of a child. Endowed with the highest genius, none was more wholly free from pride. Frank, open, and naturally without reserve, he could yet be most cautious and discreet. No less ardent than steadfast in his attachments; easily seeing and sincerely regretting the foibles or faults of his friend, he yet loved him still. Having a boundless extent of mental and moral resources, their varied display gave to the longest intimacy the interest of a recent acquaintance. With a benevolence as universal and as active as ever dwelt in the heart of a philanthropist, his treasures of knowledge were freely imparted to the world: and much of his valuable time, and of the small earnings of his honorable industry, was devoted with judicious and unostentatious liberality to the promotion of the happiness and welfare of others. Holding in slight estimation the services which he thus rendered, he manifested a lively and enduring sense of kindness received. Quick and excitable, indeed, when he saw the occasion, he was yet most placable and forgiving, and never harbored ill-will for a moment. The occasional indiscretions of an ardent temperament he redeemed by displays of the most magnanimous virtue. Devoted to the loftiest speculations, he was not neglectful of the most trifling and minute duty. Undeterred by fear, uninfluenced by any prospect of advantage, he followed

truth, and obeyed conscience; and the popular clamor, and even the coolness of some whose friendship he valued, were alike unheeded. He possessed an energy, promptness, and decision, equal to every emergency, and which insured success in each undertaking. He endeavored to save each moment of time, and apply it to the uses of eternity. Governed by the highest and purest motives, the most distinguishing and beautiful trait of his character was his perfect integrity. Never was he more truly indignant than at the want of this quality in others. Any thing, indeed, mean or dishonorable, and especially any thing like fraud, equivocation, or falsehood, always received his sternest rebuke.* It has been truly said, that, in questions of morals, you could no more becloud or mystify him than in questions of ***quantity;*** that whatever he saw in ***right*** or ***wrong***, he saw as ***clearly*** as in ***plus*** or ***minus;*** and that he carried out a ***practical obedience*** to whatever he ***believed***, alike in both cases.†

On January 1st, 1838, Dr. Bowditch, to the casual observer, seemed likely to enjoy many more years of health and strength. Nor had he himself any idea that his brief days were already numbered. To a female annuitant who then called at the office for her quarterly payment, he said he felt "very well;" but she

* Thus, many years ago, in Salem, one of his sons, at a female school, being in an apartment with one other boy, threw a ball which broke a mirror; and his comrade advised concealment. He was so much pleased when his son told the truth immediately about the affair, that, though he was then obliged to live with rigid economy, and the payment was really inconvenient to him, he bought a new mirror, and expressed far more pleasure at the son's performance of so high a duty as telling the truth, than he did regret at his carelessness.

† Christian Review, September, 1838.

was to receive her next payment from the hands of a stranger. He had attained the precise age at which two of his ancestors had been called to the tomb; and in the midst of this apparently perfect health, in the full and active enjoyment and exercise of all his faculties of body and mind, and surrounded by so much to make life desirable, his own summons came to quit it. He received it with the calmness of a Philosopher, and the cheerfulness of a Christian. After having experienced slight pain and uneasiness for three or four months, about the end of December, he mentioned his symptoms to his third son, — a physician, — who wished him immediately to submit to his prescriptions. He replied that he had not then leisure to be ill; that the affairs of the Life Insurance Company required his constant attention; and that he could not put himself under the hands of the doctors until after the payments of the first part of the month of January had been completed. As soon as possible, however, after the period thus mentioned, his son, who considered the symptoms to be of an alarming character, persuaded him to call in the aid of the same eminent medical adviser and friend,* to whose attentions his mother had been so much indebted during her protracted illness. Almost immediately it was decided that the disease under which he labored was a tumor in the abdomen, of a dangerous and probably a fatal character. The symptoms rapidly became more and more decided, and at intervals the most acute pain was experienced, lasting sometimes for twenty or thirty minutes, and from which relief could only be obtained by means of hot applications. His stomach now rejected all solid food, and could only bear the slightest

* James Jackson, M. D., now President of the American Academy of Arts and Sciences.

quantity even of liquid, and sometimes none at all. Death by starvation was in prospect. A general debility of the whole system was the unavoidable consequence of the small degree of nourishment which he was able to take. He became emaciated to a degree of which even his consulting physician, with all his extensive practice, had never before seen an instance. The disease had wholly gained the mastery over his body. But his mind seemed to acquire strength and energy as the crisis approached. He was fully apprized of his danger, arranged all his worldly affairs, and executed his will in a manner with which he expressed himself perfectly satisfied. He continued to sit in his library part of each day, until the day before his death, when he for the first time was unable to rise from his bed. He rode to his office every day until February 17, not quite four weeks before his death. It was an elevating spectacle to see such an unconquerable spirit struggling to discharge every duty, even when the body had almost refused to perform its functions, and when death was most legibly written upon the countenance. Subsequently, the secretary of the company, by his desire, came each day to his house with such papers as required his signature, or with the books for him to examine; and as lately as the 7th of March, he transmitted to the company whose affairs he had so long superintended, the complete account of the transactions of the preceding month, drawn up as usual; and with it he sent a farewell communication, which he had dictated and signed. In this he states that his declining health would probably make it the last which he should ever address to them, and takes an affectionate leave of those who had had the control of the institution, and of those who had been associated with him in

its management. He also alludes to the length of time during which the institution had been under his charge, and earnestly commends its interests "to that Providence which had seen fit to bless their efforts to make it deserving of public regard." To this letter he received a most affectionate reply, not attested as an official act by the secretary of the company, but personally signed by each of the twelve directors, who assured him in the strongest terms of their respect and regard, of their conviction of the value of his past services, and of their deep and sincere sorrow for his serious illness. The promissory note upon which he made the endorsement before mentioned, has upon it the latest specimen of his hand-writing.

In like manner he continued to correct the proof-sheets of this volume; and within a week of his death, he said that the sheet which he was then revising contained the discussion of a difficult problem; that M. Poisson thought he had made an improvement upon the method of the author, whereas he believed he had shown that, on the contrary, the supposed improved method was fairly deducible from that of La Place: and he added, "I feel that I am Nathaniel Bowditch still — only a little weaker." The last page upon which his eye was ever to rest, was *the thousandth*, though no part of the volume subsequent to the six hundred and eighty-fourth page has received that final revision which he was accustomed to bestow upon it, after the friend before alluded to had laid before him the list of typographical errata, which he had discovered.* The reader will therefore pass a charitable judgment

* Whenever one hundred and twenty pages were printed, Dr. Bowditch had them bound

upon this latter portion of the volume. Dr. Bowditch hoped to be spared to finish its few remaining pages. It called forth the last efforts of his powerful intellect, and afforded him amusement and solace almost as it were to the hour of death.

He continued to take a lively interest in all such passing events as he considered to have an important bearing upon the welfare of the community. He was able to see a few, and only a few, of his most valued friends; and he conversed with them and with his family upon his approaching separation with the utmost resignation and calmness. To two of his most intimate friends, then absent in Europe, he sent a message, assuring them of his continued attachment. Throughout his illness he was only "watched by eyes that loved him." The kind offices of others were not needed. Filial hands alone ministered to his wants; filial hearts alone anticipated his wishes. To his eldest daughter, as she stood hour after hour behind his chair, or beside his bed, gently rubbing his head in the manner which had ever been agreeable to him, he playfully remarked that her fingers were like "Perkins's Tractors," and that the process itself was "Terrible Tractoration." He said of her to one of his sons, "I feel respecting Mary to-day, as I did the day when she was born;" and to the inquiry how he then felt, he replied, "It was the happiest day of my life, for I then first had a little daughter."

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in a pamphlet form, and sent them to Professor Pierce, who, in this manner, read the work for the first time. He returned the pages with the list of errata, which were then corrected with a pen or otherwise in every copy of the whole edition.

He once said to her in a smiling manner, "You seem to my eyes to be forty years old. This expression in itself may not be flattering to you; but I mean by it, that you have compressed the services of many years into the brief period of my illness." And one day, as he was examining his papers, and burning those he thought of no value, he met a copy which he had made several years before of those beautiful lines in Scott's Marmion —

"O woman, in our hours of ease,
Uncertain, coy, and hard to please,
And variable as the shade
By the light, quivering aspen made,
When pain and anguish wring the brow,
A ministering angel thou!"

This he handed to her, saying that of the compliment contained in the two last lines she was certainly deserving.*

With no less assiduity did his younger daughter delight to discharge such kind offices as did not require the greater skill and experience possessed by her elder sister; and he who of late years had always assented to her request to be the companion of his noon-day walk upon the Sabbath, and who indeed had always regarded her with peculiar tenderness as the child of his

* During his illness, he examined and burned very many papers; and after his death, it was found that he had probably, in this manner, and at this time, destroyed all the correspondence between himself and his wife before and after marriage; and also a manuscript folio volume, in about seventy pages of which his eldest son had, several years before, recorded the details of his early life, as taken down from his own lips, and which volume had been left in his library that the narrative might be continued from time to time.

old age, now made a like affectionate return for these her efforts to please him. His eldest son he once addressed in the language of Scripture, "My first-born, my beloved." He employed him to draft his will, and all his various letters and other documents. His second son attended to his requests in regard to all matters of business, and the arrangement of his pecuniary affairs. He was particularly desirous to discharge all his debts, however trifling, before he died, or to leave the means for their instant discharge afterwards. To this son he mentioned, the day before his death, a female, of whose little concerns he had always taken care, and said to him, "I wish you to call upon her before you visit any one else, *after attending my funeral*, and inform her that I have transferred her to your charge, and that you will supply my place to her through life." His two eldest sons no longer resided under the parental roof, and as they were one evening leaving his presence, he said to them, "Farewell, my sons; my blessing goes with you." His third son had the peculiar privilege, as his medical attendant, to pass nearly all of each day, and the whole of each night, in his apartment, enjoying an unreserved intercourse with him of the most elevating character; and boundless indeed, to a degree, as he admits, far beyond his deserts, was the gratitude which owned his constant attentions. His youngest son was, like the elder ones, absent from his father's house, but upon learning his illness, each evening saw him a visiter there; and on the last night but one of his life, when an elder brother intended to act as a watcher, he asked and readily obtained his father's consent to be allowed that privilege. The teachings of that night he will never forget. He had asked his father for a kiss when leaving him upon one of these evening

visits, and received the reply — "Kiss you, my dear! Yes, if I die in the act!" At another time he said, "I leave behind me a family of love, which, I rejoice to believe, will long continue a united household, after I shall have been removed from it by death." To her who in early years had been left alone among strangers, he recalled the dying words of the sister who had then intrusted her to his care — "Promise to be a father to my child;" and he stated that he had always endeavored to redeem the pledge then solemnly given, and had never intentionally made any distinction between her and his own children; and that he had made an adequate provision for her by his will, that she might not feel herself dependent even upon them, though he doubted not for a moment, that each of them would always be ready to welcome her to his home and his heart. He then thanked her for that performance of household duties which had so much lightened the labors of his wife and himself, and added that if any occasion had ever occurred (which there had not, to his knowledge) when he had shown her less affection than the kindest parent ought to have shown to the most dutiful daughter, she must overlook and forget it as accidental. In various ways he constantly showed the most considerate affection for his family. Thus he said that he had himself found great consolation, after the death of friends, in reflecting that they met their fate with a cheerful and resigned spirit; and he added, "I am happy that I can leave to you the same consolation." And we indeed saw in him a soul perfectly calm and serene. Two nights only before his death, after awaking at midnight, and speaking a few moments very impressively respecting his approaching end to two of his sons who were

present, he yet sank again, apparently in less than five minutes, into the most tranquil sleep.

To one of his sons, who, as he thought, was not always sufficiently careful of making remarks which, though innocently intended, might give offence, he said that upon a certain occasion he had himself, in speaking to a female friend, alluded to one of her features as not handsome; and that after she had gone, his wife blamed him for doing so, because the lady in question might have received the impression that he thought her countenance disagreeable; when in reality there was scarce a being in the world, to whom they were both more attached, or upon whose face they were always more delighted to look; that this advice of his wife, dictated by the truest kindness of heart, he had often reflected upon, and, as he hoped, had been benefited by it. He then said, "There is no friendship or connection so intimate as to justify a disregard of a constant endeavor to please;" and added that upon one occasion, when his wife had appeared in the library in a new dress, and he, happening to be engaged in his studies, had not noticed the circumstance, she seemed quite disappointed, and said to him, "I purchased this dress on purpose to please you, as being of your favorite color, and now you do not seem to care the least about it." He added, "I immediately left my books, told her she must lay the blame not upon me, but upon mathematics; that the dress suited my taste exactly; and thus succeeded in restoring her cheerful looks. And ever afterwards," said he, "through life, I endeavored, whenever she came into my presence, not to omit to express towards her, outwardly, something of that pleasure which I always really felt."

To another of his sons he was speaking of truth as never to be in the slightest degree or upon any inducement disregarded, and holding up his finger, and repeating the words with most solemn emphasis, said, "Follow truth — truth — truth! Let that be the family motto." So many, indeed, are the touching incidents of his last illness which throng upon the memory of his children, that a selection is almost impossible, where each was such an exhibition of moral greatness. He had expressed the wish to be approached with smiles and cheerfulness. Feeling no melancholy in his own soul, he was averse to the manifestation of it in others. Observing, therefore, one of his family whose countenance was marked with sadness, he called for his volume of Bryant, and opening at his favorite piece, read,

> "Why weep ye, then, for him who, having won
> The bound of man's appointed years, at last, —
> Life's blessings all enjoyed, life's labor done, —
> Serenely to his final rest has passed?"

He then proceeded to read all the remaining lines, remarking upon each, that he believed or hoped it was applicable to himself, or that he thought it not so. His voice, though low, was throughout clear and firm, and the incident was a truly impressive one.

Rarely was a complaint or murmur extorted from him even by the most excruciating pain. One evening, as his eldest sons were present, he said, "Much as it usually gratifies me to see you, your presence now is unwelcome. I am suffering so much, that I cannot enjoy the society of any one. You can do nothing for my relief. I had rather you would go home." On another occasion, when the torture he experienced was almost beyond endurance,

he exclaimed, "Why was I born!" After he had obtained relief, one of his sons asked him why he had made that remark. He said that he meant, "Why was I born to suffer so much! But I see the reason. It is that I may be weaned from this world."

Happily, a few weeks before his death, he had longer intervals of ease. On one of these occasions, he asked a son if he remembered the word, derived from the Greek, signifying an easy death. Being answered in the negative, he said that in Pope's Works there was a letter from Dr. Arbuthnot, which he had not read for forty years, but which he distinctly remembered as containing this word, with a note mentioning that that *excellent* man died shortly afterwards; so that he had always associated the idea of an easy death with that of excellence of character. The book was opened, and the letter found. The writer says, "A recovery in my case, and at my age, is impossible. The kindest wish of my friends is *euthanasia*." To this subject he upon more than one occasion afterwards recurred, and, applying it to his own situation, said, "This is indeed euthanasia."

The following is an extract from the private journal of his third son, under date March 4, 1838, recording a dialogue which took place between him and his father: — "He said, 'I have left in my will the manuscript of La Place to the College. I wish I had not done so; for who will care any thing about it? It is a mere bagatelle.' I told him that, though in itself valueless, it would be interesting, perhaps, at some future period, for the lover of mathematics to look upon his original manuscript copy of so great a work. 'O,' said he, 'the work will soon become obsolete,

and nobody will look at it.' — 'Very true, it *will* become obsolete; and what work is there that will not become old? but still we honor talent, even if the labors of that talent are superseded by later writers.' — 'Yes,' replied father, 'Archimedes was of the same order of talent with Newton, and we honor him as much; and Leibnitz was equal to either of them. Euclid was a second-rate mathematician; yet I should like to see some of his hand-writing. My order of talent is very different from that of La Place. La Place originates things which it would have been impossible for me to have originated. La Place was of the Newton class; and there is the same difference between La Place and myself as between Archimedes and Euclid.'"*

Not less interesting were many incidents which occurred during his interviews with others. A young lady had been playing, by his desire, upon a harmonicon. As the strains of the music rose and

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* A similar anecdote is mentioned by Mr. Young, (Eulogy, p. 83,) of Dr. Bowditch's admitting La Place to be altogether his superior, and saying, "I *hope* I know as much about mathematics as Playfair." The word *hope* is probably a verbal mistake for *think*, since the expression otherwise seems to imply a disrespect for Playfair, such as Dr. Bowditch did not entertain, and to which, therefore, he could not, as we believe, have given utterance.

Dr. Bowditch was always of opinion that men are born with the same diversities of intellectual, as of physical powers and stature. Thus he would speak of one as "a man of small calibre," and say of another that he had reached his "couche de niveau." And he considered as wholly absurd a remark once made in his hearing, "I have no doubt that any man could become a mathematician if *he only had time!*" It seemed indeed, in his own case, that he became a mathematician *notwithstanding the want of time;* and a striking contrast is exhibited by Mr. Pickering, (Eulogy, p. 56,) between the long life of La Place, exclusively devoted to the pursuit of science, and the comparatively short life of his translator, of which so much was occupied by other important engagements.

fell upon the ear, like that of the Æolian harp, he listened intently; and when the last cadence had died away, and the musician approached to take her leave, he gave her an affectionate greeting, and after she had retired said, "You must tell her that she has been playing my dirge." A lady visited him, and as she was quitting the apartment, he said, "Good night," twice, with a tone of voice, and an expression of countenance, which indicated his conviction that he saw her for the last time; and then he immediately added, "Good morning at the resurrection."

Exactly a week before his death, the President of Harvard College, Mr. Quincy, had an interview with him, the following account of which he reduced to writing immediately afterwards: — He says, "I found him sitting in his chair, in his library, emaciated, pale, and apparently wasted by his disease to the last stage of life; his mind clear, active, and self-possessed. He spoke of his disorder as incurable; that he felt himself gradually sinking, and that he could not long survive. 'I have wished to see you,' said he, 'to take my leave, and that you might have the satisfaction of knowing that I depart willingly, cheerfully, and, as I hope, prepared. From my boyhood, my mind has been religiously impressed. I never did or could question the existence of a Supreme Being, and that he took an interest in the affairs of men. I have always endeavored to regulate my life in subjection to his will, and studied to bring my mind to an acquiescence in his dispensations; and now, at its close, I look back with gratitude for the manner in which He has distinguished

me, and for the many blessings of my lot. As to creeds of faith, I have always been of the sentiment of the poet, —

> For modes of faith let graceless zealots fight;
> His can't be wrong, whose life is in the right.' " *

Then he alluded to the lines of Hafiz, before mentioned, saying of them, " 'They are lines of which I at this moment feel all the force and consolation. I can only say, Mr. Quincy, that I am content; that I go willingly, resigned, and satisfied.' † After this he spoke to me of his works, his gratification that the four first volumes, which constituted the principal work, were so nearly completed. 'There are only about ten pages wanting; perhaps I may live to finish them. I have been to-day correcting the proofs.' He then showed me his will, explained his motives, asked me to read it, and my opinion. In every respect, his state of mind was such as at such a moment his best friends could have wished, — calm, collected, rational, resigned, — looking confidently for an existence beyond the grave, — happy in reflecting on the past, and in anticipating the future. On taking leave, he impressed a kiss on my hand, saying, 'Farewell!' " On another

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* Dr. Bowditch often repeated passages from Pope's "Essay on Man" and "Universal Prayer."

† The following lines, which he had also copied on the covers of his portfolio, are strikingly applicable to the frame of mind which he now manifested: —

> "Parent of nature, Master of the world,
> Where'er thy providence directs, behold
> My steps with cheerful resignation turn.
> Fate leads the willing, drags the backward on.
> Why should I grieve, when grieving I must bear;
> Or take with guilt, what guiltless I might share?"

Cleanthes, translated by Bolingbroke. Orig. Epist. 107.

occasion, he mentioned the early impression made on his mind by the remark of a Quaker lady, that the external symbols and observances of religion were only valuable as indicating the existence of an inward principle, and a life in accordance with it.

Among those, also, who had the happiness of a like interview, were two of his subsequent Eulogists; one of whom (Judge White) says, "Being deeply affected by his whole appearance and conversation, and absorbed in the feelings which these produced, I could not retain much of the language which he uttered, though the general impression of what he said was indelibly fixed in my mind. I recollect, however, very distinctly his expressions in speaking of his early and deep feeling of religious truth and accountability. 'I cannot remember,' he said, 'when I had not this feeling, and when I did not act from it, or endeavor to. In my boyish days, when some of my companions, who had become infected with Tom Paine's* infidelity, broached his notions in conversation with me, I battled it with them stoutly, not exactly with the logic you would get from Locke, but with the logic I found *here*, (pointing to his breast;) and here it has always been my guide and support: it is my support still.' With feelings of humility inseparable from the purest minds in such a situation, he expressed the satisfaction which he felt from having always endeavored to do his duty. . . . . 'My whole life,' he said, 'has been crowned with blessings beyond my deserts. I am still surrounded with blessings unnumbered. Why should I distrust the goodness

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* The well-known "Age of Reason," by Thomas Paine, was a work which at that time had many readers in the community.

of God? Why should I not still be grateful and happy, and confide in his goodness?' And indeed why should he not?"*

In his interview with the other, (Rev. Mr. Young,) he dwelt much upon the kindness and assistance which in early life he had received in Salem, and expressed a like affection and gratitude towards the city in which he was to end his days. Mr. Young says that every one of the friends who then visited him "will bear testimony to his calm, serene state of mind. The words which he spoke in those precious interviews they will gather up and treasure in their memory, and will never forget them so long as they live." †

During his illness, Dr. Bowditch was asked to state his particular religious belief, and replied, — "Of what importance are *my* opinions to any one? I do not wish to be made a show of." When mention was made of the various teachers of mankind, inspired and others, (Socrates, Moses, &c.,) at the name of Christ, he said, "Yes — the greatest of them all." He dwelt often upon the fitness of the gospel to purify the heart and elevate the soul; and preferred to rest its authority upon these views, rather than upon any other. A recent article in the Christian Examiner, upon the point that a belief in miracles is not *essential* to a belief in Christianity, received his approbation.

The Rev. John Brazer, D. D., of Salem, was a friend who rarely visited Boston without passing the night under his roof, and

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* Eulogy, p. 53. † Discourse, p. 94.

whose own house had often had as an inmate for several weeks a daughter of Dr. Bowditch. He, during the last illness of the latter, offered up for him within his church a prayer which, in the words of a correspondent, "touched all hearts." More than one interview left his mind also filled with the same delightful impressions. In one of them, Dr. Bowditch, after alluding to the intimacy which existed between themselves, and also between himself and certain absent friends, observed that he felt himself "capable of faithful friendship." And in a brief public notice of his decease, this clergyman observes, "And so he was, in a degree never surpassed. Aching hearts can now testify to this; and there are some who feel that there is a void left in their affections, which can only be filled by a reunion with him in another world." Dr. Bowditch had requested his children to send to Dr. Brazer a small legacy, saying, "I know that it will be grateful to my friend to be assured that I thought of him with unabated love and confidence in my dying moments."

He had through life delighted to attend to the interests and feelings of many who were comparatively alone in the world; and for these services, they now expressed the warmest gratitude. A short time before his death, he received from a young lady who, being herself an invalid, could not in person express her sentiments towards him, a letter, in which she addresses him as "her dear father," and assures him that "his kindness fell not upon stony ground, when it fell upon an orphan's heart;" and the last person who had an interview with him, (except the members of his own family,) was another lady, before alluded to, (p. 143,) who expressed the delight which it had afforded her, and said that

she never could have been happy if he had died without her having had an opportunity of acknowledging her many and great obligations to the best of friends.

He himself literally never forgot a kindness. Thus he enjoined it on his children to transmit a legacy from him to the widow of one of his early employers, as being his oldest friend, "one whose affection had ever been to him as that of a mother, knowing no interruption or abatement." And he remembered in a similar manner a near relative, from whom he had always received a sister's welcome when he visited Salem.

One little being alone stood to him in the relation of a grandchild, the daughter of his eldest son. Desirous of leaving for her some small token of his remembrance, a silver cup was made by his directions, bearing the inscription, "Elizabeth Francis Bowditch, from her grandfather, Nathaniel Bowditch, March 1st, 1838," which, a day or two afterwards, he placed in her own hands. Though the image of that affectionate relative has long since faded away from her infant memory, that visible emblem will in after years remind her of one who, on the day of his death, when his failing senses led him erroneously to believe that he was addressing her mother, said, "Give my love to the little one."

There was one who was a sister to him by marriage, as she had always been in affection. Her daily visits during his illness were ever most welcome. She was a wife, and is a widow; was a mother, and is childless. She asked him his belief in a recognition of friends after death. He said to her, that, to his

apprehension, it was not clearly revealed. She exclaimed, "Do not say so. The chief consolation I have here, is the hope of meeting my lost ones again." He saw her grief as she retired, and in the course of that day told his family to be sure to inform him when she next called, as he wished much to see her. She came again. He said to her, "Let me assure you of my conviction that if, in the future world, it will be best that we should know again the friends we have here loved, that happiness will certainly be ours. What I meant to say yesterday was, that I do not think that Almighty Wisdom has explicitly revealed to mortals its decrees in this particular. But of one thing I am certain; all will be for the best. I approach the unseen world with the same reverence as I would the Holy of Holies, and have no desire to draw aside the veil which conceals its mysteries from my sight."

He had always entertained a most important as well as just sentiment, to which he constantly recurred during his illness, namely, that the highest intellectual cultivation and acquirements are entirely worthless, when compared with moral excellence. Often have we heard the author of this Commentary, during his last days, say that the consciousness which he then felt that throughout life he had endeavored to discharge its various duties, and the humble hope that those efforts would be approved hereafter, were far sweeter to him than any praises which he had already received, or the thoughts of any reputation which might await his name in future times as having been a faithful laborer in the cause of science.

Indeed, he valued his own peculiar studies for their elevating moral tendency, and for producing, as it were, an indirect effect, more important and lasting than their immediate results. Thus, a few days only before he died, he listened with attention and pleasure to a recent publication of Mrs. Sigourney, as it was read to him by his eldest son, where that writer says, "The adoring awe and profound humility inspired by the study of the planets and their laws, the love of truth which he cherishes who pursues the science that demonstrates, *will find a response among archangels*."

His own life, indeed, which had been spent in search of *the true* and *the right*, had led to that unwavering belief and trust in the wise providence of God, and that humble and confiding submission to his will, which dispelled from the chamber of death the gloom which so often enshrouds it. His eye shone with its wonted brightness. His feeble voice inculcated, in its low and scarcely audible accents, its lessons of wisdom and love, with an earnestness and solemnity that seemed almost like inspiration, and spoke to the hearts of his hearers. Though his emaciated countenance told of many an hour of severe pain, the patient sufferer recalled the blessings he had enjoyed through life, and gratefully acknowledged those which still surrounded him. He was often, during his intervals of ease, playful and humorous in his remarks, but without any levity of thought or manner. He did not affect any indifference to life, but was perfectly willing to quit it. His was

"Earth's lingering love, to parting reconciled."

He approached his end with feelings the most becoming to the man and the Christian. His spirit was perfected by the sufferings through which he passed. Truly we esteem it a high privilege to have been present at such scenes. May the lesson of his life and his death be read by us aright!

On the morning of Friday, the sixteenth of March, at about six o'clock, when his sight was quite dim, his third son told him that he thought the time had come when he had better take leave of all his children. He answered, "I know it; I feel it." Each in succession then approached; and as the father returned the kiss he received, he inquired who it was; and in this manner he took a most affectionate farewell of his children, all of whom were gathered around his bedside. He said, ,' O! sweet and pretty are the visions that rise up before me. 'Now let thy servant depart in peace, for mine eyes have seen thy salvation.' I say these words not because I have entire love for all the * . . . . but because I love the words, and feel kindly towards all * . . . ." Upon drinking a little water, he said, "How delicious! I have swallowed a drop — a drop from

> 'Siloa's brook, that flowed
> Fast by the oracle of God.'"

Soon after this time, he fell into a tranquil sleep, from which, at about half past nine, he awoke, and once more desired to see his family assembled; then, looking round upon them, and addressing each by name, he said, "There, my children, I have known you all; have I not, perfectly? O! it is beautiful to me

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* His voice here became wholly indistinct.

to see you all about me — pretty! It is beautiful to me to bless you all. May God forever bless you, my dears! It is for the last time that your father blesses you." It pleased Heaven, after this, to afflict him with the most severe bodily suffering during nearly three hours; but about noon it left him, and the quiet, tranquil state of body and mind returned. He addressed his son with the epithet "my dear," and said, "*It is coming! I am ready.*" And at one o'clock, Death gently set his seal upon that placid countenance.

He was buried on the morning of the following Sabbath. The face of spring was hidden by the falling snow. The streets of the city were silent and deserted. Every thing seemed to feel the quiet of the day and hour. Dust was given back to dust: the spirit had returned to God who gave it.

APPENDIX.

We have thought that a few particulars respecting the library of Dr. Bowditch, and its future intended appropriation, might be of some general interest. Montaigne has said of the apartment which contained his books, that he endeavored " to sequester this corner from all society, conjugal, filial, and civil." Dr. Bowditch, however, did exactly the reverse; he selected for his library the family parlor. To us it will always be the scene of the most happy associations. It will ever present one common centre of attraction, bringing our hearts near together, and uniting us in the close and intimate circle of brotherhood. It will recall a husband never so much immersed in his studious researches, as to be forgetful of those little proofs of affection which first won and ever secured in return the affections of the wife; and a wife never so much occupied with household duties and cares, as to neglect for a moment the kindest and most considerate attentions which woman's love ever prompted. A father's advice, also, and a mother's gentleness, will speak to us from the inanimate objects around. There the present will be full of the past. Nor will it be without its interest to many others. Who, indeed, that has ever seen Dr. Bowditch in that library, will fail to acknowledge the truth as well as beauty of the description given by one who was himself only an occasional, but always a welcome visiter there: — " You saw the Philosopher, entering, with all the enthusiasm of youth, into every subject of passing interest. You saw his eye kindle with honest indignation, or light up with sportive glee; you caught the infection of his quick, sharp-toned, good-natured laugh, and felt inclined to rub your hands in unison with him at every sally of wit, or every

outbreaking of mirthfulness. Let the conversation turn in which way it might, he was always prepared to take the lead; he always seemed to enter into it with a keener zest than any one else. You were charmed and delighted; the evening passed away before you were aware, and you did not reflect, until you had returned home, that you had been conversing with unrestrained freedom with the first Philosopher in America." *

Though, of course, it cannot have the same degree of interest to others, which is felt by the children of the deceased, we are confident, then, that all who have ever been favored with an interview like that above described, will be happy to learn that it is our hope and expectation, that for very many years that apartment will remain as it was left by our father; that the chair in which he sat, the desk and the portfolio containing the last proofs of this work which were ever submitted to him, the table around which were usually seated his family and friends, and the noble array of works of science which adorn the walls of the apartment, will all long remain undisturbed. That collection is one which, in its particular department, we believe to be unsurpassed, and probably unequalled, by any in the United States; and as no one of our number has in any considerable degree inherited the peculiar tastes of his father, it is obvious that to us it will be of but little practical utility. But we knew that he himself always freely lent his books to every one having a fondness for scientific pursuits, and who had not the means of otherwise obtaining them. We remembered, also, that a free diffusion of knowledge was, indeed, ever the chief object of his own life; and we have dedicated "The Bowditch Library" to the use of the public, as far as, in the exercise of a sound discretion, we deemed consistent with the safety of the books loaned.

Many of the most rare and valuable works in this library were presents to

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* President Wayland, of Brown University. Christian Review, September, 1838.

Dr. Bowditch from various societies or authors in other countries, — a circumstance which adds greatly to the interest of the collection; and we feel assured that, containing all the volumes which he habitually consulted while preparing this work, and also all the manuscript proofs of his early industry, this library will, as long as it shall exist, remain a most interesting monument to the memory alike of the Ship-Chandler's Apprentice, and the Commentator upon La Place.

ADDITIONAL NOTES.

Note to page 28.

Dr. Bowditch, in his last illness, in answer to the direct question of the writer, replied that he had made *six* voyages; and the anecdote respecting his being in Boston in July, 1802, attending to his vessel, which was wind-bound, seems to favor the supposition of another voyage besides the five mentioned in the text. We find, however, that the ship Astrea, *Stanwood* master, arrived in Boston, from Batavia and the Isle of France, July 10, 1802. One who was an inmate of his family from the time of his second marriage, October, 1800, says that he made but one voyage afterwards. Of that the journal is extant, to speak for itself, beginning November, 1802. So that we believe the text to be correct.

Note to page 60.

There are extant several portraits of Dr. Bowditch: —

1. There are two miniatures, taken at the times of his first and second marriage, apparently by the same artist. They have no merit either as likenesses or paintings.

2. About the year 1820, portraits of Dr. Bowditch and his wife were painted by James Frothingham of Salem, which, though wanting in expression, are yet in other respects very good. It was from his portrait of Mrs. Bowditch, that, after death, Miss Lalanne painted for the writer the miniature which is engraved for this memoir; certain alterations being introduced, which have made the likeness more accurate.

3. The portrait by Gilbert Stuart was painted in 1828, and, even in its unfinished state, is, we think, far superior to any other. A friend, who admired it very much, and selected the frame for it, has written on the back, "The last work of Stuart. 'Sancte inviolateque servatum sit.'"

4. The portraits belonging to the Salem East India Marine Society and the Salem Marine Society, are by Charles Osgood, having been copied by him from Stuart's picture, with the aid of a few additional sittings.

Note to page 69.

In the farewell address of his Royal Highness the Duke of Sussex, the President, delivered at the last anniversary meeting of the Royal Society of London, November, 1838, an outline is given of Dr. Bowditch's life, with the following summary of the merits of this Translation and Commentary: —

"Every person who is acquainted with the original must be aware of the great number of steps in the demonstrations which are left unsupplied, in many cases comprehending the entire processes which connect the enunciation of the propositions with the conclusions; and the constant reference which is made, both tacit and expressed, to results and principles, both analytical and mechanical, which are co-extensive with the entire range of known mathematical science: but in Dr. Bowditch's very elaborate Commentary every deficient step is supplied, every suppressed demonstration is introduced, every reference explained and illustrated; and a work which the labors of an ordinary life could hardly master, is rendered accessible to every reader who is acquainted with the principles of the differential and integral calculus, and in possession of even an elementary knowledge of statical and dynamical principles.

"When we consider the circumstances of Dr. Bowditch's early life, the obstacles which opposed his progress, the steady perseverance with which he overcame them, and the courage with which he ventured to expose the mysterious treasures of that sealed book, which had hitherto only been approached by those whose way had been cleared for them by a systematic and regular mathematical education, we shall be fully justified in pronouncing him to have been a most remarkable example of the pursuit of knowledge under difficulties, and well worthy of the enthusiastic respect and admiration of his countrymen, whose triumphs in the field of practical science have fully equalled, if not surpassed, the noblest works of the ancient world."

Note to page 82.

The following are the children of Dr. Bowditch, mentioned in the order of their ages: —

1. Nathaniel Ingersoll Bowditch, a graduate of Harvard College, 1822, is engaged in the practice of the law in Boston.

2. Jonathan Ingersoll Bowditch, having made a number of India voyages, is now president of the American Insurance Company in Boston.

3. Henry Ingersoll Bowditch, a graduate of Harvard College, 1828, having pursued the study of medicine, is now established in that profession in Boston.

4. Charles Ingersoll Bowditch, born December 1, 1809, died February 21, 1820.

5. A son, born July 7, 1813, died the next day.

6. Mary Ingersoll Bowditch.

7. William Ingersoll Bowditch, a graduate of Harvard College, 1838, now a student at law.

8. Elizabeth Boardman Ingersoll Bowditch.

Note to page 108.

We have said that Mr. Young's Discourse contains some trifling errors. Prepared in the course of a few weeks, it could hardly have been otherwise. Upon several points, we ourselves were at first mistaken. In justice to him, it is proper to specify these errors, that no vague impression of general inaccuracy may be left on the reader's mind. It is incorrectly said, in p. 23 of his Discourse, that Dr. Bowditch's instructer was an Irishman named Ford, and that when he solved the problem so quickly, he was actually punished for lying; in p. 39, that he learned French without an instructer; and in p. 59, that his knowledge of navigation was picked up during the intervals of his voyages. The anecdote in p. 51, of the report that "La Place once remarked, 'I am sure that Dr. Bowditch comprehends my work, for he has not only detected my errors, but has also shown me how I came to fall into them,'" may be correct; but Dr. Bowditch never heard of it. The statement in p. 68, of his entire abstinence from politics, is correct only of the latter part of his life. Such was his political zeal in early life, that he once assisted in carrying an invalid upon his bed to the polls to vote. The anecdote in p. 87, respecting his magnanimity in giving up the benefit of his chart of Salem to one who had endeavored to appropriate it wrongfully to himself, is related as it was at first told to us; but we are satisfied, from subsequent inquiries, that there was but one interview

between the parties, and that the account, though it has a basis of truth, is probably very much exaggerated. So in the anecdote, p. 33, respecting his solution of a question proposed by an Englishman while at the theatre, it is not true that he proposed in return one which the latter could not solve ; as is proved by a written account of this incident entered by Dr. Bowditch in one of his common-place books at the time. In page 88, it is said that he, latterly, usually took one glass of wine after dinner, and another in the evening; and seldom or never more. He took two glasses at each time, which he called his *certain quantity.*

Notwithstanding the numerous details and anecdotes collected by Mr. Young, it is believed that the above are all the matters stated by him, relating to Dr. Bowditch, which require correction or qualification.

We have not thought it necessary to *quote* from this Discourse, in cases where the original information was obtained from conversations with us, or where the same materials were placed by others at the disposal both of Mr. Young and ourselves. A statement of some of these sources of information, will enable the reader to judge of the relative authenticity of different parts of the present memoir. During some months before the removal of the family from Salem, the writer, having a taste for antiquarian researches, spent several leisure hours of each day in examining the public records and other sources of information, for the purpose of tracing back his ancestry to the first settlement of that town. One of our number went to Salem the week after his father's death, where he remained during several days, making inquiries of those who had formerly been most nearly connected by business or friendship with the deceased. He invited Mr. Young to join him in a visit to Danvers, and the latter was thus present at the interview with the relations of Dr. Bowditch's first instructress, of which he has given an account. He likewise procured a drawing of the house his father there occupied, which Mr. Young caused to be engraved for his Discourse. Among others, Captain Prince was inquired of respecting his recollections of Dr. Bowditch. He referred us to a written account, containing anecdotes of the second, third, and fourth voyages, which his son had drawn up a day or two before, and sent to Mr. Pickering, (not taken down by Mr. Pickering from that gentleman's own lips, as was thought by Mr. Young.) This account was afterwards lent by Mr. Pickering both to Mr. Young and ourselves. The original journals of all Dr. Bowditch's voyages, except the second, are still preserved in his library, and verify the accuracy of Captain Prince's dates, &c., given in his account. The whole series of the successive editions of the Navigator are also in the library, the prefaces to which show very clearly the most important circumstances connected with the commencement and progress of that work. All Dr. Bowditch's occasional publications having been collected and bound together by him, we were enabled even to add one or two

to those discovered by Mr. Pickering, who himself added several to the list given by Mr. Young. Dr. Bowditch's manuscripts were given to the writer of the present memoir. Among them is a separate file of all the letters received by him relating to the *Mécanique Céleste*, and another containing all his diplomas, and letters offering him any appointments, either of honor or profit. Every thing connected with the printing of the *Mécanique Céleste*, and the management of the Life Insurance Company, and indeed most of the recent incidents of his life, are, it is needless to say, within our own personal knowledge; and those of his last illness, especially, are indelibly impressed upon our memory. The information which we possessed from these various sources, we were happy to communicate to Mr. Young. Every one who reads them both, will perceive that in its really important details, his Discourse agrees with the present memoir. The summary of character which is given by him we believe to be a strikingly just one, and sufficient, if nothing else had ever been written, to place before the reader quite a distinct and faithful portrait of Dr. Bowditch.

Note to page 110.

The property left by Dr. Bowditch at his death, exclusive of his dwelling-house in Boston, and the library, furniture, &c., in it, consisted of printed copies of this work,

valued in the inventory at	$5,000 00
And other personal estate, valued at	31,571 38
Total, ...	$36,571 38

*The Translator presented this Work to the Institutions and Individuals named in the following List, and perhaps to Others not known to us.**

American Academy of Arts and Sciences, Boston, Massachusetts.
Boston Athenæum.
Salem Athenæum.
Nantucket Athenæum.
Harvard University, Cambridge, Massachusetts.
Brown University, Providence, Rhode Island.
University of Vermont, Burlington, Vermont.
Philosophical Society held at Philadelphia.
New York Philosophical Society.
Professor Adrain, of New Brunswick, N. Y.
Professor Anderson, of New York.
Professor Nulty, of Philadelphia.
John Pickering, Esq., LL. D., of Boston, Massachusetts.
Professor Pierce, of Harvard University.
Professor Renwick, of New York.
Professor Strong, of New Brunswick, N. Y.
Rev. Francis Wayland, D. D., President of Brown University.

Royal Society of London.
Royal Society of Edinburgh.
Royal Irish Academy.
Royal Astronomical Society of London.
British Museum, London.
Bodleian Library, Oxford.

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* If any, to whom the translator intended to present it, have not received the work, and will inform us of an opportunity of sending it safely, it will be forwarded by us according to the wishes of our deceased father.

Trinity College, Cambridge.

George B. Airy, Esq., F. R. S., Professor at the University of Cambridge, England.

Charles Babbage, Esq., F. R. S., London.

Francis Baily, Esq., F. R. S., London.

Henry Beaufoy, Esq., South Lambeth, London.

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The late Baron Zach, Paris.

A copy is also sent to Francis Beaufort, Esq., Captain in the Royal Navy, Hydrographer to the British Admiralty.

PREFACE.

After having explained, in the two preceding books, the theories of the planets, and of the moon, it now remains to examine those of the other satellites and comets, which are the chief objects of the present volume. The satellites of Jupiter are the most interesting of all the satellites, except that of [6000] the Earth. The observation of these bodies, the first which were discovered in the heavens by the telescope, goes no farther back than two centuries; and in fact we ought not to estimate the interval of time in which their eclipses have been observed, at more than a century and a half. But, in this short interval, these bodies have presented to our view, by the rapidity of their revolutions, all those great changes which time produces with extreme slowness in the planetary orbits, the system of the satellites being an image of that of the planets. Their frequent eclipses have made known the principal inequalities of their motions, with a degree of accuracy which could never have [6000′] been attained by observations of their elongations from Jupiter. To obtain the theory of these motions, we shall develop, in the first place, the differential equations of the orbits, and then, by integrating these equations, we shall ascertain the various perturbations. These inequalities differ but little in their forms from those of the planets and moon; but the relations which exist between the mean motions and the mean longitudes of the three inner satellites of Jupiter, augment some of these inequalities so much as to give them a great influence on the whole of their theory. These mean motions are very nearly in a subduple progression, and from this peculiarity arise several very sensible

inequalities, whose periods are different; but in eclipses they are all transformed
[6001] into one of $437^{days},659$. Bradley first noticed this period in the return of the eclipses of the first and second satellites. Afterwards, Wargentin explained in a full manner the law of the inequalities on which it depends. The cause of it he attributed to the mutual attractions of these three satellites, but without submitting it to analysis, which was not then sufficiently advanced for that purpose. Mathematicians have since improved their methods, and have applied them to the investigation of the perturbations of these satellites. These great inequalities were the first results of the investigations, as they had been the first which were discovered by observation; we shall here develop them with all the detail their importance requires.

The relative positions of the orbits of the three inner satellites give rise to a singular phenomenon, which at present is *unique* in the theory of the motions of the heavenly bodies. The mean motion of the first satellite, *plus* twice that
[6002] of the third, would be equal to three times the motion of the second, if these motions were exactly in a subduple progression; but this equality is incomparably more correct than the law of the progression, and the small variations, which have been noticed by astronomers, are within the limits of the errors of the observations. Another result, which is not less remarkable, is that, since the discovery of the satellites of Jupiter, the mean longitude of the first, *minus* three times that of the second, *plus* twice that of the third, has never differed from two right angles but by a nearly insensible quantity. It would be improbable to suppose that the primitive motions of these three bodies exactly satisfied these equations. It is much more reasonable to suppose they
[6002′] were very nearly correct at the origin of the motions, and that afterwards the mutual action of the satellites was sufficient to make them rigorously accurate. This has been proved by analysis, as we have already seen in the eighth chapter of the second book. We have here resumed this interesting subject, and treated it by another method; and the results, agreeing with those of the

book just cited, serve to confirm them. We may, in the preceding equations, instead of the mean *sideral* motions and longitudes, substitute the mean [6003]
synodical motions and longitudes, and in general we may refer the motions and longitudes of the three satellites to an axis moving according to any law; whence it follows that they cannot all be eclipsed at the same time. For, in the simultaneous eclipses of the second and third, the first is always in conjunction with Jupiter; it is always in opposition in the simultaneous eclipses of the Sun produced on the surface of Jupiter by the other two [6003′]
satellites.

The mean motions and epochs form six of the twenty-four arbitrary quantities, which the integrals of the twelve differential equations of the motions of the four satellites must contain. The preceding relations establish between these constant quantities two equations of condition, which reduce them to twenty-two; but the arbitrary quantities which these equations cause to disappear, are replaced by the constant quantities of an inequality which is denoted by the name of *the libration of the satellites*, whose period rather [6004]
exceeds six years. This inequality is apportioned among the three inner satellites, according to a ratio depending on their masses and distances; but, as all the researches of Delambre, for the discovery of such an inequality by observation, have been fruitless, it must be very small. Therefore, at the origin, the motions of the three inner satellites and their epochs must have very nearly satisfied the two preceding equations. The secular equations of the mean motions of the satellites, do not alter these inequalities. In consequence of the mutual action of these bodies, these equations are so modified, that the secular equation of the first, *plus* twice that of the third, is equal to three times the secular equation of the second; and even the inequalities, which vary slowly, conform so much the more to this arrangement as their periods are longer. The libration, by which the motions of the three inner satellites are balanced according to the laws we have just mentioned, extends to their rotatory

[6005] motions, if these motions are equal to those of revolution, as appears to be the case by observation. Then the attraction of Jupiter, by causing the rotatory motions of the satellites to participate in the secular equations, modifies these motions so that the rotation of the first, *plus* twice that of the third, is always equal to three times that of the second satellite. We may here observe a great analogy between the libration of the satellites and the actual libration of the moon, whose theory has been explained in the fifth book. We have there seen that the attraction of the earth upon the lunar spheroid, produces a rigorous equality between the mean motions of rotation and of revolution; and that the two arbitrary quantities, which disappear in consequence of this
[6005'] equality, are replaced by those of an equation produced by the actual libration. We have also seen that the secular equation of the mean motion of revolution does not alter this equality, as the earth's action causes the rotatory motion of the moon to participate in that equation.

The orbits of the satellites suffer changes analogous to those of the great variations of the planetary orbits; their motions are also subjected to secular equations similar to those of the moon. We shall explain, in detail, the theory of all these inequalities, the observation of which will furnish the best data for the determination of the masses of the satellites, and of the oblateness of Jupiter. The great influence of this last element upon the motions of the nodes, determines its value with greater accuracy than direct measurement. By this means we find that the axis of revolution of Jupiter is equal to its
[6006] equatorial diameter multiplied by 0,9287; which differs but very little from the ratio of thirteen to fourteen, given by a mean of the most exact measures of the oblateness of that planet. This agreement is a new proof that the gravity of the planets towards the principal planet, is composed of the attractions of all its particles; as we have found in the seventh book, for that of the moon relative to the earth.

However perfect the theory might be, there still remained an immense task to fulfil, in order to reduce the analytical formulas into tables. Bouvard first computed the coefficients of these formulas in numbers; but, in that state, they yet contained thirty-one indeterminate constant quantities, namely, the twenty-four arbitrary quantities of the twelve differential equations of the motions of the satellites, the masses of these bodies, the oblateness of Jupiter, the inclination of its equator, and the position of its nodes. To obtain the [6006'] value of all these unknown quantities, it was necessary to discuss a very great number of eclipses of each satellite, and to combine them in the most judicious manner to obtain each element. Delambre executed this important work with the greatest success, and his tables, which represent the times of the eclipses with as much accuracy as the observations themselves, afford to the navigator a sure and easy method of finding immediately, by the eclipses of the satellites, particularly by those of the first, the longitude of the places at which he may stop.

One of the most curious results of these researches, is, the knowledge of the masses of the satellites, which seemed to be out of our power to find, because their extreme smallness renders it impossible to measure directly their diameters. We have selected for this purpose the data which appear to be the most advantageous in the present state of astronomy; and there is reason to believe that the values of the masses which have been obtained are very [6006''] near the truth. These values may hereafter be verified, when the course of time shall have made known more accurately the secular variations of the orbits. We shall now give the principal elements of the theory of each satellite, resulting from the comparison of the formulas of the theory with the observations.

The orbit of the first satellite moves upon a fixed plane, which passes always between the equator and the orbit of Jupiter, through the mutual intersection

of these two last planes, whose relative inclination is, according to observation,
[6007] equal to $3^{\circ},4352$. The inclination of this fixed plane upon the equator of Jupiter, is, by theory, only $20''$; consequently it is insensible: therefore we may consider the first satellite as being in motion upon the equator of Jupiter. No excentricity peculiar to this orbit has been discovered; it merely partakes, in a slight degree, of the excentricities of the orbits of the third and fourth satellites. For, in consequence of the mutual action of all these bodies, the excentricity peculiar to each orbit affects the others more feebly as they are more distant. The only sensible inequality of this satellite, is that which has for its argument the double of the excess of the mean longitude of the first satellite above that of the second, and which produces in the return of eclipses
[6007′] the inequality whose period is $437^{\text{days}},659$; it is one of the data which we have used in finding the masses of the satellites; and as it depends wholly on the action of the second, it determines the value of its mass with great exactness.

The observations of the eclipses of the first satellite, were, in the first instance, the cause of the discovery of the velocity of light, which has since been better determined by the phenomenon of aberration. In the present state of the theory of this satellite, when the observations of it have become very numerous, it has appeared to me that they can be used to determine this phenomenon with greater precision than by direct observation. Delambre, at my request, has cheerfully undertaken this discussion, and he has found
[6008] $62'',5 = 20^{s},25$ for the whole aberration, which is exactly the same as Bradley had determined, from a great number of very delicate observations upon the fixed stars. It is very satisfactory to perceive such a perfect agreement between two results deduced from entirely different methods. From this we may infer that the velocity of light, in all the space comprised within the earth's orbit, is the same as upon the circumference of that orbit; and this result may also be extended to all the space included within the limits of Jupiter's orbit; for, on account of its excentricity, the variation of the radius vector of that

planet is very sensible in the duration of the eclipses of the satellites; and the discussion of these eclipses proves that its effect is exactly the same as in the hypothesis of a uniform motion of light.

The orbit of the second satellite moves upon a fixed plane, which passes always between the equator and the orbit of Jupiter, through their mutual intersection, and with an inclination to that equator of 201″. The inclination [6009]
of the orbit to this fixed plane is 5152″, and its nodes move upon this plane with a retrograde motion, whose period relative to the tropics is $29^{\text{years}},9142$; this is one of the data which are used in determining the masses. No excentricity peculiar to this satellite has been perceived by observation, but it participates a little in the excentricities of the third and fourth. Its two principal inequalities depend on the actions of the first and third satellites. The relation which exists between the longitudes of the three inner satellites, reduces these inequalities to a single term, whose period relative to eclipses is $437^{\text{days}},659$, and its coefficient is the third of the data which are used in finding the masses

The orbit of the third satellite moves upon a fixed plane, which passes always between the equator and the orbit of Jupiter, through their mutual intersection; the angle of inclination of this fixed plane to Jupiter's equator being 931″. The orbit of the third satellite is inclined 2284″ to its fixed plane, and its nodes have upon this plane a retrograde tropical motion, whose period is $141^{\text{years}},739$. Astronomers at first supposed that the orbits of the [6010]
three inner satellites were in motion upon the equator of Jupiter itself; but they found that the inclination of this equator to the orbit of the planet, deduced from the eclipses of the third satellite, was less than that obtained from the eclipses of the other two satellites. The cause of this difference was unknown; but it arises, however, from the circumstance that the orbits of the satellites do not move upon Jupiter's equator, but upon different planes, which

are so much the more inclined to the equator, as the satellites are more distant from the planet. We have found a similar result for the moon, in the second chapter of the seventh book [5352]; and it is upon this that the lunar equation in latitude depends, whose value being determined by observation, has given the ellipticity of the terrestrial spheroid with as much exactness as that which is deduced from the measured degrees of the meridian.

The excentricity of the orbit of the third satellite presents some singular anomalies, the cause of which has been found from the theory. They depend on *two distinct equations of the centre.* The one peculiar to this orbit
[6011] corresponds to a perijove, whose annual sideral motion is 29010″; the other, which may be considered as an emanation of the equation of the centre of the fourth satellite, corresponds to the perijove of this last body. It is one of the data which we have used in finding the masses. These two equations, by their combination, form a variable equation of the centre, corresponding to a perijove whose motion is not uniform. They coincided, and were added together in 1682, and their sum amounted to 2458″. In 1777, the one was subtracted from the other, and their difference was only 949″. Wargentin attempted to represent these variations by means of two equations of the centre; but as he did not refer one of them to the perijove of the fourth satellite, he was compelled by observation to abandon this hypothesis, and had recourse to that of a variable equation of the centre, whose changes he determined by observation, from which he obtained nearly the same results as those we have just mentioned.

Lastly, the fourth satellite moves upon a fixed plane, which is inclined to the equator of Jupiter by 4547″, and passes through the line of nodes of that equator between this last plane and that of the orbit of the planet. The
[6012] inclination of the orbit of the satellite to the fixed plane, is 2772″; and its nodes have upon this plane a retrograde tropical motion, whose period is

531 years. In consequence of this motion, the inclination of the orbit of the fourth satellite upon Jupiter's orbit incessantly varies. It was at its *minimum* about the middle of the last century, remaining nearly stationary at about 2°,7 from the year 1680 to 1760; and during that interval its nodes upon the orbit of Jupiter had a direct annual motion of about 8′. These results, deduced from observation, were adopted by astronomers, and used for a long time with advantage in the tables of this satellite. They accord with the formulas [7365′, &c.] which make the inclination and the motion of the nodes nearly the same as those found by astronomers from the discussion of the eclipses; but, in later years, the inclination of the orbit has sensibly increased, and it would have been difficult to discover the law of its variation without the aid [6012′] of the theory. It is curious thus to see, springing up from analysis, these remarkable phenomena, which had been, in some measure, discovered by observation; but the results, being the combination of several simple inequalities, were too complicated for astronomers to be able to discover their laws. The excentricity of the orbit of the fourth satellite is much greater than those of the other orbits. Its perijove has an annual direct motion of 7959″; this is the fifth of the data which we have used in finding the masses.

Each orbit participates a little in the motions of all the others. Their planes, which are called fixed, are not rigorously so; they move very slowly with the equator and orbit of Jupiter, passing always through their mutual [6012″] intersection. The inclinations of these planes upon the equator of Jupiter vary incessantly, in proportion to the inclination of the orbit upon the equator of the planet.

The theory of the satellites being founded upon the observations of their eclipses, it is important to have the expressions of the durations of these eclipses, noticing every thing which can have any influence, particularly the ellipticity of the mass of Jupiter. We have obtained these expressions by

considering generally the figure of the shadow projected by an opaque body, from the action of a luminous one. We may from this determine the durations of the eclipses, supposing them to commence at the moment when the centres of the satellites begin to penetrate into the shadow of the planet. But their discs, though inappreciable in themselves, may become sensible by means of the times required to make them disappear in eclipses; moreover the magnitudes of the satellites, of which but little is known, their different brightness, the effect of the penumbra, and probably, also, that of the refraction of the sun's light in Jupiter's atmosphere; all these causes, whose effects it is almost impossible to estimate, make it necessary to recur to observation, to determine the mean durations of the eclipses of Jupiter's satellites in the nodes, or when the latitude above the orbit of the planet is nothing. These durations, by observation, are
[6013] $9426''$ for the first satellite, $11951''$ for the second, $14838''$ for the third; lastly, $19780''$ for the fourth.

The observations of the satellites during the times of entering and quitting the shadow of Jupiter, and those of their shadows upon Jupiter's disc, would throw much light upon the magnitudes of their discs, and upon several of the other elements of the theory of the satellites. This kind of observations, which has hitherto been too much neglected by astronomers, ought to receive more of their attention; for it appears to me that the interior contacts of the shadows
[6013'] would determine the times of conjunction with greater exactness than by eclipses. The theory of the satellites is now so much advanced, that what yet remains to be done, can be determined only by very exact observations. Therefore it becomes necessary to try new methods of observation, or at least to satisfy ourselves that those we use deserve the preference.

The extreme difficulty of making observations on the satellites of Saturn, renders their theory so imperfect, that we hardly know, with any great degree of accuracy, their revolutions and their mean distances from that planet; it is

therefore useless at present to consider their perturbations. But the position of their orbits presents a phenomenon deserving the attention of mathematicians and astronomers. The orbits of the six inner satellites appear to be in the plane of the ring, whilst the orbit of the seventh differs considerably from it. It is [6014]
natural to suppose that this depends on the action of Saturn, which, by its ellipticity, retains the six inner orbits in the plane of its equator, as it maintains in the same plane the ring by which it is surrounded. The sun's action tends to draw them from it; and as this effect increases very rapidly, being nearly as the fifth power of the radius of the orbit, it becomes quite sensible in the outer satellite. The orbits of the satellites of Saturn move, like those of the moon and the satellites of Jupiter, upon fixed planes, which pass always between the equator and the orbit of the planet through their mutual intersection, and which are so much the more inclined to that equator as the satellites are farther from Saturn. This inclination is considerable relative to the outer satellite. Its orbit is much inclined to the fixed plane which corresponds to it, and its nodes have upon this plane a retrograde motion, whose value we have [6014′]
endeavored to determine, using the observations that have already been made for this object; but, these observations being very uncertain, the results which we have given can be considered only as a very imperfect approximation.

We know yet less about the satellites of Uranus. It only appears from Herschel's observations, that they all move in a plane, nearly perpendicular to the planet's orbit; which evidently indicates a similar position in the plane of its equator. We shall show that the oblateness of the planet, combined [6014″]
with the action of the satellites, can retain these orbits very nearly in that plane. This is all that can be said relative to these bodies, since, on account of their smallness and distance, we are unable to obtain a more thorough knowledge of them.

The theory of the perturbations of the comets is the object of the ninth book.

The greatness of the excentricities and of the inclinations of their orbits, does not permit us to apply to these bodies the formulas which are used for the planets and satellites. In the present state of analysis, it is not possible to represent the motions of a comet by analytical expressions, including an
[6015] indefinite number of revolutions, and we are reduced to the necessity of determining them by parts, by means of quadratures. The most simple method of doing this, is that proposed by La Grange, in which the orbit of a comet is considered as an ellipsis which varies incessantly; each elliptical element is then expressed by the integral of a differential function, which may be obtained, very nearly, by several methods afforded by analysis. We have here given these differential functions, under the forms which appear to be the most convenient, and we have also given a very exact method of integrating them by approximation. It would have been pleasant to have made an application of this method to the return of the comet of 1759, but various occupations have prevented; we shall, however, develop the process with sufficient detail to prevent any other difficulty in its application than what arises from the substitution of the numbers.

Then we shall notice, by a particular analysis, the case of a comet which approaches so near to a planet as to have its orbit wholly changed; this singular case deserves so much the more attention, as it appears to have happened to the first comet of 1770. The fruitless attempts of astronomers
[6015′] to reduce the observations of this comet to the laws of the parabolic motion, are well known. Lexell first discovered that it had described, during its appearance, the arc of an ellipsis corresponding to a revolution of rather more than five years and a half. Burckhardt, by a profound discussion of the observations of this comet, and of the elliptical elements proper to represent them, has confirmed this remarkable result, upon which there cannot now be the least doubt. But, with so quick a revolution, this comet ought to have appeared several times; nevertheless, it was never seen before 1770, and has

not been seen since. To explain this double phenomenon, Lexell remarked, that, in 1767 and 1779, the comet passed very near to Jupiter, whose action in 1767 might have decreased its perihelion distance, so as to render it visible to the earth in 1770, instead of being invisible, as it was before; and, by a contrary effect, that action in 1779 could have increased its perihelion distance so as to render it afterwards invisible. But this explanation requires that the elements of the orbit of the comet, determined by its positions observed in 1770, should satisfy the two preceding conditions, at least by making but very slight corrections in these elements, and such as are included within the limits of the alterations which the attraction of the planets might produce in them. Now this is found to be the case, by applying these formulas to the investigation of the perturbations of the comet by the action of Jupiter at these two epochs. The possibility of this double change in the perihelion distance at those times being thus proved, the explanation given by Lexell becomes highly probable.

Of all the comets which have been observed, the one just mentioned is that which approached nearest to the earth; it must, therefore, have suffered very sensible perturbations. We find by calculation, that the earth's action increased its sideral revolution two days; and the comet, by its reaction upon the earth, [6015″]
must likewise have altered the duration of a sideral year; and by calculation it appears that the decrement of the year would be a ninth part of a day, if the mass of the comet were equal to that of the earth. The researches lately made by Delambre, to improve the solar tables, do not allow us to attribute to the comet's action a decrease of three seconds in the length of a year; we are therefore very sure that the mass of the comet is not a five-thousandth part of that of the earth. In general the correspondence between the observed places of the planets and satellites with those which are determined by noticing only the mutual actions of these bodies upon each other, proves that, notwithstanding the great number of comets which traverse the planetary system in every direction, their attraction, up to the present time, has been insensible; thus their

masses must be extremely small, and astronomers have no reason to fear that their action can alter the accuracy of the tables.

In the tenth book, several subjects are noticed relative to the system of the world. One of the most interesting, by its connection with universal gravitation, and by its influence upon the observations of the heavenly bodies,
[6016] is the theory of astronomical refractions. The air, through which we see those bodies, refracts their rays by certain laws, which it is of importance for astronomers to ascertain fully; they depend on the constitution of the atmosphere, and on the variations it suffers in its pressure and temperature. We have explained in detail the analysis of these laws, which require some particular expedients when the body is very near the horizon. Then the refraction of its light depends upon the law according to which the temperatures of the strata of the atmosphere decrease as they are more elevated. The law which we have assumed combines the advantage of an easy calculation with that of representing, at the same time, the experiments on the diminution of the heat, the observations on the refractions, and the heights of a barometer at different elevations. Fortunately, when the elevation of any heavenly body exceeds eleven or twelve degrees, its refraction depends wholly upon the state of the air in the place of the observer, and this state is indicated by our meteorological instruments. At equal temperatures, the volume of the same quantity of air is reciprocally proportional to the pressure it suffers; but, to obtain the variations of this volume, corresponding to those of a mercurial thermometer, we must know exactly how this instrument corresponds with an air
[6016′] thermometer. Gay-Lussac has made a great many very correct observations upon this subject; he has taken an extreme degree of care in graduating accurately several mercurial and air thermometers, using particular caution in drying the glass tubes; for their humidity, in the experiments upon this subject by different observers, is the chief cause of the difference of the results.

Then, by immersing these thermometers in the same vessel of water, at the temperature of melting ice, and of boiling water, he has found, by the mean of a great number of results corrected for the effect of the expansion of the glass, and for the variations of the barometer during each experiment, that a mass of air represented by unity, at the degree of melting ice, becomes 1,375 at the heat of boiling water, under a pressure of $0^{\text{metre}},76$ of the barometer. Tobias [6016″]
Mayer, who was as correct in his experiments as he was great in astronomy, found, by observations whose accuracy he warrants, that, by the same increment of temperature, the mass becomes 1,380. This differs but very little from the preceding result, with which the experiments of Dalton agree perfectly. To obtain the corresponding changes of two mercurial and air thermometers, Gay-Lussac divided exactly into two equal parts the space taken up by these two fluids in each thermometer, from the degree of melting ice to that of boiling water, which gave him the 50th degree of each thermometer. By immersing them in a vessel of water, raised to that temperature, he has observed that their differences were always extremely small, and alternately of a contrary sign, so that the mean difference determined by twenty experiments was found to be insensible; whence we may infer that, from zero to the heat of boiling water, the march of the two thermometers is very nearly the same. These results are sufficient for the theory of refractions, in which it is only necessary to know the density of the air corresponding to the indications of the barometer and thermometer. But, in the theory of heat, it is necessary to estimate the real degrees of heat indicated by those of a mercurial thermometer; and this will be given with great accuracy by the experiments just mentioned, if the increment of heat of a mass of air submitted to a constant pressure, be proportional to the increase of its volume. Now this hypothesis is at least very probable; for, if we imagine the volume of the mass of air to remain the same whilst its temperature increases, it is natural to suppose that the elastic force, of which heat is the cause, will increase in the same ratio. By submitting it [6016‴]

in this last state to the pressure it suffered in the former case, its volume will increase as its elastic force, and therefore as its temperature. Hence it appears, that an air thermometer indicates accurately the variations of heat; but, its construction being difficult, it is sufficient to have compared its march with that of a mercurial thermometer by very exact experiments.

Heretofore no use has been made of hygrometrical observations in the calculation of refraction, and it is desirable that the influence of the humidity of the air upon these phenomena should be determined by direct experiments. We have endeavored to supply this defect, by supposing that the refractive powers of water and its vapor are proportional to their respective densities. In this probable hypothesis, the refracting force of the vapor exceeds that of air
[6016''''] of the same density; but, under equal pressures, the air exceeds in density that of the aqueous vapor; hence it follows, that the refraction arising from that vapor diffused in the atmosphere, is very nearly the same as that of the air whose place it occupies, so that the effect of the humidity of the air upon the refractions is almost insensible. This is confirmed by some observations of the meridian altitudes of the sun, seen through clouds which left its limb well defined; the refraction of its light did not appear to be changed by this circumstance.

We know that the air is a compound of two gases, azote and oxygen; and it is probable that the refractive force is not the same for both of them; therefore the refractive force of the atmosphere would change if the proportion of these
[6017] gases should be altered. But it follows from the numerous and very exact experiments of Humboldt and Gay-Lussac, that this proportion remains always very nearly constant upon the surface of the earth; and Gay-Lussac, having ascended in a balloon, obtained some atmospherical air at the height of more than six thousand five hundred metres [over four miles]; the analysis of this

air has given him the same ratio between these two gases, as that found on the surface of the earth.

The refractive force of the atmosphere may be determined, either by direct experiments upon the refraction of the air, or by astronomical observations. The great number and the accuracy of these observations have induced me to prefer the last method, and I have deduced from them a formula for computing the refractions when the apparent altitudes exceed twelve degrees. I believe that this formula will give these refractions very accurately, supposing the refractive force of the air to be in the ratio of its density, and that its temperature and humidity have no sensible influence upon it; three points which ought to be verified by a great number of observations and experiments.
It appears to me that the best method of obtaining this object, is to observe, in [6017′]
the extremes of heat and cold, and in those of elevation and depression of the barometer, the meridian altitudes of some stars which do not rise more than twelve or fifteen degrees above the horizon. For this purpose, a series of observations has been commenced at the observatory at Paris, and it is proposed to continue them during a great number of years. The theory also supposes that the density is constant throughout any stratum of air concentrical to the earth; and it is possible that the winds and other causes may produce variations of density which cannot be ascertained, but may have an influence upon the refractions. It is to this we ought chiefly to attribute the small discrepancies perceived in the observations of the same star on different days. However perfect the instruments of astronomy may become, this source of error will always be an obstacle to the extreme accuracy of the observations.

The preceding researches, grounded upon the constitution of the atmosphere, have conducted to a very simple formula for measuring the height of a mountain by a barometer. In this formula we have noticed the variations of gravity

depending on the different latitudes and elevations above the level of the sea. We should be pleased to have introduced into it the effect of hygrometrical measures, but we have not sufficient experiments for that object. Ramond
[6017''] has determined, with great exactness, the principal coefficient of this formula, by means of numerous and exact observations of the barometer, which he has made upon several mountains whose heights are well known.

The atmosphere extinguishes a part of the rays of light which traverse it. We have determined the law of this extinction, which ought also to take place in the sun's atmosphere. It follows from these formulas, compared with a
[6017'''] curious experiment of Bouguer upon the intensity of light of several points of the sun's disc, that, if the sun were deprived of its atmosphere, it would appear to us twelve times more luminous than it now does.

One of the principal arguments formerly used in opposition to the motion of the earth, was the difficulty of reconciling this motion with that of bodies detached from its surface and left to themselves. Being then ignorant of the
[6018] laws of mechanics, it was supposed that the spectator ought to fly off with the whole of the velocity depending on the rotatory motion of the earth and its motion of revolution about the sun. The knowledge of these laws leaves now no doubt upon this subject; but they show that the effect of the earth's rotation upon the motion of projectiles, although of small moment, may become sensible by proper experiments. We have here explained the analysis of this motion, which agrees with the experiments heretofore made to discover the earth's diurnal motion by the fall of bodies which descend from a great height.

After having examined several cases in which the motion of a system of

bodies which attract each other, may be exactly determined, we have resumed the theory of the secular equations depending on the resistance of an ethereal fluid surrounding the sun; a theory which has been before considered at the end of the seventh chapter, but it is here extended to an unlimited time. This resistance would really take place in nature, if the solar light were produced by the vibrations of such a fluid. If it is an emanation from the sun, its impulsion upon the planets and upon the moon, being combined with the [6018′]
velocities of those bodies, will produce in their mean motions an acceleration whose analytical expression is given; but this effect is destroyed by the diminution of the sun's mass, which must take place in this hypothesis. Then, as the attractive force of that body incessantly decreases, the orbits of the planets become more and more dilated, and their motions are diminished incomparably more than they are augmented by the impulsion of the light. As no variation in the mean motion of the earth is indicated by observation, we may infer, *First*, that the sun, during the last two thousand years, has not lost a two millionth part of its substance; *Second*, that the effect of the impulsion of light upon the moon's secular equation is insensible. The analysis of this effect may be applied to gravity, considered as being the result of the impulsion of a fluid producing the effect of gravity, by moving with extreme rapidity towards the attracting body. From this it follows, that, to satisfy the phenomena, we must suppose this fluid to have an excessively great velocity, at least one million times greater than that of light. This velocity would be infinite in the hypotheses admitted by [6018″]
mathematicians relative to the action of gravity; these hypotheses may therefore be used without fear of any perceptible error. We shall here observe that these various causes of alteration in the mean motions of the planets and satellites, do not produce any effect in the positions of their apsides; and since it is evident from observation, that the motion of the moon's perigee is subjected to a very sensible secular equation, we may from

thence conclude that the moon's secular equations must not be attributed either to the resistance or to the impulsion of a fluid. We have, in the seventh book, developed the laws of these equations, and the real source of them.

Lastly, we shall conclude this volume with a supplement to the theories of the moon and planets. Jupiter, Saturn, and Uranus, compose a separate system, upon which the inferior planets have no sensible influence. This
[6018'''] system, by the mutual action of the three bodies upon each other, suffers great inequalities, which we have developed in the sixth book. The discovery of these inequalities has given to the tables of Jupiter and Saturn an unlooked-for degree of accuracy. Bouvard has again discussed, with the greatest care, all the oppositions of these two planets, from the time of Bradley to the present moment, as observed at Greenwich and Paris, by means of great transit instruments, and the best mural quadrants. On the other hand, the theories of these three planets have been reviewed with particular care, and I have discovered some additional inequalities, which render the formulas still more accordant with observation. These formulas, reduced to tables by Bouvard, represent with remarkable accuracy the modern observations, those of Flamsteed and Tycho, and even those of the Arabs and Greeks, as well as the Chaldean observations, which Ptolemy has transmitted to us in his Almageste. The singular precision with which Jupiter and Saturn, from the most remote period, have obeyed their mutual action, proves to us that the influence of causes foreign from the planetary system is insensible. One of the principal advantages of these new researches is the exact knowledge of Saturn's mass, whose value is determined by this means much better than by the elongations of the satellites. The inequalities produced by Uranus are too small to enable us to determine its mass from them; that which we have adopted in the sixth book, seems to agree very well with observations [9159*a*—*g*].

Nothing now remains, to fulfil the engagement which was entered into at the commencement of this work, but to give an historical account of the labors of mathematicians and astronomers on the system of the world; this will be the object of the eleventh and last book.*

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* Subsequently the author published this portion of his work in six books, making in all sixteen. This portion of the work contains improvements and additions to several parts of the Mécanique Céleste, together with some historical notices relative to the most important subjects, as will be seen in the fifth volume.

CONTENTS OF THE FOURTH VOLUME.

PARTICULAR THEORIES OF THE MOTIONS OF THE HEAVENLY BODIES.

EIGHTH BOOK.

THEORY OF THE SATELLITES OF JUPITER, SATURN, AND URANUS.

Object of this theory [6019] 1

CHAPTER I. EQUATIONS OF THE MOTIONS OF THE SATELLITES OF JUPITER 1

The reciprocal action of the satellites, the sun's attraction, and the oblateness of the spheroid of Jupiter, are noticed [6020—6077] .... § 1, 2

CHAPTER II. ON THE INEQUALITIES OF THE MOTIONS OF JUPITER'S SATELLITES, WHICH ARE INDEPENDENT OF THE EXCENTRICITIES AND INCLINATIONS OF THEIR ORBITS 17

Development of the equations of the motions of these satellites. Analytical expressions of the perturbations of their radii vectores and of their longitudes. The sun's action produces an inequality analogous to the *variation* in the lunar theory [6078—6125] .... § 3

Investigation of the terms of these expressions which can acquire considerable values by the divisors introduced by integration; these divisors being very small, in consequence of the nearly commensurable ratios of the mean motions of the three inner satellites. Necessity of retaining, in these small divisors, the terms depending on the product of the constant part of the disturbing force by the variation of the radius vector, this product having a sensible influence upon their values [6125′—6171] .... § 4

Effect of terms of this kind on the times of the eclipses of the three inner satellites. All the inequalities produced by such terms depend upon the same angle, having a common period of $437^{\text{days}},659$. This result is conformable to observation [6172—6193] .... § 5

CHAPTER III. ON THE INEQUALITIES OF THE MOTIONS OF THE SATELLITES DEPENDING ON THE EXCENTRICITIES OF THE ORBITS 38

Expressions of the different equations of the centre of the satellites, and of the motions of their apsides [6194—6244″] .... § 6

Investigation of the terms which can become sensible by the effect of small divisors, introduced by integration, although they may be multiplied by the excentricities, which are very small [6245—6270] .... § 7

The sun's action produces also in the motion of the satellites some sensible inequalities, depending on the excentricities. Expression of these inequalities. That which affects the longitude is composed of two parts analogous to the *evection and to the annual equation in the lunar theory* [6271—6294] .... § 8

CHAPTER IV. ON THE INEQUALITIES OF THE SATELLITES IN LATITUDE 62

Analytical expressions of the latitude of a satellite and of the motion of its nodes [6295—6336] § 9

The part of this expression which depends on the displacements of the equator and of the orbit of Jupiter, represents the latitude which each satellite would have, if it should move in an intermediate plane which passes between the equator and orbit of Jupiter, through their common intersection. This effect is analogous to that which the earth produces upon the moon, as we have seen in [5352, &c.], but it is much more sensible. Determination of its value [6337—6431] .... § 10

Investigation of the terms which acquire very small divisors by integration in the expression of the latitude, in consequence of the nearly commensurable values of the mean motions of the three inner satellites. Estimates of their values [6432—6486] .... § 11

CHAPTER V. INEQUALITIES DEPENDING ON THE SQUARES AND PRODUCTS OF THE EXCENTRICITIES AND INCLINATIONS OF THE ORBITS 102

Calculation of these inequalities. It is sufficient to notice those only which have a long period [6487—6524] .... § 12

The terms which become the most important in the secular equations of the satellites, are those which depend on the secular variations of the equator and of the orbit of Jupiter, and on the motion of the nodes of the fourth satellite. They are analogous to those which produce the moon's secular equation, and the equation of the moon's motion depending on the longitude of its nodes. Calculation of these terms [6525—6555'] .... § 13

CHAPTER VI. ON THE INEQUALITIES DEPENDING ON THE SQUARE OF THE DISTURBING FORCE. 126

The most remarkable of these inequalities has already been discussed under its general form, in Book II. [1214'—1242^{v}]. It depends on the circumstance, that, at the origin of the motion, the mean longitude of the first satellite, *minus* three times that of the second, *plus* twice that of the third, was very *nearly equal* to the semi-circumference π; and subsequently, by means of the mutual action of the three bodies upon each other, it became accurately equal to π. Development of the theory of these motions by a different method from that which is used in [1214', &c.]. From this it follows, as in [1242^{v}], that the mean motions of the three inner satellites are subjected to a species of *libration*, and it is of importance for astronomers to investigate and determine the limits of this libration by observations. Hitherto it has appeared to be insensible. From this it follows that the relation which now exists between the mean motions of the three inner satellites, will continue unchanged to future ages. Moreover, the *two* inequalities of the first satellite, arising from the attractions of the second and third

satellites, which are reduced to a single inequality, by means of the relations between the mean motions, will never be separated [6556—6662] .... § 14, 15

The ratios of the mean motions of the three inner satellites modify their inequalities, which have a long period. These ratios are not changed by their secular inequalities, which are so adjusted as to satisfy them [6663—6711′] .... § 16

The ratios of the mean motions of the three inner satellites have a sensible influence on the variations of their excentricities and of their perijoves. Examination of this influence, and of the terms produced by it. It does not in any way alter the inclinations or the nodes of the orbits [6712—6748] .... § 17

The square of the terms depending on these ratios may become sensible in the expression of the longitude [6749—6767′] .... § 18

The square of the disturbing force does not introduce any sensible term in the secular equation of the satellites of Jupiter, nor in that of the moon [6768—6777] .... § 19

CHAPTER VII. NUMERICAL VALUES OF THE PRECEDING INEQUALITIES 170

Elements of the orbits of the satellites. Numerical values of the coefficients of the inequalities [6778—6826] .... § 20

Numerical expressions of the inequalities of the radius vector, and of the longitude, independent of the excentricities and inclinations [6827—6849] .... § 21

Numerical expressions of the inequalities depending on the excentricities [6850—6886] ...... § 22

Numerical expressions of the inequalities of the satellites in latitude [6887—6934] .... § 23

Numerical expressions of the inequalities depending on the square of the excentricities and inclinations of the orbits [6935—6944′] .... § 24

Numerical expressions of the inequalities depending on the square of the disturbing force [6945—6947] .... § 25

CHAPTER VIII. ON THE DURATION OF AN ECLIPSE OF ANY SATELLITE 207

General formulas to determine the shadow projected by an opaque body of any figure. First application of this formula to the shadow projected by Jupiter, supposing its form to be spherical [6982, &c.]. Second application, noticing its ellipticity [7015, &c.]. Equation of its shadow and of its penumbra [7050, &c.]. Calculation of the arc described by the satellites in passing through it [7063, &c.]. Formulas to determine the duration of an eclipse [7077—7112′].... § 26

CHAPTER IX. DETERMINATION OF THE MASSES OF THE SATELLITES, AND THE OBLATENESS OF JUPITER 236

To determine these quantities, we must have five *data* derived from observation. Selection of the quantities which are the most proper to be used for this object in the present state of astronomy. Values of the masses of the satellites and of the oblateness of Jupiter, which result from these observations [7162—7166′]. The ratio of the equatorial to the polar axis of Jupiter is determined by this means with greater accuracy than by direct measurement; and as both methods

give the same results, it proves that the gravity of the satellites towards Jupiter is composed of the attractions of all the particles of the planet [7161, &c.] .... § 27

CHAPTER X. ON THE EXCENTRICITIES AND INCLINATIONS OF THE ORBITS OF THE SATELLITES. 244

Formation and resolution of the equations which determine the excentricities of these orbits, and the motions of their perijoves [7170—7173]. The great influence of the oblateness of Jupiter upon these elements gives to each orbit an excentricity which is peculiarly adapted to it; but it participates in the excentricities of the other orbits [7199′, &c.] .... § 28

Formation and resolution of the equations which determine the inclinations of the orbits of the satellites, and the motions of their nodes [7221—7224]. The great influence of the oblateness of Jupiter upon these elements gives to each orbit an inclination which is peculiarly adapted to it [7251]; but it participates in the inclinations of the other orbits. They all move upon planes so much the more inclined to the equator of Jupiter, as the satellite is farther distant from the planet. These planes pass always between the equator and the orbit of the planet, through the mutual intersection of these two last planes. Calculation of the inclinations of all these planes to the equator of Jupiter [7169′—7251] .... § 28

CHAPTER XI. ON THE LIBRATION OF THE THREE INNER SATELLITES OF JUPITER 254

The mean longitudes of the three inner satellites are subjected to this law, namely, that the longitude of the first, *minus* three times that of the second, *plus* twice that of the third, is exactly and constantly equal to the semi-circumference [7252]. If this law were not rigorously correct, in less than two years the longitudes would vary from it by a quarter part of the circumference [7274′]. The observations made on eclipses satisfy this law with as much accuracy as can be expected [7262]. Reasons why there may be some small variations from this law [7262′, &c.]. The libration of the three inner satellites is apportioned among them, according to a ratio depending on their masses and distances. Calculation of this ratio [7267—7279] .... § 29

CHAPTER XII. THEORY OF THE FOURTH SATELLITE 259

Determination of its motion in longitude [7354]. Determination of its motion in latitude above the orbit of Jupiter [7355]. Astronomers have found by observation, that, from the time of the discovery of the satellites till about the year 1760, the inclination of the orbit of the satellite upon the orbit of Jupiter, was nearly 2°,7 [7362]; and that the motion of its nodes was direct, and about 8′ in a year [7362, &c.]. These results of observation are a necessary consequence of the formulas. After the year 1760, the inclination increased considerably, so that these numbers could not be used in the tables. Formulas for the duration of an eclipse [7369, 7383], and [7385] .... § 30

CHAPTER XIII. THEORY OF THE THIRD SATELLITE 276

Determination of its motion in longitude [7405]. It has an excentricity peculiar to its own orbit, and participates very sensibly in that of the fourth satellite, which introduces into its motion two distinct equations of the centre, of which the one corresponds to the perijove of the third satellite, and the other to the perijove of the fourth satellite [7406, &c.]. From this there results a compound equation of the centre, with a variable excentricity. Wargentin had discovered this variable excentricity by observation [7406], but without ascertaining the law of its variations.

Determination of the satellite's motion in latitude [7427.] Formula for the duration of its eclipses [7436, 7437] .... § 31

CHAPTER XIV. THEORY OF THE SECOND SATELLITE 288

Determination of its motions in longitude [7463], and in latitude [7482]. Formula for the duration of its eclipses [7483, 7492, 7493] .... § 32

CHAPTER XV. THEORY OF THE FIRST SATELLITE 300

Determination of its motions in longitude [7513], and in latitude [7522]. Formula for the duration of its eclipses [7523, 7530, 7531] .... § 33

CHAPTER XVI. ON THE DURATION OF THE ECLIPSES OF THE SATELLITES 309

Formulas for the duration of eclipses, supposing that the satellites are eclipsed at the moment of the immersion of their centres in the shadow of Jupiter [7533—7561]. Comparison of this duration with observation [7558—7565] .... § 34

CHAPTER XVII. ON THE SATELLITES OF SATURN 315

The inequalities of the motions of these bodies have not yet been observed [7580]. The only remarkable phenomenon which they present to our view, is the constant position of all their orbits in the plane of the ring, excepting, however, that of the outer satellite, which varies sensibly from that plane [7581′]. Explanation of this last phenomenon [7582, &c.]. It depends on the circumstance that the orbit of the outer satellite moves upon a plane passing between Saturn's orbit and equator, through their mutual intersection; this orbit being very sensibly inclined to that equator. Analytical and numerical determination of the motion of the orbit of this satellite upon this plane [7583, &c.] .... § 35, 36, 37

CHAPTER XVIII. ON THE SATELLITES OF URANUS 356

The mutual action of the planet and its satellites may retain the orbits of the satellites in the plane of its equator [7771—7799′] .... § 38

NINTH BOOK.

THEORY OF COMETS.

Difficulties of this theory. The greatness of the excentricities of the orbits of the comets, and of their inclinations to the ecliptic, does not permit us to apply to their perturbations the formulas which are used for the planets [7800]. We must calculate these perturbations from point to point, in different portions of the orbit, limiting ourselves, in each operation, to a small extent .. 361

CHAPTER I. GENERAL THEORY OF THE PERTURBATIONS OF COMETS 361

General equations of the troubled orbit [7805—7812] ....§ 1

We can satisfy these equations by some analytical formulas, which include a large arc of the orbit, when the radius vector of the orbit of the comet is either very small or very great in comparison with that of the disturbing planet. In the first case, the disturbing action becomes insensible, and may be neglected; in the second case, the comet moves very nearly in an ellipsis about the common centre of gravity of the planet and sun [7814—7845]..................§ 2

General formulas to determine the perturbations of the elements of a comet [7846—7898′].....§ 3

Formula to determine the difference between two successive returns of a comet to the perihelion [7899′—7921] ....§ 4

Method of determining the numerical values of the perturbations of the elements, by means of *generating functions*. Explanation of the theory of these functions [7922—7942]..........§ 5

Reflections on the use of these formulas, and on the manner of varying the application of them to different portions of the orbit [7943—7947] ....§ 6

General analytical expressions of the perturbations, in the case where the disturbing planet is very distant [7947′—8010]...§ 7, 8

Manner of using these formulas, by applying them to the computation of the orbit of a comet, as, for example, that of the year 1759 [8010′—8013]...................................§ 9

CHAPTER II. ON THE PERTURBATIONS OF THE MOTION OF A COMET WHEN IT APPROACHES VERY NEAR TO A PLANET.. 417

We may then suppose the planet to have a sphere of attraction, within which it wholly controls the relative motion of the comet, and beyond which the comet's motion depends solely upon the sun's action [8014—8038′]..§ 10

Development of this hypothesis. Determination of the elements of the orbit of the comet when it quits the sphere of the planet's attraction [8039—8075] ....§ 11

A more simple method of obtaining these values, when the perturbations are not considerable [8076—8092] ....§ 12

Application of these results to the comet of 1770. The attraction of Jupiter could have changed its orbit in 1767, so as to render the comet visible in 1770, instead of being invisible, as it had been before. Again, the attraction of the same planet in 1779, could have changed the orbit so as to render the comet forever afterwards invisible. Calculation of the perturbations in the time of its sideral revolution by the attraction of the earth [8093—8125] ....§ 13

CHAPTER III. ON THE MASSES OF THE COMETS, AND THEIR ACTION UPON THE PLANETS.. 435

The comet of 1770 is that which approached the nearest to the earth, and its action upon the earth did not change sensibly the duration of the sideral year [8132]. Hence it follows, that the mass of this comet must be very small, and that it is less than $\frac{1}{5000}$ of that of the earth [8132]. We may obtain the same result from the consideration that this comet passed through the whole

system of Jupiter's satellites, without producing any sensible alterations in their motions [8133′]. General reflections, tending to prove that the masses of all the comets are extremely small, so that the stability of the planetary system is not disturbed by their action [8134—8136].....§ 14

TENTH BOOK.

ON SEVERAL SUBJECTS RELATIVE TO THE SYSTEM OF THE WORLD.

CHAPTER I. ON ASTRONOMICAL REFRACTIONS 438

Differential equation of the motion of light in the air, supposing all the strata of the atmosphere to be spherical, and the densities variable according to a function of their height [8141—8154] ....§ 1

Investigation of the refraction suffered by light from the various attractions of the different mediums through which it passes [supposing the light to be an emanation from the luminous body]. From this investigation it follows that the sine of incidence is to the sine of refraction, in a constant ratio, depending on the nature of the medium. The refraction changes into reflexion beyond a certain degree of obliquity. When the successive mediums are terminated by plane and parallel surfaces, the velocity of light and its direction are at each moment the same as if it passed immediately from a vacuum into each of them [8156—8235]. § 2

Application of these results to the successive attractions that the different strata of the atmosphere exert upon the luminous particles which traverse them. Differential equation of the motion of light [8236—8264′]..§ 3

Integration of this differential equation. To effect it generally, we must know the law according to which the densities of the strata of the atmosphere decrease as their heights increase. The two limits of this law are a constant density, and a density decreasing in a geometrical progression for equidistant heights. Investigation of the refractions in these two cases. The first gives a refraction which is quite too small [8265—8282]...........................§ 4

The second hypothesis supposes a uniform temperature throughout the whole extent of the atmosphere [8283]. Integration of the differential equation in this hypothesis [8286—8339]. Reduction of the integral into a continued fraction, by means of generating functions [8340—8362]. This makes the refraction too great [8370, &c.]; therefore, this hypothesis cannot be admitted. This is conformable to observations upon the heat of the atmosphere, which decreases as the height increases [8370′] ....§ 5

Integration of the differential equation, supposing the density of the strata of the atmosphere to decrease in arithmetical progression, whilst the heights follow a similar progression. This supposition makes the refraction too small, and does not satisfy the decrement of heat of the air; but it agrees better with observation than the hypothesis of a constant density. The true constitution of the atmosphere must therefore fall between these two hypotheses [8371—8404]. § 6

Integration of the differential equation in an hypothesis composed of the two preceding. The formulas which result from it, for the refractions and for the decrement of the heat of the air, agree with the observed phenomena [8406—8462′] .... § 7

Formula which gives the astronomical refractions for all altitudes which exceed 12° [8474]. In these high altitudes, the refraction depends only upon the state of the barometer and of the thermometer, in the place where the observation is made [8474′, &c.] .... § 8

Discussion of the elements which enter into this formula. These elements are the variations of the density of the air by the variations of its pressure and heat; and the refraction of atmospherical air corresponding to a given temperature and pressure. Determination of the most exact values of these elements [8484—8505] .... § 9

Examination of the influence which the humidity of the air can have upon the refractions [8506]. Theory of evaporation [8509, &c.]. Formula which represents the variations of the elastic force of the vapor, corresponding to a change of temperature [8512]. The influence of the vapor of water upon the refractive force of the air, is nearly insensible, because the excess of its refractive force above that of the air, is very nearly compensated by its decrease of density [8537, &c.] .... § 10

CHAPTER II. ON TERRESTRIAL REFRACTIONS 541

Definition of these refractions, and determination of the formulas which express them [8542—8582] .... § 11

CHAPTER III. ON THE EXTINCTION OF THE LIGHT OF THE PLANETS IN THE ATMOSPHERE OF THE EARTH, AND ON THE SUN'S ATMOSPHERE 551

Formulas which give this extinction of the light in the earth's atmosphere for different inclinations of the luminous ray to the horizon [8583, &c.]. We may suppose, in these formulas, without any sensible error, that the temperature is uniform; then the logarithm of the intensity of the light is as the astronomical refraction divided by the cosine of the apparent altitude [8600, &c.] .... § 12

Calculation of the height of the atmosphere of the sun, using the experiments of Bouguer upon the different intensities of the light near its limb and near its centre. Determination of the decrement of the light of the sun in passing through its own atmosphere. If the sun's atmosphere were removed, it would appear twelve times as luminous as it now is [8603—8646] .... § 13

CHAPTER IV. ON THE MEASURE OF HEIGHTS BY A BAROMETER 565

Relation which exists between the heights of a barometer and thermometer, and the elevation above the earth's surface. Formula for measuring heights by a barometer, adapted to every latitude, and taking into consideration the diminution of gravity on account of the height of the observer [8648—8690] .... § 14

CHAPTER V. ON THE DESCENT OF BODIES WHICH FALL FROM A GREAT HEIGHT 573

Equation of the motion of a falling body, noticing the rotatory motion of the earth, whatever be the figure of the earth and the resistance of the air. If the body falls from a point at rest, it

deviates from the vertical towards the east during its fall, but the deviation is nothing in the direction of the meridian [8691—8769′] .... § 15

Calculation of the deviation of the body when it does not fall from rest [8770, &c.]. If it be projected upwards, it will fall to the west of the vertical [8809] .... § 16

CHAPTER VI. ON SOME CASES WHERE WE CAN RIGOROUSLY ASCERTAIN THE MOTIONS OF SEVERAL BODIES WHICH MUTUALLY ATTRACT EACH OTHER 597

Conditions in which this motion may be ascertained. Determination of the curves described by the bodies when the preceding conditions are satisfied. Application of this method to the motions of three bodies. If the moon, at the origin of its motion, had been placed in opposition to the sun, and at a distance from the earth which is equal to a hundredth part of the radius of the earth's orbit; also, if the earth and moon had been projected in parallel directions, with velocities proportional to their distances from the sun; the sun and the moon would alternately have succeeded each other on the horizon, and would always have been in opposition to each other [8810—8883′] .... § 17

CHAPTER VII. ON THE ALTERATIONS WHICH THE MOTIONS OF THE PLANETS AND COMETS MAY SUFFER BY THE RESISTANCE OF THE MEDIUMS THEY PASS THROUGH, OR BY THE SUCCESSIVE TRANSMISSION OF GRAVITY 609

The effects of this resistance are to diminish the excentricity of the orbit and its greater axis, while the perihelion remains immovable [8884—8930] .... § 18

Application of this theory to the resistance caused by the impulse of light upon the heavenly bodies, whether we consider it as being produced by the vibrations of an elastic fluid, or as an emanation from the sun. From this results a secular equation of the mean motion [8931—8959] .... § 19

Comparison of the secular equations of the earth and moon depending on this cause. They are to each other as unity to 63,169 [8987′] .... § 20

Investigation of the secular equation of the earth, which would result from the diminution of the sun's mass, if the light be an emanation of its substance [8988, &c.]. This inequality is to that which depends on the impulse of light, as — 1 to 0.0002129 [9015]. It follows from this theory, that the impulse of the sun's light upon the moon, does not produce a secular equation of a quarter of a second in that satellite [9018]; it also follows, that, for more than two thousand years last past, the sun's mass has not varied a two millionth part [9023] .... § 21

Investigation of the secular equation which might be produced in the planets by the successive transmission of gravity, supposing it to be produced by the impulsion of a fluid [9024, &c.]. This equation decreases as the velocity of the gravitating fluid increases. If we were to attribute the moon's secular equation to this cause, we must suppose the fluid to have a velocity seven millions of times greater than that of light [9034]; and as it is certain that nearly the whole of this secular equation arises from the diminution of the excentricity of the earth's orbit, it follows that the successive transmission of gravity can contribute to it only an extremely small portion; which would require a velocity in the gravitating fluid at least one hundred millions times greater than that of light [9035]; so that we may consider its transmission as being altogether instantaneous. The secular equation of the earth, depending on this cause, being only a sixth part of that of the moon, must of course be nothing, or insensible [9036] .. § 22

CHAPTER VIII. SUPPLEMENT TO THE THEORIES OF THE PLANETS AND SATELLITES....... 646

Investigation of some new inequalities in the theories of Jupiter and Saturn, in consequence of the nearly commensurable ratio of their mean motions. Application of these inequalities to observation. Definitive formulas of the heliocentric motions of Saturn and Jupiter [9037—9150] .... § 23

Investigation of a small inequality of the same kind which occurs in the moon's motion [9150''—9158] .... § 24

CHAPTER IX. ON THE MASSES OF THE PLANETS AND SATELLITES 680

Manner of calculating these masses, by comparing the analytical formulas of the perturbations with a great number of very accurate observations. Table of their most exact values obtained by this method [9159—9168] .... § 25

ON ASTRONOMICAL TABLES.

Method of rectifying and improving these tables, by using the method of equations of condition [9169—9170] .... § 26

FIRST SUPPLEMENT BY THE AUTHOR.

OF THE SUBJECTS TREATED OF IN THE THEORY OF THE CAPILLARY ATTRACTION.

ON CAPILLARY ATTRACTION [9171] 685

SECTION I. Theory of capillary attraction [9222, &c.] 695

SECTION II. Comparison of the preceding theory with observation [9656, &c.] 790

SECOND SUPPLEMENT BY THE AUTHOR.

OF THE SUBJECTS CONTAINED IN THE SUPPLEMENT TO THE THEORY OF CAPILLARY ATTRACTION.

On the fundamental equation of the theory of capillary action [9757, &c.] 806

New manner of considering the capillary action [9900, &c.] 834

On the apparent attraction and repulsion of small bodies which float upon the surface of a fluid [10155, &c.] 909

On the adhesion of a plate to the surface of a fluid [10258, &c.] 947

On the figure of a large drop of mercury, and on the depression of mercury in a glass tube of a great diameter [10376, &c.] 971

General observations [10461, &c.] 1005

IN THE COMMENTARY.

Among the subjects contained in the Notes, we may mention the following:

Table of the symbols which are used in the theory of Jupiter's satellites [6021*a*—6024*t*].

Numerical values of the symbols n, n', n'', n'''; $\mathcal{N}$, $\mathcal{N}'$, $\mathcal{N}''$, $\mathcal{N}'''$; M; g, g_1, g_2, g_3; p, p_1, p_2, p_3 [6025*a*—*p*].

General integration of a linear differential equation of the second degree [6049*k*, *l*].

Times of the synodical revolutions of the satellites, by Delambre [6781*i*, &c.].

Mass of Jupiter, by Professor Airy, deduced from the elongation of the fourth satellite $\frac{1}{1048.7}$ [6787*d*]. Encke's estimate from the perturbations of Vesta, $\frac{1}{1050.117}$ [6787*d*]; Nicolai's estimate from the perturbations of Juno, $\frac{1}{1053.924}$ [6787*e*].

Computation of the values of a, a', a'', a''' [6790*k*, &c.].

Correction of the signs of some of the inequalities [6843*h*, 6845*l*, 6847*e*, 6849*e*, &c.].

Formulas of integration by quadratures, which are useful in computing the perturbations of the orbit of a comet [7929*a*—*z*]. Application of these methods to the computation of the time of the return of Halley's comet, by Pontécoulant, Damoiseau, Lubbock, &c. [8009*a*—*i*].

Formulas for computing the elements of the orbit of a comet, by means of quadratures, supposing the intervals of time to be equal to each other [8009*k*—8015*z*]. Table of the differentials of the elements of the orbit [8013*d*—*k*]. Table of formulas for computing these elements by means of quadratures [8014*b*—*p*], and [8014*u*].

Principles of the wave theory of light, by Huygens, and its application to the laws of reflexion and refraction [8137*a*—8139*n*]. In this theory, the velocity of light decreases in going from a rarer into a denser medium, which is contrary to the results in the Newtonian theory of emission, where the velocity is supposed to increase upon entering into a denser medium [8137*n*—*r*]. The demonstrations of the laws of reflexion and refraction by means of the principle of the least action, as usually given, will not apply in the wave theory [8139*h*], because in this theory the time is a minimum [8139*n*].

The fundamental equation [8262] which expresses the differential of the refraction, may be considered as a deduction from the usual law of refraction which is obtained from observation [8262*c*—*f*]. This equation may be reduced to the form [8262*x*] given by La Grange, and it becomes integrable, as he remarked, by assuming a particular relation between the elevation of any stratum of the atmosphere and its density [8262*z*, &c.]

Experiments of Biot and Arago on the refractive power of different gases, show that the refraction produced by any gas is always rigorously proportional to its density [8264*a*—*p*].

The hypothesis of a uniform density is equivalent to that proposed by Cassini [8267a]; and the simple process proposed by him gives the same form to the expression of the refraction, for altitudes exceeding ten degrees, as the more complicated hypothesis of La Place [8267t, &c.]. Formulas for calculating the refraction, by Bouguer [8374i], by Simpson [8374l], by Bradley [8383d]. Bradley's rule for correcting the refraction on account of the temperature and density of the air [8383f]; correction of this formula [8383k].

Expression of the heat of a stratum of the atmosphere in terms of the pressure and density of the air [8400c]. La Place's formulas for the density of the strata of the atmosphere [8411—8412′], give a too rapid decrease of the pressure of the atmosphere at high elevations, and make the decrement of heat too rapid, particularly in low altitudes [8435n, 8444e, i, 8445n, &c.]. La Place's formula makes the temperature at the summit of the atmosphere equal to —91°,5 of Fahrenheit's thermometer [8444x], which is a much greater depression than that which is assumed by Fourier, who estimates the temperature of the planetary space to be —58° of Fahrenheit [8444y]. Oriani's formula [8469h, 8473]. La Place's formula of the refraction in altitudes exceeding twelve degrees, is equivalent to the methods of Bradley, Simpson, and Cassini [8474e—8475c]. Names of several persons who have treated on refraction [8538c]. Leslie's hypothesis of the density of the strata of the atmosphere [8538k]. Ivory's formulas for the same purpose [8539f—l]; these give, with a great degree of accuracy, the temperature and density of the atmosphere in the two extreme cases, namely, near the surface of the earth, and at the greatest elevation to which Gay-Lussac ascended in a balloon [8540a]. The temperature of the planetary space, deduced from Mr. Ivory's hypothesis, is nearly —74° of Fahrenheit's thermometer, differing 16° from Fourier's estimate —58° [8540e, 8444y].

Expressions of the terrestrial refraction in various hypotheses of the decrement of heat [8549g—k].

Effect of the earth's atmosphere on the intensity of the sun's light [8593e].

In calculating the height of a place by a barometer, La Place uses for a hypothesis of the temperatures of the strata of the atmosphere, the method of De Luc, as modified by La Grange [8662h]. Reduction of La Place's formula for computing the height of a place, from metres to fathoms, and from degrees of the centigrade thermometer to those of Fahrenheit [8690f].

Application of the method of definite integrals to the computation of the effects of a resisting medium on the motions of a heavenly body [8908b—o]. Hypothesis of the resistance assumed by Mr. Encke, in computing the perturbations of the orbit of the comet which bears his name [8908p]. Calculation of the coefficients of the terms of the series which expresses the resistance, by means of elliptical functions [8909f, &c.].

Formulas of reduction of elliptical functions [8910h, &c.]. Determination of the quantities $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$, $b_{\frac{1}{2}}^{(0)}$, $b_{\frac{1}{2}}^{(1)}$, $b_{\frac{3}{2}}^{(0)}$, $b_{\frac{3}{2}}^{(1)}$, &c., by means of elliptical functions [8911v—z′].

Expressions of δa, δe, δn, corresponding to one revolution of a comet [8917c—l].

Application to Encke's comet [8917g—i]. Formulas used by Encke in computing the resistance suffered by this comet [8917m—8919h]. From the numerical results of the calculations in [8919a—c], it is probable that the periodical revolution of Encke's comet, which is a little over 1200 days, is decreased about one tenth of a day in each revolution, and the mean distance from the sun is decreased about $\frac{1}{16000}$ part of its value [8919e]. When this comet is at the same distance from the sun as from the earth, the action of the resisting medium upon the comet will retard its progress in the direction of the tangent of its path, about one sixth part of the attraction of the earth upon the comet [8918w, &c.]. The orbit approaches nearer to a circular form by means of the resistance of the medium [8927d].

Formulas for the heliocentric motions of Jupiter and Saturn, for January 1, 1830 [9128—9148].

Method of computing the perturbations of the planets by a double integration by quadratures, which has the important advantage of including all the powers and products of the excentricities in the values of the terms which occur under the signs of integration [9146b, &c.]. This method has been successfully applied to the computation of the inequalities of Jupiter and Saturn, from their mutual action [9146c]. Explanation of this method, with tables of formulas [9146h—9151h]. Example of the method of reducing

these formulas to numbers [9151*i*, &c.]. Comparison of the results of the calculations by Hansen and Pontecoulant, of the great inequalities of Jupiter and Saturn, by this method, and by that of La Place [9152*p*—*s*].

Table of the values of the elements of the orbits of the planets [9159*f*, *g*].

On Capillary Attraction [9171]. The capillary action is expressed by the difference between the attractive force of the particles and the repulsive force of heat [9173*c*]; and it may be attractive or repulsive [9173*f*]. This subject is treated of by Gauss [9173*g*], and by Poisson [9173*i*, &c.]. Computation of the action of a spherical mass of a homogeneous fluid upon an external column [9254], and upon an internal column [9273]. Action of a concave or convex spherical segment [9275, 9276]. The effect of noticing the change of density of the fluid near its surface and near the sides of the vessel in which it is contained, is to change the functions *K*, *H* [9259*a*, *b*], into K, H [9161*f*], respectively.

Action of an ellipsoid upon an internal column [9294, 9301]. Action of a body of any form upon an internal column [9301]. General differential equation of the surface of a fluid in a capillary tube [9318]; its form when the tube is cylindrical [9324, 9323*n*, *q*]. Radii of curvature of the surface of the fluid [9326, 9326′]. Integral of the differential equation of the surface of a fluid in a cylindric tube [9342, 9350]. Capillary elevation or depression of a fluid in a tube [9355, 9358, 9360, 9374, 9375]. Elevation of a fluid in a conical tube [9359*e*, &c.].

Elevation of a fluid between two cylindrical surfaces is the same as in a tube whose radius is equal to the width of the space between the two surfaces [9409]; and the same result holds good between two parallel planes [9412, 9453]. Differential equation of the surface of a fluid between two parallel planes [9415*f*, *g*]. Integration of this equation by means of elliptical functions [9415*m*—9417*m*.]

Extreme value of the elevation of the fluid near the planes when their distance is infinite [9431].

Figures of the surfaces of a small column of fluid contained within a conical capillary tube [9456, &c.]. Inclination of the axis of the cone to the vertical when the capillary action balances the gravity of the fluid [9472, 9474]. Differential equation of the figure of a drop between two parallel planes [9486]; its integral [9511, &c.]. Form of a drop of fluid suspended between two planes inclined to each other [9525′, &c.]. Theorem to determine the inclination when the capillary action balances the gravity [9549]. If two parallel planes be dipped into a fluid at their lower extremities, the planes will tend towards each other [9552, &c.]. Expression of this force [9580*q*, &c.]. Inquiry into the cause of the convexity or concavity of the fluid contained in a capillary tube, or between two near planes [9587, &c.]. This surface of a fluid, in a capillary tube, is very nearly that of a concave hemisphere, when the action of the tube on the fluid is equal to, or exceeds, the action of the fluid on its own particles [9629, &c.]; and when the action of the tube on the particles of the fluid is nothing, the surface will be very nearly that of a convex hemisphere [9650, &c.] Experiments of the elevation of water and oil of oranges in a capillary tube [9667—9669, 9670, 9670′], and between parallel planes [9674, &c.].

Experiments of a drop of fluid suspended between two inclined planes [9692]; comparison of this experiment with the theory [9709]. Experiments with a bent capillary tube [9719, 9722], and with a capillary siphon [9726]. Experiments on the concavity and convexity of the surfaces of fluids in tubes [9737, &c.].

Capillary effect of a barometer [9750, &c. 9753]. Experiments of M. Dulong on the change in the convexity of the surface of the mercury in a barometer, by boiling it; showing that this arises from the oxydization of the mercury [9754*n*, &c.].

Consideration of the change of density near the surface [9841*c*, &c. 9920*h*, &c.]. The formulas are not changed by this consideration, but merely the signification of the coefficients [9842*i*, *z*].

Calculations applied to a vertical cylinder of revolution [9896*g*, &c.].

Action of a parallelopiped upon a plane in a vertical direction [9974*d*]. Action of a wedge upon a plane [9976*c*, &c.]; of the successive concentrical laminæ of a homogeneous fluid [9977*k*, &c.]; when the change of density near the surface is considered [9977*r*, &c.]. Extension of the calculation from a cylinder with a circular to one with any base [9980*h*, &c.]. Angle of inclination at the side of the tube, independent of

the curvature [9980q]; when the tube has no action upon the fluid [9980r, &c.]; when it has the same action upon the fluid as upon its own particles [9980y, &c.]; when it has a greater action [9981f, &c.]. Correction of the computed pressures on opposite sides of two vertical and parallel planes dipped in a fluid [9981n, &c.].

Elevated mass proportional to the circumference of the base [9985a]. Modification of La Place's remarks relative to the action of a tube upon a fluid at its upper extremity [10012a, &c.]. Differential equations of the upper and lower surfaces of a fluid in a capillary tube investigated [10016a, &c.]; difference between theory and the experiments of Gay-Lussac [10017g, &c.].

Case of two fluids in a vertical column [10020a, &c.]; elevation of the central point in consequence of the introduction of the upper fluid [10024a, &c.]. Dr. Young's experiment explained [10025p, &c.]; common surface of the two fluids [10027a, &c.].

Effect of change of temperature from mixture of two fluids [10150e, &c.]; applied to experiments of Gay-Lussac [10150w, &c.].

Investigation of the surface of a fluid when there is a point of inflexion [10168k, &c.]; when the two branches are equal and similar [10170p, &c.]; point of inflexion must be on a level with the outer fluid [10195b, &c.]; effect of moving the planes towards each other till the point of inflexion disappears [10199i, &c.]; when the distance of the planes is small [10200b, &c.]. Case of a concave surface [10200m, &c.]; when there is a point of inflexion [10201n, &c.]. When the planes will attract, and when they will repel each other [10214m].

Case of a glass disk dipped into a vessel of water, and then gradually raised up [10280c, &c.].

Formulas for a drop of mercury, published by La Place in the year 1812 [10385e, f]. Depression of the mercury in a barometer [10442a, &c.].

Drop of water upon a surface of mercury [10450e, &c.]. Phenomenon of endosmose [10469a, &c.].

Comparison of La Place's theory with Poisson's "*Nouvelle Théorie de l'action capillaire, etc.*" [9173h, 9258y, 9259n, 9323r, 9372k, t, 9410o, 9415p, 9416n, o, 9417h, 9425i, 9485l, 9514w, 9522f, 9549c, 9580q,. 9702p, 9852e, 9976u, 9980e, m, 10024z, 10025g, 10027u', 10045f, m, 10089h, 10094e, 10103e, 10150d, 10164f, 10169x, 10170f, 10171p, 10200b, 10241d, 10264s, 10280v, 10281y, 10293k, 10355d, 10380g—k, 10439v, 10440a, c, m, q, 10442p, s, 10144a, 10450b—g, 10450h, 10452b, c, d, p, q, r'', s, 10453s, 10456e—k].

SECOND PART.

PARTICULAR THEORIES OF THE MOTIONS OF THE HEAVENLY BODIES.

EIGHTH BOOK.

THEORY OF THE SATELLITES OF JUPITER, SATURN, AND URANUS.

We propose in this book to consider the perturbations of the satellites, particularly those of Jupiter. By comparing the results of the analytical theory with many observations of their eclipses, we find, that the disturbing force of the sun, and the mutual attraction of the satellites upon each other, [6019] produce all the inequalities in their motions. The expressions of these inequalities, being reduced to numbers, produce accurate tables of the motions of the satellites.

CHAPTER I.

EQUATIONS OF THE MOTIONS OF THE SATELLITES OF JUPITER.

1. The formulas of the second and sixth books, relative to the perturbations of the planets, may be applied, in the same way, to the [6020] perturbations of Jupiter's satellites.* But the almost commensurable ratios

* (3211) Many symbols are used in this theory of Jupiter's satellites, and as the definitions are scattered in various parts of the book, we have here made an alphabetical [6021a] catalogue of the most important of them, with occasional references to the places where they are defined.

[6020′] which obtain in their mean motions, and the great ellipticity of the spheroid of Jupiter, render several of these inequalities, which can be neglected in

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[6021b] $A^{(0)}, A^{(1)}, A^{(2)}$, &c, Developed as in [6090]; $A_{\prime}^{(1)}$, in [6141]; $A'^{(0)}$, $A'^{(1)}$, &c. [6146]; $\frac{2C-A-B}{C}$ [6335, &c.];

. c] a, a', a'', a''', The mean distances of the satellites from the centre of gravity of Jupiter [6061, 6797—6800];

. c′] at, bt, ct, Arcs depending on the variations of the equator and excentricity of Jupiter's orbit, [6400–6405, 6527];

. d] a, b, c, These symbols occur in the equation of a plane touching the shadow of Jupiter [6956, &c.];

. e] B, Is the radius of the equator of Jupiter [6045]; and in [6082′] it is put $B=1$;

. f] $B^{(0)}, B^{(1)}, B^{(2)}$.... Developed in [6296];

. g] $b_s^{(0)}, b_s^{(1)}, b_s^{(2)}$..... Developed in [6305, 6802—6826]. For $\mathfrak{b}$, b, see the references in [6021 c', d];

. h] C, C', C'', Defined in [6706];

. i] c, Is annexed at the end of any number, to denote that it has been altered from the original work, to correct for some error in the calculation, as in [6843r, &c.]; ct is used in [6527, &c.];

c, Is defined in [6612′];

. k] D, Represents, in [6027], the radius vector of the sun's relative orbit about Jupiter;

. l] D', Is the mean value of the radius vector D [6104];

. m] d, d′, d″, These symbols are defined in [6055, 6566];

. n] E, Is the mean longitude of the sun at the epoch, in its relative orbit about Jupiter [6101′, 6102];

. o] e, e', e'', e''', The excentricities of the orbits of the satellites [6061′, &c., 6238, &c.];

. p] F, Defined in [6130]; F', in [6147];

. q] f, Defined in [7003 or 7026];

. r] G, Defined in [6137, 6145]; G', [6163];

. s] g, g_1, g_2, g_3, Defined in [6205, 6225, &c., 6025o]; $gt+\Gamma$, $g_1t+\Gamma_1$, $g_2t+\Gamma_2$, $g_3t+\Gamma_3$ are longitudes of the perijoves of the first, second, third and fourth satellites, from the fixed equinox;

. t] H, The ratio of the excentricity to the semi major axis of the orbit of Jupiter [6275′];

. u] h, h', h'', h''', Defined in [6205]; h_1, h_2, h_3, [6227–6228′]; h^2 is used as in the planetary theory [6067—6077];

. v] I, The longitude of the perihelion of Jupiter's orbit [6276];

. w] i, The rotatory motion of Jupiter about its axis, in the time t, is represented by it [6316′]; i is used in [6433, &c., 6775, &c.], as a general coefficient;

. w′] $\mathrm{i}t+\mathrm{o}$, Is used in [6664—6708, &c.];

. x] k, Is defined in [6093, 6122, &c.]; k, in [6609, &c.];

. y] L, L', l, l', l'', l''', Are defined in [6300]; l, l_1, l_2, l_3, in [6427]; ${}^1L$, in [6367];

. z] $Mt+E$, The mean longitude of the sun in its relative orbit about Jupiter [6102, 6025m];

[6022a] $Mt+E-I=V$, The mean anomaly of Jupiter, counted from its perihelion [6275d—g, 7313];

. b] M, The mass of Jupiter [6028]; and in [6082] $\mathrm{M}=1$; and if we neglect m in comparison with

. c] M we shall have, from $\mu=\mathrm{M}+m$ [6054], $\mu=\mathrm{M}=1$, nearly;

. d] m, m', m'', m''', The masses of the first, second, third and fourth satellites [6021, 6024, 7162—7165];

. e] m, m′, m″, m‴, $\mathrm{m}=10000.m$; $\mathrm{m}'=10000.m'$; $\mathrm{m}''=10000.m''$; $\mathrm{m}'''=10000.m'''$ [6841b—e];

. f] $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$, $n'''t+\varepsilon'''$, the mean longitudes of the satellites, viewed from the centre of gravity of Jupiter, referred to the fixed plane, and counted from the fixed vernal equinox, as in [6062′, 6091, 6175′, 6782, &c.]; $\mathrm{n}t$, $\mathrm{n}'t$, $\mathrm{n}''t$, are defined in [6176b];

. g] N^2, Defined in [6112 or 6124]; N'^2, N''^2, &c., [6134, 6162a, &c.]; $N_{\prime}^2$, in [6447]; $N_{\prime}'^2$, in [6469′]; $N_{\prime}$, $N_{\prime}'$, $N_{\prime}''$, $N_{\prime}'''$, approximate values [6485, 6486]; values computed in [6025e—l];

. h] o, Is used in connection with $\mathrm{i}t$ in forming the angle $\mathrm{i}t+\mathrm{o}$ in [6664—6708, &c.];

. i] P, Is used as a constant coefficient in [6683—6691];

. k] p, p_1, p_2, p_3. Defined and computed [6300, 6424, 6025p]; ${}^1p$, [6366, 6927];

. l] $pt+\Lambda$, $p_1t+\Lambda_1$, $p_2t+\Lambda_2$, $p_3t+\Lambda_3$, represent the distance of the nodes of the satellites from the epoch [6300, 6427, &c.];

. m] Q, Q'. Q'', Are used as coefficients in [6671—6689, 6725—6747, 6850—6862, 6873′];

the planetary theory, sufficiently great to be noticed, when we propose to determine accurately the motions of the satellites. We shall therefore [6020″]

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q, q', q'', q''', The mean values of the semi-diameters of the shadow, where the satellites pass, these discs being viewed from the centre of Jupiter, [7103—7110, 7533—7557]; q, defined in [7032]; [6022n]

R, General expression of this function [6030]; similar functions R' [6467], R'' [6565]. In [6982, 6982′, 7019-7110], R, R', are used for the semi-diameters of the sun and Jupiter; . o]

r, r', r'', r''', The radii vectors of the first, second, third and fourth satellites [6023, 6024, 6036′]; the centre of gravity of Jupiter being taken as their origin; . p]

S, The sun's mass [6025]; . q]

S', The tangent of the sun's latitude above the fixed plane [6040, 6300, 6398]; . r]

s, s', s'', s''', The tangents of the latitudes of the first, second, third and fourth satellites, above the fixed plane of xy [6033, 6036′, 6300]; these are changed, in [6426′, &c.], into the latitudes above the orbit of Jupiter; . s]

s_1, s_1', s_1'', s_1''', The latitudes of these satellites, supposing them to move in the plane of Jupiter's equator, [6051, &c.]; . t]

T, Is used in [7069—7579], for the semi-duration of an eclipse of any satellite when in the nodes; . u]

T, In [7153—7156] denotes the time of the sidereal revolution of the fourth satellite; . v]

t, Is the general expression of the time, which is expressed in Julian years in [7284, &c.]; t is connected with constant coefficients g, g_1, g_2, g_3 [6229, &c.]; M [6102]; n, n', n'', n''' [6175′, &c.]; p, p_1, p_2, p_3, [6427], &c. The value of t in eclipses corresponds with the definition in [7070]; . w]

t', Represents the whole time of duration of an eclipse [7081]; . x]

U, The angle which the projection of the sun's radius vector D, upon the fixed plane, makes with the axis of x [6041], &c.; . y]

u, Is defined in [6068 or 6072]; . z]

δV, Represents, in [6029—6052], a part of the function $-R$, depending on the ellipticity of Jupiter; [6023a]

$V=Mt+E-I$, Is used in [7313, &c.], for the mean anomaly of Jupiter, as in [6022a]; . b]

v, v', v'', v''', The longitudes of the first, second, third and fourth satellites, counted on the fixed plane, from the axis of x, [6032, 6036′], in their relative orbits about the centre of Jupiter; . c]

v_1, In the computation of eclipses, represents the angle which is described by the satellite, by its synodical motion, on the orbit of Jupiter, from the time of the conjunction [7055, &c.]; . d]

$dv_{\prime}$, Represents, [6059-6066], the angle included between the two radii $r, r+dr$, of the first satellite; . e]

v, Represents, in [6045′—6052], the sine of the declination of the first satellite m, relative to the equator of Jupiter; . f]

X, Y, Z, Represent the three co-ordinates of the sun, in its relative orbit about the centre of gravity of Jupiter, [6026—6042]; but in [6963, &c.] they represent the co-ordinates of the sun's surface; and X', Y', Z', [6971] the co-ordinates of Jupiter's surface; . g] . h]

X, In eclipses, has also another signification, as in [7071, 7376′, 7431, 7487, 7526]; and Z also in [7053′]; . i]

x, y, z; x', y', z'; x'', y'', z''; x''', y''', z'''; the rectangular co-ordinates of the four satellites [6022, 6024], the origin being the centre of gravity of Jupiter. In [6956—7061] x, y, z are taken for the rectangular co-ordinates of a plane which touches the surfaces of the sun and Jupiter. The value of x in eclipses is used according to the definition in [7056]. . k]

$\alpha=\frac{a}{a'}$, In [6801—6947] α represents the ratio of the two semi-major axes of the satellites acting upon each other; the least number, a, being the numerator, and the greatest, a', the denominator; . l]

α, In calculating eclipses, in [7047—7112], α represents the greater axis of the elliptical section of . m]

β, the shadow of Jupiter, through which the satellite passes; and in this case $\sin\beta=\frac{\alpha}{a}$, [7073], using the values of a, α, corresponding to the satellite under consideration; ϵ is used in [6917]; . n]

[6020‴] resume the differential equations of these motions, using the following symbols;

[6023o] $\beta', \beta'', \beta'''$, Defined in [6226]; $\beta_1', \beta_1'', \beta_1'''$ [6227′]; $\beta_2', \beta_2'', \beta_2'''$ [6228]; $\beta_3', \beta_3'', \beta_3'''$ [6228′]; also in a table of these symbols [6229*d*];

. *p*] $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3$, Defined in [6205, 6229—6233];

. *q*] γ, The inclination of Jupiter's orbit to the fixed plane [6313′]; the fixed plane in [6398] is the orbit of Jupiter in 1750;

. *r*] $\gamma_1, \gamma_1', \gamma_1'', \gamma_1'''$, The inclinations of the orbits of the first, second, third and fourth satellites to the fixed plane [6315, 6489, &c.];

. *s*] δ, Is the symbol to denote the variation of any quantity on account of the disturbing force [6056];

. *t*] $\varepsilon, \varepsilon', \varepsilon'', \varepsilon'''$, The longitudes of the satellites at the epoch; these being connected with nt, $n't$, $n''t$, $n'''t$, as in [6175′, &c.];

. *u*] ζ, Is used in [6558, &c.];

. *v*] ζ, Corresponding to each of the four satellites, is defined in [7528, 7490, 7434′, 7380];

. *w*] $\zeta, \zeta', \zeta'', \zeta'''$; $\zeta_1, \zeta_1', \zeta_1'', \zeta_1'''$; $\zeta_2, \zeta_2', \zeta_2'', \zeta_2'''$; $\zeta_3, \zeta_3', \zeta_3'', \zeta_3'''$; are defined in [6422—6426];

. *x*] $\Theta, \Theta', \Theta'', \Theta'''$, The mean longitudes of the satellites, as viewed from the centre of gravity of Jupiter, and counted from the earth's moveable vernal equinox [7285, 7285*a*];

. *y*] θ, The inclination of Jupiter's equator to the fixed plane [6312];

. *z*] θ', The inclination of Jupiter's equator to the variable orbit of this planet [6360];

[6024*a*] $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3$, Defined in [6300, 6427, &c.]; ${}^1\Lambda$ in [6368];

. *b*] $\lambda, \lambda', \lambda'', \lambda'''$, Defined in [6343—6346]; computed in [7206—7209];

. *b′*] $\lambda_{,}, \lambda_{,}', \lambda_{,}''$, Defined and used in [6664—6711]; ${}_{,}\lambda$, used in [6317*b*, &c.];

. *c*] λ_1, For a spherical form of Jupiter, is defined in [6999]; for a spheroid, in [7025];

. *d*] μ, Is first used for $\mathrm{M}+m$, as in [6054, &c.]; afterwards in [6963, &c.], $\mu=o$ is taken for the equation of the sun's surface; and $\mu'=o$ for that of Jupiter [6971]. Both these

. *e*] values of μ are finally eliminated, and it is used in [6863, 7151, &c.] for the ratio of $\rho-\frac{1}{2}\varphi$ to 0,0217794;

. *f*] Π, Represents, in [7307], the mean longitude of Jupiter in its orbit about the sun, counted from the earth's moveable vernal equinox; hence $\Pi+200°$ is the mean longitude of the sun in its relative orbit about Jupiter;

. *g*] $\varpi, \varpi', \varpi'', \varpi'''$, The longitudes of the perijoves of the satellites [6062, 6199′]. From [7286*c*] to the end of this theory, the longitudes are counted from the earth's moveable vernal equinox; $\varpi_{,}$, [6620];

. *h*] ρ, Represents the ellipticity of the mass of Jupiter [6044];

. *i*] ρ', Is defined in [7049];

. *k*] Σ, This symbol is defined in [6113 or 6325]; Σ' in [6324′]; $\Sigma_{,}$ in [6235*d*];

. *l*] $\daleth$, Is the longitude of the ascending node of the sun's relative orbit about Jupiter, referred to the fixed plane [6314];

. *m*] $\daleth_1, \daleth_1', \daleth_1'', \daleth_1'''$, The longitude of the ascending nodes of the orbits of the satellites, relative to the fixed plane [6316, 6489, &c.];

. *n*] φ, The ratio of the centrifugal force to the force of gravity, on the surface of Jupiter's equator [6044];

. *o*] Ψ, The retrograde motion, or precession of Jupiter's equator upon the fixed plane [6313];

. *p*] $-\Psi'$, The longitude of the descending node of Jupiter's equator upon the variable orbit of the planet [6361];

. *q*] ω, Defined in [6184];

. *r*] (0), (1), (2), (3); $\boxed{0}$, $\boxed{1}$, $\boxed{2}$, $\boxed{3}$; defined in [6216, &c., 6217*c*, *d*], valued in [6864];

. *s*] (0,1), (0,2), &c.; $\boxed{0,1}$, $\boxed{0,2}$, &c.; defined in [6213, 6214, &c., 6307]; computed in [6865—6868];

. *t*] (I), (II), (III), Defined in [6172—6175]; used in [6749—6766].

$m =$ the mass of the first satellite; [6021]

x, y, z, the rectangular co-ordinates of the first satellite m, referred to the centre of gravity of Jupiter, as the origin of the co-ordinates, supposing that centre to be at rest; [6022]

$r = \sqrt{x^2+y^2+z^2}$ [914'], the distance of the satellite m from the centre of gravity of Jupiter. [6023] The symbols m, x, y, z, r, being marked with *one*, *two* or *three accents*, correspond respectively to the *second*, *third* or *fourth* satellite; [6024]

$S =$ the sun's mass; [6025]

X, Y, Z, the rectangular co-ordinates of the sun, referred to the centre of gravity of Jupiter as their origin, supposing this centre to be at rest, and the sun to describe a relative orbit about the planet; [6026]

$D = \sqrt{X^2+Y^2+Z^2}$, is the distance of the sun from Jupiter's centre of gravity; [6027]

$\mathrm{M} =$ the mass of Jupiter. This is put equal to unity in [6082, &c.]; [6028]

$\frac{\mathrm{M}}{r} + \delta V$ represents the sum of the particles of Jupiter, divided respectively by their distances from the centre of the first satellite m. [6029]

The numerical values of n, n', n'', n''', are given in [6782, &c.]; $\mathcal{N}, \mathcal{N}', \mathcal{N}'', \mathcal{N}'''$, in [6835, &c.]; $\mathcal{M}$, in [6840]; g, g_1, g_2, g_3, in [7176—7195]; and p, p_1, p_2, p_3, in [7226—7245]. As these numbers are frequently used in this book, we have here collected them together, to the nearest second, for convenience of reference. The methods of computing them will be given in [6781a, &c.]. The numbers, n, n', &c., correspond to a Julian year of $365\frac{1}{4}$ days, and to a fixed equinox.

$n = n.\ 1{,}000000 = n'.\ 2{,}007294 = n''.\ 4{,}044090 = n'''.\ 9{,}433419;$ [6025a]

$n' = n.\ 0{,}498183 = n'.\ 1{,}000000 = n''.\ 2{,}014697 = n'''.\ 4{,}699569;$. b]

$n'' = n.\ 0{,}247274 = n'.\ 0{,}496353 = n''.\ 1{,}000000 = n'''.\ 2{,}332643;$. c]

$n''' = n.\ 0{,}106006 = n'.\ 0{,}212785 = n''.\ 0{,}428698 = n'''.\ 1{,}000000;$. d]

$\mathcal{N} = n.\ 0{,}999311 = n'.\ 2{,}005911 = n''.\ 4{,}041303 = n'''.\ 9{,}4269167;$. e]

$\mathcal{N}' = n.\ 0{,}498011 = n'.\ 0{,}999655 = n''.\ 2{,}014003 = n'''.\ 4{,}6979499;$. f]

$\mathcal{N}'' = n.\ 0{,}247239 = n'.\ 0{,}496282 = n''.\ 0{,}999857 = n'''.\ 2{,}3323090;$. g]

$\mathcal{N}''' = n.\ 0{,}105996 = n'.\ 0{,}212766 = n''.\ 0{,}428658 = n'''.\ 0{,}9999070;$. h]

$\mathcal{M} = n.\ 0{,}000408 = n'.\ 0{,}000820 = n''.\ 0{,}001651 = n'''.\ 0{,}00385196;$. i]

$n = 825826010'';\quad n' = 411412427'';\quad n'' = 204205635'';\quad n''' = 87542591'';$. k]

$\mathcal{N} = 825256713'';\quad \mathcal{N}' = 411270707'';\quad \mathcal{N}'' = 204176373'';\quad \mathcal{N}''' = 87534450'';$. l]

$\mathcal{M} = 337211'';$. m]

$n - \mathcal{M} = 825488799'';\quad n' - \mathcal{M} = 411075216'';\quad n'' - \mathcal{M} = 203868424'';\quad n''' - \mathcal{M} = 87205380'';$. n]

$g = 606990'';\quad g_1 = 178142'';\quad g_2 = 29010'';\quad g_3 = 7959'';$. o]

$p = 571389'';\quad p_1 = 133870'';\quad p_2 = 28375'';\quad p_3 = 7683''.$. p]

This being premised, we shall represent, by the symbol R, the following function;*

Function R.

[6030]

$$
\begin{aligned}
R = {} & \frac{m'.(xx'+yy'+zz')}{r'^3} - \frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}} && 1 \\
& + \frac{m''.(xx''+yy''+zz'')}{r''^3} - \frac{m''}{\{(x''-x)^2+(y''-y)^2+(z''-z)^2\}^{\frac{1}{2}}} && 2 \\
& + \frac{m'''.(xx'''+yy'''+zz''')}{r'''^3} - \frac{m'''}{\{(x'''-x)^2+(y'''-y)^2+(z'''-z)^2\}^{\frac{1}{2}}} && 3 \\
& + \frac{S.(Xx+Yy+Zz)}{D^3} - \frac{S}{\{(X-x)^2+(Y-y)^2+(Z-z)^2\}^{\frac{1}{2}}} && 4 \\
& - \delta V. && 5
\end{aligned}
$$

[6031] *This function, R [6030], contains all the disturbing forces of the motion*

[6030a] * (3212) If we suppose, in the values of R, λ, μ [913, 914, 914'], that there are five bodies m, m', m'', m''', m'''', revolving about the central body or planet M; and instead of
[6030b] m'''', with its co-ordinates x'''', y'''', z'''', we write S, X, Y, Z, respectively; we shall find, that the expression of R [913] will become the same as in [6030]; the term $-\delta V$
[6030c] being introduced for the correction arising from the elliptical form of the planet M; and the terms of [914], depending on the products $m'm''$, $m'm'''$, $m'm''''$; $m''m'''$, $m''m''''$; $m'''m''''$, being omitted. Now these last terms may evidently be neglected, because they
[6030d] contain only the co-ordinates of the disturbing bodies m', m'', m''', m'''', and their partial differentials relative to the co-ordinates x, y, z of the body m, vanish from the functions $\left(\frac{dR}{dx}\right)$, $\left(\frac{dR}{dy}\right)$, $\left(\frac{dR}{dz}\right)$, which occur in the fundamental equations [915], for computing
[6030e] the motions of the satellite, in its revolution about the primary planet M. It only remains therefore to ascertain whether the quantity δV, as it is defined in [6029], is the correction
[6030f] to be introduced in the value of R, on account of the ellipticity of the planet M. Now
[6030g] we have calculated, in the lunar theory, that the increment of Q [4773], arising from the elliptical part of the planet, is represented by $\frac{M+m}{M}.\delta V$; which, on account of the
[6030h] smallness of $\frac{m}{M}$ is very nearly equal to δV; moreover we have, in [4774a], $R=\frac{1}{r}-Q$; and as the radius vector r is supposed to be the same, whether the planet be spherical or elliptical, it is evident that the increment of R, can be derived from that of Q, by
[6030i] merely changing its sign; from δV [6030h] to $-\delta V$ [6030, line 5]; observing that the value of δV, in the notation used in [4767, 4765] is the same as in [6029]. We
[6030k] may however observe, that the author uses V for δV [6029—6052]; but as the letter
[6030l] V is used in other parts of this book [7313, &c.] for the mean anomaly of Jupiter, it was thought best to change V into δV, in order to conform to the notation in [4767, &c.].

of the satellite m; *and we have seen, in* [915, &c.], *that the differential equations of this motion depend on its partial differentials.* [6031′]

We shall refer these co-ordinates to others, which are more convenient for astronomical uses; putting

Symbols.

$v =$ the angle comprised between the axis of x, and the projection of the radius vector r, upon the plane of $x\,y$; [6032]

$s =$ the tangent of the latitude of m, above the plane of $x\,y$. [In the formulas, 6426′, &c. s is taken for the latitude of the planet above the orbit of Jupiter.] [6033]

Then we shall have,*

$$x = \frac{r.\cos.v}{\sqrt{1+ss}}; \qquad [6034]$$

$$y = \frac{r.\sin.v}{\sqrt{1+ss}}; \qquad [6035]$$

$$z = \frac{rs}{\sqrt{1+ss}}. \qquad [6036]$$

Marking successively in these expressions the quantities r, s, v, with [6036′] one, two and three accents, we obtain the expressions of x', y', z'; x'', y'', z''; [6037] x''', y''', z'''. This being premised, if we neglect quantities of the order, s^4, we shall find, for the part of R [6030], relative to the action of the satellites,† [6038]

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* (3213) Substituting the expression of u [4776] in [4777—4779], we get [6034—6036], respectively. [6034a]

† (3214) If we develop the expressions [6034—6036], according to the powers of s, neglecting terms of the order s^4, we shall get the values of x, y, z [6039a]; and by accenting the symbols we get the values of x', y', z' [6039b].

$$x = r.(1-\tfrac{1}{2}s^2).\cos.v; \qquad y = r.(1-\tfrac{1}{2}s^2).\sin.v; \qquad z = r.(1-\tfrac{1}{2}s^2).s; \qquad [6039a]$$

$$x' = r'.(1-\tfrac{1}{2}s'^2).\cos.v'; \qquad y' = r'.(1-\tfrac{1}{2}s'^2).\sin.v'; \qquad z' = r'.(1-\tfrac{1}{2}s'^2).s'. \qquad [6039b]$$

Substituting these values in the first member of [6039e], it becomes as in its second member, and by using [24] Int. it changes into [6039f]. Again, by developing the first [6039c] member of [6039g], it becomes as in its second member; and this is reduced to the form [6039h], by the substitution of the values [6039f, 6023] and the value of $r' = \sqrt{x'^2+y'^2+z'^2}$, [6039d] which is similar to [6023]. Then using, for brevity, the symbol W [6039k], it becomes as in [6039i]. The power $-\frac{1}{2}$ of this expression is given in [6039l]. Multiplying [6039f] by $\frac{m'}{r'^3}$, and [6039l] by $-m$, and adding the products, we get [6039m], which

Part of R depending on the satellites.

[6039]

$$R=\frac{m'r}{r'^2}.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s'^2).\cos.(v'-v)+ss'\}-\frac{m'}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{1}{2}}} \qquad 1$$

$$-\frac{m'.rr'.\{ss'-\frac{1}{2}(s^2+s'^2).\cos.(v'-v)\}}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}} \qquad 2$$

$$+\frac{m''r}{r''^2}.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s''^2).\cos.(v''-v)+ss''\}-\frac{m''}{\{r^2-2rr''.\cos.(v''-v)+r''^2\}^{\frac{1}{2}}} \qquad 3$$

$$-\frac{m''.rr''.\{ss''-\frac{1}{2}(s^2+s''^2).\cos.(v''-v)\}}{\{r^2-2rr''.\cos.(v''-v)+r''^2\}^{\frac{3}{2}}} \qquad 4$$

$$+\frac{m'''r}{r'''^2}.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s'''^2).\cos.(v'''-v)+ss'''\}-\frac{m'''}{\{r^2-2rr'''.\cos.(v'''-v)+r'''^2\}^{\frac{1}{2}}} \qquad 5$$

$$-\frac{m'''.rr'''.\{ss'''-\frac{1}{2}(s^2+s'''^2).\cos.(v'''-v)\}}{\{r^2-2rr'''.\cos.(v'''-v)+r'''^2\}^{\frac{3}{2}}}. \qquad 6$$

Symbols. We shall put, in the relative orbit of the sun about Jupiter,

[6040] $S'=$ the tangent of the latitude of the sun S, above the fixed plane;

[6041] $U=$ the angle which the projection of D, upon the fixed plane, makes with the axis of x.

Then if we neglect the terms of R, divided by D^4, which may be done on account of the great distance of Jupiter from the sun, in comparison with

[6041′] that of the satellite from Jupiter, we shall evidently have, for the part of R [6030], depending on the sun's action, the following expression;*

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represents the value of the two terms of R [6030, line 1], being the same as in [6039, lines 1, 2].

[6039e] $$xx'+yy'+zz'=rr'.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s'^2).(\cos.v'.\cos.v+\sin.v'.\sin.v)+ss'\}$$

[6039f] $$=rr'.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s'^2).\cos.(v'-v)+ss'\}$$

[6039g] $$(x'-x)^2+(y'-y)^2+(z'-z)^2=(x'^2+y'^2+z'^2)-2(xx'+yy'+zz')+(x^2+y^2+z^2)$$

[6039h] $$=r'^2-2rr'.\cos.(v'-v)+r^2-2rr'.\{ss'-\tfrac{1}{2}(s^2+s'^2).\cos.(v'-v)\}$$

[6039i] $$=W-2rr'.\{ss'-\tfrac{1}{2}(s^2+s'^2).\cos.(v'-v)\}$$

[6039k] $$W=r'^2-2rr'.\cos.(v'-v)+r^2$$

[6039l] $$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{-\frac{1}{2}}=W^{-\frac{1}{2}}+W^{-\frac{3}{2}}.rr'.\{ss'-\tfrac{1}{2}(s^2+s'^2).\cos.(v'-v)\}$$

[6039m] $$\frac{m'(xx'+yy'+zz')}{r'^3}-\frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}=\frac{m'r}{r'^2}.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s'^2).\cos.(v'-v)+ss'\}-m'W^{-\frac{1}{2}}$$
$$-m'.W^{-\frac{3}{2}}.rr'.\{ss'-\tfrac{1}{2}(s^2+s'^2).\cos.(v'-v)\}.$$

In like manner the terms [6030, lines 2, 3], produce these in [6039, lines 3—6]; which may also be derived from [6039, lines 1, 2], by merely accenting the letters.

* (3215) Developing the first member of [6042b], and using the values of r, D,

[6042a] [6023, 6027], we obtain [6042c]. Involving this to the power $-\frac{1}{2}$, and then multiplying it by $-S$, neglecting terms of the order mentioned in [6041′], we get [6042d];

$$R=-\frac{S}{D}-\frac{Sr^2}{4D^3}\cdot\{1-3s^2-3S'^2+3(1-s^2-S'^2).\cos.(2U-2v)+12sS'.\cos.(U-v)\}.$$ Part of R depending on the sun. [6042]

To determine the part of R, relative to the attraction of the spheroid of Jupiter, we shall observe that this part is equal to $-\delta V$ [6030c]. If we suppose this spheroid to be elliptical, and put [6043]

$\rho=$ to its ellipticity; [6044]

$\varphi=$ to the ratio of the centrifugal force, to the gravity, at the surface of the equator of Jupiter; [6044']

Symbols.

$B=$ the radius of the equator of Jupiter; which is put equal to unity in [6082']; [6045]

$\text{v}=$ the sine of the declination of the satellite m, relative to the equator of Jupiter; [6045']

then we shall have, as in § 35 of the third book,*

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whence, by transposition, we obtain [6042e];

$$(X-x)^2+(Y-y)^2+(Z-z)^2=(X^2+Y^2+Z^2)-2(Xx+Yy+Zz)+(x^2+y^2+z^2)\qquad[6042b]$$
$$=D^2-2(Xx+Yy+Zz)+r^2;\qquad[6042c]$$
$$-\frac{S}{\{(X-x)^2+(Y-y)^2+(Z-z)^2\}^{\frac{1}{2}}}=-\frac{S}{D}-\frac{S}{D^3}(Xx+Yy+Zz)+\frac{Sr^2}{2D^3}-\frac{3(Xx+Yy+Zz)^2S}{2D^5};\qquad[6042d]$$
$$\frac{S.(Xx+Yy+Zz)}{D^3}-\frac{S}{\{(X-x)^2+(Y-y)^2+(Z-z)^2\}^{\frac{1}{2}}}=-\frac{S}{D}+\frac{Sr^2}{2D^3}-\frac{3(Xx+Yy+Zz)^2S}{2D^5}.\qquad[6042e]$$

The terms in the first member of [6042e] are the same as the part of R depending on S, in [6030, line 4]; and we must reduce the corresponding expression, in the second member of [6042e], by substituting the value of $Xx+Yy+Zz$. Now this expression is easily deduced from that in [6039e, f], by changing x', y', z', r', v', s' into X, Y, Z, D, U, S', respectively; hence we get [6042h]; whose square, neglecting s^4 [6038], is as in [6042i]; and by reduction, using [20] Int. we obtain [6042k], [6042f] [6042g]

$$Xx+Yy+Zz=Dr.\{(1-\tfrac{1}{2}s^2-\tfrac{1}{2}S'^2).\cos.(U-v)+sS'\}\qquad[6042h]$$
$$(Xx+Yy+Zz)^2=D^2r^2\{(1-s^2-S'^2).\cos.^2(U-v)+2sS'.\cos.(U-v)\}\qquad[6042i]$$
$$=\tfrac{1}{2}D^2r^2\{(1-s^2-S'^2)+(1-s^2-S'^2).\cos.2(U-v)+4sS'.\cos.(U-v)\}.\qquad[6042k]$$

Substituting this last expression in the second member of [6042e], and making a slight reduction, it becomes as in [6042].

* (3216) To conform to the notation, which is used in this article, we must change $\alpha\varphi$ [1726'] into φ [6044']; and αh [1795'] into ρ [6044]; hence [1812] becomes $V=\frac{\text{M}}{r}+\frac{\text{M}}{r^3}.(\tfrac{1}{2}\varphi-\rho).(\mu^2-\tfrac{1}{3})$. The quantity V, in this expression, is defined in [1428''''], and is what is called $\frac{\text{M}}{r}+\delta V$ in [6029]; hence the expression δV, corresponding to the notation of the present article, is [6046a] [6046b]

[6046] Part of $-R$ depending on the ellipticity of Jupiter.

$$\delta V = \frac{\mathrm{M}.B^2}{r^3}.\,(\tfrac{1}{2}\varphi - \rho).\,(\mathrm{v}^2 - \tfrac{1}{3}).$$

If Jupiter be not elliptical, we shall have, by § 32 of Book III.,*

[6047]
$$\delta V = \frac{\mathrm{M}\ B^2}{r^3}\{(\tfrac{1}{2}\varphi - \rho).\,(\mathrm{v}^2 - \tfrac{1}{3}) + h.\,(1 - \mathrm{v}^2).\cos.2\varpi\};$$

h being an arbitrary quantity depending on the figure of Jupiter, and
[6048] ϖ [1616^{xx}] the angle, formed by one of the two principal axes of Jupiter situated in the plane of its equator, and the meridian of Jupiter which passes through the centre of the satellite. It is easy to prove, by the following
[6049] analysis, that the term depending on $\cos.2\varpi$, has no sensible influence upon the motion of the satellite, by reason of the rapidity with which the

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[6046c]
$$\delta V = \frac{\mathrm{M}}{r^3}.\,(\tfrac{1}{2}\varphi - \rho).(\mu^2 - \tfrac{1}{3}).$$

The symbol μ [1434′, 1430″], represents the cosine of the polar distance of the satellite;
[6046d] hence $\mu = \mathrm{v}$ [6045′]. Substituting this in [6046c], and then multiplying the expression by the square of the radius B^2 [6045], which is taken for unity in [1702″, 1812], we get [6046]. The propriety of this multiplication by B^2, is evident from the consideration
[6046e] that δV [6029] is of the order 2, relative to the co-ordinates and radii, and M is of the order 3.

* (3217) The expression of δV [6046], corresponds to an ellipsoid of revolution;
[6047a] but if we suppose the surface of the planet to be a curve of the second order, and put as in [1792, 1793] $Y^{(3)} = 0$, $Y^{(4)} = 0$, &c., $Z^{(3)} = 0$, $Z^{(4)} = 0$, &c.; changing also, as in [6046b], V into $\frac{\mathrm{M}}{r} + \delta V$; we find that the expression [1811] becomes

[6047b]
$$\frac{\mathrm{M}}{r} + \delta V = \frac{4\pi}{3r}.\int_0^1 \rho.d.a^3 + \frac{4\alpha\pi}{3r^3}.\ Y^{(2)}\int_0^1 \rho.d.a^3 - \frac{\alpha}{r^3}.\,Z^{(2)}.$$

[6047c] Substituting the value of $\mathrm{M} = \frac{4}{3}\pi.\int_0^1 \rho.d.a^3$ [1811′], and rejecting this term from both sides of the equation, we get

[6047d]
$$\delta V = \frac{4\alpha\pi}{3r^3}.Y^{(2)}.\int_0^1 \rho.\,d.\,a^3 - \frac{\alpha}{r^3}.Z^{(2)} = \frac{\alpha\mathrm{M}}{r^3}.Y^{(2)} - \frac{\alpha}{r^3}.Z^{(2)}.$$

Now $Z^{(2)}$ [1632] depends in this case on the centrifugal force, and must produce the
[6047e] same term of δV as in [6046], namely, that which is produced by φ, or $\frac{MB^2}{r^3}.\frac{1}{2}\varphi(\mathrm{v}^2 - \frac{1}{3})$.
[6047f] But $\alpha Y^{(2)}$, instead of being $-\alpha h.(\mu^2 - \frac{1}{3})$ [1792], must be $-\alpha h.(\mu^2 - \frac{1}{3}) + \alpha h''''.(1 - \mu^2).\cos.2\varpi$ [1763], and if we substitute $\alpha h = \rho$ [6046a], $\mu = \mathrm{v}$ [6046d], also $\alpha h'''' = h$, it becomes $-\rho.(\mathrm{v}^2 - \frac{1}{3}) + h.(1 - \mathrm{v}^2).\cos.2\varpi$. Substituting this for $\alpha Y^{(2)}$ in [6047d], and then multiplying
[6047g] it, as in [6046e], by B^2, using also the part depending on $Z^{(2)}$ [6047e], we get the value of δV [6047], corresponding to the general ellipsoid.

angle ϖ varies;* so that the value of δV, which is here to be used, is the same as in the hypothesis of an elliptical spheroid, whose ellipticity is ρ. [6049']
Therefore we shall suppose,

$$\delta V=(\rho-\tfrac{1}{2}\varphi).\,(\tfrac{1}{3}-\mathrm{v}^2).\,\frac{\mathrm{M}.B^2}{r^3}. \qquad [6050]$$

We have very nearly, $\mathrm{v}=s-s_1$, s_1 *denoting the tangent of the latitude* [6051]
of the satellite m *above the fixed plane, supposing it to be moved in the* s_1

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* (3218) The rotation of Jupiter about its axis is performed in about $\frac{3}{8}$ of a day; and if we compare this angular velocity with that of the first, or swiftest of the satellites, whose [6049a] revolution is performed in about $1\frac{3}{4}$ days, we shall find that it is about $4\frac{1}{2}$ times as great; so that the angle 2ϖ will vary about 9 times as swiftly, and may be considered as of the [6049b] form $9nt$; that of the satellite being of the form nt. Now a term of R, of this kind, [6049c] being introduced into the numerator of the expression of $\delta v_{,}$, [6060], in the terms which contain R explicitly, under the signs of integration; will produce by the integrations a divisor of the order $9n$, or even of a higher order; so that the resulting term will be much [6049d] decreased by this divisor; and as the term h, with which it is connected, is found also, by observation, to be much less than $\frac{1}{2}\varphi-\rho$, we may, on both accounts, neglect it, in the [6049e] terms of $\delta v_{,}$, now under consideration. Similar results are obtained from the part of $\delta v_{,}$, [6060], containing $r\delta r$, or δr; which is obtained by integrating the equation [6057], as [6049f] in [865 a, b], changing $r\delta r$ into y, &c. These last formulas being frequently referred to in this volume, we shall here insert them, with a slight modification in the symbols, so as not to interfere with the notation which is generally used in this theory. Substituting in [865a] $\alpha Q=\Sigma.K.\,{}^{\sin.}_{\cos.}(m_{,}t+\varepsilon_{,})$, and changing a into $a_{,}$, it becomes as in [6049k]; [6049g] Σ being the sign of finite integrals. The terms under the sign Σ, are similar to those in [870'], accenting the letters $m_{,}, \varepsilon_{,}, a_{,}$. The value of y, obtained from the integral of [6049h] [6049k] is given in [6049l], and is similar to that in [871, 865b] $b_{,}$, $\varphi_{,}$, being the [6049i] arbitrary constant quantities, introduced by the integration;

$$\frac{ddy}{dt^2}+a_{,}^2.y+\Sigma.K.\,{}^{\sin.}_{\cos.}(m_{,}t+\varepsilon_{,}); \qquad [6049k]$$

$$y=b_{,}.\,{}^{\sin.}_{\cos.}(a_{,}t+\varphi_{,})+\Sigma.\frac{K}{m_{,}^2-a_{,}^2}.\,{}^{\sin.}_{\cos.}(m_{,}t+\varepsilon_{,}). \qquad [6049l]$$

Now putting $y=r\delta r$ in [6049k], it becomes similar to the equation [6057]; and we see that the integral [6049l] introduces the divisor $m_{,}^2-a_{,}^2$; which by putting, as in [6049b], [6049m] $m_{,}=9n$ and $a_{,}=n$, becomes equal to $80n^2$; so that the part of $r\delta r$, and the corresponding [6049n] part of $\delta v_{,}$, [6060], is also much decreased by the integration, and may therefore, on account of its smallness, be neglected, as in [6049f]. This result of the analysis in the present [6049o] article is conformable to the remarks in [6049]; but we may observe that, without using this analytical method, proposed by the author in [6049], we may obtain the same result, [6049p] in a more simple manner, by the process used for Saturn, in [3604a—c].

[6051′] *plane of Jupiter's equator;** *therefore we have for the part* $-\delta V$ *of the expression of* R,

Part of R depending on δV.

[6052]
$$R=-\delta V=-(\rho-\tfrac{1}{2}\varphi).\left\{\tfrac{1}{3}-(s-s_1)^2\right\}.\frac{\mathrm{M}.B^2}{r^3}.$$

2. *We shall now resume the differential equations of the motions of a body which is acted upon by any forces.* Among the several forms which we have given to these equations, in the preceding books, we shall select those which lead, in the most simple manner, to the results we wish to obtain. We have, by § 46, Book II.,†

[6053]
$$0=\tfrac{1}{2}\frac{d^2.r^2}{dt^2}-\frac{\mu}{r}+\frac{\mu}{a}+2.f\mathrm{d}R+r.\left(\frac{dR}{dr}\right);\quad (A)$$

[6054] dt *is the element of the time, and this element is supposed to be constant* [397″, 416, 915, &c.]; $\mu=\mathrm{M}+m$ is the sum of the masses of Jupiter and the satellite m [914′]. We also have $r.\left(\frac{dR}{dr}\right)=x.\left(\frac{dR}{dx}\right)+y.\left(\frac{dR}{dy}\right)+z.\left(\frac{dR}{dz}\right)$

[6055] [6053e]. *The differential characteristic* d [916′], *refers only to the*

d

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* (3219) The tangents s, s_1 [6033, 6051] being very small, we may take their difference

[6051a] $s-s_1$ for the angle, or sine of the angle of elevation of the satellite above the equator of Jupiter, which is represented by v, in [6045′]; hence we get $v=s-s_1$ [6051].

[6051b] Substituting this in [6050], and changing the signs of the terms, we obtain [6052].

† (3220) The expression of R [6030] contains the quantities x, y, z; and if we

[6053a] suppose R to be a function of these quantities, and substitute their values [6034—6036], it will become a function of r, v, s. Now taking its differential in both suppositions, we have generally,

[6053b]
$$\left(\frac{dR}{dr}\right).dr+\left(\frac{dR}{dv}\right).dv+\left(\frac{dR}{ds}\right).ds=\left(\frac{dR}{dx}\right).dx+\left(\frac{dR}{dy}\right).dy+\left(\frac{dR}{dz}\right).dz.$$

Substituting, in the second member, the values of dx, dy, dz, deduced from the differentials

[6053c] of [6034—6036], the result will be identical for all values of dr, dv, ds; and if we retain in the second member only the parts depending on dr, we shall get the corresponding part of the first member, namely $\left(\frac{dR}{dr}\right).dr$. Now by considering x, r, as the variable

[6053d] quantities, in [6034], we get $dx=dr.\frac{\cos.v}{\sqrt{1+ss}}=dr.\frac{x}{r}$; and in like manner from [6035, 6036] we get $dy=dr.\frac{y}{r}$, $dz=dr.\frac{z}{r}$. Substituting these in the second member of [6053b],

[6053e] we get $\left(\frac{dR}{dr}\right).dr=\left(\frac{dR}{dx}\right).\frac{x}{r}dr+\left(\frac{dR}{dy}\right).\frac{y}{r}dr+\left(\frac{dR}{dz}\right).\frac{z}{r}dr$; multiplying this by $\frac{r}{dr}$, we get [6055]; hence the equation [917] becomes as in [6053].

co-ordinates of the satellite m. *If we denote, by* δ [917″], *the variation depending on the disturbing forces*, we shall have, by taking the differential of the preceding equation* relative to δ, δ [6056]

$$0=\frac{d^2.(r\delta r)}{dt^2}+\mu.\frac{(r\delta r)}{r^3}+2.\int\delta R+r.\left(\frac{dR}{dr}\right); \quad (1)$$ [6057]

We shall determine, by this differential equation, the perturbations of the radius vector; we may even include in its integral the excentricity of the orbit of the satellite; for by reason of the extreme smallness of that excentricity, we may neglect its square and higher powers [6057d—g], *and suppose that the variation* $2r\delta r$ *contains not only the inequalities depending on the perturbations, but also the elliptical part of* r^2. [6057′] [6058]

If we put $dv_{,}$ *for the angle included between the two radii vectores* r *and* $r+dr$, we shall have, by § 46 of the second book,† $dv_{,}$ [6059]

$$\delta v_{,}=\frac{\frac{2.d.(r\delta r)-dr\delta r}{a^2.ndt}+\frac{3a}{\mu}.\iint ndt.\mathrm{d}R+\frac{2a}{\mu}.\int ndt.r.\left(\frac{dR}{dr}\right)}{\sqrt{1-e^2}}; \quad (2)$$ [This is very nearly equal to δv [6066].] [6060] δv

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* (3221) If we put $r=a+\delta r$, and suppose δr to contain the elliptical part of the radius vector as well as the perturbations, we shall have, by neglecting the square of the excentricity and the square of the disturbing forces, $r^2=a^2+2a\delta r$, or $r^2=a^2+2r\delta r$; whose second differential is $d^2.(r^2)=2.d^2.(r\delta r)$. Moreover, $\frac{\mu}{a}=\frac{\mu}{r-\delta r}=\frac{\mu}{r}\left(1-\frac{\delta r}{r}\right)^{-1}=\frac{\mu}{r}+\frac{\mu\delta r}{r^2}$; hence $-\frac{\mu}{r}+\frac{\mu}{a}=\frac{\mu\delta r}{r^2}=\frac{\mu.(r\delta r)}{r^3}$. Substituting this and the preceding value of $d^2.(r^2)$ in [6053], we get [6057]. We may remark that the smallness of the excentricities of the orbits of the satellites is seen by the inspection of the first line in the values of v''', v'', v', v [7318, 7405, 7467, 7513], which are of the order of the quantities $2e'''$, $2e''$, $2e'$, $2e$, respectively, as is evident from the value of v [669]. Hence it appears that e''', e'', e', e, are of the order 4632″, 851″, 183″, 20″; or in parts of the radius 0,007; 0,001; 0,0003; 0,00003, respectively; which are very small in comparison with the excentricity of Jupiter's orbit $\frac{1}{2}\times 61208''$ [6882]. We may also observe that in strictness we may wholly neglect the terms depending on e', e, [6057e,f], because the terms in [7467 line 1, 7513 line 1] refer to the perijove of the third satellite ϖ''. These remarks agree with [6057′]. [6057a] [6057b] [6057c] [6057c'] [6057d] [6057e] [6057f] [6057g]

† (3222) The angle $v_{,}$, in the present article [6059], is the same as v [923 line 4], and the angle v [6032] corresponds to $v_{,}$ [923′]. Making these changes in [931] we get the value of $\delta v_{,}$ [6060]; observing that the two first terms of [931] may be reduced by putting $2r.d\delta r+dr.\delta r=2d.(r\delta r)-dr.\delta r$; which is easily proved, by developing the first term, as in [3715a]. [6060a] [6060b]

the symbols being

Symbols.

[6061] $a =$ the semi-major axis of the satellite m;

[6061′] $e =$ the ratio of the excentricity to the semi-major axis of the orbit of m [935′];

[6062] $\varpi =$ the longitude of the perijove of m;

[6062′] $nt =$ the mean motion of the satellite m.

[6063] Putting v *for the angle described by the projection of the radius vector* r *upon the fixed plane*, we shall have, by the same article,*

[6064]
$$dv = dv_{\prime} \cdot \frac{\sqrt{(1+ss)^2 - \frac{ds^2}{dv_{\prime}^2}}}{\sqrt{1+ss}};$$

[6065] s *being the tangent of the latitude of the satellite* m, *above the fixed plane* [923′]; so that if we neglect the square of s we shall have,

[6066]
$$\delta v = \delta v_{\prime}.$$

[6066′] To determine s, we shall observe, that we have in [526], by supposing dv to be constant,†

[6067]
$$0 = \left(\frac{dds}{dv^2} + s\right) \cdot \left\{1 - \frac{2}{h^2} \cdot \int \left(\frac{dR}{dv}\right) \cdot \frac{dv}{u^2}\right\} \qquad 1$$
$$- \frac{1}{h^2u^2} \cdot \frac{ds}{dv} \cdot \left(\frac{d'R}{dv}\right) + \frac{s}{h^2u} \cdot \left(\frac{d'R}{du}\right) + \frac{(1+s^2)}{h^2u^2} \cdot \left(\frac{d'R}{ds}\right); \qquad 2$$

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* (3223) The equation [6064] is the same as [925], changing reciprocally v into $v_{\prime}$,

[6064a] to conform to the notation in this article [6060a]. If we neglect s^2 and $\frac{ds^2}{dv_{\prime}^2}$, as in [6065], we obtain [6066].

† (3224) The equation [6067], is deduced from the third of the equations [526], by

[6067a] substituting $Q = -R + \frac{1}{r}$ [6030h]; and observing that the part $\frac{1}{r}$ produces nothing in

[6067b] [6067]. For if we put $Q = \frac{1}{r}$, and use the value of r [6072], we get $Q = u.(1+ss)^{-\frac{1}{2}}$, whose partial differentials give,

[6067c]
$$\left(\frac{dQ}{dv}\right) = 0; \qquad \left(\frac{dQ}{du}\right) = (1+ss)^{-\frac{1}{2}}; \qquad \left(\frac{dQ}{ds}\right) = -us.(1+ss)^{-\frac{3}{2}}.$$

Substituting these in the third of the equations [526], we find that the terms depending on this part of Q mutually destroy each other; and the remaining part of Q [6067a]

[6067d] becomes $Q = -R$; and by using this part of Q we find that this equation of [526]

[6067e] becomes as in [6067]. In this case R is considered as a function of u, v, s, as is observed in [6069], and this circumstance is noticed by the author, by annexing an accent

[6067f] to the symbol d, in the partial differential relative to s [6069]; and the same accent might have been placed in the partial differential relative to u, in [6069, 6071, 6074]; and for the sake of symmetry we have inserted it.

$\frac{1}{u}$ being the projection of the radius vector r upon the fixed plane, [6068]
[517' &c.], and h^2, a constant quantity, which, in the elliptical orbit, is represented by $h^2 = \mu a.(1-e^2)$ [534' line 1]; lastly, the partial differential $\left(\frac{d'R}{ds}\right)$ corresponds to the case where R is considered as a function of [6069]
u, v, and s. We shall now consider R as a function of r, v, s, as [6070]
in the preceding article, and we shall have,*

$$du.\left(\frac{d'R}{du}\right)+ds.\left(\frac{d'R}{ds}\right)=dr.\left(\frac{dR}{dr}\right)+ds.\left(\frac{dR}{ds}\right), \qquad [6071]$$

r being represented by $r=\frac{\sqrt{1+ss}}{u}$ [6034a, or 4776], we shall have,† [6072]

$$du=-\frac{dr}{r^2}.\sqrt{1+ss}+\frac{sds}{r.\sqrt{1+ss}}; \qquad [6073]$$

therefore we have, by comparing separately the coefficients of ds in the preceding equation,

$$\frac{us}{1+ss}.\left(\frac{d'R}{du}\right)+\left(\frac{d'R}{ds}\right)=\left(\frac{dR}{ds}\right). \qquad [6074]$$

Hence the differential equation [6067] becomes,

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* (3225) In the first member of the equation [6071], R is considered as a function of v, u, s, and its complete differential would be had by adding the term $dv.\left(\frac{dR}{dv}\right)$ to its first [6071a]
member. In the second member R is considered as a function of v, r, s; and its complete differential is found, by adding the same term $dv.\left(\frac{dR}{dv}\right)$ to the second member of [6071b]
[6071]. Rejecting this term from both members, we get [6071]. Now the differential of $u=r^{-1}.(1+ss)^{\frac{1}{2}}$[6072] is as in [6073]; and by substituting it in [6071] we must obtain an identical equation; in which we may put the coefficient of ds, in each member, separately equal to each other; and we shall get $\frac{s}{r\sqrt{1+ss}}\left(\frac{d'R}{du}\right)+\left(\frac{d'R}{ds}\right)=\left(\frac{dR}{ds}\right)$ This [6071c]
may be reduced to the form [6074], by putting $\frac{1}{r}=\frac{u}{\sqrt{1+ss}}$, as in [6072].

† (3226) Squaring r [6072] and dividing by h^2, we obtain $\frac{(1+ss)}{h^2u^2}=\frac{r^2}{h^2}$; multiplying [6074] by this, we obtain

$$\frac{s}{h^2u}.\left(\frac{d'R}{du}\right)+\frac{(1+ss)}{h^2u^2}.\left(\frac{d'R}{ds}\right)=\frac{r^2}{h^2}.\left(\frac{dR}{ds}\right). \qquad [6075a]$$

Substituting this value of the two last terms of [6067], we find that the whole expression becomes as in [6075].

[6075]
$$0=\left(\frac{dds}{dv^2}+s\right).\left\{1-\frac{2}{h^2}.\int\left(\frac{dR}{dv}\right).\frac{dv}{u^2}\right\} \qquad 1$$
$$-\frac{1}{h^2u^2}.\frac{ds}{dv}.\left(\frac{dR}{dv}\right)+\frac{r^2}{h^2}.\left(\frac{dR}{ds}\right). \qquad 2$$

If the disturbing force be nothing, we shall have,

[6076]
$$0=\frac{dds}{dv^2}+s;$$

therefore, by neglecting the square of that force; then substituting $\frac{\sqrt{1+ss}}{r}$

[6076′] for u, and neglecting the product of the disturbing force by $s^2.\frac{ds}{dv}$, we shall have,*

[6077]
$$0=\frac{dds}{dv^2}+s+\frac{r^2}{h^2}.\left(\frac{dR}{ds}\right)-\frac{r^2}{h^2}.\frac{ds}{dv}.\left(\frac{dR}{dv}\right); \quad (3).$$

The equations [6057, 6060, 6077] *give, in the most simple manner, all the perturbations of the satellites, which depend on the first power of the disturbing force.*†

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* (3227) If the disturbing force, or the value of R be nothing, the equation [6075]

[6077a] will become as in [6076]; or, in other words, the expression $\frac{dds}{dv^2}+s$ will vanish; therefore when the satellite is disturbed, this quantity must be of the order of the disturbing force;

[6077b] and its product, by the quantity $-\frac{2}{h^2}.\int\left(\frac{dR}{dv}\right).\frac{dv}{u^2}$, which is also of the same order, will be of the order of the square of the disturbing force. Neglecting this quantity, we find that the equation [6075] becomes

[6077c]
$$0=\frac{dds}{dv^2}+s+\frac{r^2}{h^2}.\left(\frac{dR}{ds}\right)-\frac{1}{h^2u^2}.\frac{ds}{dv}.\left(\frac{dR}{dv}\right).$$

Substituting, in the last term of this expression, the value [6072], $\frac{1}{u^2}=\frac{r^2}{1+ss}=r^2.(1-s^2+s^4-\&c.)$

[6077d] and neglecting terms of the order $s^2\frac{ds}{dv}$, it becomes as in [6077].

CHAPTER II.

ON THE INEQUALITIES OF THE MOTIONS OF JUPITER'S SATELLITES, WHICH ARE INDEPENDENT OF THE EXCENTRICITIES AND INCLINATIONS OF THEIR ORBITS.

3. We shall resume the differential equations [6057],

$$0 = \frac{d^2.(r\delta r)}{dt^2} + \frac{\mu.(r\delta r)}{r^3} + 2f\mathrm{d}R + r.\left(\frac{dR}{dr}\right). \quad (1)$$ [6078]

In the elliptical hypothesis, if we neglect the square of the excentricity of the orbit, the constant part of the radius vector [659] will be reduced to the semi-major axis a; we may therefore suppose $r = a$, in the preceding [6079] equation. But *for greater exactness, and for reasons which will be given hereafter* [6128], *we shall retain the product of* $r\delta r$ *by the constant parts* [6079'] *of the disturbing force.* Now this force adds to the radius r, a constant part, which we shall denote by δa; therefore, by substituting $r = a + \delta a$ [6079''] in the equation [6078], we shall have * [6080]

$$0 = \frac{d^2.(r\delta r)}{dt^2} + \frac{\mu.(r\delta r)}{a^3}.\left(1 - \frac{3.\delta a}{a}\right) + 2f\mathrm{d}R + r.\left(\frac{dR}{dr}\right);$$ [6081]

the part of R depending on the ellipticity of Jupiter's mass is equal to $-\delta V$ [6043]; and if we neglect the square of v, we shall have, by [6081'] noticing only this part, *and taking for the unit of mass that of Jupiter; or* M $= 1$ [6028], *and the semi-diameter* B *of its equator* [6045], *for the* [6082] *unit of distance;* † [6082']

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* (3228) The equation [6081] is deduced from [6078], by substituting in its second term, for $\frac{1}{r^3}$, its value $\frac{1}{(a+\delta a)^3} = \frac{1}{a^3} - \frac{3\delta a}{a^4}$. [6081a]

† (3229) Putting, as in [6082, 6081'], M $=1$, $B=1$, $\mathrm{v}^2=0$, we find that the expression [6046] becomes $\delta V = \frac{(\rho - \frac{1}{2}\varphi)}{3r^3}$; and by [6043] the corresponding part of R [6084a]

[6083] Part of R depending on the ellipticity of Jupiter.

$$R = -\frac{(\rho-\frac{1}{2}\varphi)}{3r^3}.$$

Hence we easily deduce,

[6084] $$\int dR = R; \qquad r.\left(\frac{dR}{dr}\right) = -3R;$$

therefore,

[6085] $$2\int dR + r.\left(\frac{dR}{dr}\right) = -R = \frac{(\rho-\frac{1}{2}\varphi)}{3r^3}.$$

[6086] Substituting $r^2 = a^2 + 2r\delta r$ [6057b], we shall obtain

[6087] $$2\int dR + r.\left(\frac{dR}{dr}\right) = \frac{(\rho-\frac{1}{2}\varphi)}{3a^3} - \frac{(\rho-\frac{1}{2}\varphi)}{a^5}.r\delta r.$$

If we notice only the action of the second satellite m', and neglect

[6088] the squares and products of s and s', we shall obtain from [6039],*

[6089] Part of R depending on the satellites.

$$R = \frac{m'.r}{r'^2}.\cos.(v'-v) - \frac{m'}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{1}{2}}}.$$

[6084b] is $R = -\delta V = -\frac{(\rho-\frac{1}{2}\varphi)}{3r^3}$, as in [6083]. From the value of this part of R, we easily deduce the formulas [6084, 6085]. Substituting, in the last of the expressions

[6084c] [6085], for $\frac{1}{r^3}$, its value $\frac{1}{(a+\delta r)^3} = \frac{1}{a^3} - \frac{3\delta r}{a^4}$; or as it may be written $\frac{1}{a^3} - \frac{3r\delta r}{a^5}$, it becomes as in [6087].

* (3230) If we neglect the excentricities and inclinations in the part of the expression

[6089a] of R [6039 lines 1, 2], depending on m', it will become of the same form as in (6089).

[6089b] This may be developed, as in [953′, 954], in a series of the form [6090]; $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, &c. being functions of r, r', independent of v, v'. Substituting, in [6090], the values of

[6089c] r, r', v, v' [6091], it becomes, by putting for a moment, for brevity, $T = n't - nt + \varepsilon' - \varepsilon$,

[6089d] $$R = m'.\{\tfrac{1}{2}A^{(0)} + A^{(1)}.\cos.T + A^{(2)}.\cos.2T + A^{(3)}.\cos.3T + \&c.\}.$$

Taking the differential of this expression relative to the characteristic d; then integrating and adding the constant quantity $\frac{k}{a}$, we get $\int dR$; multiplying this by 2, we obtain

[6089e] [6092]. Again, we have in [962], $r.\left(\frac{dR}{dr}\right) = a.\left(\frac{dR}{da}\right)$; hence, by taking the partial differential of R relative to a, and multiplying by $\frac{a}{da}$, we get the expression of $r.\left(\frac{dR}{dr}\right)$ [6094]. We may observe that the differential of v [6091], being multiplied by $\frac{p}{n}$, gives

[6089f] $\frac{p}{n}.dv = pdt$, which is used in [6300′].

We shall suppose this function to be developed in a series of cosines of angles which are multiples of $v'-v$, of the following form; Symbol $A^{(i)}$

$$R = m'.\{\tfrac{1}{2}A^{(0)} + A^{(1)}.\cos.(v'-v) + A^{(2)}.\cos.2(v'-v) + A^{(3)}.\cos.3(v'-v) + \&c.\}. \qquad [6090]$$

If in this series we put

$$r = a, \qquad r' = a', \qquad v = nt+\varepsilon, \qquad v' = n't+\varepsilon'; \qquad [6091]$$

nt and $n't$ being the mean motions of m and m', we shall have,

$$2\int \mathrm{d}R = \frac{2k}{a} + \frac{2n.m'}{n-n'}.\left\{\begin{array}{ll} A^{(1)}.\cos.\,(n't-nt+\varepsilon'-\varepsilon) & 1 \\ +A^{(2)}.\cos.2(n't-nt+\varepsilon'-\varepsilon) & 2 \\ +A^{(3)}.\cos.3(n't-nt+\varepsilon'-\varepsilon) & 3 \\ +\&c. & \end{array}\right\} \qquad [6092]$$

$\frac{k}{a}$ being an arbitrary constant quantity, added to the integral $\int \mathrm{d}R$. Then [6093]
we shall have,

$$r.\left(\frac{dR}{dr}\right) = m'.\left\{\begin{array}{ll} \tfrac{1}{2}a.\left(\frac{dA^{(0)}}{da}\right) & 1 \\ +a.\left(\frac{dA^{(1)}}{da}\right).\cos.\,(n't-nt+\varepsilon'-\varepsilon) & 2 \\ +a.\left(\frac{dA^{(2)}}{da}\right).\cos.2(n't-nt+\varepsilon'-\varepsilon) & 3 \\ +a.\left(\frac{dA^{(3)}}{da}\right).\cos.3(n't-nt+\varepsilon'-\varepsilon) & 4 \\ +\&c. & \end{array}\right\} \qquad [6094]$$

We can ascertain the values of $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, &c. and their partial differentials in a, a', by the formulas [963iv—1003].

As we intend to retain the constant part, depending on the square of the disturbing force and connected with the factor $r\delta r$ [6079'], we must add to the preceding expressions of $2\int \mathrm{d}R$ and $r.\left(\frac{dR}{dr}\right)$ [6092, 6094], the terms of this kind which they contain. If, in the term $\frac{1}{2}m'.A^{(0)}$ of R [6090], we substitute $r = a + \frac{r\delta r}{a}$, we shall obtain the term* $R = m'.\frac{r\delta r}{2a}.\left(\frac{dA^{(0)}}{da}\right)$; [6094'] [6095]
therefore the function $2\int \mathrm{d}R$ contains the term

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* (3231) The expression of R [6089d] contains the term $R = \frac{1}{2}m'.A^{(0)}$, which [6096a]
is a function of r, r' [6089b]. Substituting $r = a + \frac{r\delta r}{a}$ [6095] and developing, by Taylor's [6096b]
theorem [617], according to the powers of $\frac{r\delta r}{a}$, it becomes $\frac{1}{2}m'.A^{(0)} + m'.\frac{r\delta r}{2a}.\left(\frac{dA^{(0)}}{da}\right) + \&c.$, [6096b']
$A^{(0)}$ being the value of $A^{(0)}$ corresponding to $r = a$. Neglecting the second and higher

[6096] $$2\int \mathrm{d}R = m'.\frac{r\delta r}{2a}.2\left(\frac{dA^{(0)}}{da}\right).$$

Substituting, in like manner, $a+\frac{r\delta r}{a}$ for r in the function $r.\left(\frac{dR}{dr}\right)$, we find that it contains the terms,

[6097] $$r.\left(\frac{dR}{dr}\right) = m'.\frac{r\delta r}{2a}.\left\{\left(\frac{dA^{(0)}}{da}\right)+a.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}.$$

Changing successively the quantities relative to the satellite m', into those relative to the satellites m'', m''', we shall have the corresponding

[6098] parts of $2\int \mathrm{d}R$, $r.\left(\frac{dR}{dr}\right)$.

Part of R depending on the sun.

To obtain the part relative to the sun's action, we shall observe, that by noticing only this action, and neglecting the squares and products of s

[6099] and S', we shall have as in [6042],

[6100] $$R = -\frac{S}{D}-\frac{S.r^2}{4D^3}.\{1+3.\cos.(2U-2v)\}.$$

[6101] *U is the longitude of the sun, as viewed from the centre of Jupiter*

[6096c] powers of $r\delta r$, we obtain in R the additional term $R = m'.\frac{r\delta r}{2a}.\left(\frac{dA^{(0)}}{da}\right)$, as in [6095];

[6096d] which gives $\mathrm{d}R = \frac{m'}{2a}.\left(\frac{dA^{(0)}}{da}\right).\mathrm{d}.(r\delta r) = \frac{m'}{2a}.\left(\frac{dA^{(0)}}{da}\right).d.(r\delta r)$; whence $\int \mathrm{d}R = \frac{m'}{2a}.\left(\frac{dA^{(0)}}{da}\right).(r\delta r)$; as in [6096]. The same term of $R = \frac{1}{2}m'.A^{(0)}$ [6096a], being substituted in $r.\left(\frac{dR}{dr}\right)$ gives

[6096e] $r.\left(\frac{dR}{dr}\right) = \frac{1}{2}m'r.\left(\frac{dA^{(0)}}{dr}\right)$. Now substituting $a+\frac{r\delta r}{a}$ for r in $\left(\frac{dA^{(0)}}{dr}\right)$, and developing, as above by Taylor's theorem, it becomes $\left(\frac{dA^{(0)}}{dr}\right) = \left(\frac{dA^{(0)}}{da}\right)+\frac{r\delta r}{a}.\left(\frac{ddA^{(0)}}{da^2}\right)$; multiplying this

[6096f] expression by $\frac{1}{2}m'r = \frac{1}{2}m'.\left(a+\frac{r\delta r}{a}\right)$, and retaining only the terms depending on the first power of $r\delta r$, we obtain in $r.\left(\frac{dR}{dr}\right)$ the two additional terms, contained in [6097]. Adding together the expressions [6096, 6097], we obtain

[6096g] $$2\int \mathrm{d}R + r.\left(\frac{dR}{dr}\right) = m'.\frac{(r\delta r)}{2a}.\left\{3.\left(\frac{dA^{(0)}}{da}\right)+a.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}.$$

[6096h] This may be reduced to another form by substituting $\frac{1}{a} = n^2a^2$ [6110]; and then using the symbol Σ to denote the sum of the terms similar to [6096, 6097], corresponding to all the disturbing satellites m', m'', m''', with their respective values of $A^{(0)}$, we shall have,

[6096i] $$2\int \mathrm{d}R + r.\left(\frac{dR}{dr}\right) = n^2.r\delta r.\Sigma.\tfrac{1}{2}m'.\left\{3a^2.\left(\frac{dA^{(0)}}{da}\right)+a^3.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}.$$

[6042g, 6023c]; *and if we put the sideral motion of this planet equal to* Mt, [6101']
we shall have, by neglecting the excentricity of its orbit,

$$U = Mt + E. \qquad [6102]$$

Hence we have,*

$$R = -\frac{S}{D'} - \frac{S.r^2}{4D'^3}.\{1+3.\cos.(2nt-2Mt+2\varepsilon-2E)\}; \qquad [6103]$$

D' being the semi-major axis of Jupiter's orbit. We have as in [605', &c.], by neglecting Jupiter's mass in comparison with that of the sun,† [6104]

$$\frac{S}{D'^3} = M^2; \qquad [6105]$$

therefore, by neglecting the arbitrary constant quantity, arising from the integration in $2f\mathrm{d}R$, because it may be supposed to be included in the arbitrary quantity k [6093], we shall have,‡ [6105']

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* (3232) Changing $\cos.2(U-v)$ into $\cos.2(v-U)$, and then substituting the values of v, U [6091, 6102], we find that [6100] becomes as in [6103]; D being changed into D' [6027, 6104], on account of the neglect of the excentricity of the orbit of Jupiter [6101']. [6103a]

† (3233) It follows from [605', 606] that if the mean motion of a planet be nt, its mean distance from the sun a, and μ the sum of the masses of the sun and planet [530iv], we shall have $\frac{\mu}{a^3} = n^2$. To conform to the present notation, we must change n into M [6101'], a into D' [6104], and μ [530iv] into $S+\mathrm{M}$ [6025, 6028]; then the preceding equation becomes $\frac{S+\mathrm{M}}{D'^3} = M^2$. Neglecting M in comparison with S, we get [6105]. [6105a] [6105b]

‡ (3234) If we substitute [6105] in [6103], we shall get,

$$R = -\frac{S}{D'} - \tfrac{1}{4}M^2.r^2 - \tfrac{3}{4}M^2.r^2.\cos.(2nt-2Mt+2\varepsilon-2E). \qquad [6106a]$$

The partial differential of this expression, relative to dr, being multiplied by $\frac{r}{dr}$, becomes as in [6107]. The partial differential relative to d, being multiplied by 2, gives,

$$2\mathrm{d}R = -\tfrac{1}{2}M^2.\mathrm{d}.(r^2) + 3M^2.r^2.ndt.\sin.(2nt-2Mt+2\varepsilon-2E); \qquad [6106b]$$

observing that in the first term we have noticed the variableness of r^2, because its integral $-\frac{1}{2}M^2.r^2$, corresponding to the first term of [6106], produces in [6108] the term $-M^2.r\delta r$, which is noticed, for the reasons given in [6079', &c.]. But in finding the second term of [6106b] we have considered r^2 as constant and equal to a^2, on account of the smallness of the factor M^2e. The integral of this last term being taken, supposing r^2 to be constant, gives the second term of [6106]. [6106c] [6106d]

[6106] $$2\int\mathrm{d}R = -\tfrac{1}{2}M^2.r^2.\left\{1+\frac{3n}{n-M}.\cos.(2nt-2Mt+2\varepsilon-2E)\right\}$$

[6107] $$r.\left(\frac{dR}{dr}\right) = -\tfrac{1}{2}M^2.r^2.\{1+3.\cos.(2nt-2Mt+2\varepsilon-2E)\}.$$

Hence we deduce,*

[6108] $$2\int\mathrm{d}R = -\tfrac{1}{2}M^2.a^2.-M^2.r\delta r-\frac{3M^2.na^2}{2n-2M}.\cos.(2nt-2Mt+2\varepsilon-2E)$$

[6109] $$r.\left(\frac{dR}{dr}\right) = -\tfrac{1}{2}M^2.a^2.-M^2.r\delta r-\tfrac{3}{2}.M^2.a^2.\cos.(2nt-2Mt+2\varepsilon-2E).$$

This being premised, we shall connect together all these terms in the
[6109′] differential equation [6057], and divide it by a^2; putting $n^2=\frac{1+m}{a^3}$, or very nearly;†

[6110] $$n^2=\frac{1}{a^3},$$

[6111] m being a very small fraction, which is less than 0,0001 [7162, &c.]; the mass of Jupiter M being taken for unity. We shall also put, for brevity,

[6112] $$N^2=n^2.\left\{1-\frac{3\delta a}{a}-\frac{(\rho-\frac{1}{2}\varphi)}{a^2}-\frac{2.M^2}{n^2}+\Sigma.\tfrac{1}{2}m'.a^2.\left[3.\left(\frac{dA^{(0)}}{da}\right)+a.\left(\frac{ddA^{(0)}}{da^2}\right)\right]\right\};$$

[6113] the symbol Σ denotes that the sum of the terms following it, corresponding to the actions of *all* the disturbing satellites, is to be taken. Then we shall have,‡

[6108a] * (3235) Substituting $r^2=a^2+2r\delta r$ [6057b] in [6106, 6107], we get [6108, 6109] respectively, observing that $r\delta r$ is retained, as in [6079′], only in the terms with constant
[6108b] coefficients. The square root of this expression is $r=a+\frac{r\delta r}{a}$; which is used hereafter.

[6110a] † (3236) Substituting in the expression of n^2 [6105a] the value of $\mu=\mathrm{M}+m$ [6021, 6028], corresponding to the planet and satellite m; and then putting as in [6111]
[6110b] $\mathrm{M}=1$, we get $\mu=1+m$, $n^2=\frac{1+m}{a^3}$ [6109′]; and as the mass of the satellite is very small in comparison with that of the planet, we may neglect m, and we shall have

[6110c] $$\mu=1,\qquad \text{and} \qquad n^2=\frac{1}{a^3},$$

[6110d] as in [6110]. This last value of μ being substituted in h^2 [6069], gives $h^2=a.(1-e^2)$, which will be of use hereafter.

‡ (3237) Substituting in [6081] the terms of $2\int\mathrm{d}R+r.\left(\frac{dR}{dr}\right)$, which have been
[6114a] computed in [6087, 6092, 6094, 6096i, 6108, 6109], then dividing by a^2, and observing

$$0=\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2} \qquad 1$$

$$+2n^2.k+n^2.\frac{(\rho-\frac{1}{2}\varphi)}{3a^2}-M^2+\Sigma.\tfrac{1}{2}m'.n^2.a^2.\left(\frac{dA^{(0)}}{da}\right) \qquad 2$$

$$-3M^2.\frac{(2n-M)}{2n-2M}.\cos.(2nt-2Mt+2\varepsilon-2E) \qquad 3$$

Differential equation in $r\delta r$.

$$+\Sigma.m'.n^2.\left\{\begin{array}{l} \left\{\frac{2n}{n-n'}.aA^{(1)}+a^2.\left(\frac{dA^{(1)}}{da}\right)\right\}.\cos.\,(n't-nt+\varepsilon'-\varepsilon) \qquad 4 \\ +\left\{\frac{2n}{n-n'}.aA^{(2)}+a^2.\left(\frac{dA^{(2)}}{da}\right)\right\}.\cos.2(n't-nt+\varepsilon'-\varepsilon) \qquad 5 \\ +\left\{\frac{2n}{n-n'}.aA^{(3)}+a^2.\left(\frac{dA^{(3)}}{da}\right)\right\}.\cos.3(n't-nt+\varepsilon'-\varepsilon) \qquad 6 \\ +\&c. \end{array}\right\} \qquad [6114]$$

We may integrate this equation, without adding to it any arbitrary constant quantities, which may be supposed to be included in the elements of the elliptical motion. Then if we neglect M and $N-n$ in comparison with n,* in the terms depending on the sun's action, because these fractions are small, we shall have,† [6115]

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that by [6110c] we have $\frac{\mu}{a^3}=\frac{1}{a^3}=n^2$; $\frac{1}{a^5}=\frac{n^2}{a^2}$; $\frac{A^{(i)}}{a^2}=n^2a.A^{(i)}$; $\frac{1}{a}.\left(\frac{dA^{(i)}}{da}\right)=n^2a^2.\left(\frac{dA^{(i)}}{da}\right)$; [6114b]
we shall obtain the equation [6114], using the abridged symbol N^2 [6112].

* (3238) We have in [6025e, i] $N=n.0{,}999311$; $M=n.0{,}000408$; hence we [6115a]
see that $\frac{M}{n}$, $\frac{N-n}{n}$, are very small, and that N differs but little from n. [6115b]

† (3239) Putting $y=\frac{(r\delta r)}{a^2}$ and $a_{,}=N$ [6049k], we get [6116a]

$$0=\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2}+\Sigma.K.\cos.(m_{,}t+\varepsilon_{,}).$$

Comparing this with [6114], we find that $\Sigma.K.\cos.(m_{,}t+\varepsilon_{,})$ embraces the function [6116b]
contained in [6114 lines 2—6]; and from [6049k, l], we find that if any term of this part of [6114] be represented by $K.\cos.(m_{,}t+\varepsilon_{,})$, the corresponding term of y or $\frac{r\delta r}{a^2}$ will be represented by $\frac{K}{m_{,}^2-N^2}.\cos.(m_{,}t+\varepsilon_{,})$; hence we may deduce the terms in the second [6116c]
member of [6116] from the corresponding ones of [6114 lines 2—6]. For we have without any reduction the terms in [6116, lines 3, 4, 5...], from those in [6114 lines 4, 5, 6...] [6116d]
respectively. The term in [6114 line 2] being constant, we must put $m_{,}=0$, in [6049k], and then the divisor $m_{,}^2-N^2$ [6116c] becomes $-N^2$; so that we must divide the term [6116e]
[6114 line 2], by $-N^2$, to obtain the corresponding constant term in the second member of [6116 line 1]. Lastly, the term [6114 line 3] gives $m_{,}=2n-2M$, and the divisor [6116f]

General expression of $r\delta r$

[6116]

$$\frac{r\delta r}{a^2} = -\frac{n^2}{N^2}\cdot\left\{2k+\frac{(\rho-\frac{1}{2}\varphi)}{3a^2}-\frac{M^2}{n^2}+\Sigma.\tfrac{1}{2}m'.a^2.\left(\frac{dA^{(0)}}{da}\right)\right\} \qquad 1$$

$$-\frac{M^2}{n^2}\cdot\cos.(2nt-2Mt+2\varepsilon-2E). \qquad 2$$

$$+\Sigma.m'.\left(\begin{array}{l}\frac{n^2}{(n-n')^2-N^2}\cdot\left\{\frac{2n}{n-n'}.aA^{(1)}+a^2.\left(\frac{dA^{(1)}}{da}\right)\right\}.\cos.\;(n't-nt+\varepsilon'-\varepsilon) \\ +\frac{n^2}{4.(n-n')^2-N^2}\cdot\left\{\frac{2n}{n-n'}.aA^{(2)}+a^2.\left(\frac{dA^{(2)}}{da}\right)\right\}.\cos.2(n't-nt+\varepsilon'-\varepsilon) \\ +\frac{n^2}{9.(n-n')^2-N^2}\cdot\left\{\frac{2n}{n-n'}.aA^{(3)}+a^2.\left(\frac{dA^{(3)}}{da}\right)\right\}.\cos.3(n't-nt+\varepsilon'-\varepsilon) \\ +\&c.\end{array}\right) \qquad \begin{array}{l}3\\4\\5\\ \vdots\end{array}$$

[6116′] The constant part of this expression is represented by $\frac{\delta a}{a}$, in [6079″]; therefore we shall have, by observing that N^2 differs but little from n^2 [6115a],*

[6117]
$$\frac{\delta a}{a} = -2k-\frac{(\rho-\frac{1}{2}\varphi)}{3a^2}+\frac{M^2}{n^2}-\Sigma.\tfrac{1}{2}m'.a^2.\left(\frac{dA^{(0)}}{da}\right).$$

If we substitute the preceding values of $2\int dR$, $r.\left(\frac{dR}{dr}\right)$, $\frac{r\delta r}{a^2}$, in the expression [6060], we shall get the value of δv [6119]; observing that

[6118] $\frac{1+m}{a^3}=n^2$ [6110b], and $\mu=1$ nearly [6110c]; also that M is very small relative to n, and N differs but very little from n [6115a, b]†

$m_{\prime}^2-N^2$ becomes $(2n-2M)^2-N^2$, producing in [6116 line 2] a term whose coefficient

[6116g] is $-3M^2.\frac{(2n-M)}{2n-2M}\cdot\frac{1}{(2n-2M)^2-N^2}$. Now as M is very small in comparison with n [6115a], we may neglect it in the factors $2n-M$, $2n-2M$, and then the preceding coefficient will become $-3M^2.\frac{1}{4n^2-N^2}$, which, by putting $N=n$ nearly [6115b],

[6116h] becomes $-\frac{M^2}{n^2}$; as in [6116 line 2].

* (3240) Substituting $r=a+\delta a$ [6079″], in the first member of [6116], and neglecting

[6117a] the square of δa, it becomes $\frac{\delta a}{a}$, which represents very nearly the constant part of this first member. Putting this equal to the constant part of the second member [6116 line 1], and then substituting $\frac{n^2}{N^2}=1$, nearly [6115a]; we get [6117].

† (3241) In the expression of δv, or δv [6060, 6066], we may put $\mu=1$ [6110c]

[6118a] and neglect e^2. We may also neglect the term $dr.\delta r$, because dr is of the order e, and we have in this chapter neglected such terms. Hence [6060] becomes, by a slight

$$\delta v = nt.\left\{3k + \frac{(\rho-\frac{1}{2}\varphi)}{a^2} - \tfrac{7}{4}.\frac{M^2}{n^2} + \Sigma.m'.a^2.\left(\frac{d.A^{(0)}}{da}\right)\right\} \qquad 1$$

General expression of δv

$$+\tfrac{11}{8}.\frac{M^2}{n^2}.\sin.(2nt-2Mt+2\varepsilon-2E) \qquad 2$$

$$+\Sigma.\frac{m'.n}{n-n'}.\left\{\begin{array}{ll} \left\{\frac{n}{n-n'}.aA^{(1)} + \frac{2.N^2}{(n-n')^2-N^2}.\left[\frac{2n}{n-n'}.aA^{(1)} + a^2.\left(\frac{d.A^{(1)}}{da}\right)\right]\right\}.\sin.(n't-nt+\varepsilon'-\varepsilon) & 3 \\ +\tfrac{1}{2}.\left\{\frac{n}{n-n'}.aA^{(2)} + \frac{2.N^2}{4.(n-n')^2-N^2}.\left[\frac{2n}{n-n'}.aA^{(2)} + a^2.\left(\frac{dA^{(2)}}{da}\right)\right]\right\}.\sin.2(n't-nt+\varepsilon'-\varepsilon) & 4 \\ +\tfrac{1}{3}.\left\{\frac{n}{n-n'}.aA^{(3)} + \frac{2.N^2}{9.(n-n')^2-N^2}.\left[\frac{2n}{n-n'}.aA^{(3)} + a^2.\left(\frac{dA^{(3)}}{da}\right)\right]\right\}.\sin.3(n't-nt+\varepsilon'-\varepsilon) & 5 \\ +\ \&c. & . \end{array}\right. \qquad [6119]$$

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change in the arrangement of the terms,

$$\delta v = \frac{2d.\left(\frac{r\delta r}{a^2}\right)}{ndt} + \int andt.\left\{3\int dR + 2r.\left(\frac{dR}{dr}\right)\right\}. \qquad [6118b]$$

We must substitute, in this expression, the values of dR, $r.\left(\frac{dR}{dr}\right)$, which are given in [6084, 6092, 6094, 6108, 6109, 6116]; those in [6096, 6097] may be neglected, on [6118c] account of their smallness, being only used in forming the quantity N^2 [6112]. We shall first calculate the term in [6119 line 1], depending on the constant coefficient; then the term in [6119 line 2], depending on the angle $2nt-2Mt+2\varepsilon-2E$; lastly the terms of [6118d] the general form in [6119 lines 3, 4, 5..], depending on the angle $i(n't-nt+\varepsilon'-\varepsilon)$, or [6118e] iT [6089c].

First. If $3\int dR+2r.\left(\frac{dR}{dr}\right)$ contain a *constant term*, represented by C, the corresponding [6118f] term of δv [6118b], will be $\delta v = \int andt.\,C = C.ant$; hence the constant terms of the [6118g] function $3\int dR+2r.\left(\frac{dR}{dr}\right)$ must be multiplied by ant to obtain the corresponding parts [6118h] of δv. Now the terms depending on the ellipticity [6084] give

$$3\int dR+2r.\left(\frac{dR}{dr}\right) = 3R-6R = -3R = \frac{(\rho-\frac{1}{2}\varphi)}{r^3}\ [6083]; \qquad [6118i]$$

multiplying this by ant [6118h] and changing r into a, we get the term depending on $\rho-\frac{1}{2}\varphi$ in [6119 line 1]. The first term of [6092] gives, in $3\int dR$, the term $\frac{3k}{a}$; and by [6118k] multiplying it by ant, we get the first term of [6119 line 1]. The term in [6094 line 1] gives, in $2r.\left(\frac{dR}{dr}\right)$, the expression $m'.a.\left(\frac{d.A^{(0)}}{da}\right)$; multiplying this by ant [6118h] it [6118l] produces in δv the term $nt.\left\{m'.a^2.\left(\frac{dA^{(0)}}{da}\right)\right\}$, and by prefixing the sign Σ, so as to [6118l'] include all the disturbing satellites, we get the last term of [6119 line 1]. The first terms of the functions [6108, 6109] produce, in the expression [6118f], the terms $-\frac{3}{4}.M^2a^2-M^2a^2=-\frac{7}{4}.M^2a^2$. Multiplying this by ant [6118h], and substituting [6118m]

The term depending on $\sin.(2nt-2Mt+2\varepsilon-2E)$ corresponds to the
[6120] equation, which is known in the lunar theory by the name of the *variation*

[6118n] $a^3=\frac{1}{n^2}$ [6110c], we get the third term of [6119 line 1]. The rest of the functions, enumerated in [6118c], produce nothing of the required form and order.

Second. The term of [6118b] depending on the angle $2nt-2Mt+2\varepsilon-2E$, is found
[6118o] in the following manner. Multiplying the last term of [6108] by $\frac{3}{2}$, and the last term of [6109] by 2, and taking the sum of the products, it becomes, by noticing only this term, and changing the denominator $2n-2M$ into $2n$, as in [6116g],

[6118p] $$3\int \mathrm{d}R+2r.\left(\frac{dR}{dr}\right)=-\tfrac{21}{4}.M^2a^2.\cos.(2nt-2Mt+2\varepsilon-2E).$$

Multiplying this by $andt$, and then integrating, using the divisor $2n$ instead of $2n-2M$, we get, by changing a^3 into n^{-2} as above,

[6118q] $$\int andt.\left\{3\int \mathrm{d}R+2r.\left(\frac{dR}{dr}\right)\right\}=-\tfrac{21}{8}.\frac{M^2}{n^2}.\sin.(2nt-2Mt+2\varepsilon-2E).$$

Again, by noticing only the term in [6116 line 2], depending on the proposed angle, we get, by the same reduction,

[6118r] $$\frac{2.d.\left(\frac{r\delta r}{a^2}\right)}{ndt}=4.\frac{M^2}{n^2}.\sin.(2nt-2Mt+2\varepsilon-2E).$$

The sum of the expressions [6118q, r] being substituted in [6118b] gives the term of δv in [6119 line 2].

[6118s] *Third.* If we notice only the term depending on the angle $i(n't-nt+\varepsilon'-\varepsilon)$, or iT [6118e], we shall have in [6092, 6094], by neglecting, for brevity, the symbol Σ,

[6118t] $$3\int \mathrm{d}R=m'.\frac{3n}{n-n'}.A^{(i)}.\cos.iT\,; \qquad 2r.\left(\frac{dR}{dr}\right)=m'.2a.\left(\frac{dA^{(i)}}{da}\right).\cos.iT.$$

The sum of these two expressions being multiplied by $andt$, and then integrated, gives

[6118u] $$\int andt.\left\{3\int \mathrm{d}R+2r.\left(\frac{dR}{dr}\right)\right\}=\frac{m'.n}{i(n-n')}.\left\{-\frac{3n}{n-n'}.aA^{(i)}-2a^2.\left(\frac{dA^{(i)}}{da}\right)\right\}.\sin.iT.$$

Taking the differential of the general term of [6116 lines 3, 4, 5..], corresponding to the angle iT, and multiplying it by $\frac{2}{ndt}$, we get [6118v], which, by successive operations, is reduced to the form [6118x];

[6118v] $$\frac{2.d.\left(\frac{r\delta r}{a^2}\right)}{ndt}=m'n.\frac{2i.(n-n')}{i^2.(n-n')^2-N^2}.\left\{\frac{2n}{n-n'}.aA^{(i)}+a^2.\left(\frac{dA^{(i)}}{da}\right)\right\}.\sin.iT$$

[6118w] $$=\frac{m'.n}{i.(n-n')}.\frac{2i^2.(n-n')^2}{i^2.(n-n')^2-N^2}.\left\{\frac{2n}{n-n'}.aA^{(i)}+a^2.\left(\frac{dA^{(i)}}{da}\right)\right\}.\sin.iT$$

[6118x] $$=\frac{m'.n}{i.(n-n')}.\left(2+\frac{2N^2}{i^2.(n-n')^2-N^2}\right).\left\{\frac{2n}{n-n'}.aA^{(i)}+a^2.\left(\frac{dA^{(i)}}{da}\right)\right\}.\sin.iT.$$

Adding together the functions [6118u, x] we get the part of δv [6118b] depending on the

[5556 line 2]; but it is less sensible in the theory of Jupiter's satellites, because the ratio $\frac{M^2}{n^2}$ is much smaller than the corresponding expression in the lunar theory. [6120']

The mean motion of the satellite m, *being supposed equal to* nt [6062']; *the coefficient of* t *must vanish from the preceding expression of* δv [6119 line 1]; hence we get,* [6121]

$$k = -\frac{(\rho - \frac{1}{2}\varphi)}{3a^2} + \tfrac{7}{12}\cdot\frac{M^2}{n^2} - \tfrac{1}{3}.\Sigma.m'.a^2.\left(\frac{dA^{(0)}}{da}\right). \qquad [6122]$$

Substituting this value of k in that of $\frac{\delta a}{a}$ [6117], we obtain,

$$\frac{\delta a}{a} = \frac{(\rho - \frac{1}{2}\varphi)}{3a^2} - \tfrac{1}{6}\cdot\frac{M^2}{n^2} + \tfrac{1}{6}.\Sigma.m'a^2.\left(\frac{dA^{(0)}}{da}\right); \qquad [6123]$$

hence we deduce,†

$$N^2 = n^2.\left\{1 - 2.\frac{(\rho - \frac{1}{2}\varphi)}{a^2} - \tfrac{3}{2}.\frac{M^2}{n^2} + \Sigma.m'.\left[a^2.\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^3.\left(\frac{ddA^{(0)}}{da^2}\right)\right]\right\}. \qquad [6124]$$

We shall obtain the values of $\frac{r'\delta r'}{a'^2}$, $\delta v'$; $\frac{r''\delta r''}{a''^2}$, $\delta v''$; $\frac{r'''\delta r'''}{a'''^2}$, $\delta v'''$; *by changing, in the preceding expressions of* $\frac{r\delta r}{a^2}$ *and* δv [6116, 6119, &c.], *the quantities relative to the first satellite, into those relative to the second, third and fourth successively, and the contrary.* [6125]

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angle iT. Now by inspection we see that the term of [6118x] containing N^2, is the same as the corresponding terms in [6119 lines 3, 4, 5..]. Of the two remaining terms of [6118x], independent of N^2, that which contains $dA^{(i)}$ is destroyed by the similar term in [6118u]; and that which depends on $A^{(i)}$ [6118x], being added to the corresponding term of [6118u], becomes $\frac{m'.n}{i.(n-n')}.\left\{\frac{n}{n-n'}.aA^{(i)}\right\}$; being of the same form as the first terms of [6119 lines 3, 4, 5..]; therefore the expression of δv [6119] is correct, within the limits of the present approximation. [6118y] [6118z]

* (3242) The mean motion of the satellite m is assumed in [6062'] to be nt, and this is supposed to comprise the *whole* of its value; consequently the part of it, contained in [6119 line 1], must vanish. Putting therefore the coefficient of nt [6119 line 1] equal to nothing, and dividing by 3, we get the value of the arbitrary constant quantity k [6122]. [6122a] [6122b]

† (3243) Substituting the value of k [6122] in [6117], we get [6123]; then substituting the value of $\frac{\delta a}{a}$ [6123] in [6112], we obtain [6124]. [6124a]

4. *The ratios, which obtain in the mean motions of the three first satellites, make some of the terms of the preceding expressions very large. These terms*
[6125′] *deserve particular attention, because they are the source of the principal inequalities, which are observed in the motions of the three first satellites.* The mean motion of the first satellite is very nearly double that of the second; and the mean motion of the second is nearly double that of the
[6125″] third [6025*a*]. Hence it follows, that the term of the expression of $\frac{r\delta r}{a^2}$, depending on the angle $2n't-2nt+2\varepsilon'-2\varepsilon$, must become very great, on
[6126] account of its divisor $4.(n-n')^2-N^2=(2n-2n'+N).(2n-2n'-N)$
[6127] [6116 line 4]. For as N and $2n'$ differ but very little from n [6025*e*], the divisor $2n-2n'-N$ will be very small; consequently the term in question will acquire a considerable value. At the same time we perceive
[6127′] the necessity of determining N with great precision, as we have done in [6124]; because the difference between N and n, which depends on the disturbing forces, although it is very small, becomes sensible in the divisor $2n-2n'-N$; particularly by means of the term $-n.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}$ which N
[6128] contains [6124]; and this is the reason why we have retained, in [6079′, &c.], the terms depending on the disturbing force, in which $r\delta r$ is multiplied by constant quantities; these terms having an influence on the value of N. In the other divisors, which are not very small, we may suppose, without
[6129] any sensible error, that $N=n$. Therefore, if we put

[6130] F
$$F=-\frac{2n}{n-n'}.aA^{(2)}-a^2.\left(\frac{dA^{(2)}}{da}\right),$$

and notice only the term depending on the cosine of $2n't-2nt+2\varepsilon'-2\varepsilon$,
[6130′] observing that we may, in the factor $2n-2n'+N$, suppose $2n'$ and N equal to n [6127], we shall have,*

[6131]
$$\frac{r\delta r}{a^2}=-\frac{m'.nF}{2.(2n-2n'-N)}.\cos.(2nt-2n't+2\varepsilon-2\varepsilon').$$

The expression of δv gives, by noticing only the terms which have the

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* (3244) Substituting $2n'=n$, $N=n$ [6127] in the first factor of the second
[6131*a*] member of [6126], we get $4.(n-n')^2-N^2=2n.(2n-2n'-N)$. Using this, and the value of F [6130], we find that the term [6116 line 4] becomes as in [6131]. We have already seen, in [1227, &c.], the great importance of the terms treated of in this article.

divisor $2n-2n'-N$, *

$$\delta v = \frac{m'.nF}{2n-2n'-N}.\sin.(2nt-2n't+2\varepsilon-2\varepsilon'). \qquad [6132]$$

This part of δv is the most sensible inequality of the motion of the first satellite; it is the only one which has been discovered by observation. [6133]

If, in the theory of the second satellite, *we denote, by N', the quantity which corresponds to N [6124], in the theory of the first; and if we put $A_{\prime}^{(1)}$ for what corresponds to $A^{(1)}$ in the perturbations of the first by the second satellite;* it will follow, from what has been said, that the expression of $\frac{r'\delta r'}{a'^2}$ contains the term,† [6134] Symbols N' $A_{\prime}^{(1)}$ [6134′]

$$\frac{r'\delta r'}{a'^2} = \frac{m.n'^2}{(n-n')^2-N'^2}.\left\{\frac{2n'}{n'-n}.a'A_{\prime}^{(1)}.+a'^2.\left(\frac{dA_{\prime}^{(1)}}{da'}\right)\right\}.\cos.(nt-n't+\varepsilon-\varepsilon'). \qquad [6135]$$

The divisor $(n-n')^2-N'^2$ is equal to $(n-n'+N').(n-n'-N')$. Now we have very nearly, as in [6025*f*, *a*], $N'=n'$, $n=2n'$; therefore the divisor $n-n'-N'$ is very small; and the preceding term [6135] becomes important. We shall put, [6135′] [6136] Symbol G

$$G = \frac{2n'}{n-n'}.a'A_{\prime}^{(1)} - a'^2.\left(\frac{dA_{\prime}^{(1)}}{da'}\right); \qquad [6137]$$

and by substituting $n=2n'$ and $N'=n'$ [6136] in the factor $n-n'+N'$, we shall obtain, [6138]

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* (3245) The term in question is that depending on N^2 in [6119 line 4]; which, by using the value of F [6130] and that of the denominator [6131*a*], becomes

$$-\frac{m'.n}{n-n'}.\tfrac{1}{2}.\frac{2N^2}{2n.(2n-2n'-N)}.F.\sin.2.(n't-nt+\varepsilon'-\varepsilon). \qquad [6132a]$$

Now we have very nearly, $\frac{n}{n-n'}.\tfrac{1}{2}=1$, $2N^2=2n^2$ [6131*a*]; also [6132*b*]

$$\sin.2.(n't-nt+\varepsilon'-\varepsilon) = -\sin.2.(nt-n't+\varepsilon-\varepsilon');$$

hence the preceding term of δv becomes as in [6132]. The expressions [6131, 6132] correspond to the values of δr, δv [1227].

† (3246) Changing reciprocally the elements of m into those of m', in the term [6116 line 3], we get [6135]; observing, in the denominator, that $(n'-n)^2=(n-n')^2$. [6135*a*] If we use the values of n, N' [6138], we get

$$(n-n')^2-N'^2 = (n-n'+N').(n-n'-N') = 2n'.(n-n'-N'), \text{ nearly}. \qquad [6135b]$$

Substituting this and G [6137] in [6135], we obtain [6139]; observing that in the first term of G, we may change $n-n'$ into $-(n'-n)$, to correspond with the form in [6135]. This expression [6139] is equivalent to the term depending on G in [1227]. [6135*c*]

[6139] $$\frac{r'\delta r'}{a'^2}=-\frac{m.n'.G}{2.(n-n'-N')}.\cos.(nt-n't+\varepsilon-\varepsilon').$$

Then we shall have, by noticing only the terms which have $n-n'-N'$ for a divisor,*

[6140] $$\delta v'=\frac{m.n'.G}{n-n'-N'}.\sin.(nt-n't+\varepsilon-\varepsilon').$$

We may here observe that†

[6141] $$A_{,}^{(1)}=\frac{a'}{a^2}-\frac{a}{a'^2}+A^{(1)};$$

hence we get,‡

[6142] $$G=\frac{3n'-n}{n-n'}.\frac{a'^2}{a^2}-\frac{2n}{n-n'}.\frac{a}{a'}+\frac{2n'}{n-n'}.a'A^{(1)}-a'^2.\left(\frac{dA^{(1)}}{da'}\right).$$

Now we have,§

[6143] $$\frac{a'^2}{a^2}=\frac{a'^3}{a^3}.\frac{a}{a'};\qquad\qquad \frac{a'^2}{a^2}=\frac{n^2}{n'^2}.\frac{a}{a'};$$

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* (3247) Changing reciprocally the elements of m into those of m', in the part of the term [6119 line 3] which contains N^2, we obtain in $\delta v'$ the following term;

[6140a] $$\delta v'=\frac{m.n'}{n'-n}.\frac{2N'^2}{(n'-n)^2-N'^2}.\left\{\frac{2n'}{n'-n}.a'A_{,}^{(1)}+a'^2.\left(\frac{dA_{,}^{(1)}}{da'}\right)\right\}.\sin.(nt-n't+\varepsilon-\varepsilon').$$

Changing $n'-n$ into $-(n-n')$, and substituting the value of G [6137], it becomes

[6140b] $$\delta v'=\frac{mn'}{n-n'}.\frac{2N'^2}{(n-n')^2-N'^2}.G.\sin.(nt-n't+\varepsilon'-\varepsilon).$$

[6140c] Now we have nearly $N'=n'$, and $\frac{n'}{n-n'}=1$ [6138], substituting these, and that in [6135b], in [6140b], we get [6140].

[6141a] † (3248) We have in [997, 1008'] $A^{(1)}=\frac{a}{a'^2}-\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}$; $A_{,}^{(1)}=\frac{a'}{a^2}-\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}$; subtracting the first of these expressions from the second, we get [6141].

‡ (3249) The partial differential of [6141], relative to a', gives

[6142a] $$\left(\frac{dA_{,}^{(1)}}{da'}\right)=\frac{1}{a^2}+\frac{2a}{a'^3}+\left(\frac{dA^{(1)}}{da'}\right).$$

Multiplying this by $-a'^2$, and [6141] by $\frac{2n'a'}{n-n'}$, then adding the products, we get, for the value of G [6137], the same expression as in [6142].

§ (3250) Neglecting the masses m, m', of the satellites, in comparison with that of

[6143a] Jupiter, we get as in [6110] $n^2a^3=1$, $n'^2a'^3=1$; hence we easily deduce the expressions [6143].

hence we obtain,*

$$G = 2a'.A^{(1)} - a'^2.\left(\frac{dA^{(1)}}{da'}\right) - \frac{n.(n-2n')}{n'^2}.\frac{a}{a'} - \frac{2.(n-2n')}{n-n'}.a'A^{(1)};$$ [6144]

and as $n-2n'$ is extremely small [6138], the preceding equation will become very nearly,

$$G = 2a'.A^{(1)} - a'^2.\left(\frac{dA^{(1)}}{da'}\right);$$ [6145]

but for greater accuracy we shall use [6144], in the numerical calculation of G.

The preceding values of $\frac{r'\delta r'}{a'^2}$ and $\delta v'$ [6139, 6140], correspond to the action of the first satellite. The action of the third, produces also sensible [6145′] terms in $\frac{r'\delta r'}{a'^2}$, $\delta v'$. For the motion of the second satellite being very nearly double that of the third, there must occur, in these expressions, some terms analogous to those which are produced by the action of the second Symbols $A'^{(i)}$ satellite, in the values of $\frac{r\delta r}{a^2}$ and δv. We shall put $A'^{(0)}$, $A'^{(1)}$, $A'^{(2)}$, &c., [6146] relative to the second and third satellites, for what we have denoted by $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, &c., relative to the first and second. Then we shall suppose, [6146′]

$$F' = -\frac{2n'}{n'-n''}.a'A'^{(2)} - a'^2.\left(\frac{dA'^{(2)}}{da'}\right);$$ [6147] Symbol F'

we shall have, by the action of the third satellite,†

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* (3251) The coefficient of $a'A^{(1)}$ in [6142], is $\frac{2n'}{n-n'}$, which is easily reduced to [6144a] the form $2 - \frac{2.(n-2n')}{n-n'}$, as in [6144]. The term depending on $dA^{(1)}$ is the same in both expressions [6142, 6144]. Lastly, substituting the second value of $\frac{a'^2}{a^2}$ [6143] in the first term of [6142], we find that the terms independent of $A^{(1)}$ and $dA^{(1)}$ are [6144b]

$$\frac{a}{a'}.\left\{\frac{3n'-n}{n-n'}.\frac{n^2}{n'^2} - \frac{2n}{n-n'}\right\} = \frac{a}{a'}.\frac{n}{n-n'}.\left\{\frac{(3n'-n).n}{n'^2} - 2\right\} = \frac{a}{a'}.\frac{n}{n-n'}.\left\{\frac{3nn'-n^2-2n'^2}{n'^2}\right\}$$

$$= \frac{a}{a'}.\frac{n}{n-n'}.\left\{\frac{(n'-n).(n-2n')}{n'^2}\right\} = -\frac{a}{a'}.\frac{n.(n-2n')}{n'^2};$$ [6144c]

and this last expression is the same as the third term of [6144]. If we neglect the terms which are multiplied by the very small factor $n-2n'$, in the expression of G [6144], [6144d] we shall obtain [6145].

† (3252) The relative situations of the first and second satellites are similar to those of the second and third; and the formulas [6130, 6131, 6132], corresponding to the action of [6148a]

[6148] $$\frac{r'\delta r'}{a'^2} = -\frac{m''.n'.F'}{2.(2n'-2n''-N')}.\cos.(2n't-2n''t+2\varepsilon'-2\varepsilon'');$$

[6149] $$\delta v' = \frac{m''.n'.F'}{2n'-2n''-N'}.\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon'').$$

Connecting these values with those in [6139, 6140] we shall obtain the most sensible terms of $\frac{r'\delta r'}{a'^2}$, $\delta v'$.

A very remarkable relation which obtains between the mean motions of the three first satellites, permits us to unite, in one single term, the two terms

[6150] of each of these expressions, depending on the actions of the first and third

[6150'] satellites. We have observed, in [6125'', &c.], that *the mean motion of the first satellite is nearly double that of the second; and the mean motion of the*

[6150''] *second is nearly double that of the third; so that we shall have, very nearly,*

[6151] $$n = 2n'; \qquad n' = 2n''; \qquad n = 4n''.$$

Hence we deduce,*

[6152] $$n - 3n' + 2n'' = 0.$$

First law of La Place relative to the motion of Jupiter's satellites.

This last equation is much more accurate than the two equations from which we have deduced it; and conforms with observation to so great a degree that no sensible value has been perceived in its first member since

[6153] the first discovery of Jupiter's satellites; therefore we may suppose it to be nothing, at least during one century. We shall see, in [6628, &c.], that the mutual action of the satellites renders the expression $n-3n'+2n''$ rigorously equal to nothing. Hence we get accurately,

[6154] $$2n'-2n'' = n-n'; \qquad n'-2n'' = n-2n'.$$

Second law of La Place.

It has been found by all the observations, since the discovery of the satellites, that the mean longitude of *the first, minus three times that of the second,*

[6155] *plus twice that of the third, is equal to the semi-circumference, or* 200°; so

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the second satellite on the first, will therefore give those in [6147, 6148, 6149] respectively;

[6148b] corresponding to the action of the third upon the second; by merely increasing the accents

[6148c] by unity, in the symbols m', a, n, n', &c.

* (3253) We have already remarked, in [1226a, b], that the values [6025a—d] give very nearly,

[6151a] $n-2n'=n'''.0{,}034281=n.0{,}003634$; $n'-2n''=n'''.0{,}034283=n'.0{,}007294$; $n-3n'+2n''=-n'''.0{,}000002$;

hence it is evident that the equation [6152] is much nearer to the truth than either of the equations [6151]; and we shall have as in [6151, 6025a], nearly,

[6151b] $$n = 2n' = 4n''; \qquad n' = \tfrac{1}{2}n = 2n''; \qquad n'' = \tfrac{1}{4}n = \tfrac{1}{2}n'; \qquad n''' = \tfrac{3}{7}n''.$$

that during the interval of at least one century, we may suppose,*

$$nt - 3n't + 2n''t + \varepsilon - 3\varepsilon' + 2\varepsilon'' = 200^\circ ;$$ [6156]

consequently,

$$2n't - 2n''t + 2\varepsilon' - 2\varepsilon'' = nt - n't + \varepsilon - \varepsilon' - 200^\circ .$$ [6157]

We shall see, in [6627], that these equations are rigorously correct. Hence the terms of $\frac{r'\delta r'}{a'^2}$ and $\delta v'$ [6148, 6149], depending on the action of the third satellite, become

$$\frac{r'\delta r'}{a'^2} = \frac{m''.n'.F'}{2.(n-n'-N')}.\cos.(nt-n't+\varepsilon-\varepsilon')$$ [6158]

$$\delta v' = -\frac{m''.n'.F'}{n-n'-N'}.\sin.(nt-n't+\varepsilon-\varepsilon') ;$$ [6159]

therefore, by the combined action of the first and third satellites, we shall have

$$\frac{r'\delta r'}{a'^2} = \frac{-n'}{2.(n-n'-N')}.\{mG-m''F'\}.\cos.(nt-n't+\varepsilon-\varepsilon') ;$$ [6160]

$$\delta v' = \frac{n'}{n-n'-N'}.\{mG-m''F'\}.\sin.(nt-n't+\varepsilon-\varepsilon').$$ [6161]

The action of the second satellite produces, in the theory of the third, some terms, analogous to those which the action of the first produces in the theory of the second; therefore, by putting†

$A_{\prime}'^{(1)}$

$$A_{\prime}'^{(1)} = A'^{(1)} + \frac{a''}{a'^2} - \frac{a'}{a''^2} ;$$ [6162]

$$G' = \frac{2n''}{n'-n''}.a''.A_{\prime}'^{(1)} - a''^2.\left(\frac{dA_{\prime}'^{(1)}}{da''}\right) ;$$ [6163]

G'

we shall have

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* (3254) The mean longitudes of the three first satellites are $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$ [6022f]; substituting these in the theorem [6155], we get

$$(nt+\varepsilon)-3(n't+\varepsilon')+2(n''t+\varepsilon'')=200^\circ ;$$ [6158a]

which is easily reduced to the form [6156]; and by transposition, to the form [6157]. Substituting this in [6148, 6149], and changing also the divisor $2n'-2n''-N'$ into $n-n'-N'$, in conformity with [6154], we get [6158, 6159]; adding these, to the terms in [6139, 6140], we obtain [6160, 6161] respectively, for the action of both satellites m and m'', upon m'. [6158b]

† (3255) The equations [6162, 6163, 6164, 6165] are derived from [6141, 6137, 6139, 6140] respectively, by adding one accent to each of the symbols $r', v', a, a', n, n', \varepsilon, \varepsilon', m, m', N', A^{(1)}, A_{\prime}^{(1)}, G$; as in [6148c]. [6162a]

[6164] $$\frac{r''\delta r''}{a''^2} = -\frac{m'.n''.G'}{2.(n'-n''-N'')}.\cos.(n't-n''t+\varepsilon'-\varepsilon'');$$

[6165] $$\delta v'' = \frac{m'.n''.G'}{n'-n''-N''}.\sin.(n't-n''t+\varepsilon'-\varepsilon'').$$

[6165′] The values of $\frac{r''\delta r''}{a''^2}$, $\delta v''$, may also acquire some sensible terms from the action of the fourth satellite; but its mean motion, being sensibly less than the half of that of the third satellite, these terms must be but of little importance. We shall however notice them in the course of this work.

We may here observe, that, as n differs but little from $2n'$, and n'

[6166] but little from $2n''$ [6025a], $\frac{a}{a'}$ will differ but little from $\frac{a'}{a''}$. For*

[6167] $$\frac{a}{a'} = \left(\frac{n'}{n}\right)^{\frac{2}{3}} = \left(\frac{n-(n-2n')}{2n}\right)^{\frac{2}{3}} = (\tfrac{1}{2})^{\frac{2}{3}}.\left\{1-\left(\frac{n-2n'}{n}\right)\right\}^{\frac{2}{3}}.$$

[6168] This last quantity is very nearly equal to $(\frac{1}{2})^{\frac{2}{3}}.\left\{1-\frac{2}{3}.\frac{(n-2n')}{n}\right\}$. In like manner we have,

[6169] $$\frac{a'}{a''} = (\tfrac{1}{2})^{\frac{2}{3}}.\left\{1-\tfrac{2}{3}.\frac{(n'-2n'')}{n'}\right\} = (\tfrac{1}{2})^{\frac{2}{3}}.\left\{1-\tfrac{2}{3}.\left(\frac{n-2n'}{n'}\right)\right\};$$

[6170] so that $\frac{a}{a'}$ and $\frac{a'}{a''}$ differ but very little from $(\frac{1}{2})^{\frac{2}{3}}$; now $A^{(0)}$, $A^{(1)}$, &c. being of the degree -1, in a, a' [1001′], F and G [6130, 6137] will

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[6171a] * (3256) We have, as in [6143a], $a = n^{-\frac{2}{3}}$; $a' = n'^{-\frac{2}{3}}$; hence we get, by successive reductions,

[6171a'] $$\frac{a}{a'} = \left(\frac{n'}{n}\right)^{\frac{2}{3}} = (\tfrac{1}{2})^{\frac{2}{3}}.\left(\frac{2n'}{n}\right)^{\frac{2}{3}} = (\tfrac{1}{2})^{\frac{2}{3}}.\left\{1-\frac{(n-2n')}{n}\right\}^{\frac{2}{3}} = (\tfrac{1}{2})^{\frac{2}{3}}.\left\{1-\tfrac{2}{3}.\left(\frac{n-2n'}{n}\right)-\&c.\right\}.$$

This agrees with [6167, 6168], neglecting the square and higher powers of $n-2n'$. In like manner, by adding another accent to the symbols, we get

[6171b] $$\frac{a'}{a''} = (\tfrac{1}{2})^{\frac{2}{3}}.\left\{1-\tfrac{2}{3}.\left(\frac{n'-2n''}{n'}\right)-\&c.\right\};$$

[6171c] and by substituting $n'-2n'' = n-2n'$ [6154], it becomes as in [6169]; and as $n-2n'$

[6171d] is very small, we shall have very nearly $\frac{a}{a'} = \frac{a'}{a''} = (\frac{1}{2})^{\frac{2}{3}}$. Now having shown, in

[6171e] [6169, 6170], that F, G are functions of $\frac{a}{a'}$ of the order 0; and F', G'

[6171f] [6147, 6163] similar functions of $\frac{a'}{a''}$; also $\frac{a}{a'} = \frac{a'}{a''}$ nearly; it follows that we must have, very nearly, $F' = F$ and $G' = G$, as in [6171].

be of the degree 0, or functions of $\frac{a}{a'}$; F' and G' are similar functions of $\frac{a'}{a''}$; therefore we have, very nearly, $F'=F$, and $G'=G$. But for greater accuracy we shall notice the difference of these quantities. [6171]

5. *We shall now consider the law of the preceding inequalities, in the eclipses of the satellites.* For this purpose, we shall put the values of δv, $\delta v'$, $\delta v''$ [6132, 6161, 6165] under the following forms;*

$$\delta v = \quad (\text{I}) \,.\, \sin.(2nt-2n't+2\varepsilon-2\varepsilon'); \qquad [6172]$$

$$\delta v' = -(\text{II}) \,.\, \sin.(nt-n't+\varepsilon-\varepsilon'); \qquad [6173]$$

$$\delta v'' = -(\text{III}) \,.\, \sin.(n't-n''t+\varepsilon'-\varepsilon''); \qquad [6174]$$

the coefficients (I), (II) and (III), are positive, as we shall hereafter find [6175]
[6172b, &c.]. *Instead of referring the angles* $nt+\varepsilon$, $n't+\varepsilon'$, *and* $n''t+\varepsilon''$ [6175']
to a fixed line, we may refer them to a moveable axis; because the position of this axis vanishes from the angles $2nt-2n't+2\varepsilon-2\varepsilon'$, $nt-n't+\varepsilon-\varepsilon'$, [6176]
$n't-n''t+\varepsilon'-\varepsilon''$.† *We shall take, for this axis, the radius vector of Jupiter, supposing it to move uniformly about the sun. In this case, the angles* $\mathrm{n}t$, $\mathrm{n}'t$, $\mathrm{n}''t$ *denote the mean synodical motions of the three first satellites.* [6176']
We shall also suppose $\varepsilon=0$, $\varepsilon'=0$; or, in other words, that at the origin [6177]
of the time t, the two first satellites are in conjunction. The equation,

$$\mathrm{n}t-3\mathrm{n}'t+2\mathrm{n}''t+\varepsilon-3\varepsilon'+2\varepsilon''=200^{\circ} \text{ [6156]}, \qquad [6178]$$

which holds good relative to the synodical motions, gives‡ $\varepsilon''=100^{\circ}$; hence [6179]

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* (3257) The values [6172, 6173, 6174] represent the parts of δv, $\delta v'$, $\delta v''$ [6172a]
[6842 line 2; 6844 lines 1, 7; 6846 line 4], changing $-\sin.2(n't-nt+\varepsilon'-\varepsilon)$ into $+\sin.2(nt-n't+\varepsilon-\varepsilon')$; also $\sin.(2n''t-2n't+2\varepsilon''-2\varepsilon')$ into $\sin.(nt-n't+\varepsilon-\varepsilon')$, as in [6157]; hence it appears that (I), (II), (III), are positive in [6172—6174]. [6172b]

† (3258) This is easily proved, as in [1240b]; where it is shown that the equation [6176a]
[6156] holds good when we use the synodical motions as in [6176—6178]; therefore, in the equations [6172—6174], we may use the synodical motions; and the same is to be observed in the equations [6156, 6157, &c.]. To distinguish these synodical motions, we shall use the Roman letter n, denoting them in this article by $\mathrm{n}t$, $\mathrm{n}'t$, $\mathrm{n}''t$, as in [6176']; [6176b]
and it is evident that we make the same changes in the equation [6152], and by this means we shall obtain $\mathrm{n}-3\mathrm{n}'+2\mathrm{n}''=0$. [6176c]

‡ (3259) Multiplying [6176c] by $-t$, and adding the product to [6178], we get $\varepsilon-3\varepsilon'+2\varepsilon''=200^{\circ}$; substituting the values of ε, ε' [6177], and dividing by 2, we get [6179a]
$\varepsilon''=100^{\circ}$, as in [6179]; and by using these values of ε, ε', ε'', we find that the equations [6179b]
[6172—6174] become as in [6180—6182].

the expressions of δv, $\delta v'$, $\delta v''$ become

[6180] $$\delta v = (\text{I}) \cdot \sin.(2\text{n}t - 2\text{n}'t);$$

[6181] $$\delta v' = -(\text{II}) \cdot \sin.(\text{n}t - \text{n}'t);$$

[6182] $$\delta v'' = (\text{III}) \cdot \cos.(\text{n}'t - \text{n}''t).$$

In the eclipses of the first satellite, at the moment of its mean conjunction,
[6183] $\text{n}t$ is nothing, or a multiple of 400°. We shall put,

[6184] $$2\text{n} - 2\text{n}' = \text{n} + \omega, \qquad \text{or} \qquad \text{n} - 2\text{n}' = \omega;$$

and then we shall have,*

[6185] $$\delta v = (\text{I}) \cdot \sin. \omega t.$$

In the eclipses of the second satellite, at the moment of its mean conjunction
[6186] $\text{n}'t$ is nothing, or a multiple of 400°; then we shall have,†

[6187] $$\delta v' = -(\text{II}) \cdot \sin. \omega t.$$

Lastly, in the eclipses of the third satellite, at the instant of its mean
[6188] conjunction, $\text{n}''t + \varepsilon''$ is nothing, or a multiple of 400°; then we shall
[6189] have, by means of the equations $\text{n}' - 2\text{n}'' = \text{n} - 2\text{n}' = \omega$ [6176c, 6184], and $\varepsilon'' = 100°$ [6179],‡

[6190] $$\delta v'' = (\text{III}) \cdot \sin. \omega t.$$

[6191] *Hence we see that the preceding values of* δv, $\delta v'$, $\delta v''$ [6185, 6187, 6190] *in eclipses, depend solely upon the angle* ωt. The period of these inequalities is therefore the same, being equal to the duration of the synodical revolution
[6192] of the first satellite, multiplied by§ $\frac{\text{n}}{\text{n} - 2\text{n}'}$; $\text{n}t$ and $\text{n}'t$ being the mean

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[6185a] * (3260) Substituting $2\text{n} - 2\text{n}' = \text{n} + \omega$ [6184] in [6180], we get $\delta v = (\text{I}).\sin.(\text{n}t + \omega t)$, and at the time of the mean conjunction $\text{n}t = 0$ [6183]; hence δv becomes as in [6185].

[6187a] † (3261) From [6184] we obtain $\text{n}t - \text{n}'t = \text{n}'t + \omega t$; substituting this in [6181] and then putting, as in [6186], $\text{n}'t = 0$, we get [6187].

[6190a] ‡ (3262) From [6189] we get $\text{n}'t - 2\text{n}''t = \omega t$, or $\text{n}'t - \text{n}''t = \text{n}''t + \omega t$; hence [6182] becomes $\delta v'' = (\text{III}) . \cos.(\text{n}''t + \omega t)$. Now at the time of the mean conjunction we have $\text{n}''t + \varepsilon'' = 0$ [6188,] or $\text{n}''t = -\varepsilon'' = -100°$ [6179]; hence the preceding
[6190b] expression of $\delta v''$ becomes $\delta v'' = (\text{III}).\cos.(\omega t - 100°) = (\text{III}).\sin.\omega t$, as in [6190].

[6192a] § (3263) The mean synodical motion of the first satellite, in the time t, is here represented by $\text{n}t$ [6176']; and if we put T for the value of t, corresponding to a
[6192b] synodical revolution of this satellite, we shall have $\text{n}T = 400°$. Moreover, if we put T' for the time required to complete the period of the inequality, depending on the angle ωt,

synodical motions of the first and second satellites. Substituting the values of n, n', we find that this period is equal to $437^{\mathrm{days}},659$.* All these results [6193]
agree perfectly with observation; and it was by observation that these inequalities were discovered, before they had been indicated by the theory.

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we shall have $400^{\circ} = \omega T' = (\mathrm{n}-2\mathrm{n}').T'$ [6184]; hence $(\mathrm{n}-2\mathrm{n}').T' = \mathrm{n}T$; or [6192c]
$T' = \frac{\mathrm{n}}{\mathrm{n}-2\mathrm{n}'}.T$, as in [6192]. [6192d]

* (3264) The mean motion of Jupiter about the sun is $Mt = n't.0,000820$ [6025i].
Subtracting this from the mean motions of the first and second satellites $n't.2,007294$ [6193a]
[6025a], and $n't$ [6025b], we get their synodical motions $\mathrm{n}t = n't.2,006474$, [6193b]
$\mathrm{n}'t = n't.0,999180$; which are to each other, as $2,006474$ to $0,999180$; or as $2,008121$ [6193c]
to 1 nearly; and these express the ratio of n to n'; hence we have [6193d]

$$\frac{\mathrm{n}}{\mathrm{n}-2\mathrm{n}'} = \frac{2,008121}{0,008121} = 247,28. \qquad [6193e]$$

Substituting this in [6192d], we get $T' = 247,28.T$. To obtain the time T, of the [6193f]
synodical revolution of the first satellite, we have the synodical motion in a Julian year, or
$365^{\mathrm{days}},25$, $\mathrm{n} = n - M = 825488799''$ [6025n]. Therefore the time T of describing [6193g]
the whole circumference $4000000''$, by the synodical motion, is

$$T = 365^{\mathrm{days}},25 \times \frac{4000000''}{825488799''} = 1^{\mathrm{days}},76986. \qquad [6193h]$$

Substituting this in the expression of T' [6193f], we get $T' = 247,28.T = 437^{\mathrm{days}},65$, as in [6193] nearly.

CHAPTER III.

ON THE INEQUALITIES OF THE MOTIONS OF THE SATELLITES, DEPENDING ON THE EXCENTRICITIES OF THE ORBITS.

6. WE shall now consider the parts of the radius vector and of the longitude of the satellites, depending on the excentricities of the orbits. These excentricities are very small [6057e]; therefore, by substituting
[6194] $r^2 = a^2 + 2r\delta r$ [6108a], in the equation [6053], we may suppose that $2r\delta r$ *represents, not only the perturbations of* r^2 *depending on the disturbing forces, but also the part of* r^2 *depending on the elliptical motion* [6057a, b].
[6195] Then the differential equation [6057], into which the equation [6053] is transformed, by neglecting the square of δr, gives, by its integration, not only the perturbations of the radius vector, but also the elliptical part depending upon the arbitrary quantities introduced by the integrations. In this case the expression of δv, given by the equation [6060], contains the
[6196] elliptical part of v, and this part is evidently represented by* $\delta v = \frac{2.d.(r\delta r)}{a^2.ndt}$; *neglecting the square of the excentricity of the orbit*, and considering only the elliptical part of $r\delta r$.

Those terms of the differential equation [6057], where $r\delta r$ is multiplied
[6196′] by constant quantities, also those depending on the sines and cosines of $nt+\varepsilon$ require particular attention, *because on them depend the secular*

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* (3265) If we notice only the purely elliptical parts of $r\delta r$, δv, we must neglect R
[6196a] in [6060], which depends on the disturbing force; and as dr, δr, [669, 6200a], are each of the order e, we may neglect, as in [6057′], the term $dr.\delta r$ [6060], which is of the order e^2, and then δv [6060] becomes as in [6196]. Connecting this term of v, with
[6196b] its mean longitude $nt+\varepsilon$ [6022f], we get for these parts of v the following expression;

[6196c] $$v = nt + \varepsilon + \frac{2.d.(r\delta r)}{a^2.ndt}.$$

variations of the excentricity of the orbit and its perijove. We have computed, in [6112, 6114 line 1], the terms containing $r\delta r$, multiplied by constant quantities. To determine the others, we shall consider the term $m'.A^{(1)}.\cos.(v'-v)$ of the expression of R [6090], and shall substitute in it, [6197]

$$r'=a'+\frac{r'\delta r'}{a'}\ [6108b]\,;\qquad v'=n't+\varepsilon'+\frac{2.d.(r'\delta r')}{a'^2.n'dt}\ [6196c]\,; \qquad [6198]$$

hence we obtain the following terms of R;*

$$R=m'.\frac{r'\delta r'}{a'}.\left(\frac{dA^{(1)}}{da'}\right).\cos.(n't-nt+\varepsilon'-\varepsilon)-\frac{2m'.d.(r'\delta r')}{a'^2.n'dt}.A^{(1)}.\sin.(n't-nt+\varepsilon'-\varepsilon)\,; \qquad [6199]$$

$a'e'$ being the excentricity of the orbit of m', and ϖ' the longitude of its [6199'] perijove [6061—6062, &c.]. The elliptical part of $\frac{r'\delta r'}{a'^2}$ is represented by $\frac{r'\delta r'}{a'^2}=-e'.\cos.(n't+\varepsilon'-\varpi')$.† Substituting it in [6199], we obtain a term [6200] depending on the cosine of the angle $nt+\varepsilon-\varpi'$, and it is evident that this [6200'] term is the only one of the same kind, which arises from the development of the part of R, depending on the actions of the satellite m'. The two

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* (3266) The values of r', v' [6198] are similar to those of r, v [6108b, 6196c]; and if we substitute them in the term $R=m'.A^{(1)}.\cos.(v'-v)$ [6090], we can easily [6198a] develop it, according to the powers of $\frac{r'\delta r'}{a'^2}$ and its differentials, by means of the formulas [610—615]. Then if we retain only the first power of this quantity, we shall get, by [6198b] using the abridged symbol T [6089c], and the values of r, v [6091],

$$R=m'.A^{(1)}.\cos.T+m'.\frac{r'\delta r'}{a'}.\left(\frac{dA^{(1)}}{da'}\right).\cos.T-\frac{2m'.d.(r'\delta r')}{a'^2.n'dt}.A^{(1)}.\sin.T\,; \qquad [6198c]$$

the value of $A^{(1)}$, in this expression, being found by putting $r'=a'$ in its general expression. We may observe that it will not be necessary to retain the first term [6198d] $m'.A^{(1)}.\cos.T$ [6198c], as it is of no use in the calculation in [6204, &c.]; the other two terms are as in [6199].

† (3267) Neglecting e^2 in the expression of r [669], we get $\frac{r}{a}=1-e.\cos.(nt+\varepsilon-\varpi)$; [6200a] hence $\frac{\delta r}{a}=-e.\cos.(nt+\varepsilon-\varpi)$ [6057a]. Multiplying these two equations together, and neglecting e^2, we get $\frac{r\delta r}{a^2}=-e.\cos.(nt+\varepsilon-\varpi)$; and by accenting the symbols we [6200b] obtain $\frac{r'\delta r'}{a'^2}=-e'.\cos.(n't+\varepsilon'-\varpi')$, as in [6200]. Substituting this in [6199] and [6200c] reducing the products, by [17, 20] Int., we obtain terms depending on the angle $nt+\varepsilon-\varpi'$, as in [6200'].

satellites m'' and m''' produce similar terms in R; but it is evident,
[6201] from the expression of R [6042], that the sun's action does not produce any, if we neglect the terms divided by D'^4, as in [6041'].*

[6202] Now if we notice only the terms depending on nt, we shall have† $\int \mathrm{d}R = R$; therefore by retaining only these terms, we shall have,

[6203] $$2\int\mathrm{d}R+r.\left(\frac{dR}{dr}\right)=\Sigma.\left\{\begin{array}{l}\frac{m'.r'\delta r'}{a'^2}.\left\{2a'.\left(\frac{dA^{(1)}}{da'}\right)+aa'.\left(\frac{ddA^{(1)}}{dada'}\right)\right\}.\cos.(n't-nt+\varepsilon'-\varepsilon)\\ -\frac{m'.d.(r'\delta r')}{a'^2.n'dt}.\left\{4A^{(1)}+2a.\left(\frac{dA^{(1)}}{da}\right)\right\}.\sin.(n't-nt+\varepsilon'-\varepsilon)\end{array}\right\}.$$

[6203'] The differential equation [6057] will therefore become, by noticing only the terms in which $r\delta r$ is multiplied by constant quantities, and those which depend on the sine and cosine of nt; observing also that $n^2=\frac{1}{a^3}$ very nearly [6110]; ‡

[6201a] * (3268) Neglecting terms of the order D'^{-4}, we get the expression of R, depending on the sun's action, as in [6042]. If we substitute the elliptical value of r [6200a], and the similar one of D [6275], it will not produce, in R, any sensible term
[6201b] depending on angles of the form $nt+\varepsilon$, similar to that in [6200']; as is evident by mere inspection and reference to the values of s, s', &c. [6300, &c.], also to the calculation in [6271, &c.].

† (3269) If we retain, in the development of R, only the terms depending on nt,
[6202a] similar to those in [6200'], we may put it under the form $R=\Sigma.G.\cos.(nt+\varepsilon-\varpi_{,})$. Hence $\mathrm{d}R=-\Sigma.nG.\sin.(nt+\varepsilon-\varpi_{,})$ and $\int\mathrm{d}R=\Sigma.G.\cos.(nt+\varepsilon-\varpi_{,})=R$. We have
[6202b] also, as in [6089c], $r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right)$; hence we deduce,

[6202c] $$2\int\mathrm{d}R+r.\left(\frac{dR}{dr}\right)=2R+a.\left(\frac{dR}{da}\right).$$

The partial differential of R [6199], relative to a, being multiplied by $\frac{a}{da}$, gives, by using the symbol T [6089c], and retaining terms of the same order as those in [6199],

[6202d] $$a.\left(\frac{dR}{da}\right)=m'.\frac{r'\delta r'}{a'}.a.\left(\frac{ddA^{(1)}}{dada'}\right).\cos.T-\frac{2m'.d.(r'\delta r')}{a'^2.n'dt}.a.\left(\frac{dA^{(1)}}{da}\right).\sin.T.$$

Substituting this and R [6199], in the second member of [6202c], we get [6203].

‡ (3270) Dividing the equation [6057] by a^2, we find that its first and second terms,
[6204a] which contain $r\delta r$, must be the same as in [6114], or in [6204]. The third and fourth terms of [6057] are the same as in the first member of [6203]; and if we divide, as above,
[6204b] the second member of [6203] by a^2, which is equivalent to multiplying it by n^2a [6110], we obtain the terms of [6204], contained under the sign Σ.

$$0=\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2}+\Sigma.m'.n^2.\left\{\begin{array}{l}\frac{(r'\delta r')}{a'^2}.\left\{2aa'.\left(\frac{dA^{(1)}}{da'}\right)+a^2a'.\left(\frac{ddA^{(1)}}{dada'}\right)\right\}.\cos.(n't-nt+\varepsilon'-\varepsilon)\\ -\frac{d.(r'\delta r')}{a'^2.n'dt}.\left\{4aA^{(1)}+2a^2.\left(\frac{dA^{(1)}}{da}\right)\right\}.\sin.(n't-nt+\varepsilon'-\varepsilon)\end{array}\right\} \qquad [6204]$$

The most simple method for integrating this equation, is to suppose*

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* (3271) The equation [6204] contains the four quantities $r\delta r$, $r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$,
and the second differential of $r\delta r$; all of them being under a *linear* form. If we change [6205a]
reciprocally the elements of the satellite m, into those of m', we shall get a *second*
equation, similar to the preceding, containing the same four variable quantities, with the [6205b]
second differential of $r'\delta r'$. In like manner we have a *third* linear equation, containing the
second differential of $r''\delta r''$; and a *fourth* linear equation, containing the second differential
of $r'''\delta r'''$. The integrals of these *four linear equations* can be found, by the usual methods [6205c]
of proceeding with such equations, as in the similar example [1096—1102], where we have
integrated the linear equations of the second order [1089]. This process consists, in the [6205d]
present case, in finding *four particular integrals*, or values, of each one of the
unknown quantities $r\delta r$, $r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$, which satisfy the *four* differential equations
abovementioned. Then multiply these particular values of any one of these quantities,
as, for example, those of $r\delta r$, by arbitrary constant quantities, and take the sum of the
products for its general value, as in [6229]. In like manner we get the expressions of [6205e]
$r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$, [6230—6232], and we shall soon see that these may be considered as
the general values of these quantities, because the number of arbitrary constant quantities is
exactly what is required for the complete integrals, as in [6233']. For a very slight
attention will make it evident, that we can assume, as particular values of $r\delta r$, $r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$, [6205f]
any quantities which are similar to those in [6200b, e]; supposing *the longitudes of the
perijoves* ϖ, ϖ', ϖ'', ϖ''', *counted from the fixed axis to be represented by* $gt+\Gamma$, [6205g]
$g_1t+\Gamma_1$, $g_2t+\Gamma_2$, $g_3t+\Gamma_3$, *respectively, changing also the excentricities* e, e', e'', e''' [6205h]
into $-h$, $-h'$, $-h''$, $-h'''$ *respectively*; and by this means they will become as in
[6205]. Now if, for brevity, we put $W=nt+\varepsilon-gt-\Gamma$, we shall have, from the [6205i]
first of the equations [6205],

$$\frac{(r\delta r)}{a^2}=h.\cos.W;\quad\text{hence,}\quad\frac{d^2.(r\delta r)}{a^2.dt^2}=-h(n-g)^2.\cos.W; \qquad [6205k]$$

so that if we neglect terms of the order g^2, we shall get,

$$\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2}=h.(N^2+2ng-n^2).\cos.W. \qquad [6205l]$$

Substituting the value of $\frac{r'\delta r'}{a'^2}$ [6205], in the *first* members of [6205o, p], and reducing the
products by means of [17, 20] Int., retaining only the terms depending on the angle W, [6205m]
we shall get the *second* members of these equations. If we neglect the very small term $\frac{g}{n'}$, [6205n]

[6205]

$$\frac{r\delta r}{a^2} = h.\cos.(nt+\varepsilon-gt-\Gamma); \qquad \frac{r'\delta r'}{a'^2} = h'.\cos.(n't+\varepsilon'-gt-\Gamma); \qquad 1$$

$$\frac{r''\delta r''}{a''^2} = h''.\cos.(n''t+\varepsilon''-gt-\Gamma); \qquad \frac{r'''\delta r'''}{a'''^2} = h'''.\cos.(n'''t+\varepsilon'''-gt-\Gamma); \qquad 2$$

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in the last of these equations, which produces only a term of the order of the square of the

[6205n'] disturbing force in [6204], we shall have $\frac{n'-g}{n'} = 1$, and the expression [6205p] will become, by successive reductions, as in [6205r],

[6205o] $$\frac{r'\delta r'}{a^2}.\cos.(n't-nt+\varepsilon'-\varepsilon) = \tfrac{1}{2}h'.\cos.(nt+\varepsilon-gt-\Gamma) = \tfrac{1}{2}h'.\cos.W,$$

[6205p] $$-\frac{d.(r'\delta r')}{a^2.n'dt}.\sin.(n't-nt+\varepsilon'-\varepsilon) = h'.\left(\frac{n'-g}{n'}\right).\sin.(n't+\varepsilon'-gt-\Gamma).\sin.(n't-nt+\varepsilon'-\varepsilon)$$

[6205q] $$= \tfrac{1}{2}h'.\left(\frac{n'-g}{n'}\right).\cos.(nt+\varepsilon-gt-\Gamma)$$

[6205r] $$= \tfrac{1}{2}h'.\cos.W.$$

Substituting the values [6205l, o, r], and the similar terms depending on the satellites m'', m''',

[6205s] in [6204], and then dividing by $\cos.W$, we get the equation [6208]; which is afterwards reduced to the form [6217]. Hence we see that the values of g, h, h', h'', h''', corresponding to a particular value of $r\delta r$, must satisfy the equation [6217]. In like

[6205s'] manner the differential equations in $r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$ [6205b, &c.] give the three equations [6220—6222]. From these four equations we get, by the usual rules of elimination, four values of g; which are represented, in [6225—6228'], by g, g_1, g_2, g_3, with

[6205t] corresponding coefficients h, h', &c., so as to form the complete integrals of the unknown quantities $r\delta r$, $r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$ [6229—6232].

We may also remark that the whole of this calculation is similar to that in [1102, &c.], relative to the planets; and this ought evidently to be the case; since the action of the

[6205u] satellites upon each other, in their revolutions about Jupiter, must produce variations, in the excentricities and inclinations of the orbits; also in the motions of the perijoves and nodes; similar to those which are produced in the solar system, by the action of the planets

[6205v] upon each other; and we may deduce the expressions [6205], from those in [1102, 1102a]. For if we use the sign Σ of finite integrals, in the values of h, l [1102, 1102a], we shall have, by putting these values equal to those in [1022],

[6205w] $$e.\sin.\varpi = \Sigma.N.\sin.(gt+\beta); \qquad e.\cos.\varpi = \Sigma.N.\cos.(gt+\beta).$$

Multiplying the first of these equations by $-\sin.(nt+\varepsilon)$, the second by $-\cos.(nt+\varepsilon)$; adding the products, and reducing by means of [24] Int., we get,

[6205x] $$-e.\cos.(nt+\varepsilon-\varpi) = -\Sigma.N.\cos.(nt+\varepsilon-gt-\beta).$$

The first member of this equation is the same as the value of $\frac{r\delta r}{a^2}$ [6200b], and the second member is of the same form, as the expression of $\frac{r\delta r}{a^2}$ [6205], changing h into $-N$,

[6205y] also Γ into β, and prefixing the sign Σ. Hence we see, that the forms adopted in [6205] are easily deduced from those which are used in [1102, &c.].

*g being a very small coefficient, of the order of the disturbing forces upon which it depends.** Substituting these values in the preceding differential equation, and retaining only the terms depending on $\cos.(nt+\varepsilon-gt-\Gamma)$, we find that the comparison of these terms will give, by neglecting the square of g, † [6206] [6207]

$$0=h.\{N^2+2ng-n^2\}+\Sigma.\tfrac{1}{2}m'.n^2.h'.\left\{\begin{array}{l}2aa'.\left(\frac{dA^{(1)}}{da'}\right)+a^2a'.\left(\frac{ddA^{(1)}}{dada'}\right)\\ +4aA^{(1)}+2a^2.\left(\frac{dA^{(1)}}{da}\right)\end{array}\right\}. \qquad [6208]$$

Substituting for N^2 its value [6124], we obtain,

$$\begin{array}{ll}0=h.\left\{g-\frac{(\rho-\frac{1}{2}\phi)}{a^2}.n-\frac{3}{4}.\frac{M^2}{n}+\frac{1}{2}.\Sigma.m'n.\left[a^2.\left(\frac{dA^{(0)}}{da}\right)+\frac{1}{2}a^3.\left(\frac{ddA^{(0)}}{da^2}\right)\right]\right\} & 1\\ +\frac{1}{2}.\Sigma.m'n.h'.\left\{2aA^{(1)}+a^2.\left(\frac{dA^{(1)}}{da}\right)+aa'.\left(\frac{dA^{(1)}}{da'}\right)+\frac{1}{2}a^2a'.\left(\frac{ddA^{(1)}}{dada'}\right)\right\}. & 2\end{array} \qquad [6209]$$

$A^{(1)}$ being a homogeneous function in a, a', of the dimension -1, [6170], we have, by the nature of these functions [1002], [6210]

$$a.\left(\frac{dA^{(1)}}{da}\right)+a'.\left(\frac{dA^{(1)}}{da'}\right)=-A^{(1)}; \qquad [6211]$$

hence the preceding equation [6209] becomes, ‡

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* (3272) The values of g depend on the disturbing forces, as is evident from the equations [6217, 6220—6222]; their numerical values are given in [6025o]. The corresponding value of gt, during a revolution of the first satellite $1^{\text{days}},769$ [6778], is less than $30'$; so that though this greatest value of g [6025o], appears rather large, when taken for a year, its effect in the motion of the perijove, during one revolution of the satellite, is comparatively small; as in the lunar theory. The relative values of g, n, are seen also by the inspection of the annual movements of the angles gt, nt, or values of g, n [6025o, k]; these quantities being nearly represented by 60° and 82582°, making the fraction $\frac{g}{n}$ less than $\frac{1}{1300}$. [6206a] [6206b] [6206c] [6206d]

† (3273) The equation [6208] is found as in [6205s]. Substituting in this the value of N^2 [6124], then dividing by $2n$, and altering a little the arrangements of the terms, we get [6209]. We may observe that the equations [6209, 6212], in the original work, are divided by n, which makes them of a different form from that in [6217]; we have therefore omitted this divisor. [6209a]

‡ (3274) The equation [6211] is the same as in [1002]; multiplying it by a, and transposing, we get [6212a]. Multiplying the second of the equations [1003] by $\frac{1}{2}a^2$,

[6212]
$$0=h.\left\{g-\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.n-\tfrac{3}{4}.\frac{M^2}{n}+\tfrac{1}{2}.\Sigma.m'n.\left[a^2.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}.a^3.\left(\frac{ddA^{(0)}}{da^2}\right)\right]\right\} \quad 1$$
$$+\tfrac{1}{2}.\Sigma.m'n.h'.\left\{aA^{(1)}-a^2.\left(\frac{dA^{(1)}}{da}\right)-\tfrac{1}{2}.a^3.\left(\frac{ddA^{(1)}}{da^2}\right)\right\}. \quad 2$$

Now substituting as in [1073, 1076, 1082],

[6213] $$(0,1)=-\tfrac{1}{2}.m'n.\left\{a^2.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}.a^3.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}=-\frac{3m'.n.\alpha^2.b^{(1)}_{-\frac{3}{2}}}{4(1-\alpha^2)^2};$$

[6214] $$\boxed{0,1}=\tfrac{1}{2}.m'n.\left\{aA^{(1)}-a^2.\left(\frac{dA^{(1)}}{da}\right)-\tfrac{1}{2}.a^3.\left(\frac{ddA^{(1)}}{da^2}\right)\right\}=-\frac{3m'.n.\alpha.\{(1+\alpha^2).b^{(1)}_{-\frac{3}{2}}+\frac{1}{2}\alpha b^{(0)}_{-\frac{3}{2}}\}}{2(1-\alpha^2)^2};$$

[6215] also putting $(0,2)$, $\boxed{0,2}$; $(0,3)$, $\boxed{0,3}$; for what $(0,1)$, $\boxed{0,1}$ become respectively, when we change successively what relates to m' into the corresponding quantities relative to m'' or m'''; and lastly putting,

[6216] $$(0)=\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.n; \qquad \boxed{0}=\tfrac{3}{4}.\frac{M^2}{n};$$

we shall obtain from [6212] the following equation; *

[6217] $$0=h.\left\{g-(0)-\boxed{0}-(0,1)-(0,2)-(0,3)\right\}+\boxed{0,1}.h'+\boxed{0,2}.h''+\boxed{0,3}.h'''. \quad (i)$$

In like manner, if we consider the perturbations of the motions of m', m'' or
[6217′] m''', it is evident that we shall obtain another equation, similar to [6217],

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we get [6112*b*]. Substituting these expressions in [6209, line 2], and reducing, we get [6212],

[6212*a*] $$aa'.\left(\frac{dA^{(1)}}{da'}\right)=-aA^{(1)}-a^2.\left(\frac{dA^{(1)}}{da}\right)$$

[6212*b*] $$\tfrac{1}{2}a^2a'.\left(\frac{ddA^{(1)}}{dada'}\right)= \quad -a^2.\left(\frac{dA^{(1)}}{da}\right)-\tfrac{1}{2}a^3.\left(\frac{ddA^{(1)}}{da^2}\right).$$

* (3275) Substituting in [6212] the values [6213, 6214], we get,

[6217*a*] $$0=h.\left\{g-\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.n-\tfrac{3}{4}.\frac{M^2}{n}-\Sigma.(0,1)\right\}+\Sigma.\boxed{0,1}.h';$$

and by using the values (0), $\boxed{0}$, [6216], it becomes as in [6217]. For convenience of future reference, we shall insert, in the following table, the expressions of the functions
[6217*b*] (0), (1), &c., and $\boxed{0}$, $\boxed{1}$, &c.

[6217*c*] $$(0)=\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.n; \qquad (1)=\frac{(\rho-\frac{1}{2}\varphi)}{a'^2}.n'; \qquad (2)=\frac{(\rho-\frac{1}{2}\varphi)}{a''^2}.n''; \qquad (3)=\frac{(\rho-\frac{1}{2}\varphi)}{a'''^2}.n'''.$$

[6217*d*] $$\boxed{0}=\tfrac{3}{4}.\frac{M^2}{n}; \qquad \boxed{1}=\tfrac{3}{4}.\frac{M^2}{n'}; \qquad \boxed{2}=\tfrac{3}{4}.\frac{M^2}{n''}; \qquad \boxed{3}=\tfrac{3}{4}.\frac{M^2}{n'''}.$$

and which may be deduced from it, by changing successively the quantities relative to m, into those relative to m', m'', or m''', respectively, and the [6217''] contrary. Therefore, by inserting in the functions (0), $\boxed{0}$, $(0,1)$ $\boxed{0,1}$, [6218] &c., for 0, the number of the disturbed satellite, and the quantities corresponding to it; and for 1, the number of the disturbing satellite, and [6219] the quantities corresponding to it; we shall obtain the following equations;

$$0=h'.\left\{g-(1)-\boxed{1}-(1,0)-(1,2)-(1,3)\right\}+\boxed{1,0}.h+\boxed{1,2}.h''+\boxed{1,3}.h''';\qquad (i') \qquad [6220]$$

$$0=h''.\left\{g-(2)-\boxed{2}-(2,0)-(2,1)-(2,3)\right\}+\boxed{2,0}.h+\boxed{2,1}.h'+\boxed{2,3}.h''';\qquad (i'') \qquad [6221]$$

$$0=h'''.\left\{g-(3)-\boxed{3}-(3,0)-(3,1)-(3,2)\right\}+\boxed{3,0}.h+\boxed{3,1}.h'+\boxed{3,2}.h''.\qquad (i''') \qquad [6222]$$

We may observe that we have, as in [1093, 1094],

$$(0,1).m.\sqrt{a}=(1,0).m'.\sqrt{a'}; \qquad [6223]$$

$$\boxed{0,1}.m.\sqrt{a}=\boxed{1,0}.m'.\sqrt{a'}. \qquad [6224]$$

These equations hold good relative to the similar quantities corresponding to any two satellites whatever; hence we have a very simple method of [6224'] deriving these functions from each other.

The four equations [6217, 6220, 6221, 6222] between, h, h', h'', h''', are similar to the equations [1097], and may be solved in the same manner as [6224''] in [1100''—1102, 1102a]. *They give a final equation in* g *of the fourth degree** [1100^{v}]. We shall represent its four roots by g, g_1, g_2, g_3, [6225]

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* (3276) In order to have, at one view, the different values of β, β', β'', β''' and h, h', h'', h''', corresponding to the four different values of g, or to the four different [6229a] satellites; according to the notation adopted in [6225—6228'], we have collected these symbols together in the following table. The first column contains the values of h, for [6229b] the first satellite, and for each of the roots g, g_1, g_2, g_3. In like manner, the second column contains the values of h', for the second satellite; the third column, the values of [6229c] h'', for the third satellite; and the fourth column, the values of h''', for the fourth satellite.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	1	
Satellite m.	Satellite m'.	Satellite m''.	Satellite m'''.	Value of g.	2	
$h=h$;	$h'=\beta'.h$	$h''=\beta''.h$	$h'''=\beta'''.h$	g	3	
$h=h_1$;	$h'=\beta_1'.h_1$	$h''=\beta_1''.h_1$	$h'''=\beta_1'''.h_1$	g_1	4	[6229d]
$h=h_2$;	$h'=\beta_2'.h_2$	$h''=\beta_2''.h_2$	$h'''=\beta_2'''.h_2$	g_2	5	
$h=h_3$;	$h'=\beta_3'.h_3$	$h''=\beta_3''.h_3$	$h'''=\beta_3'''.h_3$	g_3	6	

The values of h', h'', h''', in the third horizontal line of this table, are the same as those

[6025o]; then for the root g, we shall suppose the values of h, h', h'', h''' to be represented by

[6226] $$h' = \beta'.h\,; \qquad h'' = \beta''.h\,; \qquad h''' = \beta'''.h\,;$$

and the preceding equations [6217, 6220, 6221, 6222] will give $\beta', \beta'', \beta'''$,
[6227] in functions of g; and h will be the corresponding arbitrary constant
[6227′] quantity. In like manner we shall suppose $\beta_1', \beta_1'', \beta_1'''$, to represent the values of $\beta', \beta'', \beta'''$, corresponding to the root g_1, and h_1, the arbitrary constant quantity. With the root g_2, we shall suppose $\beta_2', \beta_2'', \beta_2'''$, to
[6228] represent the values of $\beta', \beta'', \beta'''$, and h_2 the corresponding arbitrary constant quantity. Lastly, with the root g_3, we shall suppose $\beta_3', \beta_3'', \beta_3'''$, to represent the values of $\beta', \beta'', \beta'''$, and h_3 the corresponding arbitrary
[6228′] constant quantity. Then we shall have, by the nature of linear differential equations,

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[6229e] in [6226]; and if we substitute them in the *four* equations [6217, 6220, 6221, 6222], and divide by the common factor h, we shall obtain *four* equations which are *linear* in
[6229f] $\beta', \beta'', \beta'''$. From three of these equations we may determine $\beta', \beta'', \beta'''$, in terms of g, and by substituting these values in the fourth equation, we obtain an equation of the fourth
[6229f′] degree in g, whose roots are marked as in col. 5, g, g_1, g_2, g_3. Two examples of this process of calculation have already been given, in [1097b—1097c]. Each of these
[6229g] roots, g, g_1, g_2, g_3, may be combined with the corresponding quantities $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3$,
[6229h] so as to produce a *particular* value of $\frac{r\delta r}{a^2}$ [6205], and the sum of these four expressions will be the general value [6229], which satisfies the differential equation [6204]. In like manner, by substituting in the expression of $\frac{r'\delta r'}{a'^2}$ [6205], the values of h' [6229d, col. 2],
[6229i] namely $\beta'h, \beta_1'.h_1, \beta_2'.h_2, \beta_3'.h_3$, corresponding to the four roots of g, we shall get the four particular values, whose sum gives the general expression of $\frac{r'\delta r'}{a'^2}$ [6230]. In the
[6229k] same way we obtain the general values of $\frac{r''\delta r''}{a''^2}, \frac{r'''\delta r'''}{a'''^2}$ [6231, 6232]. Finally we may observe, that, as the differential equation [6204] is of the *second* order, its finite integral
[6229l] requires, by the usual rules of integration, *two* arbitrary constant quantities; and as there are *four* equations of this kind [6205a—c], they must require eight arbitrary constant quantities; being the same number as is found in the four integral expressions [6229–6232],
[6229m] namely, h, h_1, h_2, h_3; $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3$, as in [6233].

$$\frac{r\delta r}{a^2} = h.\cos.(nt+\varepsilon-gt-\Gamma)+h_1.\cos.(nt+\varepsilon-g_1t-\Gamma_1) \qquad 1$$
$$+h_2.\cos.(nt+\varepsilon-g_2t-\Gamma_2)+h_3.\cos.(nt+\varepsilon-g_3t-\Gamma_3)\,; \qquad 2 \qquad [6229]$$

$$\frac{r'\delta r'}{a'^2} = \beta'.h.\cos.(n't+\varepsilon'-gt-\Gamma)+\beta_1'.h_1.\cos.(n't+\varepsilon'-g_1t-\Gamma_1) \qquad 1$$
$$+\beta_2'.h_2.\cos.(n't+\varepsilon'-g_2t-\Gamma_2)+\beta_3'.h_3.\cos.(n't+\varepsilon'-g_3t-\Gamma_3)\,; \qquad 2 \qquad [6230]$$

$$\frac{r''\delta r''}{a''^2} = \beta''.h.\cos.(n''t+\varepsilon''-gt-\Gamma)+\beta_1''.h_1.\cos.(n''t+\varepsilon''-g_1t-\Gamma_1) \qquad 1$$
$$+\beta_2''.h_2.\cos.(n''t+\varepsilon''-g_2t-\Gamma_2)+\beta''_3.h_3.\cos.(n''t+\varepsilon''-g_3t-\Gamma_3)\,; \qquad 2 \qquad [6231]$$

$$\frac{r'''\delta r'''}{a'''^2} = \beta'''.h.\cos.(n'''t+\varepsilon'''-gt-\Gamma)+\beta_1'''.h_1.\cos.(n'''t+\varepsilon'''-g_1t-\Gamma_1) \qquad 1$$
$$+\beta_2'''.h_2.\cos.(n'''t+\varepsilon'''-g_2t-\Gamma_2)+\beta_3'''.h_3.\cos.(n'''t+\varepsilon'''-g_3t-\Gamma_3)\,; \qquad 2 \qquad [6232]$$

Γ, Γ_1, Γ_2, Γ_3 being four arbitrary constant quantities. *These expressions are complete, since they contain eight arbitrary constant quantities* [6229m]; *that is to say, twice as many arbitrary quantities as there are differential equations* [6204, 6205c, &c.] of the second order in $r\delta r$, $r'\delta r'$, $r''\delta r''$, $r'''\delta r'''$. [6233] [6233']

These arbitrary quantities take the place of the elliptical elements of the satellites. For if we consider these orbits as ellipses, in which the excentricities and positions of the apsides are variable, and put, as in [6061—6062], [6233'']

ae = the excentricity of the orbit of the first satellite, [6234]

ϖ = the longitude of its perijove, counted from the fixed axis, taken for the origin of the angles, [6234']

we shall have, as in [6200b],

$$\frac{r\delta r}{a^2} = -e.\cos.(nt+\varepsilon-\varpi). \qquad [6235]$$

Comparing this with the equation [6229], we get,*

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* (3277) From [24] Int. we easily obtain,

$$\cos.(nt+\varepsilon-\varpi) = \cos.\varpi.\cos.(nt+\varepsilon)+\sin.\varpi.\sin.(nt+\varepsilon)\,; \qquad [6235a]$$

$$\cos.(nt+\varepsilon-gt-\Gamma) = \cos.(gt+\Gamma).\cos.(nt+\varepsilon)+\sin.(gt+\Gamma).\sin.(nt+\varepsilon). \qquad [6235b]$$

Substituting [6235a] in the second member of [6235], also [6235b], and the similar expressions relative to the roots g_1, g_2, g_3, in [6229]; and then putting these two expressions of $\frac{r\delta r}{a^2}$ equal to each other, we shall obtain an equation which must hold good for all values of $nt+\varepsilon$. Putting therefore the terms in each member, containing $\cos.(nt+\varepsilon)$, [6235c] [6235c']

[6236] $e.\cos.\varpi = -h.\cos.(gt+\Gamma) - h_1.\cos.(g_1t+\Gamma_1) - h_2.\cos.(g_2t+\Gamma_2) - h_3.\cos.(g_3t+\Gamma_3)$;

[6237] $e.\sin.\varpi = -h.\sin.(gt+\Gamma) - h_1.\sin.(g_1t+\Gamma_1) - h_2.\sin.(g_2t+\Gamma_2) - h_3.\sin.(g_3t+\Gamma_3)$;

hence we easily obtain e and ϖ. We shall have by the same process

[6238] e', ϖ', &c. The method in [1275, &c., 6235n], will give the same values; but the preceding analysis is rather more simple.

[6239] The elliptical part of v is $2e.\sin.(nt+\varepsilon-\varpi)$; and if we put this equal to δv, we shall have,*

[6240] $$\delta v = 2e.\sin.(nt+\varepsilon-\varpi) = 2e.\cos.\varpi.\sin.(nt+\varepsilon) - 2e.\sin.\varpi.\cos.(nt+\varepsilon);$$

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separately equal to nothing, we get [6236]; and the terms depending on $\sin.(nt+\varepsilon)$ give [6237].

[6235d] If we suppose the symbol $\Sigma_{,}$ *to include the four terms depending on the four roots* g, g_1, g_2, g_3, we may put the expressions [6236, 6237], under the abridged forms

[6235e] [6235i], which are convenient for reference. In like manner we obtain the expressions [6235k, l, m] corresponding to the satellites m', m'', m''', respectively. We may incidentally observe, that the expressions [6235i] may be derived from the second member of [6229],

[6235f] by putting successively $nt+\varepsilon=200^\circ$, $nt+\varepsilon=300^\circ$, and making a slight reduction, as is evident by inspection. In like manner, if we put successively $n't+\varepsilon'=200^\circ$,

[6235g] $n't+\varepsilon'=300^\circ$, and $\beta'h=h'$ [6226], in [6230], we get [6235k]; also $n''t+\varepsilon''=200^\circ$, $n''t+\varepsilon''=300^\circ$, being substituted with $\beta''h=h''$ [6226], in [6231], give [6235l];

[6235h] lastly, $n'''t+\varepsilon'''=200^\circ$ and $n'''t+\varepsilon'''=300^\circ$, being substituted with $\beta'''h=h'''$ [6226], in [6232], give [6235m].

	Column 1.	Column 2.
[6235i]	$e.\cos.\varpi = -\Sigma_{,}.h.\cos.(gt+\Gamma)$;	$e.\sin.\varpi = -\Sigma_{,}.h.\sin.(gt+\Gamma)$;
[6235k]	$e'.\cos.\varpi' = -\Sigma_{,}.h'.\cos.(gt+\Gamma)$;	$e'.\sin.\varpi' = -\Sigma_{,}.h'.\sin.(gt+\Gamma)$;
[6235l]	$e''.\cos.\varpi'' = -\Sigma_{,}.h''.\cos.(gt+\Gamma)$;	$e''.\sin.\varpi'' = -\Sigma_{,}.h''.\sin.(gt+\Gamma)$;
[6235m]	$e'''.\cos.\varpi''' = -\Sigma_{,}.h'''.\cos.(gt+\Gamma)$;	$e'''.\sin.\varpi''' = -\Sigma_{,}.h'''.\sin.(gt+\Gamma)$.

[6235n] The same results are obtained by the method in [1275, &c., or 1089, &c.], as is shown in [6205u—y]. This agrees with [6238].

* (3278) Neglecting terms of the order e^2, as in [6196], we obtain, from [669 line 2], the following expression of v;

[6240a] $$v = nt+\varepsilon+2e.\sin.(nt+\varepsilon-\varpi);$$

[6240b] the elliptical part, or that which depends upon e, being $\delta v = 2e.\sin.(nt+\varepsilon-\varpi)$. Developing this, by [22] Int., we get [6240]. Multiplying the expression of $e.\cos.\varpi$

[6240c] [6235i] by $2\sin.(nt+\varepsilon)$; also that of $e.\sin.\varpi$ [6235i], by $-2\cos.(nt+\varepsilon)$; then taking the sum of these products, we get the expression of δv [6240], as in [6240d]. This is

hence we get,

$$\delta v = -2h.\sin.(nt+\varepsilon-gt-\Gamma)-2h_1.\sin.(nt+\varepsilon-g_1t-\Gamma_1) \qquad 1$$
$$-2h_2.\sin.(nt+\varepsilon-g_2t-\Gamma_2)-2h_3.\sin.(nt+\varepsilon-g_3t-\Gamma_3). \qquad 2 \qquad [6241]$$

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easily reduced to the form [6240e], by means of [22] Int. The last expression of δv is the same as that in [6241],

$$\delta v = 2e.\cos.\varpi.\sin.(nt+\varepsilon)-2e.\sin.\varpi.\cos.(nt+\varepsilon) = -\Sigma_{,}.2h.\{\sin.(nt+\varepsilon).\cos.(gt+\Gamma)-\cos.(nt+\varepsilon).\sin.(gt+\Gamma)\} \quad [6240d]$$
$$\delta v = 2e.\sin.(nt+\varepsilon-\varpi) = -\Sigma_{,}.2h.\sin.(nt+\varepsilon-gt-\Gamma). \quad [6240e]$$

In like manner we may obtain the values of $\delta v'$, $\delta v''$, $\delta v'''$. We may also deduce them from [6241], by changing successively the elements relative to m, into those which correspond to m', m'', m''' respectively, and the contrary, as in the table [6229d]; by this means we obtain the following expressions, which agree with those in [6242, 6243, 6244]; [6240f]

$$\delta v' = 2e'.\sin.(n't+\varepsilon'-\varpi') = -\Sigma_{,}.2h'.\sin.(n't+\varepsilon'-gt-\Gamma); \quad [6240g]$$
$$\delta v'' = 2e''.\sin.(n''t+\varepsilon''-\varpi'') = -\Sigma_{,}.2h''.\sin.(n''t+\varepsilon''-gt-\Gamma); \quad [6240h]$$
$$\delta v''' = 2e'''.\sin.(n'''t+\varepsilon'''-\varpi''') = -\Sigma_{,}.2h'''.\sin.(n'''t+\varepsilon'''-gt-\Gamma). \quad [6240i]$$

We may observe that any one of the angles which occur in [6229—6232, 6241—6244], as, for example, $nt+\varepsilon-gt-\Gamma$ *is the difference of two angles* $nt+\varepsilon$ and $gt+\Gamma$, counted from the *same* point; it must therefore be a matter of indifference whether this point be *fixed*, as is supposed in [6022f], or *variable*, as in [6023x]. In other words, these angles may both be counted from the *fixed* equinox, or both from the *variable* equinox, at pleasure. Now if we suppose that the mean longitudes of the satellites, counted from the fixed equinox, $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$, $n'''t+\varepsilon'''$ [6022f], are changed respectively into Θ, Θ', Θ'', Θ''', from the variable equinox [6023x]; and the longitudes of the perijoves $gt+\Gamma$, $g_1t+\Gamma_1$, $g_2t+\Gamma_2$, $g_3t+\Gamma_3$, from the fixed equinox [6021s], into ϖ, ϖ', ϖ'', ϖ''', from the variable equinox [6024g]; we shall obtain the differences of these angles, as in [6240p—s]. Changing in like manner the mean longitude of the sun, in its relative orbit about Jupiter $Mt+E$ [6021z] from the fixed equinox, into $\Pi+200^{\circ}$ from the variable equinox [6024f], we get [6240s']; observing that the whole circumference, 400°, may be neglected. [6240k] [6240l] [6240m] [6240n] [6240o]

$$n't-nt+\varepsilon'-\varepsilon=\Theta'-\Theta; \quad n''t-nt+\varepsilon''-\varepsilon=\Theta''-\Theta; \quad n'''t-nt+\varepsilon'''-\varepsilon=\Theta'''-\Theta; \quad [6240p]$$
$$n''t-n't+\varepsilon''-\varepsilon'=\Theta''-\Theta'; \quad n'''t-n't+\varepsilon'''-\varepsilon'=\Theta'''-\Theta'; \quad n'''t-n''t+\varepsilon'''-\varepsilon''=\Theta'''-\Theta''; \quad [6240q]$$
$$nt+\varepsilon-gt-\Gamma=\Theta-\varpi; \quad nt+\varepsilon-g_1t-\Gamma_1=\Theta-\varpi'; \quad [6240r]$$
$$nt+\varepsilon-g_2t-\Gamma_2=\Theta-\varpi''; \quad nt+\varepsilon-g_3t-\Gamma_3=\Theta-\varpi'''; \quad [6240s]$$
$$2nt-2Mt+2\varepsilon-2E=2\Theta-2\Pi; \quad nt-2Mt+\varepsilon-2E+gt+\Gamma=\Theta+\varpi-2\Pi \quad [6240s']$$

We may also change n, ε, Θ successively into n', ε', Θ'; n'', ε'', Θ''; n''', ε''', Θ''' respectively.

Substituting the values [6240r, s] in [6240e, g, h, i], and using the symbols [6229d], we obtain the following expressions of the equations of the centre of the four satellites, δv, $\delta v'$, $\delta v''$, $\delta v'''$;

In like manner we obtain,

[6242] $\delta v' = -2\beta'.h.\sin.(n't+\varepsilon'-gt-\Gamma)-2\beta_1'.h_1.\sin.(n't+\varepsilon'-g_1t-\Gamma_1)$ 1

$-2\beta'_2.h_2.\sin.(n't+\varepsilon'-g_2t-\Gamma_2)-2\beta'_3.h_3.\sin.(n't+\varepsilon'-g_3t-\Gamma_3)$; 2

[6243] $\delta v'' = -2\beta''.h.\sin.(n''t+\varepsilon''-gt-\Gamma)-2\beta_1''.h_1.\sin.(n''t+\varepsilon''-g_1t-\Gamma_1)$ 1

$-2\beta_2''.h_2.\sin.(n''t+\varepsilon''-g_2t-\Gamma_2)-2\beta_3''.h_3.\sin.(n''t+\varepsilon''-g_3t-\Gamma_3)$; 2

[6244] $\delta v''' = -2\beta'''.h.\sin.(n'''t+\varepsilon'''-gt-\Gamma)-2\beta_1'''.h_1.\sin.(n'''t+\varepsilon'''-g_1t-\Gamma_1)$ 1

$-2\beta_2'''.h_2.\sin.(n'''t+\varepsilon'''-g_2t-\Gamma_2)-2\beta_3'''.h_3.\sin.(n'''t+\varepsilon'''-g_3t-\Gamma_3)$. 2

		Col. 1.	Col. 2.	Col. 3.	Col. 4.
[6240t]	$\delta v =$	$-2h.\sin.(\Theta-\varpi)$	$-2h_1.\sin.(\Theta-\varpi')$	$-2h_2.\sin.(\Theta-\varpi'')$	$-2h_3.\sin.(\Theta-\varpi''')$;
[6240u]	$\delta v' =$	$-2\beta'.h.\sin(\Theta'-\varpi)$	$-2\beta'_1.h_1.\sin.(\Theta'-\varpi')$	$-2\beta'_2.h_2.\sin(\Theta'-\varpi'')$	$-2\beta'_3.h_3.\sin.(\Theta'-\varpi''')$;
[6240v]	$\delta v'' =$	$-2\beta''.h.\sin.(\Theta''-\varpi)$	$-2\beta''_1.h_1.\sin.(\Theta''-\varpi')$	$-2\beta''_2.h_2.\sin.(\Theta''-\varpi'')$	$-2\beta''_3.h_3.\sin.(\Theta''-\varpi''')$;
[6240w]	$\delta v''' =$	$-2\beta'''.h.\sin.(\Theta'''-\varpi)$	$-2\beta'''_1.h_1.\sin.(\Theta'''-\varpi')$	$-2\beta'''_2.h_2.\sin.(\Theta'''-\varpi'')$	$-2\beta'''_3.h_3.\sin.(\Theta'''-\varpi''')$.

If we use the same values of h', h'', h''', as in [7177—7179], corresponding to the first root g [6229d line 3], we shall find that the coefficients, in the first column of the table
[6240x] [6240t—w], will become $-2h$, $-2h'$, $-2h''$, $-2h'''$, respectively; so that we shall have,

[6240y] $-2h = -2h$; $\quad -2h' = -2\beta'.h$; $\quad -2h'' = -2\beta''.h$; $\quad -2h''' = -2\beta'''.h$;

and by comparing these with the expressions [7177—7179], we obtain,

[6241a] $-2h = -2h$; $-2\beta'.h = -0{,}0185238.2h$; $-2\beta''.h = 0{,}0034337.2h$; $-2\beta'''.h = 0{,}00001735.2h$.

Again, by using the values of h, h'', h''', as in [7184—7186], corresponding to the second root g_1, [6229d line 4], we find that the coefficients in the second column of the
[6241b] table [6240t—w], will again be represented by $-2h$, $-2h'$, $-2h''$, $-2h'''$, as in [6240x], and by comparing these with the symbols in the second column of the table [6240t—w], we get,

[6241c] $-2h = -2h_1$; $\quad -2h' = -2\beta_1'.h_1$; $\quad -2h'' = -2\beta_1''.h_1$; $\quad -2h''' = -2\beta_1'''.h_1$.

Hence, by using the values [7184—7186], we shall obtain,

[6241d] $-2h_1 = 0{,}0375392\times 2h'$; $\quad -2\beta_1'.h_1 = -2h'$; $\quad -2\beta_1''.h_1 = 0{,}0436686.2h'$; $\quad -2\beta_1'''.h_1 = -0{,}00004357.2h'$.

In like manner, by comparing the results in [7191—7193], corresponding to the third root g_2, [6229d line 5], with the expressions in the third column of the table [6240t—w], we get,

[6241e] $-2h_2 = -0{,}0238111.2h''$; $\quad -2\beta_2'.h_2 = -0{,}\ 2152920.2h''$; $\quad -2\beta_2''.h_2 = -2h''$; $\quad -2\beta_2'''.h_2 = 0{,}1291564.2h''$.

Finally, comparing [7196—7198], corresponding to the fourth root g_3, [6229d line 6], with the fourth column of the table [6240t—w], we find,

[6241f] $-2h_3 = -0{,}0020622.2h'''$; $\quad -2\beta_3'.h_3 = -0{,}0173350.2h'''$; $\quad -2\beta_3''.h_3 = -0{,}0816578.2h'''$; $\quad -2\beta_3'''.h_3 = -2h'''$.

Substituting the expressions [6241a, d, e, f] in [6240t—w], we obtain,

The whole process is now reduced to the formation and reduction of the equations [6217—6222]; but we shall see, in [6744—6747], that they are [6244′]

$$\delta v = -1{,}0000000.2h.\sin.(\Theta-\varpi)+0{,}0375392.2h'.\sin.(\Theta-\varpi') -0{,}0238111.2h''.\sin.(\Theta-\varpi'')-0{,}0020622.2h'''.\sin.(\Theta-\varpi''');$$ [6241g]

$$\delta v' = -0{,}0185238.2h.\sin.(\Theta'-\varpi)-1{,}0000000.2h'.\sin.(\Theta'-\varpi') -0{,}2152920.2h''.\sin.(\Theta'-\varpi'')-0{,}0173350.2h'''.\sin.(\Theta'-\varpi''');$$ [6241h]

$$\delta v'' = 0{,}0034337.2h.\sin.(\Theta''-\varpi)+0{,}0436686.2h'.\sin.(\Theta''-\varpi') -1{,}0000000.2h''.\sin.(\Theta''-\varpi'')-0{,}0816578.2h'''.\sin.(\Theta''-\varpi''');$$ [6241i]

$$\delta v''' = 0{,}00001735.2h.\sin.(\Theta'''-\varpi)-0{,}00004357.2h'.\sin.(\Theta'''-\varpi') +0{,}1291564.2h''.\sin.(\Theta'''-\varpi'')-1{,}0000000.2h'''.\sin.(\Theta'''-\varpi''').$$ [6241k]

We shall see hereafter that $2h$, $2h'$, are insensible [7441′]; $2h''=-1709''.05$ [7499], [6241l] and $2h'''=-9265''.56$ [7502]; so that all the terms, except those depending on h'', h''', may be neglected. The coefficient of $2h$, neglecting the signs, is greater in the expression of δv [6241g], than in those of $\delta v'$, $\delta v''$, $\delta v'''$ [6241h, i, k]; therefore the coefficient [6241m] $-2h$, or $2e$ [6205h], with the corresponding perijove ϖ, is considered as the excentricity and perijove *peculiar to the first satellite.* For similar reasons $2h'$, $2e'$, ϖ', [6241n] are considered as *peculiar to the second satellite*; $2h''$, $2e''$, ϖ'', *peculiar to the third*; and $2h'''$, $2e'''$, ϖ''', *peculiar to the fourth.* This assignment of *peculiar* values to each [6241o] satellite, is very natural, and is analogous to that in the planetary system, as we may see in [6205x, &c.]; or more simply, by adding together the parts of $\frac{r}{a}$ depending on the first power of e, in [669, 1020 line 2], which give, when we have only two planets m, m',

$$\frac{r}{a} = 1-(1+m'f).e.\cos.(nt+\varepsilon-\varpi)-m'f'.e'.\cos.(nt+\varepsilon-\varpi').$$ [6241p]

Here we see that the elliptical part of $\frac{r}{a}$, independent of the action of m', is $-e.\cos.(nt+\varepsilon-\varpi)$; moreover, the coefficient $-e$ is changed into $-(1+m'f).e$, by the action of m; and this large coefficient, with its perihelion ϖ, is considered as [6241q] *peculiar* to the planet m; while the other part of the equation of the centre, $-m'f'.e'.\cos\,(nt+\varepsilon-\varpi')$ [6241p], has the coefficient $-m'f'.e'$, and refers to the perihelion [6241r] ϖ' of the planet m'.

If the values of h, h', h'', were much enlarged, so as, for example, to be nearly equal to h''', we should find, in this system of satellites, this singular result, that the equation [6241s] of the centre of each satellite would consist of four sensible parts, each part referring to a [6241t] different perijove. This occurs, in some respects, in the system as it is now constituted; for each of the satellites has two distinct and sensible equations of the centre, referring to the perijoves ϖ'', ϖ''', of the third and fourth satellites; as we shall find hereafter in [6241u] [7318 lines 1, 9; 7405 lines 1, 2; 7467 lines 1, 2; 7513 lines 1, 2].

[6244″] incomplete, and that the ratios which obtain, between the mean motions of the three first satellites, add to these equations some sensible terms, although they depend on the squares and products of the disturbing forces.

[6245] 7. The terms of the double integral $\frac{3a}{\mu}.\iint ndt.\mathrm{d}R$, in the expression of δv [6060], which depend on the angle $nt-2n't+\varepsilon-2\varepsilon'$, acquire, by integration, the divisor $(n-2n')^2$; and as n differs but very little from

[6245′] $2n'$ [6151a], this divisor is very small; therefore these terms may become very sensible, although they are multiplied by the small excentricities of the orbits; we shall now proceed to determine them.

[6246] We shall consider the term $m'.A^{(1)}.\cos.(v'-v)$, in the expression of R [6090]. Substituting in it the values [6198, &c.],

[6247] $$r'=a'+\frac{r'\delta r'}{a'}; \qquad v=n't+\varepsilon'+\frac{2d.(r'\delta r')}{a'^2.n'dt}; \qquad r=a; \qquad v=nt+\varepsilon;$$

we shall find, as in [6199], that this part of R contains the following terms;

[6248] $$R=m'.\frac{r'\delta r'}{a'^2}.a'.\left(\frac{dA^{(1)}}{da'}\right).\cos.(n't-nt+\varepsilon'-\varepsilon)-\frac{2m'.d(r'\delta r')}{a'^2.n'dt}.A^{(1)}.\sin.(n't-nt+\varepsilon'-\varepsilon).$$

[6249] $\frac{r'\delta r'}{a'^2}$ contains, as in [6205], the term $h'.\cos.(n't+\varepsilon'-gt-\Gamma)$, and if we substitute it for $\frac{r'\delta r'}{a'^2}$, in [6248], neglecting quantities of the order $m'g$, we shall obtain the following terms of R;*

[6250] $$R=\frac{m'.h'}{2a'}.\left\{a'^2.\left(\frac{dA^{(1)}}{da'}\right)-2a'.A^{(1)}\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

We have, as in [6145],

[6251] $$G=2a'.A^{(1)}-a'^2.\left(\frac{dA^{(1)}}{da'}\right);$$

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* (3279) If we substitute the term of $\frac{r'\delta r'}{a'^2}$ [6249], in the first members of [6250a,b], and reduce the products, by means of [20, 17] Int., retaining only the part depending on the angle $nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma$, we shall get,

[6250a] $$\frac{r'\delta r'}{a'^2}.\cos.(n't-nt+\varepsilon'-\varepsilon)=\tfrac{1}{2}h'.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma);$$

[6250b] $$\frac{d.(r'\delta r')}{a'^2.n'dt}.\sin.(n't-nt+\varepsilon'-\varepsilon)=\tfrac{1}{2}h'.\left(\frac{n'-g}{n'}\right).\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

[6250c] Substituting the expressions [6250a, b], in [6248]; then putting as in [6205n'], $\frac{n'-g}{n'}=1$, we get [6250]; and by using [6251], it becomes as in [6252].

hence the term [6250] becomes,

$$R = -\frac{m'.Gh'}{2a'} . \cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma). \qquad [6252]$$

We shall now consider the term $m'.A^{(2)}.\cos.(2v'-2v)$, in the expression of R [6090]. Substituting in it,* [6253]

$$r = a + \frac{r\delta r}{a}; \qquad v = nt+\varepsilon+\frac{2d.(r\delta r)}{a^2.ndt}; \qquad r' = a'; \qquad v' = n't+\varepsilon', \qquad [6253']$$

we shall find that this term contains the function,†

$$R = m'.\frac{r\delta r}{a^2}.a.\left(\frac{dA^{(2)}}{da}\right).\cos.(2n't-2nt+2\varepsilon'-2\varepsilon)+m'.\frac{4d.(r\delta r)}{a^2.ndt}.A^{(2)}.\sin.(2n't-2nt+2\varepsilon'-2\varepsilon). \qquad [6254]$$

Substituting in this function the expression $\frac{r\delta r}{a^2} = h.\cos.(nt+\varepsilon-gt-\Gamma)$ [6255]
[6205], we obtain the following terms of R;

$$R = \frac{m'.h}{2a}.\left\{4aA^{(2)}+a^2.\left(\frac{dA^{(2)}}{da}\right)\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma). \qquad [6256]$$

The value of F [6130], by putting as in [6151] $n=2n'$, becomes

$$F = -4aA^{(2)}-a^2.\left(\frac{dA^{(2)}}{da}\right); \qquad [6257]$$

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* (3280) The expressions, in [6253′], are similar to those in [6247], changing reciprocally the elements of the satellite m into those of m'. [6252a]

† (3281) The development of the term $R = m'.A^{(2)}.\cos.(2v'-2v)$ [6253], is made like that in [6198a, c], by the substitution of the values of r, v, &c. [6253′]; then developing it, according to the powers of $\frac{r\delta r}{a^2}$, retaining only the first power of this quantity; by this means we get the two terms of R [6254]. Now if we substitute the value of $\frac{r\delta r}{a^2}$ [6255], in the first members of [6253c, d], and reduce the products, by means of [20, 17] Int.; retaining only the terms depending on the angle $nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma$; [6253b′] we shall get, by neglecting terms of the order $\frac{g}{n}$, as in [6205n, &c.], [6253a] [6253b]

$$\frac{(r\delta r)}{a^2}.\cos.(2n't-2nt+2\varepsilon'-2\varepsilon) = \frac{h}{2}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma); \qquad [6253c]$$

$$\frac{4.d.(r\delta r)}{a^2.ndt}.\sin.(2n't-2nt+2\varepsilon'-2\varepsilon) = \frac{h}{2}.4.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma). \qquad [6253d]$$

Multiplying [6253c] by $m'.a.\left(\frac{dA^{(2)}}{da}\right)$, also [6253d] by $m'.A^{(2)}$, and taking the sum of the products, we obtain, for the expression of R [6254], the value [6256]; a slight reduction being made, by multiplying the numerator and denominator by a. [6253e]

hence the expression [6256] is reduced to the form,

[6258] $$R=-\frac{m'.Fh}{2a}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

Adding together the expressions in [6252, 6258], we obtain, in R, the following terms;

[6259] $$R=-\frac{m'}{2a}.\left\{Fh+\frac{a}{a'}.Gh'\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma);$$

and it is evident that the action of m' upon m, produces no other terms of this kind.

Now if we observe that μ [6110c] may be supposed equal to unity, in

[6260] the function $\frac{3a}{\mu}.\iint ndt.\mathrm{d}R$ [6245], which occurs in the expression of δv, we shall find, that the term [6259] produces, in δv, the following inequality;*

[6261] $$\delta v=\frac{-3m'.n^2}{2.(n-2n'+g)^2}.\left\{Fh+\frac{a}{a'}.Gh'\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

n' being very nearly equal to $2n''$ [6151], it is evident that the action of m'' upon m', produces, in $\delta v'$, an inequality analogous to the preceding, and represented by

[6262] First part of $\delta v'$. $$\delta v'=\frac{-3m''.n'^2}{2.(n'-2n''+g)^2}.\left\{F'h'+\frac{a'}{a''}.G'h''\right\}.\sin.(n't-2n''t+\varepsilon'-2\varepsilon''+gt+\Gamma).$$

The action of m upon m', produces also, in $\delta v'$, an inequality of the same kind, which may be easily found by means of [1208]; for we have, in that article, by noticing only the proposed terms,†

[6263] $$\delta v'=-\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.\delta v;$$

which gives, for this part of $\delta v'$, the following expression;

[6264] Second part of $\delta v'$. $$\delta v'=\frac{3m.n^2\sqrt{a}}{2.(n-2n'+g)^2.\sqrt{a'}}.\left\{Fh+\frac{a}{a'}.Gh'\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

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* (3282) Substituting R [6259], in $\delta v=3a\iint ndt.\mathrm{d}R$ [6260], we get [6261] nearly,

[6261a] by neglecting terms of the order $\frac{g}{n}$. Increasing the accents on the elements by unity, we get the expression of $\delta v'$ [6262], corresponding to the action of the satellite m'' upon m', and to the same angle $gt+\Gamma$.

† (3283) The equation [6263] is the same as [1208]; changing ζ into δv, and ζ'

[6263b] into $\delta v'$, in order to conform to the present notation. Substituting, in this, the value of δv [6261], we get [6264].

We may connect this term with that in [6262], by observing that we have, as in [6156],

$$n't-2n''t+\varepsilon'-2\varepsilon''=nt-2n't+\varepsilon-2\varepsilon'-200^{\circ};$$ [6265]

moreover, we have very nearly $n=2n'$, [6151]; also $\left(\frac{a}{a'}\right)^3=\left(\frac{n'}{n}\right)^2$ [6167]; [6266]

therefore we shall have, very nearly,*

$$\frac{3n^2.\sqrt{a}}{2\sqrt{a'}}=3n'^2.\frac{a'}{a}.$$ [6267]

Hence we get, by adding together the terms [6262, 6264],

$$\delta v'=\frac{3n'^2}{(n-2n'+g)^2}.\left\{m.\left(\frac{a'}{a}.Fh+Gh'\right)+\tfrac{1}{2}m''.\left(F'h'+\frac{a'}{a''}.G'h''\right)\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$ [6268]

Whole value of $\delta v'$.

Lastly, the action of m' upon m'' produces, in the motion of m'', an inequality analogous to that which the action of m upon m' produces in the motion of m'; therefore it will be represented by †

$$\delta v''=-\frac{3m'.n''^2}{(n'-2n''+g)^2}.\left\{\frac{a''}{a'}.F'h'+G'h''\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$ [6269]

The preceding inequalities correspond to the root g. It is evident that each of the other three roots, g_1, g_2, g_3, produce, in the motion of the [6270] three inner satellites, similar inequalities. *These are the only sensible terms which depend, at the same time, on the action of the satellites and on the excentricities of the orbits.*

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* (3284) The square root of the second equation [6143] gives $\frac{n}{n'}.\frac{\sqrt{a}}{\sqrt{a'}}=\frac{a'}{a}$. [6267a] Multiplying it by $\frac{3}{2}n\,n'=3n'^2$ nearly [6266], we get [6267]. Substituting this in [6264], we obtain the terms of $\delta v'$ [6268], which are multiplied by m, and arising from the action of m upon m'. Again, from [6154] we get $n'-2n''+g=n-2n'+g$; [6267b] substituting this and [6265], in [6262], we get the part of [6268], which is multiplied by m'', arising from the action of m'' upon m'.

† (3285) The part of $\delta v'$ [6268], which is multiplied by m, represents the effect of the action of the satellite m upon m'. Increasing the accents on these elements by unity, [6269a] we get the similar term of $\delta v''$ [6269b], arising from the action of m' upon m'', and corresponding to the same value of g;

$$\delta v''=\frac{3n''^2}{(n'-2n''+g)^2}.\left\{m'.\left(\frac{a''}{a'}.F'h'+G'h''\right)\right\}.\sin.(n't-2n''t+\varepsilon'-2\varepsilon''+gt+\Gamma).$$ [6269b]

Substituting the expression $n't-2n''t+\varepsilon'-2\varepsilon''=nt-2n't+\varepsilon-2\varepsilon-200^{\circ}$ [6265], and [6269c] reducing, we get [6269].

8. The sun's action may also produce, in the motion of the satellites, some sensible inequalities, although they depend on the excentricities of the orbits. The value of R relative to this action, contains the term

[6271] $R = -\frac{3S.r^2}{4D^3}.\cos.(2v-2U)$ [6042]. Substituting, in it,*

[6272] $r^2 = a^2.\{1+2h.\cos.(nt+\varepsilon-gt-\Gamma)\}$; $v = nt+\varepsilon-2h.\sin.(nt+\varepsilon-gt-\Gamma)$; $D = D'$ [6021*l*]; $\frac{S}{D'^3} = M^2$ [6105]; we shall obtain, in R, the following term;†

[6273] $$R = -\tfrac{9}{4}.M^2.a^2h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma).$$

[6274] The value of R, relative to the sun's action, contains also the term $-\frac{S.r^2}{4D^3}$ [6042]. Substituting

[6275] $$D = D'.\{1-H.\cos.(Mt+E-I)\};\ddagger$$

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* (3286) The values of r, v [6272] are like those in [6253′], using the expression of

[6272*a*] $\frac{r\delta r}{a^2}$ [6205], and neglecting in v, a term of the order $\frac{g}{n}$, as in [6253*b*′]. Moreover, by neglecting the excentricity of Jupiter's orbit, we may put the distance D, equal to the mean distance D' [6021*l*].

† (3287) If we put $\delta.(r^2)$ and δv, for the parts of r^2, v, respectively depending on

[6273*a*] h, we shall have, as in [6272], $r^2 = a^2+\delta.(r^2)$, $v = nt+\varepsilon+\delta v$. Substituting these in the term of R [6271], then developing it by the formula [610], retaining only the first power of δr, δv, and using M^2 [6272], we get, in R, the terms [6273*b*]. These are easily reduced to the form [6273*c*], by the substitution of the values of U, v [6102, 6247];

[6273*b*] $$R = -\tfrac{3}{4}M^2.\delta.(r^2).\cos.(2v-2U)+\tfrac{3}{4}M^2.r^2.2\delta v.\sin.(2v-2U)$$

[6273*c*] $$= -\tfrac{3}{4}M^2.\delta.(r^2).\cos.(2nt-2Mt+2\varepsilon-2E)+\tfrac{3}{4}M^2.a^2.2\delta v.\sin.(2nt-2Mt+2\varepsilon-2E).$$

Now by using the values of $\delta.(r^2)$, δv [6272, 6273*a*], and reducing by means of [17, 20]

[6273*d*] Int., retaining only the terms depending on the angle $nt-2Mt+\varepsilon-2E+gt+\Gamma$, which, for brevity, we shall call W'', we get,

[6273*e*] $$\delta.(r^2)\cos.(2nt-2Mt+2\varepsilon-2E) = 2a^2h.\cos.(nt+\varepsilon-gt-\Gamma).\cos.(2nt-2Mt+2\varepsilon-2E) = a^2h.\cos.W''$$

[6273*f*] $$2\delta v.\sin.(2nt-2Mt+2\varepsilon-2E) = -4h.\sin.(nt+\varepsilon-gt-\Gamma).\sin.(2nt-2Mt+2\varepsilon-2E) = -2h.\cos.W''.$$

Substituting the values [6273*e*, *f*] in [6273*c*], we get

[6273*g*] $$R = -\tfrac{3}{4}M^2.a^2h.\cos.W''-\tfrac{3}{2}M^2.a^2h.\cos.W'' = -\tfrac{9}{4}M^2.a^2h.\cos.W'',$$

as in [6273].

[6275*a*] ‡ (3288) This expression of D is easily deduced from that of r [6200*a*]; changing r, a, e, n, ε, ϖ, into D, D', H, M, E, I, to conform to the elements of the orbit of

H being the ratio of the excentricity, to the semi-major axis of Jupiter's orbit; [6275′]
and I, the longitude of its perihelion; we shall obtain the following [6276]
terms of R;

$$R = -\tfrac{3}{4} M^2 . a^2 . H . \cos.(Mt+E-I). \qquad [6277]$$

If we neglect the term $\frac{S}{D}$ *of the expression of* R [6042], *since it would* [6277′]
be useless to notice it,* we shall have,

$$r . \left(\frac{dR}{dr}\right) = 2R. \qquad [6278]$$

This being premised, the differential equation [6078] becomes, by considering only the terms which depend on the cosines of the angles

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Jupiter [6021c—6024g]. Substituting this value of D, in the term [6274], it becomes, by development, using M^2 [6272],

$$R = -\frac{S . r^2}{4D^3} = -\frac{S\, r^2}{4D'^3} . \{1 - H . \cos.(Mt+E-I)\}^{-3} = -\tfrac{1}{4} M^2 . r^2 . \{1 + 3H . \cos.(Mt+E-I) + \&c\}. \qquad [6275b]$$

If we retain only the term depending on the first power of H, using $r = a$ [6247], it [6275c]
becomes as in [6277]. We may remark that the angle $Mt+E-I$, in the expression of [6275d]
D [6275], represents, at pleasure, either the mean anomaly of Jupiter, in its actual orbit about the sun, or the mean anomaly of the sun, in its relative orbit about Jupiter, supposing the planet to be at rest. This angle is represented, for brevity, by V in [7313, &c.]; so that we shall have $V = Mt+E-I$. We may however observe, that, if we suppose [6275e]
$Mt+E$ to represent, as in [6101′, 6102], the mean longitude of the sun, in its relative orbit about Jupiter, we must put I equal to the longitude of the sun's perihelion as seen [6275f]
from Jupiter; but if we take $Mt+E$ to represent the mean longitude of Jupiter, in its actual orbit about the sun, we must put I equal to the longitude of the perihelion as seen from the sun. [6275g]

* (3289) The term $R = \frac{S}{D}$ [6277′], gives $\mathrm{d}R = 0$, $r . \left(\frac{dR}{dr}\right) = 0$; which produce nothing in the equation [6057], for the determination of the value of $r\delta r$, or in the value of δv [6060]; therefore it may be neglected, and the expression [6042] will become [6278a]
as in [6278c]. Taking its partial differential relative to r, and multiplying it by $\frac{r}{dr}$, we get $r . \left(\frac{dR}{dr}\right)$ [6278d], which is evidently equal to $2R$ [6278c], as in [6278]. [6278b]

$$R = -\, r^2 . \frac{S}{4D^3} . \{1 - 3s^2 - 3S'^2 + 12sS' . \cos.(U-v) + 3(1 - s^2 - S'^2) . \cos.(2U-2v)\}. \qquad [6278c]$$

$$r . \left(\frac{dR}{dr}\right) = -2r^2 . \frac{S}{4D^3} . \{1 - 3s^2 - 3S'^2 + 12sS' . \cos.(U-v) + 3(1 - s^2 - S'^2) . \cos.(2U-2v)\}. \qquad [6278d]$$

[6278′] $nt-2Mt+\varepsilon-2E+gt+\Gamma$, $Mt+E-I$, and observing that M and g are very small in comparison with n [6025m, o, k],*

[6279]
$$0=\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2}.\{1-3h.\cos.(nt+\varepsilon-gt-\Gamma)\} \qquad 1$$
$$-9M^2.h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma)-\tfrac{3}{2}.M^2.H.\cos.(Mt+E-I). \qquad 2$$

We have seen, in [6116 line 2], that the expression of $\frac{r\delta r}{a^2}$ contains the

[6280] term $-\frac{M^2}{n^2}.\cos.(2nt-2Mt+2\varepsilon-2E)$. Hence it appears that the following term in the equation [6279],

[6281]
$$-3N^2.\frac{r\delta r}{a^2}.h.\cos.(nt+\varepsilon-gt-\Gamma),$$

[6282] will produce the expression $\frac{3}{2}.M^2.h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma)$ [6282a]; N^2 being very nearly equal to n^2 [6279b]. Hence the differential equation

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* (3290) From the equation [6078], we have deduced [6081], by substituting $a+\delta a$ for r; and if we suppose the term depending on h [6205], to be included in this part of

[6279a] δa, it will produce, in $\frac{\delta a}{a}$, the term $\frac{\delta a}{a}=h.\cos.(nt+\varepsilon-gt-\Gamma)$ [6272]. This produces, in [6081], the term $-\frac{\mu.(r\delta r)}{a^3}.3h.\cos.(nt+\varepsilon-gt-\Gamma)$; and we have $\frac{\mu}{a^3}=n^2$ [6110c],

[6279b] and $n^2=N^2$, nearly [6025e]; hence this term becomes $-N^2.r\delta r.3h.\cos.(nt+\varepsilon-gt-\Gamma)$.

[6279c] Substituting it in [6081], and then dividing by a^2, we obtain the following expression, which is similar to [6114], using N^2 [6124];

[6279d]
$$0=\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2}.\{1-3h.\cos.(nt+\varepsilon-gt-\Gamma)\}+\frac{1}{a^2}.\left\{2\int dR+r.\left(\frac{dR}{dr}\right)\right\}.$$

Now we have, from [6273, 6277], the terms of R [6279e], whence we deduce dR [6279f] nearly; also $\frac{1}{2}r.\left(\frac{dR}{dr}\right)$ [6278, 6279e];

[6279e] $$R=-\tfrac{9}{4}.M^2.a^2h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma)-\tfrac{3}{4}.M^2.a^2H.\cos.(Mt+E-I)=\tfrac{1}{2}.r.\left(\frac{dR}{dr}\right)$$

[6279f] $$dR=\tfrac{9}{4}.M^2.a^2h.ndt.\sin.(nt-2Mt+\varepsilon-2E+gt+\Gamma).$$

[6279g] Integrating this last expression and changing the divisor $n-2M+g$, into n, on account of the smallness of M, g, in comparison with n, we get,

[6279h]
$$\int dR=-\tfrac{9}{4}.M^2.a^2h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma).$$

Multiplying the sum of the expressions [6279h, e] by 2, we get,

[6279i] $$2\int dR+r.\left(\frac{dR}{dr}\right)=-9M^2.a^2h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma)-\tfrac{3}{2}.M^2.a^2H.\cos.(Mt+E-I).$$

Substituting this in [6279d], we get [6279].

[6279] becomes,*

$$0=\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2} \qquad 1$$
$$-\tfrac{15}{2}.M^2.h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma) \qquad 2 \qquad [6283]$$
$$-\tfrac{3}{2}.M^2.H.\cos.(Mt+E-I). \qquad 3$$

In this expression we may neglect M and g, in comparison with n, [6278'], except in the small divisor $2M+N-n-g$; and we shall obtain,† [6283']

$$\frac{r\delta r}{a^2}=\frac{15.M^2.h}{4n.(2M+N-n-g)}.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma) \qquad 1$$
$$+\frac{3M^2.H}{2n^2}.\cos.(Mt+E-I). \qquad 2 \qquad [6284]$$

The expression of δv [6060], gives very nearly, by substituting $\frac{1}{n^2}$ for a^3 [6110], ‡

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* (3291) Substituting, in [6281], the term of $\frac{r\delta r}{a^2}$ [6280]; and reducing the product by [20] Int., we obtain the term [6282]; observing that $\frac{N^2}{n^2}=1$, nearly [6279b]. [6282a] Connecting the term [6282], with the similar one in [6279 line 2], we get [6283].

† (3292) The terms of $\frac{r\delta r}{a^2}$ [6284], are deduced from those containing h, H, in [6283], by the method in [6049l]; namely, by dividing these terms respectively by $m_{\prime}^2-N^2$; $m_{\prime}$ being the coefficient of t in these terms. Thus, in the term depending on h [6283 line 2], we have $m_{\prime}=n-2M+g$; hence $m_{\prime}-N=-(2M+N-n-g)$; [6284a] [6284b] $m_{\prime}+N=n+N-2M+g=2n$ nearly [6278']; therefore,

$$m_{\prime}^2-N^2=(m_{\prime}-N).(m_{\prime}+N)=-2n.(2M+N-n-g). \qquad [6284b']$$

Now dividing the term in [6283 line 2], by this factor, we get that in [6284 line 1]. In the term depending on H [6283 line 3], we have $m_{\prime}=M$; hence $m_{\prime}^2-N^2=M^2-N^2=-n^2$ [6284c] nearly [6278']; hence, by dividing the term [6283 line 3], by $-n^2$, we get the [6284d] corresponding term in [6284 line 2].

‡ (3293) Multiplying [6279h] by 3, and [6279e] by 4, then adding the products, we obtain,

$$3\textstyle\int dR+2r.\left(\frac{dR}{dr}\right)=-\tfrac{63}{4}.M^2.a^2h.\cos.(nt-2Mt+\varepsilon-2E+gt+\Gamma)-3M^2.a^2H.\cos.(Mt+E-I). \qquad [6285a]$$

Substituting these in δv [6118b], and performing the integrations, we find that the term depending on H, is $-3M.a^3nH.\sin.(Mt+E-I)$; which, by substituting $a^3n=\frac{1}{n}$ [6285b] [6110], becomes as in [6285 line 2]. The other term of [6285a], depending on h, being

Evection. [6285] Annual equation.

$$\delta v = -\frac{15.M^2.h}{2n.(2M+N-n-g)}.\sin.(nt-2Mt+\varepsilon-2E+gt+\Gamma) \qquad 1$$

$$-\frac{3M}{n}.H.\sin.(Mt+E-I). \qquad 2$$

[6286] *The first of these inequalities* [6285 line 1] *corresponds to the evection* in the lunar theory; but it is not the only one, since it is evident that each of the three roots* g_1, g_2, g_3, *produces a similar equation. In eclipses, this*
[6287] *inequality, like the evection, is confounded with the equation of the centre, and decreases it.* For, in these eclipses, the sun's longitude, seen from Jupiter's
[6288] centre, is less than that of the satellite by 200°; so that we shall have as in [$6021z$, $6022f$] nearly,

[6289] $$2Mt+2E+400^\circ = 2nt+2\varepsilon;$$

hence the inequality [6285 line 1] becomes,†

[6290] $$\frac{15.M^2.h}{2n.(2M+N-n-g)}.\sin.(nt+\varepsilon-gt-\Gamma).$$

The corresponding equation of the centre [6241 line 1], is

[6291] $$-2h.\sin.(nt+\varepsilon-gt-\Gamma).$$

Hence the value of h, determined by eclipses, is less than its true value,

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multiplied by the very small quantity of the order M^2, may be neglected. The
[6285c] remaining term of δv [$6118b$] is $\frac{2d.\left(\frac{r\delta r}{a^2}\right)}{ndt}$, in which we must substitute the expression
[6285d] [6284]. Now the term in [6284 line 2] depending on H, produces, in [$6285c$], a term of the order M^3, which may be neglected on account of its smallness. The term in [6284 line 1], depending on h, being substituted in [$6285c$], produces the term in
[6285e] [6285 line 1]; observing that on account of the smallness of M, g, in comparison with n [6283'], we may change the factor $n-2M+g$ into n.

[6286a] * (3294) The evection in [5551 line 6] has for its argument (2☽ long.−2☉ long.−☽ Anom.); and the argument of the inequality [6285 line 1], may be put under the following form;

[6286b] $$nt-2Mt+\varepsilon-2E+gt+\Gamma = 2(nt+\varepsilon)-2(Mt+E)-(nt+\varepsilon-gt-\Gamma)$$
[6286c] $$=2(\text{long. satellite})-2(\text{Jup. long.})-(\text{mean anom. satellite});$$

which is similar to that of the lunar evection [$6286a$].

[6289a] † (3295) Changing the signs of all the terms of [6289]; then adding, to both members, $nt+\varepsilon+gt+\Gamma$; we get $nt-2Mt+\varepsilon-2E+gt+\Gamma-400^\circ = -(nt+\varepsilon-gt-\Gamma)$; hence
[6289b] we obtain, $\sin.(nt-2Mt+\varepsilon-2E+gt+\Gamma) = -\sin.(nt+\varepsilon-gt-\Gamma)$. Substituting this in the term of δv [6285 line 1], it becomes as in [6290].

in the ratio of $1 - \frac{15.M^2}{4n.(2M+N-n-g)}$ to unity.* [6292]

The second inequality [6285 line 2] *corresponds to the annual equation of the moon's motion* [5551 line 1]. *Its period being very long, we shall see, in* [6696, &c.], *that it is sensibly modified by the terms depending on the square of the disturbing force.* [6293]

Changing successively what relates to m, into the corresponding quantities relative to m', m'', and m''', we shall obtain the similar inequalities of the other satellites. [6294]

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* (3296) Adding together the two terms [6290, 6291], we obtain for their sum,

$$-2h.\left\{1-\frac{15.M^2}{4n.(2M+N-n-g)}\right\}.\sin.(nt+\varepsilon-gt-\Gamma)\,; \qquad [6292a]$$

which represents very nearly the value of the equation of the centre in eclipses; and this quantity is to its value [6291], in the ratio given in [6292].

CHAPTER IV.

ON THE INEQUALITIES OF THE SATELLITES IN LATITUDE.

9. We shall now resume the differential equation [6077],

[6295]
$$0=\frac{dds}{dv^2}+s-\frac{r^2}{h^2}\cdot\left\{\left(\frac{dR}{dv}\right)\cdot\frac{ds}{dv}-\left(\frac{dR}{ds}\right)\right\};$$

and shall suppose, as in [956],

[6296]
$$\frac{1}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}}=\tfrac{1}{2}.B^{(0)}+B^{(1)}.\cos.(v'-v)+B^{(2)}.\cos.2(v'-v)+\&c.$$

We shall have in [6039], by neglecting the excentricity of the orbit, which

[6296′] is the same as to suppose $r=a$, *

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* (3297) The terms of R [6039 lines 1, 2], depending on m', being multiplied by a, may be put under the following form ;

[6297a]
$$aR=\frac{m'.ra}{r'^2}\cdot\{ss'-\tfrac{1}{2}.(s^2+s'^2).\cos.(v'-v)\} \qquad 1$$
$$+a.\left\{\frac{m'.r}{r'^2}.\cos.(v'-v)-\frac{m'}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{1}{2}}}\right\} \qquad 2$$
$$-\frac{m'.rr'a.\{ss'-\frac{1}{2}.(s^2+s'^2).\cos.(v'-v)\}}{\{r^2-2rr'.\cos.(v'-v)+r'^2\}^{\frac{3}{2}}}. \qquad 3$$

[6297b] The excentricities of the orbits of the satellites being small [6057e, f], they produce no terms, of importance, of the forms adopted in [6300], and may therefore be neglected in [6297c] the preceding expression, by putting $r=a$, $r'=a'$ [6200a, &c.]. Now substituting the values [6089, 6090, 6296], we get the terms of [6297 lines 1, 2, 3], depending on m'; and by prefixing the sign Σ, it includes the other disturbing satellites. The terms of aR [6297 line 4] depending on S, are found by multiplying R [6042] by a, putting [6297d] $r=a$, and using D' [6104], M^2 [6105], $a^3=n^{-2}$ [6110]. Again, if we substitute $\mathrm{M}=1$, $B=1$ [6082], and $r=a$, in the part of aR [6052] depending on the ellipticity, it becomes as in [6297 line 5].

$$aR = \Sigma . m' . \left\{ \begin{array}{l} \frac{a^2}{a'^2} . \{ ss' - \frac{1}{2}.(s^2 + s'^2).\cos.(v'-v) \} \\ + \frac{1}{2}.aA^{(0)} + aA^{(1)}.\cos.(v'-v) + aA^{(2)}.\cos.2(v'-v) + \&c. \\ -a^2a'.\{ ss' - \frac{1}{2}.(s^2+s'^2).\cos.(v'-v) \}.\{ \frac{1}{2}.B^{(0)} + B^{(1)}.\cos.(v'-v) + B^{(2)}.\cos.2(v'-v) + \&c. \} \end{array} \right\} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \quad [6297]$$

$$- \frac{Sa}{D} - \frac{M^2}{4n^2}.\{ 1 - 3s^2 - 3S'^2 + 3.(1 - s^2 - S'^2).\cos.(2v - 2U) + 12sS'.\cos.(v - U) \} \quad 4$$

$$- \frac{(\rho - \frac{1}{2}\varphi)}{a^2} . \{ \tfrac{1}{3} - (s - s_1)^2 \} . \quad 5$$

If we consider only the terms multiplied by s, *and those which depend* [6297'] *on the sines and cosines of* v, *upon which the secular variations of the elements of the orbit depend;** observing also that we have very nearly [6298]

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* (3298) This is similar to what we have seen in the planetary theory [1037—1039, 1041], where the secular variations of the planet, and the chief term of the latitude s of the planet, depend on the angle $nt+\varepsilon$, *which is the mean value of* v, [6298a] [6091]; and the terms of this form are therefore retained, as being by far the most important. In like manner, in the lunar theory, we find that the greatest part of the moon's [6298b] latitude is contained in the first line of the function [5308], namely, $18542^s.79.\sin.(gv-\theta)$, or $18542^s.79.\sin.\{v+(g-1).v-\theta\}$; the coefficient $g-1$ being of the same order [6298c] as the disturbing force of the sun upon the moon [4828e, 4842]; and $(gv-\theta)$ *representing, as in* [5388a], *the moon's distance from the ascending node of her orbit upon the fixed plane*; [6298d] and $(g-1)v$ [4817] being the retrograde motion of the moon's node. If we represent the mean value of $(g-1).v-\theta$ by $pt+\Lambda$, the preceding expression of the chief part [6298d'] of the moon's latitude [6298c] becomes $18542^s.79.\sin.(v+pt+\Lambda)$; being of the same form as that which is assumed for the first satellite in [6300 line 1]; so that $v+pt+\Lambda$ represents, as in [6298d], the *distance of the satellite from the ascending node.* The [6298e] values of s', s'', s''', depending on the same angle $pt+\Lambda$, must evidently have similar forms to that of s; and these are found by changing v into v', v'', v'''; also l into [6298f] l', l'', l''', respectively, as in [6300 lines 2, 3, 4]. We shall see in [6323—6333] the methods by which S', s_1, are reduced to the forms [6300 lines 5, 6]. We may also observe that the form of s [6300 line 1] may be obtained by the same process as that [6298g] which is used in the planetary theory [1133, 1137''], where we have found,

$$q = \Sigma.N.\cos.(gt+\beta); \quad p = \Sigma.N.\sin.(gt+\beta); \quad s = q.\sin.(nt+\varepsilon) - p.\cos.(nt+\varepsilon). \quad [6298h]$$

Substituting these values of q, p, in that of s [6298h] and reducing, by [22] Int., we get $s = \Sigma.N.\sin.(nt+\varepsilon-gt-\beta)$. Now changing N, $-g$, $-\beta$, into l, p, Λ, [6298i] respectively; also $nt+\varepsilon$, which is the mean value of v, into v [6298a]; it becomes of the same form as in [6300 line 1]. In like manner we may deduce the forms of [6298k] s', s'', &c. [6300] from those of q', p', &c. [1133]; observing also that, in this notation, the quantity g, *or the equivalent expression* $-p$ [6298i], *is of the order of the* [6298l] *disturbing forces,* as in [1097c].

We shall now make a few additional remarks on the process of integration, which is used

[6298'] $h^2=a$ [6110d], we shall find that the preceding differential equation

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by the author in this chapter. We shall see, in the next note, that by retaining only the
[6298m] terms mentioned in [6298], the equation [6295] can be reduced to the form [6299]; which is *linear* in terms of s, s', s'', s''', S', s_1. *This differential equation corresponds to the*
[6298n] *satellite* m; *and there are three similar linear equations, corresponding to the satellites* m', m'', m'''; *also an equation relative to the motion of Jupiter in its orbit*; *and another relative to that of the equator of Jupiter*. Now if we suppose s to contain a term of the form $s=l.\sin.(v+pt+\text{A})$, it will produce, in [6299 line 1], some terms of the form
[6298o] $l_1.\sin.(v+pt+\text{A})$; which must be destroyed by similar terms in [6299 line 2], so that the terms in this last line must produce the quantity $-l_1.\sin.(v+pt+\text{A})$, which depends on the same angle $v+pt+\text{A}$, but has a coefficient $-l_1$, with a different sign from that
[6298p] which is produced by the terms in [6299 line 1]. We must therefore assume such forms for s', s'', s''', S', s_1, as will, by substitution in [6299 line 2], produce terms depending on $\sin.(v+pt+\text{A})$, and this is obtained by means of the forms assumed in [6300 lines 2–6],
[6298q] as is easily proved in the manner pointed out in the note in [6302b, c, &c.] There is a perfect symmetry in all these expressions, since the argument, or angle, under the sign *sin.*, is found, in every case, by adding the same angle $pt+\text{A}$, to the longitude of the object, whose latitude is to be computed by that formula. Thus, the longitude of the
[6298r] second satellite, v', is added to $pt+\text{A}$, in [6300 line 2], to find the argument of the
[6298s] latitude s' of the second satellite; the sun's longitude U, is used in [6300 line 5], in finding the sun's latitude S'; and the longitude v, of the first satellite, is used in
[6298t] [6300 line 6], in finding the latitude s_1 of that satellite, supposing it to move in the plane of the equator of Jupiter [6051]. In the same manner as we have found the six expressions
[6298u] [6300], corresponding to the angle $pt+\text{A}$, we may obtain similar ones, relative to any other angle $p_1t+\text{A}_1$, which satisfies the proposed differential equations, enumerated in
[6298v] [6298n]; and by changing the coefficients l, l', l'', &c. into l_1, l_1', &c., we obtain another system of equations $s=l_1.\sin.(v+p_1t+\text{A}_1)$, $s'=l_1'.\sin.(v'+p_1t+\text{A}_1)$, &c., similar to [6300]. A third angle, $p_2t+\text{A}_2$, will in like manner give $s=l_2.\sin.(v+p_2t+\text{A}_2)$;
[6298w] $s'=l_2'.\sin.(v'+p_2t+\text{A}_2)$, &c.; and so on for other values of p. Finally, it is plain, from the *linear* form of the equations [6298m, &c.], that we may add together all these values of s, for the complete value of s; all the values of s', for the complete value of s', &c.; so that we shall finally have, by using the symbol Σ' as in [6324'],

[6298x]

$$s=l.\sin.(v+pt+\text{A})+l_1.\sin.(v+p_1t+\text{A}_1)+l_2.\sin.(v+p_2t+\text{A}_2)+\&c.=\Sigma'.l.\sin.(v+pt+\text{A})\,; \quad 1$$

$$s'=l'.\sin.(v'+pt+\text{A})+l_1'.\sin.(v'+p_1t+\text{A}_1)+l_2'.\sin.(v'+p_2t+\text{A}_2)+\&c.=\Sigma'.l'.\sin.(v'+pt+\text{A})\,; \quad 2$$

$$s''=l''.\sin.(v''+pt+\text{A})+l_1''.\sin.(v''+p_1t+\text{A}_1)+l_2''.\sin.(v''+p_2t+\text{A}_2)+\&c.=\Sigma'.l''.\sin.(v''+pt+\text{A})\,; \quad 3$$

$$s'''=l'''.\sin.(v'''+pt+\text{A})+l_1'''.\sin.(v'''+p_1t+\text{A}_1)+l_2'''.\sin.(v'''+p_2t+\text{A}_2)+\&c.=\Sigma'.l'''.\sin.(v'''+pt+\text{A})\,; \quad 4$$

$$S'=L'.\sin.(U+pt+\text{A})+L_1'.\sin.(U+p_1t+\text{A}_1)+L_2'.\sin.(U+p_2t+\text{A}_2)+\&c.=\Sigma'.L'.\sin.(U+pt+\text{A})\,; \quad 5$$

$$s_1=L.\sin.(v+pt+\text{A})+L_1.\sin.(v+p_1t+\text{A}_1)+L_2.\sin.(v+p_2t+\text{A}_2)+\&c.=\Sigma'.L.\sin.(v+pt+\text{A})\,. \quad 6$$

becomes, * [6298″]

These forms agree with those in [6427—6430]; the terms depending on the angles $pt+\text{A}$, $p_1t+\text{A}_1$, $p_2t+\text{A}_2$, $p_3t+\text{A}_3$, arising from the mutual action of the sun, satellites, and the ellipticity of Jupiter, are explicitly retained, in [6427—6430]. The remaining terms depending on the angles $p_4t+\text{A}_4$, $p_5t+\text{A}_5$, &c., arising from the displacement of Jupiter's equator and orbit, are reduced, in [6342—6414], to a single term of the form [6362], corresponding to those in the first line of each of the expressions [6427—6430]. [6298y] [6298z]

* (3299) Substituting $r=a$ [6296′], and $h^2=a$ [6298′], in [6295], it becomes, [6299a]

$$0=\frac{dds}{dv^2}+s+a.\left(\frac{dR}{ds}\right)-\frac{ds}{dv}.a.\left(\frac{dR}{dv}\right). \qquad [6299b]$$

If we introduce into this expression the value of aR [6297], and retain only the terms depending on $\sin.(v+pt+\text{A})$, as in [6301], it will become as in [6299]. For the terms $\frac{dds}{dv^2}+s$ [6299b], are the same as the two first terms in [6299 line 1]; and we shall now proceed to prove that the last term of [6299b] may be neglected, and that the remaining term, $a.\left(\frac{dR}{ds}\right)$, produces the rest of the terms of [6299]. The chief terms of aR [6297] containing v, are of the form $aR=\Sigma.A.\cos.i(v'-v)$; which gives, [6299c] [6299d]

$$a.\left(\frac{dR}{dv}\right)=\Sigma.Ai.\sin.i(v'-v).$$

Multiplying this by $\frac{ds}{dv}=l.\cos.(v+pt+\text{A})$ nearly, [6300 line 1], it produces terms depending on the angles $iv'-(i\pm1)v\mp pt\mp\text{A}$; which cannot be reduced to the form $v+pt+\text{A}$ [6299c], by using, for i, any of the integral numbers 1, 2, 3, &c.; therefore it may be neglected. The other terms of $a.\left(\frac{dR}{dv}\right)$ [6297] are of the order s, or of the forms which are neglected in [6297′]. We shall now compute the terms of [6299] depending on the remaining part $a.\left(\frac{dR}{ds}\right)$ of the equation [6299b]; which, by substituting [6297], becomes, [6299e] [6299f] [6299g]

$$a.\left(\frac{dR}{ds}\right)=\Sigma.m'.\frac{a^2}{a'^2}.\{s'-s.\cos.(v'-v)\}-\Sigma.m'.a^2a'.\{s'-s.\cos.(v'-v)\}.\{\tfrac{1}{2}B^{(0)}+B^{(1)}.\cos.(v'-v)+\&c\} \quad 1$$
$$-\frac{M^2}{4n^2}.\{-6s-6s.\cos.(2v-2U)+12S'.\cos.(v-U)\}+\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.\{2s-2s_1\}. \quad 2 \qquad [6299h]$$

As we retain only the terms depending on the angle $v+pt+\text{A}$, after substituting the values [6300], we may neglect all the terms of the first line of [6299h], except that depending on $B^{(1)}$; and this term produces $\Sigma.m'.a^2a'.s.\cos.(v'-v).B^{(1)}.\cos.(v'-v)$, or by reduction, using [20] Int., $\frac{1}{2}\Sigma.m'.a^2a'.B^{(1)}.s$, which is the last term in [6299 line 1]. It also produces the last term of [6299 line 2], namely, $-\Sigma.m'.a^2a'\,B^{(1)}.s'.\cos.(v'-v)$. We may also neglect the term depending on $\cos.(2v-2U)$ [6299h], since it does not [6299i] [6299k]

[6299]
$$0=\frac{dds}{dv^2}+s.\left\{1+2.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}+\tfrac{3}{2}.\frac{M^2}{n^2}+\tfrac{1}{2}.\Sigma.m'.a^2a'.B^{(1)}\right\} \qquad 1$$
$$-\frac{2.(\rho-\frac{1}{2}\varphi)}{a^2}.s_{\prime}-\frac{3M^2}{n^2}.S'.\cos.(U-v)-\Sigma.m'.a^2a'.B^{(1)}.s'.\cos.(v'-v). \qquad 2$$

To integrate this equation, we shall suppose, as in [6298c–l],

[6300]
$$s\;=l.\;\sin.(v\;+pt+\text{A})\;; \qquad 1$$
$$s'\;=l'.\;\sin.(v'+pt+\text{A})\;; \qquad 2$$
$$s''=l''.\sin.(v''+pt+\text{A})\;; \qquad 3$$
$$s'''=l'''.\sin.(v'''+pt+\text{A})\;; \qquad 4$$
$$S'=L'.\sin.(U+pt+\text{A})\;;\;[6329] \qquad 5$$
$$s_{\prime}=L.\sin.(v\;+pt+\text{A})\;.\;[6323] \qquad 6$$

[6300′] Putting $pdt=\frac{p}{n}.dv$ [6089f], and substituting the assumed values [6300]

[6301] in [6299]; then comparing together the coefficients of $\sin.(v+pt+\text{A})$,

[6302] neglecting the square of p; *p being a very small quantity of the same order as the disturbing forces* [6298l]; we shall obtain,*

[6303]
$$0=l.\left\{\frac{p}{n}-\frac{(\rho-\frac{1}{2}\varphi)}{a^2}-\tfrac{3}{4}.\frac{M^2}{n^2}-\tfrac{1}{4}.\Sigma.m'.a^2a'.B^{(1)}\right\}$$
$$+\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.L+\tfrac{3}{4}.\frac{M^2}{n^2}.L'+\tfrac{1}{4}.\Sigma.m'.a^2a'.B^{(1)}.l'.$$

If we put, as in [963^{iv}, 964],

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produce a term of the form [6299c]. The two terms in the second line of [6299h],

[6299l] depending on $-6s$, $+12S'$, produce the terms in [6299 lines 1, 2], which are multiplied by M^2; and the terms of [6299h], depending on $(\rho-\frac{1}{2}\varphi)$, produce those in [6299] having the same factor.

* (3300) Substituting the values of S', s' [6300 lines 5, 2] in the first members of

[6302a] [6302 b, c], and then reducing, by means of [18] Int., retaining only the terms depending on the angle $v+pt+\text{A}$, we get the second members of these equations;

[6302b] $$S'.\cos.(U-v)=L'.\sin.(U+pt+\text{A}).\cos.(U-v)=\tfrac{1}{2}L'.\sin.(v+pt+\text{A})$$

[6302c] $$s'.\cos.(v'-v)=l'.\sin.(v'+pt+\text{A}).\cos.(v'-v)=\tfrac{1}{2}l'.\sin.(v+pt+\text{A}).$$

Again, the second differential of s [6300 line 1], taken relative to v, using pdt [6300′]

[6302d] gives $\frac{dds}{dv^2}=-l\left(1+\frac{p}{n}\right)^2.\sin.(v+pt+\text{A})$; adding this to the same value of s, and

[6302e] neglecting the term of the order p^2, we get $\frac{dds}{dv^2}+s=-2l.\frac{p}{n}.\sin.(v+pt+\text{A})$.

Substituting the values [6302b, c, e; 6300 lines 1, 6], in [6299], and then dividing by $-2.\sin.(v+pt+\text{A})$, we get [6303].

$$\frac{a}{a'} = \alpha ;$$ [6304]

$$(1-2\alpha.\cos.\theta+\alpha^2)^{-s} = \tfrac{1}{2}.b_s^{(0)}+b_s^{(1)}.\cos.\theta+b_s^{(2)}.\cos.2\theta+\&c. ;$$ [6305]

we shall obtain, as in [1128],

$$a^2a'.B^{(1)} = \alpha^2.b_{\frac{3}{2}}^{(1)}.$$ [6306]

Now we have, as in [1076, 1130],*

$$(0,1) = -\frac{3m'.n\alpha^2.b_{-\frac{1}{2}}^{(1)}}{4.(1-\alpha^2)^2} = \tfrac{1}{4}.m'.n.\alpha^2.b_{\frac{3}{2}}^{(1)} = \tfrac{1}{4}.m'.n.a^2a'.B^{(1)} ;$$ [6307]

hence we get,†

$$0=l.\left\{p-(0)-\boxed{0}-(0,1)-(0,2)-(0,3)\right\}+(0).L+\boxed{0}L'+(0,1).l'+(0,2).l''+(0,3).l'''.$$ [6308]

We shall find, in the same manner,

$$0=l'.\left\{p-(1)-\boxed{1}-(1,0)-(1,2)-(1,3)\right\}+(1).L+\boxed{1}.L'+(1,0).l+(1,2).l''+(1,3).l''';$$ [6309]

$$0=l''.\left\{p-(2)-\boxed{2}-(2,0)-(2,1)-(2,3)\right\}+(2).L+\boxed{2}.L'+(2,0).l+(2,1).l'+(2,3).l''';$$ [6310]

$$0=l'''.\left\{p-(3)-\boxed{3}-(3,0)-(3,1)-(3,2)\right\}+(3).L+\boxed{3}.L'+(3,0).l+(3,1).l'+(3,2).l''.$$ [6311]

There is, between the quantities p, l, l', l'', l''', L, L', an equation, *which depends on the displacing of Jupiter's equator, by the combined actions of the sun and the satellites.* To obtain this equation, we must [6311'] determine the precession of Jupiter's equinoxes and the nutation of its axis relative to the fixed plane, using, as in § 5 of the fifth book, the following symbols ; ‡

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* (3301) The first and third values of (0,1) [6307], are the same as in [1130]; and by substituting the values of $a^2a'.B^{(1)}$ [6306], in the last expression, we obtain the second form [6307]. [6307a]

† (3302) Multiplying [6303] by n, and substituting $\frac{1}{4}.m'.n.a^2a'.B^{(1)} = (0,1)$ [6307], with the similar values (0,2), (0,3), &c.; also the values of (0), $\boxed{0}$ [6216]; we get [6308a] [6308]. Changing successively the elements relative to m, into those relative to m', m'', &c., we get [6309—6311], corresponding to the other satellites.

‡ (3303) These symbols are similar to those which are used in figure 58, page 803, of the second volume, in computing the precession and nutation of the earth's axis. The [6313a] symbol γ [6314] represents the longitude of the ascending node of Jupiter's orbit, relative [6313b]

Symbols.

[6312] θ = the inclination of Jupiter's equator to the fixed plane;

[6313] Ψ = the retrograde motion, or precession, of the descending node of Jupiter's equator upon the fixed plane, and counted from the fixed axis of x;

[6313′] γ = the inclination of the orbit of Jupiter to the fixed plane, [being the same as that of the relative orbit of the sun about Jupiter];

[6314] Π = the longitude of the ascending node of Jupiter's orbit, counted from the fixed axis of x; [being the same as the longitude of the ascending node of the relative orbit of the sun about Jupiter] [6313g];

[6315] γ_1 = the inclination of the orbit of the satellite m, upon the fixed plane;

[6316] Π_1 = the longitude of the ascending node of the satellite m, upon the fixed plane, and counted from the fixed axis of x;

[6316′] it = the rotatory motion of Jupiter in the time t.

[6316″] Then we shall have, as in § 5 of the fifth book, by neglecting the square of θ, *

[6313c] to the fixed plane, and counted from the axis of x; observing that this longitude is the same, whether we consider the actual orbit of Jupiter in its motion about the sun, or the
[6313d] relative motion of the sun about Jupiter, supposing this planet to be at rest. It being evident, that when the sun is considered as at rest, the heliocentric longitude of Jupiter, at
[6313e] the moment of passing the *ascending node of its actual orbit*, will be represented by Π;
[6313f] and that if the planet be considered as at rest, the longitude of the sun, at the time of passing the *ascending node in its apparent orbit about Jupiter, will also be represented*
[6313g] *by* Π, as seen from the planet, and *counted from the same axis* x, *or point in the heavens.* This symbol Π is equivalent to $\Gamma_{\prime}$, in the notation which is used for the earth in
[6313h] [3067, &c.]. The longitude Π_1 [6316] is counted from the fixed axis of x; and if we add to it the precession Ψ [6313], from the same axis, we shall get $\Pi_1 + \Psi$, *for the*
[6313i] *longitude of the ascending node of the satellite* m, *upon the fixed plane, and counted from the moveable vernal equinox of Jupiter.*

[6313k] We may observe, that the planet Jupiter revolves about its axis in $0^{\text{days}},41377$ [6920];
[6313l] hence its motion in a Julian year is $i = 353094^{\circ}\ 71'$ nearly, [6021w]. Comparing this with the values of n, n', n'', n''' [6025k], we get the following results, which will be of use hereafter;

[6313m] $$i = n.\ 4{,}27565 = n'.\ 8{,}58250 = n''.\ 17{,}29113 = n'''.\ 40{,}33405.$$

* (3304) The expression of $\frac{d\theta}{dt}$, corresponding to Jupiter and its satellites, is easily
[6317a] deduced from the value of θ [3089], relative to the earth and moon. For *if we neglect*

$$\frac{d\theta}{dt} = \frac{3.(2C-A-B)}{4i.C} \cdot \left\{ M^2.\gamma.\sin.(\daleth+\Upsilon) + \Sigma.m.n^2.\gamma_1.\sin.(\daleth_1+\Upsilon) \right\}. \qquad [6317]$$

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the terms of [3089] *depending on* $\cos.2v$, $\cos.2v'$, which are very small [3377]; then take the differential of the remaining terms relative to t, and divide by dt, *placing an accent below the letters* m, n, λ, to distinguish them from the same letters m, n, λ, which are used in [6022d,f; 6024b, &c.] for other purposes, we shall get, [6317b] [6317b′]

$$\frac{d\theta}{dt} = \frac{3}{4n_{,}} \cdot \left(\frac{2C-A-B}{C} \right) \cdot \left\{ (m_{,}^2 + {}_{,}\lambda m_{,}^2).\cos.\theta.\Sigma.c.\sin.(ft+\beta) - {}_{,}\lambda m_{,}^2.c'.\cos.\theta.\sin.(f't+\beta') \right\}. \qquad [6317c]$$

To obtain the similar expression for Jupiter and its first satellite, we must change, as in [3060],

$$m^2 = \frac{\text{sun's mass}}{(\text{dist. sun and earth})^3} \quad \text{into} \quad m_{,}^2 = \frac{\text{sun's mass}}{(\text{dist. sun and Jupiter})^3} = M^2 \ [6105] \text{ nearly}. \qquad [6317d]$$

We have also, as in [3079], by changing what relates to the earth into the corresponding terms relative to Jupiter; and using the values of a, m, n^2 [6021c, 6022d, 6110];

$$ {}_{,}\lambda m_{,}^2 = \frac{\text{mass of the satellite}}{(\text{distance of satellite and planet})^3} = \frac{m}{a^3} = m.n^2. \qquad [6317e]$$

The rotatory motion of the planet nt [3015], is changed into it, in [6316′]; so that $n_{,} = i$. We have in [3073d], by changing $\text{tang}.\gamma$ into γ, on account of its smallness, [6317f]
$\Sigma.c.\sin.(ft+\beta) = \gamma.\sin.A$. Substituting $A = \Gamma_{,} + \Upsilon$ [3069], and then changing $\Gamma_{,}$ [6317g]
into $\daleth$, as in [6313g], to conform to the system now under consideration, we get $\Sigma.c.\sin.(ft+\beta) = \gamma.\sin.(\daleth+\Upsilon)$. The quantity c' [3086], is changed into $\text{tang}.\gamma_1$ [6317h]
[6315], or γ_1 nearly, so that for the satellite m, we shall have $c' = \gamma_1$. The [6317i]
expression $-f't-\beta'$ [3086] represents the longitude of the ascending node of the orbit of the satellite, counted from the *moveable* vernal equinox, and this is changed into $\daleth_1 + \Upsilon$ [6317k]
in [6313i].

Now substituting the values [6317d, e, f, h, k] in [6317c], and putting as in [6316″], $\cos.\theta = 1$, we get the following expression of $\frac{d\theta}{dt}$, corresponding to the system of Jupiter and its first satellite; [6317l]

$$\frac{d\theta}{dt} = \frac{3.(2C-A-B)}{4i.C} \cdot \left\{ (M^2 + m.n^2).\gamma.\sin.(\daleth+\Upsilon) + m.n^2.\gamma_1.\sin.(\daleth_1+\Upsilon) \right\}. \qquad [6317m]$$

If we take, as in [6398], the orbit of Jupiter in 1750 for the fixed plane, γ will be of the order of the disturbing force [6313′]; and by neglecting quantities of the order of the square of this force $m.\gamma$, we may reject the term $m.n^2.\gamma.\sin.(\daleth+\Upsilon)$ [6317m], and we shall have, [6317n]

$$\frac{d\theta}{dt} = \frac{3.(2C-A-B)}{4i.C} \cdot \left\{ M^2.\gamma.\sin.(\daleth+\Upsilon) + m.n^2.\gamma_1.\sin.(\daleth_1+\Upsilon) \right\}. \qquad [6317o]$$

If in the last term of this expression, depending on the satellite m, we change successively the elements of m, into those corresponding to m', m'', m''', we shall get the parts [6317p]
depending on these satellites; and by adding them to those in [6317o], by means of the symbol Σ, we shall get the complete value of $\frac{d\theta}{dt}$, as in [6317].

[6318] We shall also have,*

[6319] $$\theta.\frac{d\Psi}{dt}=\frac{3.(2C-A-B)}{4i.C}.\{(M^2+\Sigma.m.n^2).\theta+M^2\gamma.\cos.(\gamma+\Psi)+\Sigma.m.n^2.\gamma_1.\cos.(\gamma_1+\Psi)\}.$$

Multiplying [6317] by $\sin.\Psi$, and [6319] by $\cos.\Psi$; then taking the sum of these products, we get,†

[6320] $$\frac{d.(\theta.\sin.\Psi)}{dt}=\frac{3.(2C-A-B)}{4i.C}.\{(M^2+\Sigma.m.n^2).\theta.\cos.\Psi+M^2\gamma.\cos.\gamma+\Sigma.m.n^2.\gamma_1.\cos.\gamma_1\}.$$

In like manner we shall have, ‡

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* (3305) Neglecting the terms of [3096] depending on $\sin.2v$, $\sin.2v'$, as in [6317b],
[6319a] then putting as in [6316″], $\cos.\theta=1$, $\sin.\theta=\theta$, we get, by multiplying by θ, and accenting m, n, as in [6317b′];

[6319b] $$\theta.\frac{d\Psi}{dt}=\frac{3.(2C-A-B)}{4i.C}.\{(m_{,}^2+{}_{,}\lambda m_{,}^2).\theta+(m_{,}^2+{}_{,}\lambda m_{,}^2).\Sigma.c.\cos.(ft+\beta)+{}_{,}\lambda m_{,}^2.c'.\cos.(f't+\beta')\}.$$

Substituting the values of $m_{,}^2$, ${}_{,}\lambda m_{,}^2$, $n_{,}$, c', $-f't-\beta'$ [6317d, e, f, i, k]; also putting successively, as in [3075, 3069, 6317g],

[6319c] $$\Sigma.c.\cos.(ft+\beta)=\gamma.\cos.\Lambda=\gamma.\cos.(\Gamma_{,}+\Psi)=\gamma.\cos.(\gamma+\Psi);$$

we obtain,

[6319d] $$\theta.\frac{d\Psi}{dt}=\frac{3.(2C-A-B)}{4i.C}.\{(M^2+m.n^2).\theta+(M^2+m.n^2).\gamma.\cos.(\gamma+\Psi)+m.n^2.\gamma_1.\cos.(\gamma_1+\Psi)\}.$$

Neglecting the term $m.n^2.\gamma.\cos.(\gamma+\Psi)$, as in [6317n]; and prefixing the symbol Σ to
[6319e] the terms depending on m, so as to obtain the sum corresponding to all the satellites, we get [6319].

† (3306) The first member of this sum is $\frac{d\theta.\sin.\Psi+\theta\,d\Psi.\cos.\Psi}{dt}$, which is evidently
[6320a] equal to the development of the differential expression in the first member of [6320]. The second member of this sum may be reduced, by means of the following expressions, which are easily obtained from [24] Int.;

[6320b] $$\sin.(\gamma+\Psi).\sin.\Psi+\cos.(\gamma+\Psi).\cos.\Psi=\cos.\{(\gamma+\Psi)-\Psi\}=\cos.\gamma;$$

[6320c] $$\sin.(\gamma_1+\Psi).\sin.\Psi+\cos.(\gamma_1+\Psi).\cos.\Psi=\cos.\{(\gamma_1+\Psi)-\Psi\}=\cos.\gamma_1;$$

hence the second member of this sum becomes as in the second member of [6320].

‡ (3307) Multiplying the equation [6317] by $\cos.\Psi$, and [6319] by $-\sin.\Psi$, then taking the sum of the products, we find that the first member of this sum is
[6321a] $\frac{d\theta.\cos.\Psi-\theta\,d\Psi.\sin.\Psi}{dt}$, which is easily reduced to the form in the first member of [6321]. Substituting in the second member the two following expressions, which are easily deduced from [22] Int.,

$$\frac{d.(\theta.\cos.\Psi)}{dt}=\frac{3.(2C-A-B)}{4i.C}.\{-(M^2+\Sigma.m.n^2).\theta.\sin.\Psi+M^2\gamma.\sin.\Gamma+\Sigma.m.n^2.\gamma_1.\sin.\Gamma_1\}. \quad [6321]$$

To integrate these two equations, we shall observe, that *the latitude of the first satellite, supposing it to move in the plane of Jupiter's equator, is* $-\theta.\sin.(v+\Psi)$ *above the fixed plane.** But we have already shown that this latitude is equal to a series of terms of the form $L.\sin.(v+pt+\text{A})$, and we shall denote this series by [6322] [6323]

$$-\theta.\sin.(v+\Psi)=\Sigma'.L.\sin.(v+pt+\text{A}). \quad [6324]$$

The characteristic Σ' *signifies that the sum of all the terms of the proposed function, which are similar to those under this sign, is to be taken as in* [6298x]. [6324′] Σ'.

The characteristic Σ [6118l'], *denotes that the sum of the terms relative to the different satellites is to be taken.* [6325] Σ.

Therefore we shall have,†

$$\sin.(\Gamma+\Psi).\cos.\Psi-\cos.(\Gamma+\Psi).\sin.\Psi=\sin.\{(\Gamma+\Psi)-\Psi\}=\sin.\Gamma; \quad [6321b]$$
$$\sin.(\Gamma_1+\Psi).\cos.\Psi-\cos.(\Gamma_1+\Psi).\sin.\Psi=\sin.\{(\Gamma_1+\Psi)-\Psi\}=\sin.\Gamma_1; \quad [6321c]$$

we obtain, for the second member of the sum, the same result as in the second member of [6321].

* (3308) The motion of the satellite m is direct, and represented by v [6032], counted from the axis of x. The *retrograde* motion of the descending node of Jupiter's equator is Ψ [6313], counted from the same axis x. Therefore the distance of the satellite, from that *descending* node, is $(v+\Psi)$; consequently its distance, from the *ascending* node of the equator above the *fixed plane*, is $v+\Psi+200^{\circ}$. Multiplying its sine, by the inclination θ of the equator to the fixed plane [6312], we get very nearly, as in [533a], the elevation or distance of the satellite from the fixed plane, supposing it to move in that equator; hence this distance is $\theta.\sin.(v+\Psi+200^{\circ})$, or $-\theta.\sin.(v+\Psi)$, as in [6322]. Now this height is represented by s_1 [6051]; therefore $s_1=-\theta.\sin.(v+\Psi)$; and as s_1 is represented in [6298x line 6] by a series of terms of the form $L.\sin.(v+pt+\text{A})$, whose sum is represented by prefixing the symbol Σ' [6324′], we shall have $-\theta.\sin.(v+\Psi)=\Sigma'.L.\sin.(v+pt+\text{A})$, as in [6324]. [6323a] [6323b] [6323c] [6323d] [6323e] [6323f]

† (3309) Developing the first members of [6326a, b], by means of [21] Int., we obtain,

$$\sin.(v+\Psi)=\sin.v.\cos.\Psi+\cos.v.\sin.\Psi; \quad [6326a]$$
$$\sin.(v+pt+\text{A})=\sin.v.\cos.(pt+\text{A})+\cos.v.\sin.(pt+\text{A}). \quad [6326b]$$

Substituting these in [6324], and putting the terms depending on $\cos.v$, $\sin.v$, separately equal to each other, we get, by changing the signs of all the terms, the two equations [6326, 6327]. [6326c]

[6326] $$\theta.\sin.\Psi = -\Sigma'.L.\sin.(pt+\mathrm{A});$$

[6327] $$\theta.\cos.\Psi = -\Sigma'.L.\cos.(pt+\mathrm{A}).$$

[6328] In like manner the sun's latitude, above the fixed plane, is $\gamma.\sin.(U-\Pi)$;*
[6329] but this latitude is equal to $\Sigma'.L'.\sin.(U+pt+\mathrm{A})$; hence we get,

[6330] $$\gamma.\sin.\Pi = -\Sigma'.L'.\sin.(pt+\mathrm{A});$$

[6331] $$\gamma.\cos.\Pi = \Sigma'.L'.\cos.(pt+\mathrm{A}).$$

We have likewise,†

[6332] $$\gamma_1.\sin.\Pi_1 = -\Sigma'.l.\sin.(pt+\mathrm{A});$$

[6333] $$\gamma_1.\cos.\Pi_1 = \Sigma'.l.\cos.(pt+\mathrm{A}).$$

If we substitute these values in [6320, 6321], we shall obtain, by comparing separately the coefficients of the same sines,‡

[6328a] * (3310) The projection of the radius vector of the sun's relative orbit about Jupiter, makes the angle U with the axis x [6041]; and Π [6314] is the longitude of the ascending node of the same orbit, counted from the same axis of x. Hence $U-\Pi$ is
[6328b] the argument of latitude of the sun's relative orbit; and as γ [6313′] is the inclination of
[6328c] that orbit to the fixed plane, the sun's latitude above that fixed plane will be found, as in
[6328d] [533a, &c.], to be $\gamma.\sin.(U-\Pi)$ nearly; being the same as in [6328]. Therefore we have very nearly, for the sun's latitude S' [6040], the following expression; $S'=\gamma.\sin.(U-\Pi)$. But by using the symbol Σ' [6324′], we get, from [6298x line 5], $S'=\Sigma'.L'.\sin.(U+pt+\mathrm{A})$; hence we obtain,

[6328e] $$S'=\gamma.\sin.(U-\Pi)=\Sigma'.L'.\sin.(U+pt+\mathrm{A}).$$

Now by development, using [22, 21] Int., we have,

[6328f] $$\sin.(U-\Pi) = \sin.U.\cos.\Pi-\cos.U.\sin.\Pi;$$

[6328g] $$\sin.(U+pt+\mathrm{A}) = \sin.U.\cos.(pt+\mathrm{A})+\cos.U.\sin.(pt+\mathrm{A}).$$

Substituting these last expressions in [6328e], and then putting separately the coefficients of $-\cos.U$, $\sin.U$, equal to each other, we get [6330, 6331].

[6332a] † (3311) The equations [6332, 6333] are obtained in the same manner as we have found [6330, 6331] in the preceding note; changing the elements γ, Π, S', L', U, of the sun's relative orbit, into γ_1, Π_1, s, l, v, respectively; so as to conform to the orbit of the satellite m [6313′—6316]. By this means the expression of S' [6328e] changes
[6332b] into $s=\gamma_1.\sin.(v-\Pi_1)=\Sigma'.l.\sin.(v+pt+\mathrm{A})$; which represents *the latitude of the satellite m above the fixed plane* [6298x line 1]; moreover [6330] changes into [6332], and [6331] into [6333].

‡ (3312) Substituting the values [6326, 6327, 6331, 6333] in [6320], then putting the terms depending on $\cos.(pt+\mathrm{A})$ equal to nothing, and dividing by $\cos.(pt+\mathrm{A})$, also
[6334a] transposing the terms to the second member, we get [6334]. In like manner, by substituting [6326, 6327, 6330, 6332] in [6321], transposing and dividing by $\sin.(pt+\mathrm{A})$, we get the same equation [6334].

$$0=pL+\frac{3.(2C-A-B)}{4i.C}.\{M^2.(L'-L)+\Sigma.m.n^2.(l-L)\}.$$ [6334]

We may here observe that, if we suppose Jupiter to be an ellipsoid, we shall have, as in [3408, &c.],*

$$\frac{2C-A-B}{C}=\frac{2.(\rho-\frac{1}{2}\varphi).\int_0^1\Pi_{,}R^2.dR}{\int_0^1\Pi_{,}R^4.dR},$$ [6335]

$\Pi_{,}$ *being the density of the elliptical stratum, whose radius is* R. [6336]

10. We shall now consider particularly the equations [6308–6311, 6334], putting them under the following forms; †

$$0=\left\{p-(0)-\boxed{0}-(0,1)-(0,2)-(0,3)\right\}.(L-l) \qquad 1$$
[6337]
$$+(0,1).(L-l')+(0,2).(L-l'')+(0,3).(L-l''')+\boxed{0}.(L-L')-pL; \qquad 2$$

$$0=\left\{p-(1)-\boxed{1}-(1,0)-(1,2)-(1,3)\right\}.(L-l') \qquad 1$$
[6338]
$$+(1,0).(L-l)+(1,2).(L-l'')+(1,3).(L-l''')+\boxed{1}.(L-L')-pL; \qquad 2$$

(H)

$$0=\left\{p-(2)-\boxed{2}-(2,0)-(2,1)-(2,3)\right\}.(L-l'') \qquad 1$$
[6339]
$$+(2,0).(L-l)+(2,1).(L-l')+(2,3).(L-l''')+\boxed{2}(L-L')-pL; \qquad 2$$

$$0=\left\{p-(3)-\boxed{3}-(3,0)-(3,1)-(3,2)\right\}.(L-l''') \qquad 1$$
[6340]
$$+(3,0).(L-l)+(3,1).(L-l')+(3,2).(L-l'')+\boxed{3}(L-L')-pL; \qquad 2$$

$$0=pL-\frac{3.(2C-A-B)}{4i.C}.\{M^2.(L-L')+m.n^2.(L-l)+m'.n'^2.(L-l')+m''.n''^2.(L-l'')+m'''.n'''^2.(L-l''')\}.$$ [6341]

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* (3313) If we change $\alpha\varphi$ into φ, and ah into ρ, as in [6046a]; also the density ρ [2947], into $\Pi_{,}$, [6336]; and the radius a into R, [2942, 6336]; we shall find that the equation [3408] becomes as in [6335]. We have placed an accent below $\Pi_{,}$, which is not in the original work, to distinguish it from the usual signification of Π, or the longitude of Jupiter [6024f]. [6335a]

† (3314) The terms of [6337], which are multiplied by L, excepting that which depends on (0), mutually destroy each other, as is evident by inspection. Neglecting these terms and changing the signs of the rest, it becomes as in [6308]. In like manner we find that [6338, 6339, 6340] are equivalent respectively to [6309, 6310, 6311]; observing also [6337a]

We must connect these equations with those given by the displacing of Jupiter's orbit [6347—6350], which give the values of p and L', depending
[6341′] upon this cause; observing that the action of the satellites has no sensible influence upon this secular change in the orbit.

[6342] *The values of p, corresponding to the displacing of Jupiter's equator, are much less than those which depend on the mutual action of Jupiter's satellites,*
[6342′] *as we shall hereafter see; therefore we may neglect p in comparison with* (0), (1), &c.; (0,1), &c.* In this case, if we suppose,

[6343] $$L-l\ =\lambda\,.\,(L-L')\,; \qquad \text{[For the satellite } m\text{]}$$

[6344] $$L-l'\ =\lambda'.\,(L-L')\,; \qquad \text{[For the satellite } m'\text{]}$$

[6345] $$L-l''\ =\lambda''.\,(L-L')\,; \qquad \text{[For the satellite } m''\text{]}$$

[6346] $$L-l'''=\lambda'''.(L-L')\,; \qquad \text{[For the satellite } m'''\text{]}$$

we shall obtain from [6337—6340], the four following equations;†

[6347] $$0=\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.\lambda-(0,1).\lambda'-(0,2).\lambda''-(0,3).\lambda'''-\boxed{0}\,;$$

[6348] $$0=\left\{(1)+\boxed{1}+(1,0)+(1,2)+(1,3)\right\}.\lambda'-(1,0).\lambda-(1,2).\lambda''-(1,3).\lambda'''-\boxed{1}\,;$$

[6349] $$0=\left\{(2)+\boxed{2}+(2,0)+(2,1)+(2,3)\right\}.\lambda''-(2,0).\lambda-(2,1).\lambda'-(2,3).\lambda'''-\boxed{2}\,;$$

[6350] $$0=\left\{(3)+\boxed{3}+(3,0)+(3,1)+(3,2)\right\}.\lambda'''-(3,0).\lambda-(3,1).\lambda'-(3,2).\lambda''-\boxed{3}\,.$$

We can determine, by means of these equations, the values of λ, λ', λ'', λ'''.

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that [6338, 6339, 6340] may be derived from [6337], by changing reciprocally the elements
[6337b] of m into those of m', m'', m'''. The equation [6334] is easily reduced to the form [6341], by developing the terms included under the symbol Σ, and altering a little the arrangement of the terms. It gives the following value of pL, which will be of use hereafter;

[6337c] $$pL=\frac{3.(2C-A-B)}{4i.C}.\{M^2.(L-L')+m.n^2.(L-l)+m'.n'^2.(L-l')+m''.n''^2.(L-l'')+m'''.n'''^2.(L-l''')\}.$$

* (3315) This is evident by comparing the coefficients of t, in the values of θ', Ψ'
[6342a] [6928, 6929], corresponding to the motion of the equator, and representing the values of p, with the values of (0), (1), (2), (3), &c. [6864—6868], depending on the disturbing forces of the sun and satellites, or the ellipticity of Jupiter. The former being much smaller
[6342b] than the latter, may be neglected, as in [6342′], in finding the equations depending on these values of p.

† (3316) Putting $p=0$, in the equations [6337—6340], then substituting the
[6347a] values [6343—6346], and dividing by $-(L-L')$, we get [6347—6350] respectively.

The latitude of the satellite m, above the orbit of Jupiter, is represented by a series of terms of the form $(l-L').\sin.(v+pt+\text{A})$; therefore we shall have,* [6351]

$$\Sigma'.(l-L').\sin.(v+pt+\text{A}) = \text{latitude of the satellite } m, \text{ above Jupiter's orbit.} \qquad [6352]$$

If we include only the part of this expression which depends upon the displacing of Jupiter's equator and orbit, we shall have, as we have seen in [6343], $L-l=\lambda.(L-L')$; hence we deduce,† [6353]

$$l-L'=(1-\lambda).(L-L'). \qquad [6354]$$

Therefore we shall have, for the part of the latitude of m, above the orbit of Jupiter, and including the terms depending on the displacing of Jupiter's equator and orbit; the following expression; [6354']

$$\Sigma'.(l-L').\sin.(v+pt+\text{A}) = (1-\lambda).\Sigma'.(L-L').\sin.(v+pt+\text{A}). \qquad [6355]$$

If the satellite move in the plane of Jupiter's equator, its latitude above Jupiter's orbit will be, ‡ [6355']

$$\Sigma'.(L-L').\sin.(v+pt+\text{A}). \qquad [6356]$$

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* (3317) The latitude of the satellite m above the fixed plane, in [6298x line 1], is $\Sigma'.l.\sin.(v+pt+\text{A})$; and if it were moving in the plane of the orbit of Jupiter, it would be [6352a] $\Sigma'.L'.\sin.(v+pt+\text{A})$, as in [6298$x$ line 5]. Subtracting this expression from the preceding, we get $\Sigma'.(l-L').\sin.(v+pt+\text{A})$, for the latitude of the satellite m, above Jupiter's orbit; [6352b] as in [6352].

† (3318) The equation [6343] gives $l=L-\lambda.(L-L')$; subtracting L' from both sides of the equation, we get [6354]. Multiplying this by $\sin.(v+pt+\text{A})$, and prefixing [6355a] the sign Σ', we obtain for the latitude [6352] the expression [6355]. We may observe that the equations [6344, 6345, 6346] may be deduced from [6343] by increasing the [6355b] accents on l, λ, by *one*, *two* or *three* marks respectively; and by performing the same process on [6354], which was deduced from [6343], we obtain similar expressions for the [6355c] second, third and fourth satellites. Thus, by changing l into l''', and λ into λ''', the equation [6343] becomes like [6346]; and [6354] changes into $l'''-L'=(1-\lambda''').(L-L')$; [6355d] which is used in [7343].

‡ (3319) If the satellite m be supposed to move in the plane of the *equator* of Jupiter, we shall have, by neglecting terms of the order θ^2 as in [6316''], s_1 [6051] for its latitude above the fixed plane; and this is represented by $\Sigma'.L.\sin.(v+pt+\text{A})$ [6356a] [6298x line 6]. In like manner, if the satellite be supposed to move in the plane of the *orbit* of Jupiter, its latitude, above the fixed plane, can be obtained by changing U into v, [6356b] in [6298x line 5], which gives $\Sigma'.L'.\sin.(v+pt+\text{A})$. Subtracting this from the expression [6356a], we get [6356], which represents very nearly the latitude of the satellite m, above [6356c]

The part

[6357] $$(1-\lambda).\Sigma'.(L-L').\sin.(v+pt+\mathrm{A}),$$

of the expression of the latitude of the satellite, above Jupiter's orbit [6356], *is therefore the latitude which it would have, on the supposition that it moves upon an intermediate plane, passing between the planes of the equator and the*
[6358] *orbit of Jupiter, through the common intersection of those two planes; the inclination of this intermediate plane, to the plane of Jupiter's orbit, being to the inclination of Jupiter's equator to the same orbit, as* $1-\lambda$ *to* 1. We
[6359] shall suppose, as in [3108, 3104, &c.], that

[6360] θ' = the inclination of the equator of Jupiter to its variable orbit;

[6361] $-\Psi'$ = the longitude of the descending node of the equator of Jupiter, on its variable orbit; this longitude being counted from the axis of x.

Thus the part of the latitude of the satellite m, *above Jupiter's orbit, including the effect of the changes in this orbit and in the equator* [6354'], *is**

[6362] $$(\lambda-1).\theta'.\sin.(v+\Psi').$$

[6356d] Jupiter's orbit, supposing this satellite to move in the plane of Jupiter's equator. The latitude [6356] corresponds to the inclination θ' [6360]; and if this inclination be decreased
[6356e] in the ratio of $1-\lambda$ to 1, as in [6359], the corresponding latitude will be decreased in nearly the same ratio, and will become as in [6357].

[6357a] * (3320) If we repeat the calculation in [6323a—f], changing the *fixed* plane into the plane of Jupiter's orbit, it will change θ [6312] into θ' [6360], and Ψ [6313] into Ψ'
[6357b] [6361] nearly, as in [3104—3105]. By this means the expression $-\theta.\sin.(v+\Psi)$ [6323d], which represents the elevation of the satellite m above the *fixed plane*, upon the
[6357c] supposition that the satellite moves in the plane of the equator of Jupiter, will change into
[6357d] $-\theta'.\sin.(v+\Psi')$, *representing the elevation of the satellite* m *above the variable orbit of Jupiter, the motion of the satellite being, as before, in the plane of the equator.* Putting the expression [6357d] equal to the same latitude, found in [6356], we get,

[6357e] $$\Sigma'.(L-L').\sin.(v+pt+\mathrm{A}) = -\theta'.\sin.(v+\Psi').$$

Multiplying this by $1-\lambda$, and substituting the result in [6355], we get,

[6357f] $$\Sigma'.(l-L').\sin.(v+pt+\mathrm{A}) = (1-\lambda).\Sigma'.(L-L').\sin.(v+pt+\mathrm{A}) = (\lambda-1).\theta'.\sin.(v+\Psi').$$

The first member of this expression represents, as in [6354', 6355], the part of the latitude
[6357g] of the satellite m, above Jupiter's orbit, including the terms arising from the displacing of Jupiter's equator and orbit; therefore it must also be represented by the last member of the
[6357h] expression [6357f], as in [6362]. This expression [6362] corresponds, in the lunar theory, to that in [5351 or 5357]. But there is this very essential difference, that in the lunar theory the coefficient $-6^s,487$, [5357], is but a very small part of the inclination of the
[6357i] earth's orbit to its equator, so that $\lambda-1$ must be very small; but in the theory of this satellite $(\lambda-1).\theta'$ is nearly equal to $-\theta'$; λ being very small [6923].

This result is analogous to that which we have found in the lunar theory [5352]; *observing that for the moon,* $1-\lambda$ *is very small; but for the* [6363]
satellites of Jupiter, $1-\lambda$ *differs but little from unity* [6357*i*].

We shall now determine the values of θ, Ψ, θ', Ψ', *which depend upon the displacing of the equator and the orbit of Jupiter.* We shall in the first [6364]
place observe, that we may satisfy, very nearly, the equations [6337–6341], by putting,*

$$
\begin{array}{llll}
L'=0, & \text{or} \quad L-L'=L; & 1 & \\
l\ =(1-\lambda\ ).L, & L-l\ \ =\lambda\, L; & 2 & \\
l'\ =(1-\lambda'\ ).L, & L-l'\ =\lambda'\, L; & 3 & [6365] \\
l''=(1-\lambda'').L, & L-l''\ =\lambda''\, L; & 4 & \\
l'''=(1-\lambda''').L, & L-l'''=\lambda'''\, L; & 5 &
\end{array}
$$

then the equation [6341] gives a value of p, which we shall denote by ${}^{1}p$, [6365']
and we shall have,

$$
{}^{1}p=\frac{3}{4i}\cdot\frac{(2C-A-B)}{C}\cdot\{M^2+m.n^2.\lambda+m'.n'^2.\lambda'+m''.n''^2.\lambda''+m'''.n'''^2.\lambda'''\}. \qquad [6366]
$$

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* (3321) Taking as in [6023*q* or 6398], the orbit of Jupiter in 1750 for the fixed plane, we shall have the sun's latitude S' [6040] of the same order as the disturbing forces, which act upon the planet's orbit; and if we neglect for a moment this latitude, we shall have $S'=0$; whence $L'=0$ [6300 line 5], as in [6365 line 1]. Substituting [6365*a*]
this in [6343—6346], and making a slight reduction, we get the assumed expressions of l, l', l'', l''', [6365 lines 2—5]. From these we easily deduce the values of $L-L'$, $L-l$, &c. [6365]; and by substituting them in [6337–6340], then dividing by $-L$, they [6365*b*]
become nearly satisfied by using [6347–6350]. Substituting the same values in [6341], then dividing by L, we get p, or ${}^{1}p$ [6366]; corresponding to the precession of the equinoxes of Jupiter's equator, arising from the action of the sun and of the satellites, upon the ellipsoidal figure of the planet. By means of this value of ${}^{1}p$, the argument $v+{}^{1}pt+\Lambda$ [6365*c*]
[6300 line 6], for finding the latitude s_1, supposing the satellite to move in the plane of the equator, is augmented by the quantity ${}^{1}pt$ in the time t; and as this argument must evidently represent, as in [533*a*], the distance of the satellite from the equinoctial point. [6365*d*]
The part ${}^{1}pt$ must express the precession of the equinoxes [6323*a*, &c.]; as this contains no variable part, like that neglected in [6317*a*], it may *be considered as the mean precession arising from the action of the sun and satellites, on the ellipsoid of Jupiter*; but not [6365*e*]
including the effect of the change in the plane of Jupiter's orbit, from the value of L' depending on the disturbing forces of the planets; so that ${}^{1}pt$ corresponds to lt in the [6365*f*]
expression of the precession of the equinoxes of the earth's orbit [3100]. The effect of the change in Jupiter's orbit, from the action of the planets, is noticed in [6372, &c., or 6928, 6929], being similar to that which changes ψ [3100] into ψ' [3107]. [6365*g*]

[6367] The value of L remains arbitrary, and we shall denote it by $-{}^1L$. Hence we have, by noticing only this value, the latitude of the satellite
[6368] above the fixed plane, equal to $-{}^1L.\sin.(v+{}^1pt+{}^1\Lambda)$,* ${}^1\Lambda$ being the arbitrary constant quantity corresponding to ${}^1p$. But this latitude is also
[6369] equal to $-\theta.\sin.(v+\Upsilon)$ [6322]; hence we deduce,

[6370] $$\theta.\sin.\Upsilon={}^1L.\sin.({}^1pt+{}^1\Lambda);$$

[6371] $$\theta.\cos.\Upsilon={}^1L.\cos.({}^1pt+{}^1\Lambda).$$

[6372] *${}^1pt$ denotes the mean precession of the equinoxes of Jupiter* † [6365e]; but the true precession is modified by the displacing of Jupiter's orbit, as we have
[6372'] seen, in [3111—3115], that the change of the ecliptic modifies the precession of the equinoxes relative to the earth. To determine these modifications, we shall observe that the equation [6341] gives,‡

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* (3322) This expression of s_1 is deduced from its assumed form [6300 line 6],
[6368a] changing L into $-{}^1L$ [6367], and p into ${}^1p$ [6365'], in order to conform to the present notation. Putting the value of s_1 [6368] equal to that in [6323d], we get,

[6368b] $$-\theta.\sin.(v+\Upsilon)=-{}^1L.\sin.(v+{}^1pt+{}^1\Lambda).$$

Now by means of [21] Int., we have,

$$\sin.(v+\Upsilon)=\sin.v.\cos.\Upsilon+\cos.v.\sin.\Upsilon;$$

[6368c] $$\sin.(v+{}^1pt+{}^1\Lambda)=\sin.v.\cos.({}^1pt+{}^1\Lambda)+\cos.v.\sin.({}^1pt+{}^1\Lambda).$$

[6368d] Substituting these developments in the equation [6368b], and then putting separately the coefficients of $-\cos.v$, $-\sin.v$, in both members, equal to each other, we get [6370, 6371].

† (3323) We have observed, in [6365f], that the mean precession of Jupiter's equinox
[6372a] ${}^1pt$, deduced from [6366], is similar to that of the earth lt [3100]; or, in other words, that ${}^1p$, in Jupiter's theory [6366], is equivalent to l [3098], in the theory of the earth; and for the sake of illustration, it may not be amiss to compare these two expressions.
[6372b] Now the first term of l [3098], depending on the sun's action, is $\frac{3m^2}{4n}.\left(\frac{2C-A-B}{C}\right).\cos.h$; and by changing m into M [3059, 6101']; also n into i [6317f], to conform to
[6372c] the present notation; then putting $\cos.h=1$, on account of the smallness of h [7319], we get the first term of [6366], depending on M^2. In like manner, the term of [3098], containing λ, which depends on the moon's action upon the ellipsoid of the earth, is
[6372d] similar to those in [6366], containing λ, λ', λ'', λ''', and depending on the action of the satellites upon the ellipsoid of Jupiter.

‡ (3324) Multiplying [6366] by $L-L'$, and substituting the values [6343—6346], we get,

[6373a] $${}^1p.(L-L')=\frac{3}{4i}.\frac{(2C-A-B)}{C}.\{M^2.(L-L')+m.n^2.(L-l)+m'.n'^2.(L-l')+m''.n''^2.(L-l'')+m'''.n'''^2.(L-l''')\}.$$

$$(p-{}^1p).L+{}^1p\,L'=0\,;$$ [6373]

p being, in this case, one of the values of p, relative to the displacing of Jupiter's orbit. This equation gives, [6374]

$$L=-\frac{{}^1p.L'}{p-{}^1p}.$$ [6375]

Therefore, if we notice only the values of p relative to the displacing of Jupiter's orbit, we shall have, [6375′]

$$\theta.\sin.\Psi={}^1p.\Sigma'.\frac{L'.\sin.(pt+\Lambda)}{p-{}^1p}\,;$$ [6376]

$$\theta.\cos.\Psi={}^1p.\Sigma'.\frac{L'.\cos.(pt+\Lambda)}{p-{}^1p}\,.$$ [6377]

Connecting these two values with those in [6370, 6371], we shall have,*

$$\theta.\sin.\Psi={}^1L.\sin.({}^1pt+{}^1\Lambda)+{}^1p.\Sigma'.\frac{L'.\sin.(pt+\Lambda)}{p-{}^1p}\,;$$ [6378]

$$\theta.\cos.\Psi={}^1L.\cos.({}^1pt+{}^1\Lambda)+{}^1p.\Sigma'.\frac{L'.\cos.(pt+\Lambda)}{p-{}^1p}\,.$$ [6379]

Hence we easily obtain,†

The second member of this equation is the same as the value of pL [6337e]; hence we obtain ${}^1p.(L-L')=pL$; and by transposition we get [6373]; whence we deduce L [6375]; and by substituting it in [6326, 6327], we obtain [6376, 6377]. The symbol Σ' being supposed to include the angles $pt+\Lambda$, depending on the displacement of Jupiter's orbit [6324′]. [6373b] [6373c] [6373d]

* (3325) The sum of the terms [6370, 6376] gives the value of $\theta.\sin.\Psi$ [6378]; and the sum of those in [6371, 6377], gives $\theta.\cos.\Psi$ [6379]. Observing that the values [6370, 6371] are founded on $L'=0$ [6365a], corresponding to a *fixed* orbit of Jupiter; and the terms [6376, 6377] depend on the *changes of that orbit*; consequently these sums represent the whole values of $\theta.\sin.\Psi$, $\theta.\cos.\Psi$. [6378a] [6378b]

† (3326) From [22] Int. we get $\sin.A.\cos.{}^1pt-\cos.A.\sin.{}^1pt=\sin.(A-{}^1pt)$; hence if we multiply [6378] by $\cos.{}^1pt$, and [6379] by $-\sin.{}^1pt$, then take the sum of the products, we shall get, by using [6380a], [6380a]

$$\theta.\sin.(\Psi-{}^1pt)={}^1L.\sin.{}^1\Lambda+{}^1p.\Sigma'.\frac{L'.\sin.(pt-{}^1pt+\Lambda)}{p-{}^1p},$$ [6380b]

which is easily reduced to the integral form in [6380], as is evident by taking the integral, indicated by the sign f. The advantage of this form is seen in [6386], where we are enabled to introduce the symbol γ. In like manner, by multiplying [6378] by $\sin.{}^1pt$, also [6379] by $\cos.{}^1pt$, and reducing by means of the formula [6380c] [6380d]

$$\sin.A.\sin.{}^1pt+\cos.A.\cos.{}^1pt=\cos.(A-{}^1pt)\quad [24]\ \text{Int.},$$ [6380d′]

[6380] $$\theta.\sin.(\varpi - {}^1pt) = {}^1L.\sin.{}^1A + {}^1p.\Sigma'.\int L'dt.\cos.(pt - {}^1pt + A);$$

[6381] $$\theta.\cos.(\varpi - {}^1pt) = {}^1L.\cos.{}^1A - {}^1p.\Sigma'.\int L'dt.\sin.(pt - {}^1pt + A).$$

Now we have, as in [6330, 6331],

[6382] $$\gamma.\sin.\Pi = -\Sigma'.L'.\sin.(pt + A);$$

[6383] $$\gamma.\cos.\Pi = \Sigma'.L'.\cos.(pt + A);$$

and from these we deduce,*

[6384] $$\gamma.\sin.(\Pi + {}^1pt) = -\Sigma'.L'.\sin.(pt - {}^1pt + A);$$

[6385] $$\gamma.\cos.(\Pi + {}^1pt) = \Sigma'.L'.\cos.(pt - {}^1pt + A);$$

therefore,

[6386] $$\theta.\sin.(\varpi - {}^1pt) = {}^1L.\sin.{}^1A + {}^1p.\int\gamma dt.\cos.(\Pi + {}^1pt);$$

[6387] $$\theta.\cos.(\varpi - {}^1pt) = {}^1L.\cos.{}^1A + {}^1p.\int\gamma dt.\sin.(\Pi + {}^1pt).$$

[6388] *By means of these two equations we may obtain the excess*† $\varpi - {}^1pt$ *of the*
[6389] *true precession of Jupiter's equinoxes above the mean precession; and the inclination* θ *of the equator of this planet to the fixed plane.*

The latitude of the satellite m, supposing it to move in the equator
[6390] of Jupiter, is $-\theta.\sin.(v+\varpi)$ [6322]; and its latitude above the same plane,
[6391] supposing it to move upon the orbit of Jupiter, is $\gamma.\sin.(v - \Pi)$; ‡

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we get,

[6380e] $$\theta.\cos.(\varpi - {}^1pt) = {}^1L.\cos.A + {}^1p.\Sigma'.\frac{L'.\cos.(pt - {}^1pt + A)}{p - {}^1p};$$

which is the same as [6381], as is evident by taking the integral of its last term, as in
[6380f] [6380c]. This integration procures the same advantage as above, of introducing the symbol γ in [6387].

[6386a] * (3327) Multiplying [6382, 6383] by $\cos.{}^1pt$, $\sin.{}^1pt$, respectively; taking the sum of the products and reducing, by [21, 22] Int.; we get [6384]. Again, multiplying [6382] by $-\sin.{}^1pt$, and [6383] by $\cos.{}^1pt$; taking the sum of the products and
[6386b] reducing by [23, 24] Int., we get [6385]. Substituting the values [6385, 6384] in [6380, 6381], they become respectively as in [6386, 6387].

[6388a] † (3328) The true precession being represented by ϖ [6313], and the mean precession by ${}^1pt$ [6372]; the excess of the former, above the latter, will be $\varpi - {}^1pt$, as in [6388].

[6391a] ‡ (3329) The inclination of the sun's relative orbit about Jupiter, to the fixed plane, is γ [6313′]; and the longitude of its *ascending* node is Π [6314]. Hence $v - \Pi$ is the angular distance of the satellite from that node; and by proceeding as in [6328c], we obtain
[6391b] $\gamma.\sin.(v - \Pi)$, for the latitude of the satellite m above the fixed plane, supposing it to move

subtracting this last expression from the preceding one, we get the latitude [6391′]
of the satellite above the orbit of Jupiter, *supposing the satellite to move in the plane of the equator; but this last latitude is* $-\theta'.\sin.(v+\Psi')$ [6357d]; [6392]
therefore we have,

$$-\theta'.\sin.(v+\Psi') = -\theta.\sin.(v+\Psi)-\gamma.\sin.(v-\Pi)\,; \qquad [6393]$$

v being indeterminate. If we put successively $v=-{}^{1}pt$ and $v=100^{\circ}-{}^{1}pt$, [6394]
we obtain from the preceding equation,

$$\theta'.\sin.(\Psi'-{}^{1}pt) = \theta.\sin.(\Psi-{}^{1}pt)-\gamma.\sin.(\Pi+{}^{1}pt)\,; \qquad [6395]$$

$$\theta'.\cos.(\Psi'-{}^{1}pt) = \theta.\cos.(\Psi-{}^{1}pt)+\gamma.\cos.(\Pi+{}^{1}pt). \qquad [6396]$$

These equations will make known the precession Ψ' [6361], *and the inclination* θ' [6360], *of the equator, referred to Jupiter's orbit.* [6397]

Fixed plane and axis.

It is sufficient, for the uses of astronomy, to have the values of these quantities in a converging series for two or three centuries. We shall take, for the fixed plane, the orbit of Jupiter in 1750; *and that epoch for the origin of the time* t. [6398] *We shall also take, for the axis of* x, *the line of the vernal equinox of Jupiter at the same epoch.* [6398′] Then reducing the expression to series, and neglecting the second and higher powers of t, we [6399]
shall have,*

$$\gamma.\sin.\Pi = at\,; \qquad [6400]$$

$$\gamma.\cos.\Pi = bt\,; \qquad [6401]$$

a and b being constant quantities, which are easily found, as in [6901, 6902], [6402]

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in the plane of the sun's relative orbit, or in the plane of the orbit of Jupiter. Subtracting this from the expression in [6390], we get the latitude of the satellite m above Jupiter's orbit, supposing it to move in the plane of Jupiter's equator, as in the second member of [6391c]
[6393]. Putting this equal to the value of the same latitude, found in [6357d], we get the [6391d]
first member of [6393]. Substituting in [6393] $v=200^{\circ}-{}^{1}pt$, we get [6395]; and
by putting $v=300^{\circ}-{}^{1}pt$, we obtain [6396], by making a few reductions. The same [6391e]
results are obtained by using the values of v [6394].

* (3330) The developments in [6400, 6401] are similar to those in the earth's orbit [4332], neglecting terms of the order t^2; observing that the assumed values of p'', q'' [6400a]
[4249] for the earth, are similar to the expressions $\gamma.\sin.\Pi$, $\gamma.\cos.\Pi$, in the present [6400b]
theory; because φ'', θ'', [4249, 4082, 4083], or rather $\varphi''_{\prime}$, $\theta''_{\prime}$ [4238e] are changed into γ, Π, [6313′, 6314]. The analytical values of a, b, are given in [6901, 6902]; [6400c]
and their numerical values in [6906—6908, or 6928, 6929].

[6402′] by means of the expression of the motions of Jupiter's orbit, given in [4246.] From the preceding equations we obtain,*

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* (3331) Putting $z=\daleth$, and $\alpha={}^1pt$, in [60, 61] Int., we obtain the developments
[6403a] of $\sin.(\daleth+{}^1pt)$, $\cos.(\daleth+{}^1pt)$, according to the powers of t. Substituting these expressions in the terms under the sign $\int$, in [6386, 6387], and then the values of
[6403b] $\gamma.\sin\daleth$, $\gamma.\cos.\daleth$ [6400, 6401], we find that these integrals are of the order t^2, or of a higher order, which are neglected in [6399]. Therefore we may reject the terms under the sign $\int$, in [6386, 6387], and we shall have,

[6403c] $$\theta.\sin.(\Psi-{}^1pt)={}^1L.\sin.{}^1\Lambda\,; \qquad \theta.\cos.(\Psi-{}^1pt)={}^1L.\cos.{}^1\Lambda.$$

[6403d] The sum of the squares of these two equations, gives $\theta^2={}^1L^2$, or $\theta={}^1L$, as in [6404]; substituting this in the first of the equations [6403c], and then dividing by ${}^1L$, we obtain,

[6403e] $$\sin.(\Psi-{}^1pt)=\sin.{}^1\Lambda\,; \qquad \text{whence} \quad \Psi-{}^1pt={}^1\Lambda.$$

Now we find, in [6398′], that the line drawn from the centre of Jupiter, in the direction of
[6403f] the vernal equinox of Jupiter's orbit, at the epoch 1750, is taken for the axis of x; and in [6313] Ψ represents the retrograde motion of the node from this axis, after that epoch; so that when $t=0$, we shall have $\Psi=0$. Substituting these values of t and Ψ,
[6403g] in the first member of the second equation [6403e], we obtain $0={}^1\Lambda$, as in [6403]; and by using this value of the constant quantity ${}^1\Lambda$, we find that the second equation
[6403h] [6403e] becomes generally, for any value of t, $\Psi-{}^1pt=0$, or $\Psi={}^1pt$, as in [6404].

In finding the values of Ψ', θ', from the equations [6395, 6396], we must observe that
[6403i] the equation $\Psi-{}^1pt=0$ [6403h] gives $\theta.\sin.(\Psi-{}^1pt)=0\,;$ $\theta.\cos.(\Psi-{}^1pt)=\theta={}^1L$ [6403d]. Moreover, by using the developments [6403a], in connexion with the equations [6400, 6401], and neglecting terms of the order t^2, we have,

[6403k] $$\gamma.\sin.(\daleth+{}^1pt)=at\,; \qquad \gamma.\cos.(\daleth+{}^1pt)=bt.$$

Substituting the expressions [6403i, k] in the equations [6395, 6396], we obtain,

[6403l] $$\theta'.\sin.(\Psi'-{}^1pt)=-at\,; \qquad \theta'.\cos.(\Psi'-{}^1pt)={}^1L+bt.$$

[6403m] Dividing the first of these equations by the second, we get $\tan g.(\Psi'-{}^1pt)=\frac{-at}{{}^1L+bt}$. If we neglect terms of the order t^2, we may put the second member of this equation under
[6403n] the form $\frac{-at}{{}^1L}=\tan g.\left(\frac{-at}{{}^1L}\right)$ nearly, [45] Int. Hence the equation [6403m] becomes,

[6403o] $$\tan g.(\Psi'-{}^1pt)=\tan g.\left(\frac{-at}{{}^1L}\right); \quad \text{whence} \quad \Psi'-{}^1pt=-\frac{at}{{}^1L}.$$

From the last of these equations we easily deduce the value of Ψ' [6405]. Taking the
[6403p] cosine of both members of the same equation [6403o], and developing the second member, according to the powers of t, by means of [44] Int., neglecting terms of the order t^2, we shall have $\cos.(\Psi'-{}^1pt)=1$. Substituting this in the first member of the second of the
[6403q] equations [6403l], we obtain $\theta'={}^1L+bt$, as in [6405]; and when $t=0$, it becomes
[6403r] $\theta'={}^1L$, as in [6406, 6360], corresponding to the epoch 1750.

$$^{1}\Lambda = 0\,; \qquad [6403g] \qquad [6403]$$

$$\Psi = {}^{1}pt\,; \qquad \theta = {}^{1}L\,; \qquad [6403h, d] \qquad [6404]$$

$$\Psi' = {}^{1}pt - \frac{at}{{}^{1}L}\,; \qquad \theta' = {}^{1}L + bt. \quad [6403o, q] \qquad [6405]$$

so that ^{1}L *is the inclination of the equator to the orbit of Jupiter, in* [6406]
1750 [6403r].

Lastly, if we put γ_1 *for the inclination of the orbit of the satellite* m, *to* [6407]
the fixed plane [6315]; *and* $\daleth_1$ *for the longitude of its ascending node*
[6316]; we shall have, when we consider only the quantities relative to the [6408]
displacing of the orbit and equator of Jupiter [6342, 6343],

$$L - l = \lambda.(L - L')\,; \qquad [6409]$$

and by transposition we easily deduce the following expression of l;

$$l = (1-\lambda).L + \lambda.L'. \qquad [6410]$$

Hence we obtain,*

$$\gamma_1.\sin.\daleth_1 = (1-\lambda).\theta.\sin.\Psi + \lambda.\gamma.\sin.\daleth\,; \qquad [6411]$$

$$\gamma_1.\cos.\daleth_1 = (\lambda-1).\theta.\cos.\Psi + \lambda.\gamma.\cos.\daleth\,. \qquad [6412]$$

Thus, by noticing only the displacing of the equator and orbit of Jupiter, we shall have,†

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* (3332) Substituting the value of l [6410], in [6332], we get [6411b]; and by using the values [6326, 6330] it becomes as in [6411]. In like manner, the substitution [6411a] of l [6410], in [6333], gives [6411c]; and by using the values [6327, 6331], we obtain [6412].

$$\gamma_1.\sin.\daleth_1 = -(1-\lambda).\Sigma'.L.\sin.(pt+\Lambda) - \lambda.\Sigma'.L'.\sin.(pt+\Lambda)\,; \qquad [6411b]$$

$$\gamma_1.\cos.\daleth_1 = \quad (1-\lambda).\Sigma'.L.\cos.(pt+\Lambda) + \lambda.\Sigma'.L'.\cos.(pt+\Lambda)\,. \qquad [6411c]$$

† (3333) If we neglect terms of the order t^2, we shall obtain, from the first of the equations [6404], $\sin.\Psi = {}^{1}pt$, $\cos.\Psi = 1$. Substituting these and $\theta = {}^{1}L$ [6404] in [6413a] the terms of [6411, 6412], which have the factor $(1-\lambda)$ or $(\lambda-1)$, we get the terms of [6413, 6414], having the same factor. Lastly substituting the values of $\gamma.\sin.\daleth$, [6413b] $\gamma.\cos.\daleth$, [6400, 6401], in the terms of [6411, 6412], containing γ, we get the terms of [6413, 6414], containing a or b. The formulas [6413, 6414] correspond to the satellite [6413c] m; and by accenting the symbols γ_1, $\daleth_1$, λ, we get the following expressions for the satellite m';

$$\gamma_1'.\sin.\daleth_1' = (1-\lambda').{}^{1}L.{}^{1}pt + \lambda'.at\,; \qquad [6413d]$$

$$\gamma_1'.\cos.\daleth_1' = (\lambda'-1).{}^{1}L. \quad + \lambda'.bt\,. \qquad [6413e]$$

By adding one more accent we obtain the values corresponding to m''; and with another accent they represent the values for m'''. [6413f]

[6413] $$\gamma_1 . \sin . \Pi_1 = (1-\lambda).{}^1L.{}^1pt + \lambda . at ;$$

[6414] $$\gamma_1 . \cos . \Pi_1 = (\lambda-1).{}^1L \quad + \lambda . bt .$$

Relative to the values of p, which depend upon the mutual action of the
[6415] satellites, *we may put* L' *equal to nothing; since the orbit of Jupiter is*
[6415′] *not sensibly displaced by the action of the satellites.* We may also, for these values of p, neglect the value of L, in comparison with the corresponding values of l, l', &c. [6417g]. For it follows, from the equation [6341],
[6416] that the value of pL is multiplied by the small factor $\frac{2C-A-B}{C}$; therefore it is of the order of the product of the ellipticity of Jupiter, by the masses of the satellites; and such quantities we have heretofore neglected. Hence
[6417] we may neglect pL and $(0).L$,* in comparison with $(0).l$, $(1).l$, &c.; then the equations [6337—6340] become,

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[6417a] * (3334) We have in [6313m], $i = 40n'''$ nearly; hence we obtain from [6919] $\frac{3.(2C-A-B)}{4i.C} = \frac{1}{362.n'''}$ nearly. Substituting this, together with the values of n, n', n'', M, in terms of n''' [6025a, b, c, i], and those of m, m', m'', m''' [7162—7165], in [6337c],
[6417b] and retaining only the first significant figure of the results, for the mere object of ascertaining the order of the terms, we get,

[6417c] $$pL = \frac{1}{362}.n'''.\{0,00001.(L-L')+0,002.(L-l)+0,0005.(L-l')+0,0005.(L-l'')+0,00004.(L-l''')\}.$$

[6417d] Now we have, as in [6025k], $\frac{1}{362}.n''' = 241830''$; substituting this in the preceding expression, and transposing the terms depending on L to the first member, we get, by a rough calculation, neglecting L' as in [6415],

[6417e] $$\{p-735''\}.L = -483''.l - 121''.l' - 121''.l'' - 10''.l'''.$$

Substituting the values of l', l'', l''' [7226—7229], corresponding to the first angle p, we
[6417f] find that L is less than $\frac{1}{1000}.l$; similar results are obtained by using the values [7233—7236], relative to the second angle p_1; those in [7238—7241], relative to the
[6417g] third angle p_2; or those in [7245—7248], relative to the fourth angle p_3. Hence it is evident that we may neglect L, as in [6415′].

$$0=\left\{p-(0)-\boxed{0}-(0,1)-(0,2)-(0,3)\right\}.l \qquad 1$$
$$+(0,1).l'+(0,2).l''+(0,3).l''' ; \qquad 2 \qquad [6418]$$

$$0=\left\{p-(1)-\boxed{1}-(1,0)-(1,2)-(1,3)\right\}.l' \qquad 1$$
$$+(1,0).l+(1,2).l''+(1,3).l''' ; \qquad 2 \qquad [6419]$$

$$0=\left\{p-(2)-\boxed{2}-(2,0)-(2,1)-(2,3)\right\}.l'' \qquad 1$$
$$+(2,0).l+(2,1).l'+(2,3).l''' ; \qquad 2 \qquad [6420]$$

$$0=\left\{p-(3)-\boxed{3}-(3,0)-(3,1)-(3,2)\right\}.l''' \qquad 1$$
$$+(3,0).l+(3,1).l'+(3,2).l''. \qquad 2 \qquad [6421]$$

If we suppose,

$$l'=\zeta'.l\,; \qquad l''=\zeta''.l\,; \qquad l'''=\zeta'''.l\,; \qquad [6422]$$

l will vanish from the preceding equations, and we shall obtain four equations Symbols.
between the indeterminate quantities ζ', ζ'', ζ''', and p; whence we may [6423]
find p by an equation of the fourth degree. We shall put p, p_1, p_2, p_3, [6424]
for the four roots of the equation, and

$$\zeta_1', \ \zeta_1'', \ \zeta_1''' ; \qquad \zeta_2', \ \zeta_2'', \ \zeta_2''' ; \qquad \zeta_3', \ \zeta_3'', \ \zeta_3''' ; \qquad [6425]$$

for what ζ', ζ'', ζ''' become, when we change successively p into p_1, p_2, p_3. *We shall then suppose that* s, s', s'', s''', *instead of expressing as in* [6033, 6036'] *nearly the latitudes of the satellites above the fixed plane,* [6426]
express their latitudes above Jupiter's orbit, these last latitudes being particularly required in the calculation of their eclipses; in this case we shall have,* [6426']

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* (3335) The latitude of the satellite m is expressed in [6298x line 1], by a series of terms of the form $s=\Sigma'.l.\sin.(v+pt+\text{A})$; and we have observed in [6298y] that the [6419a] terms depending on the angles $pt+\text{A}$, $p_1t+\text{A}_1$, $p_2t+\text{A}_2$, $p_3t+\text{A}_3$, arise from the mutual action of the sun and satellites, and on the ellipticity of Jupiter. These terms are [6419b] explicitly retained in the expression of s [6427 lines 2, 3, 4, 5]. We have also observed, in [6298z], that the remaining terms, depending on the angles $p_4t+\text{A}_4$, $p_5t+\text{A}_5$, &c., and arising from the displacement of Jupiter's equator and orbit, are reduced to the single term $(\lambda-1).\theta'.\sin.(v+\Psi')$ [6362]; which is the same as that in [6427 line 1], referred [6419c] to the variable orbit of Jupiter [6360—6362]; so that the expressions [6427—6430] will

[6427]

Latitudes above the variable orbit of Jupiter.

$$\begin{array}{lll}
s = (\lambda - 1).\theta'.\sin.(v + \Psi') & & 1 \\
\quad + l.\sin.(v + pt + \Lambda) & \text{[Satellite } m\text{]} & 2 \\
\quad + l_1.\sin.(v + p_1 t + \Lambda_1) & & 3 \\
\quad + l_2.\sin.(v + p_2 t + \Lambda_2) & & 4 \\
\quad + l_3.\sin.(v + p_3 t + \Lambda_3) & & 5
\end{array}$$

[6428]

$$\begin{array}{lll}
s' = (\lambda' - 1).\theta'.\sin.(v' + \Psi') & & 1 \\
\quad + \zeta'.l.\sin.(v' + pt + \Lambda) & \text{[Satellite } m'\text{]} & 2 \\
\quad + \zeta_1'.l_1.\sin.(v' + p_1 t + \Lambda_1) & & 3 \\
\quad + \zeta_2'.l_2.\sin.(v' + p_2 t + \Lambda_2) & & 4 \\
\quad + \zeta_3'.l_3.\sin.(v' + p_3 t + \Lambda_3) & & 5
\end{array}$$

[6429]

$$\begin{array}{lll}
s'' = (\lambda'' - 1).\theta'.\sin.(v'' + \Psi') & & 1 \\
\quad + \zeta''.l.\sin.(v'' + pt + \Lambda) & \text{[Satellite } m''\text{]} & 2 \\
\quad + \zeta_1''.l_1.\sin.(v'' + p_1 t + \Lambda_1) & & 3 \\
\quad + \zeta_2''.l_2.\sin.(v'' + p_2 t + \Lambda_2) & & 4 \\
\quad + \zeta_3''.l_3.\sin.(v'' + p_3 t + \Lambda_3) & & 5
\end{array}$$

[6430]

$$\begin{array}{lll}
s''' = (\lambda''' - 1).\theta'.\sin.(v''' + \Psi') & & 1 \\
\quad + \zeta'''.l.\sin.(v''' + pt + \Lambda) & \text{[Satellite } m'''\text{]} & 2 \\
\quad + \zeta_1'''.l_1.\sin.(v''' + p_1 t + \Lambda_1) & & 3 \\
\quad + \zeta_2'''.l_2.\sin.(v''' + p_2 t + \Lambda_2) & & 4 \\
\quad + \zeta_3'''.l_3.\sin.(v''' + p_3 t + \Lambda_3). & & 5
\end{array}$$

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be very nearly the latitudes referred to the variable orbit, as in [6426′]. Hence it appears
[6419d] that the expression of s [6427], contains the sensible terms of the proposed forms $v+pt+\Lambda$, $v+p_1t+\Lambda_1$, &c., depending on the mutual action of the bodies and the secular equations of the orbit and equator of Jupiter. In like manner we obtain from the expressions of s', s'', s''' [6298x lines 2, 3, 4], the corresponding values [6428, 6429, 6430].

We may incidentally remark, that if we neglect the first term in each of the expressions
[6419e] of s, s', s'', s''' [6427—6430], depending on Ψ', we shall find that the four remaining terms of each of these expressions become of similar forms to those in the values of
[6419f] δv, $\delta v'$, $\delta v''$, $\delta v'''$ [6241—6244], changing g, Γ, h, β, &c. into $-p$, $-\Lambda$, l, ζ, &c. respectively, and retaining, in every instance, the same number of accents on the corresponding symbols. Hence we shall have a table of symbols, similar to that in [6229d],
[6419g] and corresponding to the four roots p, p_1, p_2, p_3; and we may form, in like manner, a table of the values of these parts of s, s', s'', s''', similar to that in [6240t–w, or 6241g–k]. From these we deduce analogous results to those in [6241m, &c.]; namely, that the first
[6419h] satellite has a *peculiar* inclination l, and longitude of the node $-pt-\Lambda$; the second satellite has a *peculiar* inclination l', and longitude of the node $-p_1t-\Lambda_1$; the third

The expressions l, $l_{,}$, l_2, l_3; Λ, Λ_1, Λ_2, Λ_3, are the eight arbitrary constant quantities, which can be determined only by observation. *If we wish to obtain the latitudes of the satellites, above the fixed plane, we must add to the preceding expressions of s, s', s'', s''', their values, upon the supposition* [6431]
that the satellites move in the plane of the orbit of Jupiter.

11. *We shall now consider those inequalities of the motions of the satellites in latitude, depending on their mutual configuration, which acquire very small divisors by the integrations.* It is evident that the terms of the differential equation [6295], depending upon an angle which differs but little from* v, [6432]
will acquire such divisors. Now if we consider only the first power of the inclinations of the orbits, we shall find that all the angles of the different terms of this equation will be included in the form† $i.(v-v')\pm v'$; and v' [6433]

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satellite has a *peculiar* inclination l'', and longitude of the node $-p_2t-\Lambda_2$; and that the fourth satellite has a *peculiar* inclination l''', and longitude of the node $-p_3t-\Lambda_3$. [6419i]
Moreover we see, as in [6241t—u], that each of these latitudes may be considered as the sum of four distinct latitudes, computed for four different orbits, passing through the places [6419k]
of the nodes of the four satellites respectively.

* (3336) The equation [6295], for the determination of s, can be reduced to a form like that in [6446]; in which the coefficient of s is nearly equal to unity, as in [6447]. [6432a]
This equation is integrated like that in [6049k,l], changing t into v; observing that the coefficient $a_{,}^2$ is nearly equal to unity, so that the divisor $m_{,}^2-a_{,}^2$ [6049l], becomes $m_{,}^2-1$, nearly; which is very small when $m_{,}$ is nearly equal to unity, or when the angle [6432b]
$m_{,}v+\varepsilon_{,}$ becomes nearly equal to $v+\varepsilon_{,}$, as in [6432].

† (3337) The action of the satellites upon each other being similar to that of the planets, it is evident that the forms of the terms of δs relative to the satellites, will be [6434a]
similar to those which we have computed for the planets in [1030]. Now the general term of δs, [1030 line 3], depends on the angle $i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi$; and by [6434b]
neglecting terms of the second order in e, γ, we may substitute, in this angle, the values $nt+\varepsilon=v$, $n't+\varepsilon'=v'$ [953]; hence it becomes of the form $i.(v'-v)+v$; *i being* [6434c]
any integer, positive or negative, from $i=-\infty$ to $i=\infty$ [1028'], *including also* [6434d]
$i=0$, *in the term* [1030 line 2]. Then the corresponding term of [1030 line 3], has the divisor $n^2-\{n-i.(n-n')\}^2$, which is easily reduced to the form, [6434e]

$$\{n+[n-i.(n-n')]\}.\{n-[n-i.(n-n')]\}=\{2n-i.(n-n')\}.i.(n-n'). \qquad [6434f]$$

Now it is evident, from the definition of i [6434d], that we may change i into $i+1$; and [6434g]
by this means the angle $i.(v'-v)+v$ [6434c], changes into $(i+1).(v'-v)+v=i.(v'-v)+v'$; [6434h]

[6434] differs but very little from $\frac{1}{2}v$ [6150′]; so that the angle $i.(v-v')\pm v'$ will differ but very little from v, if $\frac{1}{2}(i\pm 1)=1$; which gives either $i=1$,
[6435] or $i=3$. In the case of $i=1$, the proposed angle becomes v; and in
[6436] the case of $i=3$, this angle is reduced to $3v-4v'$. *We have just examined the first of these cases*, upon which the secular variations of the
[6436′] orbit depend, in [6293, &c.]. It now remains to consider the inequalities depending on the angle $3v-4v'$.

[6437] The expression of R [6297] contains the term $R=m'.A^{(4)}.\cos.(4v-4v')$;
[6438] therefore the term $-\frac{r^2}{h^2}\cdot\frac{ds}{dv}\cdot\left(\frac{dR}{dv}\right)$, of the differential equation [6295], produces the expression [6440], by the substitution of $s=l.\sin.(v+pt+\text{A})$
[6439] [6300], and putting $t=\frac{v}{n}$ [6439a–c],* $\frac{r^2}{h^2}=a$ [6299a],

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[6434i] and the second form of the divisor [6434f], becomes $\{2n-(i+1).(n-n')\}.(i+1).(n-n')$. The form of the angle $i.(v'-v)+v'$ [6434h], is evidently equivalent to that in [6433],
[6434k] using the values of i [6434d]. Again, the divisor [6434i] becomes small when the first
[6434l] of its factors is nearly equal to nothing, or $2n-(i+1).(n-n')=0$, nearly; and if we substitute $n'=\frac{1}{2}n$ [6151], it becomes $n.\{2-\frac{1}{2}.(i+1)\}=\frac{1}{2}n.(3-i)$; which vanishes
[6434m] when $i=3$, as in [6435]. The same divisor [6434i] also vanishes when $i+1=0$, or $i=-1$, as in [6435]; the difference in the sign being of no importance, considering that
[6434n] i [6434d] may be made positive or negative, as well as the term $\pm v'$ [6433]. When $i=3$, the angle $i.(v'-v)+v'$ [6434h] becomes $4v'-3v$, as in [6436]; and when
[6434o] $i=-1$, it becomes $-(v'-v)+v'=v$, as in [6435]. The terms depending on v are
[6434p] noticed in [6298, &c.]; the angle $4v'-3v$, or $3v-4v'$, is treated of in [6437, &c.].

We find, in like manner, for the action of the third satellite m'' upon m, that the
[6434q] angle [6434h] becomes $i.(v''-v)+v''$, and the divisor [6434i] changes into
[6434r] $\{2n-(i+1).(n-n'')\}.(i+1).(n-n'')$. This factor vanishes when $i+1=0$; and then the angle $i.(v''-v)+v''$ [6434q], becomes v, as in [6435]; which is treated of in
[6434s] [6298, &c.] The other factor $2n-(i+1).(n-n'')$ [6434r], does not become small, because $n''=\frac{1}{4}n$, nearly [6151]; hence this factor becomes $n.\{2-(i+1).\frac{3}{4}\}$, which
[6434t] vanishes when $i=\frac{5}{3}$; but i being an integer [6434d], we may neglect this term. For similar reasons we may neglect the terms depending on the action of the satellite
[6434u] m''', and on the sun S; since they do not produce, by the integrations, terms having small divisors, like that depending on the angle $3v-4v'$ [6434p], which is computed in [6437, &c.].

[6439a] * (3338) The values of v, v' [6091], give $t=\frac{v}{n}-\frac{\varepsilon}{n}$; $v'=n't+\varepsilon'=\frac{n'v}{n}-\frac{n'\varepsilon}{n}+\varepsilon'$. These differ from the expressions in [6439, 6441], by the constant parts of t, v'; but this

$$4m'.aA^{(4)}.l.\sin.(4v-4v').\cos.\left(v+\frac{p}{n}v+\lambda\right).$$ [6440]

If we substitute $v'=\frac{n'v}{n}$ [6439a—c], which may always be done, when we neglect the excentricities of the orbits, the function [6440] will give, by its development, the term, [6441]

$$2m'.aA^{(4)}.l.\sin.\left(3v-\frac{4n'}{n}.v-\frac{p}{n}.v-\lambda\right).$$ [6442]

The term $\frac{r^2}{h^2}.\left(\frac{dR}{ds}\right)$ of the differential equation [6295], produces the following;* [6443]

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produces no effect in the subsequent calculations, because the differentials of the expressions [6439a], which are used in [6439d, &c.], are the same as the differentials of the expressions given by the author in [6439, 6441]; and the re-substitution of the assumed values of t, v', [6455'] corrects for this neglect of the constant terms. The part of R [6437] gives $-\left(\frac{dR}{dv}\right)=4m'.A^{(4)}.\sin.(4v-4v')$; and the value of s [6439] gives, [6439b] [6439c]

$$\frac{ds}{dv}=l.\left(1+\frac{p}{n}\right).\cos.\left(v+\frac{p}{n}.v+\lambda\right)=l.\cos.\left(v+\frac{p}{n}.v+\lambda\right) \text{ nearly.}$$ [6439d]

Substituting these and $\frac{r^2}{h^2}=a$ [6439], in the term [6438], it becomes as in [6440]. Reducing this by [18] Int. we get the term [6442], using the value of v' [6441]. The angle $3v-\frac{4n'}{n}.v-\frac{p}{n}.v$ [6442], is very nearly equal to v, because $\frac{p}{n}$ is very small, and $\frac{n'}{n}=\frac{1}{2}$ nearly [6151]. The other terms of [6090] may be neglected, in computing the term [6438], because they produce no angle which is nearly equal to v. Thus the terms depending on $A^{(2)}$, $A^{(3)}$, &c., produce the angles $v-\frac{2n'}{n}.v-\frac{p}{n}.v$, $2v-\frac{3n'}{n}.v-\frac{p}{n}.v$, &c.; and neither of these angles is equal to v, or nearly equal to it. [6439e] [6439f] [6439g]

* (3339) Taking the differential of aR [6297] relative to s, and retaining only the terms producing the same angle as that in [6442], we find that the part [6297 line 3] gives,

$$a.\left(\frac{dR}{ds}\right)=-m'.a^2a'.\{s'-s.\cos.(v'-v)\}.\{\tfrac{1}{2}B^{(0)}+B^{(1)}.\cos.(v'-v)+B^{(2)}.\cos.(2v'-2v)+B^{(3)}.\cos.(3v'-3v)+\&c\}.$$ [6444a]

Substituting $a=\frac{r^2}{h^2}$ [6439], in the first member, it becomes the same as the term in [6443]; and its second member contains the terms in [6444]. For the term $-m'.a^2a'.s'$, being multiplied by the term $B^{(3)}.\cos.(3v'-3v)$, produces the term of [6444] depending on s'; moreover the term $+m'.a^2a'.s.\cos.(v'-v)$ being multiplied by the terms [6444b] [6444c]

[6444] $$m'.a^2a'.\left\{\tfrac{1}{2}s.\{B^{(3)}+B^{(5)}\}.\cos.(4v-4v')-B^{(3)}.s'.\cos.(3v-3v')\right\};$$

which introduces, into the equation [6295], the following term,*

[6445] $$\tfrac{1}{2}m'.a^2a'.\{B^{(3)}.l'-\tfrac{1}{2}.(B^{(3)}+B^{(5)}).l\}.\sin.\left(3v-\frac{4n'}{n}.v-\frac{p}{n}.v-\text{A}\right).$$

Hence the equation [6295] becomes, by noticing only the terms depending on the angle $3v-4v'$,†

[6446] $$0=\frac{dds}{dv^2}+N_{,}^2.s \qquad 1$$
$$+m'.\{\tfrac{1}{2}a^2a'.B^{(3)}.l'+[2aA^{(4)}-\tfrac{1}{4}a^2a'.(B^{(3)}+B^{(5)})].l\}.\sin.\left(3v-\frac{4n'}{n}.v-\frac{p}{n}.v-\text{A}\right); \qquad 2$$

$N_{,}^2$ being equal to the coefficient of s in [6299 line 1], namely,

[6447] $$N_{,}^2=1+\frac{2.(\rho-\frac{1}{2}\varphi)}{a^2}+\tfrac{3}{2}.\frac{M^2}{n^2}+\tfrac{1}{2}.\Sigma.m'.a^2a'.B^{(1)}.$$

We have, by means of [996, 1006],‡

[6448] $$2aA^{(4)}-\tfrac{1}{4}.a^2a'.\{B^{(3)}+B^{(5)}\}=-2\alpha.b_{\frac{1}{2}}^{(4)}-\tfrac{1}{4}.\alpha^2.b_{\frac{3}{2}}^{(3)}-\tfrac{1}{4}.\alpha^2.b_{\frac{3}{2}}^{(5)};$$

and the formulas [966, 971], give,§

$B^{(3)}.\cos.(3v'-3v)+B^{(5)}.\cos.(5v'-5v)$, in the last factor of [6444a], produces, by using [6444d] [20] Int., the terms $m'.a^2a'.\frac{1}{2}s.\{B^{(3)}+B^{(5)}\}.\cos.(4v'-4v)$, as in [6444]. These are the only quantities which are necessary to be retained in finding terms of the form [6442].

* (3340) Substituting the values of s, s' [6300 lines 1, 2] in [6444], reducing the [6445a] products by means of [19] Int., it produces terms depending on the angle $(3v-4v'-pt-\text{A})$; and by using the values of t, v' [6439, 6441], it becomes as in [6445].

† (3341) The differential equation [6295] is reduced to the form [6299]; and the [6446a] terms in [6299 line 1], are the same as in [6446 line 1], using for brevity the symbol $N_{,}^2$ [6447]. These terms, being connected with those in [6442, 6445], produce the equation [6446].

[6448a] ‡ (3342) We have, in [996, 963iv], $2a.A^{(4)}=-\frac{2a}{a'}.b_{\frac{1}{2}}^{(4)}=-2\alpha.b_{\frac{1}{2}}^{(4)}$. Also from [1006, 963iv] we obtain,

[6448b] $$-\tfrac{1}{4}a^2a'.B^{(3)}=-\tfrac{1}{4}.\frac{a^2}{a'^2}.b_{\frac{3}{2}}^{(3)}=-\tfrac{1}{4}\alpha^2.b_{\frac{3}{2}}^{(3)};\qquad -\tfrac{1}{4}a^2a'.B^{(5)}=-\tfrac{1}{4}.\frac{a^2}{a'^2}.b_{\frac{3}{2}}^{(5)}=-\tfrac{1}{4}\alpha^2.b_{\frac{3}{2}}^{(5)}.$$

Substituting these in the first member of [6448], it becomes as in its second member.

[6449a] § (3343) Substituting $i=5$, $s=\frac{3}{2}$ in [966], we get [6449]; also $i=4$, and $s=\frac{1}{2}$, being substituted in [971], give [6450]. Using the values [6449, 6450] in the second member of [6448], it becomes by reduction equal to $-\frac{1}{2}\alpha^2.b_{\frac{3}{2}}^{(3)}$, as in [6451].
[6449b] Substituting this and $\frac{1}{2}a^2a'.B^{(3)}=\frac{1}{2}\alpha^2.b_{\frac{3}{2}}^{(3)}$ [6448b], in [6446], we get [6452].

$$b_{\frac{3}{2}}^{(5)} = \frac{8.(1+\alpha^2).b_{\frac{3}{2}}^{(4)} - 9\alpha.b_{\frac{3}{2}}^{(3)}}{7\alpha};$$ [6449]

$$b_{\frac{1}{2}}^{(4)} = \frac{2\alpha.b_{\frac{3}{2}}^{(3)} - (1+\alpha^2).b_{\frac{3}{2}}^{(4)}}{7}.$$ [6450]

Hence we deduce,

$$-2\alpha.b_{\frac{1}{2}}^{(4)} - \tfrac{1}{4}\alpha^2.b_{\frac{3}{2}}^{(3)} - \tfrac{1}{4}\alpha^2.b_{\frac{3}{2}}^{(5)} = -\tfrac{1}{2}\alpha^2.b_{\frac{3}{2}}^{(3)};$$ [6451]

therefore the differential equation [6446] becomes,

$$0 = \frac{dds}{dv^2} + N_{\prime}^2.s + \tfrac{1}{2}m'.\alpha^2.b_{\frac{3}{2}}^{(3)}.(l'-l).\sin.\left(3v - \frac{4n'}{n}.v - \frac{p}{n}.v - \mathrm{A}\right).$$ [6452]

This gives, by integration,*

$$s = \frac{\tfrac{1}{2}.m'.\alpha^2.b_{\frac{3}{2}}^{(3)}.(l'-l).\sin.\left(3v - \frac{4n'}{n}.v - \frac{p}{n}.v - \mathrm{A}\right)}{\left(3 - \frac{4n'}{n} - \frac{p}{n}\right)^2 - N_{\prime}^2}.$$ [6453]

This divisor is equal to $\left\{3 - \frac{4n'}{n} - \frac{p}{n} + N_{\prime}\right\}.\left\{3 - \frac{4n'}{n} - \frac{p}{n} - N_{\prime}\right\}.$ [6454]

Now $\frac{p}{n}$ is very small [6302]; $N_{\prime}$ is nearly equal to 1 [6447], and n is nearly equal to $2n'$ [6151]; therefore the factor $3 - \frac{4n'}{n} - \frac{p}{n} - N_{\prime}$ is very small, and the factor $3 - \frac{4n'}{n} - \frac{p}{n} + N_{\prime}$ is very nearly equal to 2. Hence [6455] we get, *by re-substituting the values* $v' = \frac{n'}{n}.v$, $t = \frac{v}{n}$, [6441, 6439], [6455']

$$s = \frac{m'.\alpha^2.b_{\frac{3}{2}}^{(3)}.(l'-l).\sin.(3v - 4v' - pt - \mathrm{A})}{4.\left(3 - \frac{4n'}{n} - \frac{p}{n} - N_{\prime}\right)}.$$ [6456]

It is evident that the different values of p, l, l', give, in the expression of s, an equal number of terms, similar to the preceding.

These inequalities of s, by reason of the smallness of the divisors, considerably exceed the others, which are produced by the action of the satellites. They are the only ones which require any notice; and we shall [6457]

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* (3344) Integrating the equation [6452] by the method in [6049k, l], we get [6453]; and by substituting for the denominator of [6453] its value $2.\left(3 - \frac{4n'}{n} - \frac{p}{n} - N_{\prime}\right)$ nearly, [6456a] it becomes as in [6456].

hereafter find, in [6931, &c.], that even these are insensible.* The sun's action produces, in the value of s, an inequality which might become
[6457′] sensible, by reason of the smallness of its divisor. This inequality depends on the angle $v-2U$; and we easily find, by § 9, that the differential equation in s becomes, by noticing this term only,†

[6457a] * (3345) This is evident from the expression of s [6931 line 1], using the values of m' [7143]; from which it appears that the coefficient of this inequality is of the order $0{,}0008.(l'-l)$, which is very small in comparison with l.

† (3346) The expression of aR [6297 line 4], depending on the sun's action, produces, in $a.\left(\frac{dR}{ds}\right)$, the terms given below in [6458b], depending on the angles $v-U$, $2v-2U$.
[6458a] Substituting the values of s, S', [6300], we obtain the terms in [6458c]; and, by reduction, those in [6458d], depending on the angle $v-2U-pt-\text{A}$. These terms increase considerably, in consequence of the divisors introduced by the integrations [6049k, l];

[6458b] $$a.\left(\frac{dR}{ds}\right)=-\frac{M^2}{4n^2}.\{-6s.\cos.(2v-2U)+12S'.\cos.(v-U)\}$$

[6458c] $$=-\frac{M^2}{4n^2}.\{-6l.\sin.(v+pt+\text{A}).\cos.(2v-2U)+12L'.\sin.(U+pt+\text{A}).\cos.(v-U\}$$

[6458d] $$=-\frac{M^2}{4n^2}.\{3l.\sin.(v-2U-pt-\text{A})-6L'.\sin.(v-2U-pt-\text{A})\}.$$

The terms in [6458c] produce also, by means of [18, 19] Int., some terms depending on
[6458e] the angles $3v-2U+pt+\text{A}$, $v+pt+\text{A}$. This last angle has already been noticed in [6300], and the term depending on the other angle is not increased by the integration [6049k, l].

Again, the same part of the value of aR [6297 line 4] gives

[6458f] $$-a.\left(\frac{dR}{dv}\right)=-\frac{M^2}{4n^2}.6.\sin.(2v-2U);$$

[6458g] neglecting the other terms. Multiplying this by $\frac{ds}{dv}=l.\cos.(v+pt+\text{A})$ [6439d], then reducing, by [18] Int., and retaining only the term depending on the angle $v-2U-pt-\text{A}$, we get,

[6458h] $$-a.\left(\frac{dR}{dv}\right).\frac{ds}{dv}=-\frac{3M^2}{4n^2}.l.\sin.(v-2U-pt-\text{A}).$$

Adding together the expressions [6458d, h], and substituting, in the first member of the sum, $a=\frac{r^2}{h^2}$ [6439], we get,

[6458i] $$-\frac{r^2}{h^2}.\left\{\left(\frac{dR}{dv}\right).\frac{ds}{dv}-\left(\frac{dR}{ds}\right)\right\}=\frac{3M^2}{2n^2}.(L'-l).\sin.(v-2U-pt-\text{A}).$$

Substituting this in [6295], and changing the second term s into $N_{\prime}^2 s$, as in [6446a], it

$$0=\frac{dds}{dv^2}+N_{\prime}^2.s+\frac{3M^2}{2n^2}.(L'-l).\sin.\left(v-\frac{2M}{n}.v-\frac{p}{n}.v-\text{A}\right); \qquad [6458]$$

whence we obtain, by integration,

$$s=-\frac{\frac{3M^2}{4n^2}.(L'-l).\sin.\left(v-\frac{2M}{n}.v-\frac{p}{n}.v-\text{A}\right)}{\frac{2M}{n}+\frac{p}{n}+N_{\prime}-1}. \qquad [6459]$$

Connecting together the parts of s, which depend upon the configurations of the satellites and the action of the sun, we obtain,

Periodical terms of s

$$s=\frac{m'.a^2.b_{\frac{3}{2}}^{(3)}.(l'-l).\sin.(3v-4v'-pt-\text{A})}{4.\left(3-\frac{4n'}{n}-\frac{p}{n}-N_{\prime}\right)} \qquad 1$$

$$-\frac{\frac{3M^2}{4n^2}.(L'-l).\sin.(v-2U-pt-\text{A})}{\frac{2M}{n}+\frac{p}{n}+N_{\prime}-1}; \qquad 2 \qquad [6460]$$

each of these terms being supposed to represent the sum of the similar terms, corresponding to the different values of p. [6460′]

In the eclipses of the satellite m, U is very nearly equal to $v-200^{\circ}$ [6032, 6041]; *hence the inequality* [6460 line 2] *is reduced to the following form;* * [6461]

$$\frac{\frac{3M^2}{4n^2}.(L'-l).\sin.(v+pt+\text{A})}{\frac{2M}{n}+\frac{p}{n}+N_{\prime}-1}. \qquad [6462]$$

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becomes as in [6458]. Integrating this equation, as in [6049k, l], we get the term of s [6459]; observing that the divisor $m_{\prime}^2-a_{\prime}^2$ [6049l], becomes, in the present case, by using the values of U, t [6102, 6155′], [6458k]

$$\left(1-\frac{2M}{n}-\frac{p}{n}\right)^2-N_{\prime}^2=\left(1-\frac{2M}{n}-\frac{p}{n}+N_{\prime}\right).\left(1-\frac{2M}{n}-\frac{p}{n}-N_{\prime}\right)=2.\left(1-\frac{2M}{n}-\frac{p}{n}-N_{\prime}\right), \qquad [6458l]$$

nearly; or, as it may be written, $-2.\left(\frac{2M}{n}+\frac{p}{n}+N_{\prime}-1\right)$, as in [6459]. Connecting together the two terms of s [6456, 6459], it becomes as in [6460]; taking care to re-substitute $2U$ for $\frac{2M}{n}.v$, and pt for $\frac{p}{n}.v$ [6455′, &c.]. [6458m]

* (3347) The longitude of the sun, as seen from Jupiter, is U [6041]; and in eclipses of the first satellite, the longitude of the satellite v is equal to this quantity increased, or decreased, by 200°; hence $U=v-200^{\circ}$, as in [6461]. Substituting this in the term [6460 line 2], and reducing, it becomes as in [6462]; observing that, [6461a]

$$\sin.\{v-2(v-200^{\circ})-pt-\text{A}\}=\sin.(v-2v-pt-\text{A})=\sin.(-v-pt-\text{A})=-\sin.(v+pt+\text{A}). \qquad [6461b]$$

When the values of p are relative to the motions of the equator and of the orbit of Jupiter, we may neglect p [6342] in comparison with M
[6463] [6927, 6025m]. Moreover the sum of all the terms $(L'-l).\sin.(v+pt+\Lambda)$, is then equal to $-(\lambda-1).\theta'.\sin.(v+\Psi')$ [6357f]; and the preceding inequality becomes,*

[6464]
$$\frac{-(\lambda-1).\frac{3M^2}{4n^2}.\theta'.\sin.(v+\Psi')}{\frac{2M}{n}+N_{\prime}-1}.$$

[6462a] * (3348) Prefixing the symbol Σ' [6324'] to the function [6462], so as to include all
[6462b] the terms of that form, it becomes as in the first member of [6462c]; substituting the value [6357f], we get its second member, neglecting $\frac{p}{n}$. This second member is the same as
[6462b'] the function [6464]; the sign *being changed from what it is in the original work, to correct for a mistake.*

[6462c]
$$\frac{3M^2}{4n^2}.\frac{\Sigma'.(L'-l).\sin.(v+pt+\Lambda)}{\frac{2M}{n}+\frac{p}{n}+N_{\prime}-1}=-\frac{3M^2}{4n^2}.\frac{(\lambda-1).\theta'.\sin.(v+\Psi')}{\frac{2M}{n}+N_{\prime}-1}.$$

Adding this to the chief term of the expression of the latitude of the satellite m above
[6462d] Jupiter's orbit $(\lambda-1).\theta'.\sin.(v+\Psi')$ [6427 line 1], we get the following expression of the part of the latitude depending on this argument $v+\Psi'$, in eclipses,

[6462e]
$$\left\{1-\frac{3M^2}{4n^2}.\frac{1}{\frac{2M}{n}+N_{\prime}-1}\right\}.(\lambda-1).\theta'.\sin.(v+\Psi').$$

The ratio of the general value [6462d], to the value in eclipses [6462e], is as 1 to
[6462f] $1-\frac{3M^2}{4n^2.\left(\frac{2M}{n}+N_{\prime}-1\right)}$, as in [6465]. Hence it is evident that if we determine this
[6462g] inclination by means of eclipses, we must *increase* it, in the ratio mentioned in [6465].
[6462h] We may observe that, in the original work, the word *increased* [6465] is printed *decreased*, and we have changed the sign of the term depending on M^2 to correct for the mistake [6465]. The same phenomenon occurs in the lunar theory. For in eclipses,
[6462i] when 2☽long.—2☉long. $=0°$, or $400°$, we find that the two chief terms of the moon's
[6462k] latitude [5595 lines 1, 3], give for the general inclination $18524^s.5$.sin.arg.lat.; and in eclipses $(18524^s.5-528^s.4)$.sin.arg.lat. The ratio of these two expressions is as 1 to $1-\frac{1}{35}$ nearly. This is nearly the same as the ratio [6465]; for by putting $N_{\prime}=1$,
[6462l] which is nearly correct in the lunar theory [5049 lines 1–4], it becomes as 1 to $1-\frac{3}{8}.\frac{M}{n}$.
[6462m] In this expression $\frac{M}{n}$ is equivalent to $m=0{,}0748..$ [5117]; hence the preceding ratio becomes as 1 to $1-0{,}028$, or 1 to $1-\frac{1}{35}$, as in [6462l].

Hence the inclination θ' *of the equator to the orbit of Jupiter, deduced from the eclipses of the satellite* m, *ought to be increased in the ratio of* 1 *to* $1 - \dfrac{3M^2}{4n^2 \cdot \left(\frac{2M}{n} + N_{\prime} - 1\right)}$. [6465]

We shall now consider, in the same manner, the periodical inequalities of the motions of the second satellite in latitude. For this purpose, we shall resume the differential equation [6295], which becomes, relatively to the second satellite,*

$$0 = \frac{dds'}{dv'^2} + s' - a' \cdot \frac{ds'}{dv'} \cdot \left(\frac{dR'}{dv'}\right) + a' \cdot \left(\frac{dR'}{ds'}\right); \qquad [6466]$$

Symbol R'

R' being what R becomes relative to this satellite. The terms of this differential equation, which depend on the angle $2v - 3v'$, acquire a small divisor, because v differs but little from $2v'$ [6151], so that the coefficient of v', in the angle $2v - 3v'$, differs but very little from unity; therefore it is important to consider these terms. If we notice these terms only, it will be evident from § 9, that the preceding differential equation becomes,† [6467] [6468]

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* (3349) Substituting $\frac{r^2}{h^2} = a$ [6439], in the third term of [6295], and then changing reciprocally the elements of m into those of m', it becomes as in [6466]; R' being the value of R deduced from [6297], by changing, in the same manner, the elements of m into those of m', and the contrary; by this means we obtain, [6466a] [6466b]

$$a'R' = \Sigma . m . \left\{ \begin{array}{l} \frac{a'^2}{a^2} \cdot \{ss' - \frac{1}{2} . (s^2 + s'^2) . \cos.(v' - v)\} \\ + \frac{1}{2} a' A^{(0)} + a' A^{(1)} . \cos.(v' - v) + a' A^{(2)} . \cos.(2v' - 2v) + \&c. \\ - aa'^2 . \{ss' - \frac{1}{2} . (s^2 + s'^2) . \cos.(v' - v)\} . \{\frac{1}{2} B^{(0)} + B^{(1)} . \cos.(v' - v) + B^{(2)} . \cos.(2v - 2v') + \&c\} \end{array} \right\} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \qquad [6466c]$$

$$- \frac{Sa'}{D} - \frac{M^2}{4n'^2} . \{1 - 3s'^2 - 3S'^2 + 3 . (1 - s'^2 - S'^2) . \cos.(2v' - 2U) + 12 s'S' . \cos.(v' - U)\} \quad 4$$

$$- \frac{(\rho - \frac{1}{2}\varphi)}{a'^2} . \{\tfrac{1}{3} - (s' - s_{\prime}')^2\}; \quad 5$$

it being evident from [6089, 6090, 6296] that the values $A^{(i)}$, $B^{(i)}$, *are not changed, except in the term* $A^{(1)}$, *as is observed in* [997, 1008']; *but the term* $A^{(1)}$ *is not used in the subsequent part of the calculation; therefore this difference is not noticed.* [6466d]

† (3350) We have found, by developing the equation [6295], as in [6299], that the two first terms become $\frac{dds}{dv^2} + N_{\prime}^2 . s$; $N_{\prime}^2$ being used for brevity, as in [6447]. The same process being performed with the similar equation [6466], we obtain, for its two first terms, $\frac{dds'}{dv'^2} + N_{\prime}'^2 . s'$, as in [6469]; $N_{\prime}'^2$ being deduced from $N_{\prime}^2$ [6447], by changing [6469a] [6469b]

[6469] $$0=\frac{dds'}{dv'^2}+N_{\prime}'^2.s' \qquad 1$$

$$+m.\{\tfrac{1}{2}aa'^2.B^{(3)}.l+[-a'A^{(2)}-\tfrac{1}{4}aa'^2.(B^{(1)}+B^{(3)})].l'\}.\sin.\left(2v-3v'-\frac{p}{n'}.v'-\text{A}\right); \qquad 2$$

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reciprocally the elements of m into those of m'; hence we have,

[6469c] $$N_{\prime}'^2=1+2.\frac{(\rho-\frac{1}{2}\varphi)}{a'^2}+\tfrac{3}{2}.\frac{M^2}{n'^2}+\tfrac{1}{2}.\Sigma.m.aa'^2.B^{(1)}.$$

[6469d] The remaining terms of the equation [6466] depend on R', and in substituting the value of R' [6466c], it will only be necessary to retain the terms depending on angles which differ but little from v', because they increase considerably by integration, as in the similar case, relative to the first satellite, treated of in [6432a—b]. Now by proceeding as

[6469e] in [6434a—u], we find, that by changing the elements of m, into those of m', and the contrary, the form of the angle [6434h], $i.(v'-v)+v'$, will become $i.(v-v')+v$; and

[6469f] the corresponding divisor [6434i] will change into $\{2n'-(i+1).(n'-n)\}.(i+1).(n'-n)$.

[6469g] This becomes 0, when $i+1=0$; and then the angle $i.(v-v')+v$ [6469f], is equal to v', producing terms analogous to those of v [6436']; it also becomes 0 when the

[6469h] factor $2n'-(i+1).(n'-n)=0$, [6469f]. Substituting $n=2n'$ [6151], in this last expression, it becomes $n'.(2+i+1)$; which vanishes when $i=-3$, and then the

[6469i] angle $i.(v-v')+v$ [6469f] becomes $3v'-2v$, or $2v-3v'$, as in [6468]. The satellite m'' produces in s' a similar term. For the factor [6469h] changes into $2n'-(i+1).(n'-n'')$; and by putting $n''=\frac{1}{2}n'$, it becomes nearly $n'.\{2-\frac{1}{2}i-\frac{1}{2}\}$.

[6469k] This factor is very small when $i=3$, and then the angle [6469i] changes into $3.(v''-v')+v''=4v''-3v'$, or $3v'-4v''$, being of the same form as that which is

[6469l] computed in [6476]. The satellite m''' produces nothing of importance, because the factor [6469h] changes into $2n'-(i+1).(n'-n''')$; and by putting $n'''=\frac{1}{4}n'$, it becomes nearly $n'.\{2-\frac{3}{4}i-\frac{3}{4}\}$; being small only with fractional values of i, which is contrary to

[6649m] the supposition in [6434d]; so that these terms may be neglected. We shall now proceed to compute the terms of [6466], connected with the angle $2v-3v'$ [6469i].

[6469n] From [6300 line 2] we have $\frac{ds'}{dv'}=l'.\cos.(v'+pt+\text{A})$; observing that $pt=\frac{pv'}{n'}$ nearly, [6439, 6441], and $\frac{p}{n'}$ is so small that it may be neglected, as in [6439d]. Moreover,

[6469o] from [6466c] we get $-a'.\left(\frac{dR'}{dv'}\right)=2m.a'A^{(2)}.\sin.(2v'-2v)$; neglecting the other quantities, which do not produce the required angle $2v-3v'$, connected with sensible terms. Multiplying these two expressions, and reducing, by [19] Int., we obtain the following term;

[6469p] $$-a'.\frac{ds'}{dv'}.\left(\frac{dR'}{dv'}\right)=-m.l'.a'A^{(2)}.\sin.(2v-3v'-pt-\text{A}).$$

Again, the expression of $a'R'$ [6466c] produces, in $a'.\left(\frac{dR'}{ds'}\right)$, the terms in the second

[6469q] member of [6469r, s]. Reducing them successively, by means of [18—20] Int.; then

$N_{\prime}'^2$ [6469c] being what $N_{\prime}^2$ [6447] becomes relative to m', and the values of $A^{(2)}$, $B^{(1)}$, $B^{(2)}$, &c. [996, 1006], being the same for R' as for R [6466d]. Therefore we shall have, by means of [996, 1006],* [6469′] [6470]

$$0=\frac{dds'}{dv'^2}+N_{\prime}'^2.s'$$
$$+m.\{\tfrac{1}{2}\alpha.b_{\frac{3}{2}}^{(3)}.l+[b_{\frac{1}{2}}^{(2)}-\tfrac{1}{4}\alpha.b_{\frac{3}{2}}^{(1)}-\tfrac{1}{4}\alpha.b_{\frac{3}{2}}^{(3)}].l'\}.\sin.\left(\frac{2n}{n'}.v'-3v'-\frac{p}{n'}.v'-\text{A}\right). \quad [6471]$$

We have, by using [971, 966],

$$b_{\frac{1}{2}}^{(2)}-\tfrac{1}{4}\alpha.b_{\frac{3}{2}}^{(1)}-\tfrac{1}{4}\alpha.b_{\frac{3}{2}}^{(3)}=-\tfrac{1}{2}\alpha.b_{\frac{3}{2}}^{(3)}; \quad [6472]$$

hence [6471] becomes,†

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substituting the values of s, s' [6300], and retaining only those terms which produce the proposed angle $2v-3v'-pt-\text{A}$; we get [6469t, u].

$$a'.\left(\frac{dR'}{ds'}\right)=-m.aa'^2.\{s-s'.\cos.(v'-v)\}.\{\tfrac{1}{2}B^{(0)}+B^{(1)}.\cos.(v'-v)+B^{(2)}.\cos.(2v'-2v)+\&c.\} \quad [6469r]$$
$$=-m.aa'^2.s.B^{(3)}.\cos.(3v'-3v)+m.aa'^2.s'.\cos.(v'-v).\{B^{(1)}.\cos.(v'-v)+B^{(3)}.\cos.(3v'-3v)\} \quad [6469s]$$
$$=-m.aa'^2.s.B^{(3)}.\cos.(3v'-3v)+\tfrac{1}{2}m.aa'^2.s'.\{B^{(1)}+B^{(3)}\}.\cos.(2v'-2v) \quad [6469t]$$
$$=\{\tfrac{1}{2}m.aa'^2.l.B^{(3)}-\tfrac{1}{4}m.aa'^2.l'.(B^{(1)}+B^{(3)})\}.\sin.(2v-3v'-pt-\text{A}). \quad [6469u]$$

Substituting the values [6469p, u] in [6466], and the value of pt [6469n], we obtain the terms of [6469] depending on the angle now under consideration.

* (3351) From [996] we get the first of the following equations. In like manner we obtain the second and third, from [1006], by substituting $\frac{a}{a'}=\alpha$ [963ⁱᵛ]. [6470a]

$$-a'A^{(2)}=b_{\frac{1}{2}}^{(2)}; \quad aa'^2.B^{(1)}=\frac{a}{a'}.b_{\frac{3}{2}}^{(1)}=\alpha.b_{\frac{3}{2}}^{(1)}; \quad aa'^2.B^{(3)}=\alpha.b_{\frac{3}{2}}^{(3)}. \quad [6470b]$$

Substituting these expressions in [6469] we get [6471]; observing that $2v=\frac{2nv'}{n'}$ nearly [6439a, &c.]; neglecting the constant parts depending on ε, ε', as in [6439, &c.]. [6470c]

† (3352) Putting $i=2$ and $s=\frac{1}{2}$, in [971], we get $b_{\frac{1}{2}}^{(2)}=\frac{2}{3}\alpha.b_{\frac{3}{2}}^{(1)}-\frac{1}{3}.(1+\alpha^2).b_{\frac{3}{2}}^{(2)}$. [6472a]
Substituting this in the first member of [6472] it becomes, by reduction,

$$\tfrac{5}{12}\alpha.b_{\frac{3}{2}}^{(1)}-\tfrac{1}{3}.(1+\alpha^2).b_{\frac{3}{2}}^{(2)}-\tfrac{1}{4}\alpha.b_{\frac{3}{2}}^{(3)}. \quad [6472b]$$

Now putting $i=3$ and $s=\frac{3}{2}$ in [966], and transposing the terms into the first member, we get $b_{\frac{3}{2}}^{(3)}-\frac{4}{3\alpha}.(1+\alpha^2).b_{\frac{3}{2}}^{(2)}+\frac{5}{3}.b_{\frac{3}{2}}^{(1)}=0$. Multiplying this by $-\frac{1}{4}\alpha$, and adding the product to [6472b], we find that the terms depending on $b_{\frac{3}{2}}^{(1)}$, $b_{\frac{3}{2}}^{(2)}$ mutually destroy each other, and by reduction the sum becomes $-\frac{1}{2}\alpha.b_{\frac{3}{2}}^{(3)}$, as in [6472]. Substituting this last expression in [6471], we get [6473]. [6472c] [6472d]

[6473] $$0=\frac{dds'}{dv'^2}+N_{\prime}'^2.s'+\tfrac{1}{2}m.\alpha.b^{(3)}_{\frac{3}{2}}.(l-l').\sin.\left(\frac{2n}{n'}.v'-3v'-\frac{p}{n'}.v'-\text{A}\right).$$

whence we get, by integration,*

[6474] $$s'=\frac{m.\alpha.b^{(3)}_{\frac{3}{2}}.(l-l').\sin.\left(\frac{2n}{n'}.v'-3v'-\frac{p}{n'}.v'-\text{A}\right)}{4.\left(\frac{2n}{n'}-3-\frac{p}{n'}-N_{\prime}'\right)}.$$

The action of the third satellite adds also to the expression of s', a term which may become sensible by its small divisor. This term is analogous to that which the action of m' upon m, produces in s; therefore, by
[6475] putting $b'^{(3)}_{\frac{3}{2}}$ for what $b^{(3)}_{\frac{3}{2}}$ becomes, relative to the second satellite, compared with the third; we shall have, for the part of s' depending on the action of m'',†

[6476] $$s'=\frac{m''.a'^2.b'^{(3)}_{\frac{3}{2}}.(l''-l').\sin.(3v'-4v''-pt-\text{A})}{4a''^2.\left(3-\frac{4n''}{n'}-\frac{p}{n'}-N_{\prime}'\right)}.$$

We may therefore unite, into one term, the two terms of the expression of

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* (3353) The equation [6473] being integrated as in [6049k, l], acquires the divisor which is given in the first member of the following expression, and this is easily separated into two factors, as in the second member;

[6474a] $$\left(\frac{2n}{n'}-3-\frac{p}{n'}\right)^2-N_{\prime}'^2=\left(\frac{2n}{n'}-3-\frac{p}{n'}-N_{\prime}'\right).\left(\frac{2n}{n'}-3-\frac{p}{n'}+N_{\prime}'\right).$$

Now we have nearly $\frac{2n}{n'}=4$, $N_{\prime}'=1$ [6151, 6469c]; therefore the factor

$$\frac{2n}{n'}-3-\frac{p}{n'}+N_{\prime}'=2 \quad \text{nearly};$$

[6474b] hence the divisor [6474a] becomes $2.\left(\frac{2n}{n'}-3-\frac{p}{n'}-N_{\prime}'\right)$. Dividing the last term of [6473] by this quantity, according to the directions in [6049l], we get the expression of s' [6474].

† (3354) If we change, in [6456], the elements of m, m', into those of m', m''
[6476a] respectively, the expression $\alpha=\frac{a}{a'}$ [6470a], will change into $\frac{a'}{a''}$; $N_{\prime}$ will become $N_{\prime}'$ [6447, 6469c], &c.; and the value of s [6456], will change into s' [6476]; using $b'^{(3)}_{\frac{3}{2}}$ [6475]. Multiplying [6477] by -2, and adding $2v-3v'$ to both members of the
[6476b] product, we get $3v'-4v''=2v-3v'-400°$; substituting this in the first member of [6478], we get its second member.

s', which depend on the action of the first and third satellites. For we have very nearly, as we have seen in [6155],

$$v-3v'+2v''=200^{\circ};$$ [6477]

which gives, as in [6476b],

$$\sin.(3v'-4v''-pt-\text{A})=\sin.(2v-3v'-pt-\text{A}).$$ [6478]

If we connect this term with that which depends on the sun's action, observing also that the equation $n-3n'+2n''=0$ [6152], gives [6478′] $\frac{2n}{n'}-3=3-\frac{4n''}{n'}$; we shall have, for the expression of the inequalities of [6479] the second satellite in latitude, relative to the mutual configurations of the satellites and the sun, the following expression;*

Periodical terms of s' [6480]

$$s'=\frac{\left\{m.\frac{a}{a'}.(l-l').b_{\frac{3}{2}}^{(3)}+m''.\frac{a'^2}{a''^2}.(l''-l').b'^{(3)}_{\frac{3}{2}}\right\}}{4.\left(\frac{2n}{n'}-3-\frac{p}{n'}-N'_{\prime}\right)}.\sin.(2v-3v'-pt-\text{A}) \qquad 1$$

$$-\frac{3M^2.(L'-l').\sin.(v'-2U-pt-\text{A})}{4n'^2.\left(\frac{2M}{n'}+\frac{p}{n'}+N'_{\prime}-1\right)}. \qquad 2$$

We shall find in the same manner, for the expression of the corresponding inequalities of the third satellite in latitude,†

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* (3355) Multiplying the equation [6478′, or 6152] by $\frac{2}{n'}$, and adding $3-\frac{4n''}{n'}$, [6479a] to both members, we get the equation [6479]. Substituting this value of $3-\frac{4n''}{n'}$, in the divisor of the expression [6476] we get $3-\frac{4n''}{n'}-\frac{p}{n'}-N'_{\prime}=\frac{2n}{n'}-3-\frac{p}{n'}-N'_{\prime}$, [6479b] which is the same as that in the divisor of [6474]. Substituting also in [6476] the expression [6478], we get the term of [6480] depending on m''; that depending on m' being the same as in [6474], using the value of α [6476a]. Lastly, the term of [6480] [6479c] depending on the sun's action, is easily obtained from [6459], by changing the elements of [6479d] m into those of m'; by this means it becomes as in [6480 line 2].

† (3356) The satellite m'' is situated, relative to m', in the same manner as m' is relative to m; hence it is evident that we may deduce the value of s'', arising from the [6481a] action of the satellite m', and the sun; by adding an accent to each of the symbols $m, a, a', l, l', v, n, n', N'_{\prime}, b^{(3)}_{\frac{3}{2}}$. In this way we find that the terms depending on m, [6481b] [6480 line 1], produce those in [6481 line 1]; and those in [6480 line 2] give the terms depending on the sun's action in [6481 line 2]; observing that the angle $2v-3v'-pt-\text{A}$, [6481c] [6480 line 1], is the correct form under which this argument appears, in [6474], after

Periodical terms of s''

[6481]
$$s''=\frac{m'.a'.(l'-l'').b'^{(3)}_{\frac{3}{2}}}{4a''.\left(\frac{2n'}{n''}-3-\frac{p}{n''}-N''_{,}\right)}.\sin.(2v'-3v''-pt-\Lambda) \qquad 1$$
$$-\frac{3M^2.(L'-l'').\sin.(v''-2U-pt-\Lambda)}{4n''^2.\left(\frac{2M}{n''}+\frac{p}{n''}+N''_{,}-1\right)}. \qquad 2$$

Periodical terms of s'''

Lastly the same expression becomes, relatively to s''',*

[6482]
$$s'''=-\frac{3M^2.(L'-l''').\sin.(v'''-2U-pt-\Lambda)}{4n'''^2.\left(\frac{2M}{n'''}+\frac{p}{n'''}+N'''_{,}-1\right)};$$

[6483] $N''_{,}$ and $N'''_{,}$ *being what* $N_{,}$ [6447] *becomes, relative to the third and fourth satellites.* We must apply to the terms of s', s'', s''', contained in [6480 line 2, 6481 line 2, 6482], what we have said upon the corresponding term of s; namely, that in eclipses each of these terms is confounded with
[6484] the corresponding term, depending upon the inclination of the equator to the orbit of Jupiter; and that, on this account, it *increases*† the inclination

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re-substituting the assumed value of $\frac{v'}{n'}=t$, &c., in the terms where it had been introduced.

[6481d] It is not necessary to notice the action of the two satellites m and m''' upon m''; for the factors similar to $2n-(i+1).(n-n')$ [6434i], corresponding to the action of the two satellites m, m''', become respectively $2n''-(i+1).(n''-n)$; $2n''-(i+1).(n''-n''')$.
[6481e] Now we have very nearly, in [6151b], $n=4n''$, $n'''=\frac{3}{7}n''$; hence the preceding factors become $n''.(5+3i)$, and $n''.(\frac{10}{7}-\frac{4}{7}i)$; which do not vanish with the integral
[6481f] values of i [6434d]; therefore we may neglect these terms.

* (3357) Changing the elements of m'' into those of m''', in the terms depending
[6482a] on the sun's action [6481 line 2], we obtain the terms of s''' [6482], corresponding to the sun's action on the satellite m'''. We may neglect the action of the three satellites
[6482b] m, m', m'' upon m'''; because the factor $2n-(i+1).(n-n')$ [6434i] becomes, for these three satellites respectively,

[6482c] $$2n'''-(i+1).(n'''-n); \qquad 2n'''-(i+1).(n'''-n'); \qquad 2n'''-(i+1).(n'''-n'').$$

[6482d] Now, by [6151b], we have nearly $n=\frac{28}{3}n'''$; $n'=\frac{14}{3}n'''$; $n''=\frac{7}{3}n'''$; hence the
[6482e] factors [6482c] become respectively $\frac{1}{3}.(31+25i).n'''$, $\frac{1}{3}.(17+11i).n'''$, $\frac{1}{3}.(10+4i).n'''$; and neither of them vanish with the integral values of i [6434d]; therefore these terms may be neglected.

† (3358) In the original work this is said to *decrease* the inclination; we have altered
[6484a] it to correct for the mistake of the author, as in [6462f, &c.].

which is deduced from observations of eclipses. We may also observe, that we can put, in all these expressions, without any sensible error,* Approximate values of $N_{,}$ $N'_{,}$ $N''_{,}$ $N'''_{,}$

$$N_{,}=1+\frac{(\rho-\frac{1}{2}\varphi)}{a^2}; \qquad N'_{,}=1+\frac{(\rho-\frac{1}{2}\varphi)}{a'^2}; \qquad [6485]$$

$$N''_{,}=1+\frac{(\rho-\frac{1}{2}\varphi)}{a''^2}; \qquad N'''_{,}=1+\frac{(\rho-\frac{1}{2}\varphi)}{a'''^2}. \qquad [6486]$$

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* (3359) We have, in [6796, 6797], $2.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}=2.\frac{0,0217794}{(5,698491)^2}=0,001$ nearly, and [6485a]
from [6025i] $\frac{3}{2}.\frac{M^2}{n^2}=\frac{3}{2}.(0,0004)^2$ is a very small fraction in comparison with the preceding; therefore it may be neglected in [6447]. Multiplying the second of the [6485b]
equations [6470b], by $m'.\frac{a}{a'}=m'.\alpha$ [6470a], we get

$$m'.a^2a'.B^{(1)}=m'.\alpha^2.b^{(1)}_{\frac{3}{2}}=-3m'.\left(\frac{\alpha}{1-\alpha^2}\right)^2.b^{(1)}_{-\frac{1}{2}} \quad [992]; \qquad [6485c]$$

and by substituting the values of α, $b^{(1)}_{-\frac{1}{2}}$ [6801, 6802], it becomes of the order $2m'$ or 0,00005 [7163]; so that this term and the similar ones depending on m'', m''', are very [6485d]
small in comparison with that in [6485a]; therefore if we neglect these terms of $N_{,}^2$ [6485e]
[6447], we shall have $N_{,}^2=1+2.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}$, whose square root gives very nearly
$N_{,}=1+\frac{(\rho-\frac{1}{2}\varphi)}{a^2}$, as in the first of the equations [6485]. Changing successively a into [6485f]
a', a'', a''', we get the other expressions in [6485, 6486].

CHAPTER V.

INEQUALITIES DEPENDING ON THE SQUARES AND PRODUCTS OF THE EXCENTRICITIES AND INCLINATIONS OF THE ORBITS.

[6487] 12. *It will suffice, in calculating these inequalities, to notice the secular variations analogous to those which we have determined for the planets*, in § 5,
[6487'] Book VI. It follows, from this article, that if we notice only the action of m' upon m, the part of $an.R$, depending wholly upon the secular

Action of the satellite. inequalities, is*

[6488] $$an.R=-\tfrac{1}{2}.(0,1).\{e^2+e'^2\}+\boxed{0,1}.ee'.\cos.(\varpi'-\varpi)+\tfrac{1}{2}.(0,1).\{\gamma_1'^2-2\gamma_1.\gamma_1'.\cos.(\Pi_1'-\Pi_1)+\gamma_1^2\};$$

[6489] γ_1 and γ_1' *being the inclinations of the orbits of* m, m' *to the fixed plane*, Π_1, Π_1', *the longitudes of their ascending nodes upon this plane.*

The part of $an.R$, depending on the sun's action, corresponding to the

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[6488a] * (3360) That part of $an.R$, depending on the squares and products of the excentricities and inclinations, and which is independent of the configuration of the bodies, affects the secular inequalities, and is as in [3765];

[6488b] $$an.R=-\tfrac{1}{2}.(0,1)\{h^2+l^2+h'^2+l'^2\}+\boxed{0,1}.(hh'+ll')+\tfrac{1}{2}.(0,1).\{(p'-p)^2+(q'-q)^2\}.$$

Now we have, in [3756*b*, *e*], the equations [6488*e*]. From [1032] we easily deduce the
[6488c] expressions [6488*f*], by using [24] Int.; observing also that φ, φ', θ, θ' [1030', 1030''], are changed into γ_1, γ_1', Π_1, Π_1', respectively, in [6489]; and that on account of the smallness of γ_1, γ_1', we may take these arcs instead of their tangents. Substituting the
[6488d] values [6488*f*] in the development of the first member of [6488*g*], we get successively its second member [6488*g*, or 6488*h*]. Finally, substituting [6488*e*, *h*] in [6488*b*], we get [6488].

[6488e] $$h^2+l^2=e^2;\qquad h'^2+l'^2=e'^2;\qquad hh'+ll'=ee'.\cos.(\varpi'-\varpi);$$

[6488f] $$p^2+q^2=\gamma_1^2;\qquad p'^2+q'^2=\gamma_1'^2;\qquad pp'+qq'=\gamma_1.\gamma_1'.\cos.(\Pi_1'-\Pi_1);$$

[6488g] $$(p'-p)^2+(q'-q)^2=(p'^2+q'^2)-2(pp'+qq')+(p^2+q^2)$$

[6488h] $$=\gamma_1'^2-2\gamma_1.\gamma_1'.\cos.(\Pi_1'-\Pi_1)+\gamma_1^2.$$

secular inequalities, is by § 1,* Action of the sun.

$$an.R=-\tfrac{1}{2}.\boxed{0}.\{e^2+H^2-\gamma_1^2+2\gamma_1.\gamma.\cos.(\Pi_1-\Pi)-\gamma^2\}. \qquad [6490]$$

Lastly, the part of $an.R$, depending on the ellipticity of the spheroid of Jupiter, is by [6052, &c.],† Effect of the ellipticity.

$$an.R=\tfrac{1}{2}.(0).\{\theta^2+2\theta.\gamma_1.\cos.(\Psi+\Pi_1)+\gamma_1^2-e^2\}. \qquad [6491]$$

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* (3361) These terms of R may be deduced from those in [6042], neglecting $\frac{S}{D}$, as in [6277'], and substituting the elliptical values of r, D, together with those of s, S', &c. [6332b, 6328e, &c.], retaining only the terms independent of the configurations. [6490a] We may also derive [6490] from [6488], by changing the elements of the satellite m', into those of the sun's relative orbit about Jupiter. This is done by changing m', e', ϖ', [6490b] γ_1', Π_1', n, α, into S, H, I, γ, Π, M, $\frac{a}{D'}$, respectively, as appears by the comparison [6490c] of the definitions of these quantities, in [6021c—6024m]. Now if these changes be made in the expression $m'.\alpha^3$, it becomes $S.\frac{a^3}{D'^3}=M^2.a^3$ [6105]; and by substituting the [6490d] value $a^3=n^{-2}$ [6110], we find that $m'.\alpha^3$ changes into $\frac{M^2}{n^2}$; so that this quantity, [6490e] multiplied by α or $\frac{a}{D'}$, must be very small [6485b], and may be neglected; or, in other words, we may neglect terms of the order α^4. In this case it will only be necessary to take the first terms of the developments in [989], which give $b_{-\frac{1}{2}}^{(0)}=2$; $b_{-\frac{1}{2}}^{(1)}=-\alpha$. [6490f] Substituting these in [1082], developing according to the powers of α, and neglecting α^4, we get $\boxed{0{,}1}=-\frac{3}{2}\alpha.m'.n.\{-\alpha+\alpha\}=0$; moreover [1076] becomes, by [6490g] neglecting quantities of the same order, $(0{,}1)=\frac{3}{4}.m'.\alpha^3.n$. Now changing, as in [6490e], $m'.\alpha^3$ into $M^2.n^{-2}$, it becomes $(0{,}1)=\frac{3}{4}.\frac{M^2}{n}=\boxed{0}$ [6216]. Hence it appears [6490h] that the change in the symbols, mentioned in [6490c], makes $\boxed{0{,}1}$ vanish, and [6490i] $(0{,}1)=\boxed{0}$. Substituting these and the values [6490c] in [6488], changing also [6490k] $\cos.(\Pi-\Pi_1)$ into $\cos.(\Pi_1-\Pi)$, we get the expression [6490], corresponding to the part of $an.R$, which depends on the sun's action.

† (3362) The part of R, depending on the ellipticity, is given in [6052]. We must substitute, in it, the values $M=1$, $B=1$ [6082, 6282']; and the part of the [6491a] elliptical value of r^{-3} [3702c], which is independent of the configuration, namely [6491b] $r^{-3}=a^{-3}.(1+\frac{3}{2}e^2)$; then multiplying by an, and using the value of (0) [6216], we

[6491′] Therefore we shall have,*

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obtain the first and second expressions of $an.R$ [6491d]. Multiplying together the factors of this last expression, and neglecting terms of the fourth order in e, s, s_1, we get

[6491c] [6491e]. Developing this expression and neglecting the term $-\frac{1}{3}.(0)$, which produces nothing in $an.\mathrm{d}R$ [6492], we obtain [6491f].

[6491d] $$an.R=-an.(\rho-\tfrac{1}{2}\varphi).\{\tfrac{1}{3}-(s-s_1)^2\}.a^{-3}.(1+\tfrac{3}{2}e^2)=-(0).\{\tfrac{1}{3}-(s-s_1)^2\}.\{1+\tfrac{3}{2}e^2\}$$

[6491e] $$=-(0)\{\tfrac{1}{3}+\tfrac{1}{2}e^2-(s-s_1)^2\}$$

[6491f] $$=\quad(0)\{s_1^2-2s_1s+s^2-\tfrac{1}{2}e^2\}.$$

Now we have, as in [6323d, 6332b],

[6491g] $$s_1=-\theta.\sin.(v+\Psi)\;;\qquad s=\gamma_1.\sin.(v-\theta_1).$$

[6491h] Taking successively the squares and product of these values of s, s_1, and reducing, by means of [17] Int., we get the following expressions, by retaining only those terms which are independent of the configurations;

[6491i] $$s_1^2=\tfrac{1}{2}\theta^2\;;\qquad s^2=\tfrac{1}{2}\gamma_1^2\;;\qquad s_1s=-\tfrac{1}{2}\theta\gamma_1.\cos.(\Psi+\theta_1).$$

Substituting these last values in [6491f], we get [6491].

[6492a] * (3363) Adding together the three functions [6488, 6490, 6491], we obtain the complete value of that part of $an.R$, which is taken into consideration in this article. The differential of this expression, relative to the characteristic d, is taken in [6492],

[6492b] *considering the elements of the satellite* m *as the only variable quantities,* as in [6055]; so that we must suppose e, ϖ, γ_1, θ_1 [6061, &c.; 6315, &c.] to be the variable quantities, a being constant as in [1044″]; also e', ϖ', γ_1', θ_1', H, γ, θ, θ, Ψ, are considered as

[6492b′] constant, relative to the characteristic d. As e is connected with $\sin.\varpi$, or $\cos.\varpi$, in

[6492c] [6493, 6494]; and γ_1 with $\sin.\theta_1$ or $\cos.\theta_1$, in [6499, 6500]; the quantities $(e.\sin.\varpi)$, $(e.\cos.\varpi)$, $(\gamma_1.\sin.\theta_1)$, $(\gamma_1.\cos.\theta_1)$, are considered as the variable quantities in [6492];

[6492d] and the parts of $an.R$ [6488, 6490, 6491] may be made to contain these terms, by the substitution of the following expressions, which are easily deduced from [23, 24] Int.;

[6492e] $$e^2=(e.\cos.\varpi)^2+(e.\sin.\varpi)^2\;;\quad e'^2=(e'.\cos.\varpi')^2+(e'.\sin.\varpi')^2\;;\quad \gamma_1^2=(\gamma_1.\cos.\theta_1)^2+(\gamma_1.\sin.\theta_1)^2\;;$$

[6492f] $$ee'.\cos.(\varpi'-\varpi)=(e.\cos.\varpi).(e'.\cos.\varpi')+(e.\sin.\varpi).(e'.\sin.\varpi')\;;$$

[6492g] $$\gamma_1\gamma.\cos.(\theta_1-\theta)=(\gamma_1.\cos.\theta_1).(\gamma.\cos.\theta)+(\gamma_1.\sin.\theta_1).(\gamma.\sin.\theta)\;;$$

[6492h] $$\theta\gamma_1.\cos.(\Psi+\theta_1)=(\gamma_1.\cos.\theta_1).(\theta.\cos.\Psi)-(\gamma_1.\sin.\theta_1).(\theta.\sin.\Psi).$$

Substituting these developments in [6488, 6490, 6491], and then taking the differentials, considering the variable quantities to be as in [6492c], we shall get the terms of [6492] depending on (0), $\boxed{0}$, (0,1), $\boxed{0{,}1}$. Adding one accent to the terms between the

[6492i] braces, connected with (0,1), $\boxed{0{,}1}$, we get those depending on (0,2), $\boxed{0{,}2}$; and by adding another accent, we get those depending on (0,3), $\boxed{0{,}3}$.

$$an.\mathrm{d}R = -d.(e.\cos.\varpi).\left\{\begin{array}{ll}\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\} e.\cos.\varpi & 1 \\ -\boxed{0,1}.e'.\cos.\varpi'-\boxed{0,2}.e''.\cos.\varpi''-\boxed{0,3}.e'''.\cos.\varpi''' & 2\end{array}\right.$$

$$-d.(e.\sin.\varpi).\left\{\begin{array}{ll}\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.e.\sin.\varpi & 3 \\ -\boxed{0,1}.e'.\sin.\varpi'-\boxed{0,2}.e''.\sin.\varpi''-\boxed{0,3}.e'''.\sin.\varpi''' & 4\end{array}\right. \qquad [6492]$$

$$+d.(\gamma_1.\cos.\Pi_1).\left\{\begin{array}{ll}\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\} \gamma_1.\cos.\Pi_1 & 5 \\ +(0).\theta.\cos.\Psi-\boxed{0}.\gamma.\cos.\Pi-(0,1).\gamma_1'.\cos.\Pi_1'-(0,2).\gamma_1''.\cos.\Pi_1''-(0,3).\gamma_1'''.\cos.\Pi_1''' & 6\end{array}\right.$$

$$+d.(\gamma_1.\sin.\Pi_1).\left\{\begin{array}{ll}\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.\gamma_1.\sin.\Pi_1 & 7 \\ -(0).\theta.\sin.\Psi-\boxed{0}.\gamma.\sin.\Pi-(0,1).\gamma_1'.\sin.\Pi_1'-(0,2).\gamma_1''.\sin.\Pi_1''-(0,3).\gamma_1'''.\sin.\Pi_1''' & 8\end{array}\right.$$

We have, as in [6236, 6237],

$$e.\sin.\varpi = -h.\sin.(gt+\Gamma)-h_1.\sin.(g_1t+\Gamma_1)-, \text{ \&c.}; \qquad [6493]$$

$$e.\cos.\varpi = -h.\cos.(gt+\Gamma)-h_1.\cos.(g_1t+\Gamma_1)-, \text{ \&c.}; \qquad [6494]$$

hence we obtain, from [6217—6222], the following expressions; *

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* (3364) Multiplying [6217] by $\cos.(gt+\Gamma)$, and transposing its first term, we get,

$$-hg.\cos.(gt+\Gamma)=\left\{-\left[(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right].h+\boxed{0,1}.h'+\boxed{0,2}.h''+\boxed{0,3}.h'''\right\}.\cos.(gt+\Gamma). \qquad [6495a]$$

We may obtain similar results for all the other values of g; namely, g_1, g_2, g_3. Taking the sum of these expressions, which is done by prefixing the sign $\Sigma_{\prime}$ [6235d], we get,

$$-\Sigma_{\prime}.hg.\cos.(gt+\Gamma)=\Sigma_{\prime}.\left\{-\left[(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right].h+\boxed{0,1}.h'+\boxed{0,2}.h''+\boxed{0,3}.h'''\right\}.\cos.(gt+\Gamma). \qquad [6495b]$$

The first member of this last equation is the same as the differential of the second member of [6493] divided by dt; hence we get,

$$\frac{d.(e.\sin.\varpi)}{dt}=\Sigma_{\prime}.\left\{-\left[(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right].h+\boxed{0,1}.h'+\boxed{0,2}.h''+\boxed{0,3}.h'''\right\}.\cos.(gt+\Gamma). \qquad [6495c]$$

Now the expressions of $e.\cos.\varpi$, $e'.\cos.\varpi'$, $e''.\cos.\varpi''$, $e'''.\cos.\varpi'''$ [6235i—m], being substituted in the second member of [6495c], we get [6495]. [6495d]

In like manner, if we multiply [6217] by $-\sin.(gt+\Gamma)$, then transpose its first term, and prefix the sign $\Sigma_{\prime}$, we get the following equation, which is similar to [6495b];

$$\Sigma_{\prime}.hg.\sin.(gt+\Gamma)=\Sigma_{\prime}.\left\{\left[(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right].h-\boxed{0,1}.h'-\boxed{0,2}.h''-\boxed{0,3}.h'''\right\}.\sin.(gt+\Gamma). \qquad [6495e]$$

The first member of this equation is equal to the differential of the expression of $e.\cos.\varpi$, [6491], divided by dt; and if we substitute in the second member of [6495e], the values [6495f] $\Sigma_{\prime}.h.\sin.(gt+\Gamma) = -e.\sin.\varpi$, &c. [6235i, k, l, m], it becomes as in [6496].

[6495] $$\frac{d.(e.\sin.\varpi)}{dt} = \left\{ (0) + \boxed{\;0\;} + (0,1) + (0,2) + (0,3) \right\} . e.\cos.\varpi \qquad 1$$

$$-\boxed{0,1}.e'.\cos.\varpi' - \boxed{0,2}.e''.\cos.\varpi'' - \boxed{0,3}.e'''.\cos.\varpi''' \,; \qquad 2$$

[6496] $$\frac{d.(e.\cos.\varpi)}{dt} = -\left\{ (0) + \boxed{\;0\;} + (0,1) + (0,2) + (0,3) \right\} . e.\sin.\varpi \qquad 1$$

$$+\boxed{0,1}.e'.\sin.\varpi' + \boxed{0,2}.e''.\sin.\varpi'' + \boxed{0,3}.e'''.\sin.\varpi'''. \qquad 2$$

Then we have, as in [6332b],

[6497] $$s = \gamma_1.\sin.(v - \Pi_1).$$

Comparing this equation with the following, which is similar to [6300 line 1, 6427],

[6498] $$s = l.\sin.(v + pt + \text{A}) + l_1.\sin.(v + p_1 t + \text{A}_1) + \&\text{c.},$$

we obtain,*

[6499] $$\gamma_1.\sin.\Pi_1 = -l.\sin.(pt + \text{A}) - l_1.\sin.(p_1 t + \text{A}_1) - , \&\text{c.}\,;$$

[6500] $$\gamma_1.\cos.\Pi_1 = \quad l.\cos.(pt + \text{A}) + l_1.\cos.(p_1 t + \text{A}_1) + , \&\text{c.}$$

Therefore the equations [6308—6311] will give the following expressions ;†

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* (3365) We obtain from [21, 22] Int. the following developments ;

[6498a] $$\sin.(v - \Pi_1) = \sin.v.\cos.\Pi_1 - \cos.v.\sin.\Pi_1\,;$$

[6498b] $$\sin.(v + pt + \text{A}) = \sin.v.\cos.(pt + \text{A}) + \cos.v.\sin.(pt + \text{A}),\ \&\text{c.}$$

[6498c] Substituting these expressions in the values of s [6497, 6498] ; and then comparing separately the coefficients of $-\cos.v$ and $\sin.v$, we obtain the equations [6499, 6500] ; and if we use the symbol Σ', in like manner as in [6298x], to include the terms depending on all the roots $p,\ p_1,\ p_2,\ p_3$, &c. [6498z], these equations may be put under the forms [6498d]. From these last expressions we easily derive the similar ones, corresponding to the satellites m', m'', m''', as in [6498e—g] respectively ; the whole system of equations being similar to those in [6235i—m].

	Column 1.	Column 2.
[6498d]	$\gamma_1.\sin.\Pi_1 = -\Sigma'.l.\sin.(pt + \text{A})\,;$	$\gamma_1.\cos.\Pi_1 = \Sigma'.l.\cos.(pt + \text{A})\,;$
[6498e]	$\gamma_1'.\sin.\Pi_1' = -\Sigma'.l'.\sin.(pt + \text{A})\,;$	$\gamma_1'.\cos.\Pi_1' = \Sigma'.l'.\cos.(pt + \text{A})\,;$
[6498f]	$\gamma_1''.\sin.\Pi_1'' = -\Sigma'.l''.\sin.(pt + \text{A})\,;$	$\gamma_1''.\cos.\Pi_1'' = \Sigma'.l''.\cos.(pt + \text{A})\,;$
[6498g]	$\gamma_1'''.\sin.\Pi_1''' = -\Sigma'.l'''.\sin.(pt + \text{A})\,;$	$\gamma_1'''.\cos.\Pi_1''' = \Sigma'.l'''.\cos.(pt + \text{A}).$

[6501a] † (3366) Multiplying [6308] by $\cos.(pt + \text{A})$, transposing its first term, and prefixing the sign Σ' [6498c], we get,

$$\frac{d.(\gamma_1.\sin.\Pi_1)}{dt} = -\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.\gamma_1.\cos.\Pi_1 \quad 1$$

$$-(0).\theta.\cos.\Upsilon+\boxed{0}.\gamma.\cos.\Pi+(0,1).\gamma_1'.\cos.\Pi_1'+(0,2).\gamma_1''.\cos.\Pi_1''+(0,3).\gamma_1'''.\cos.\Pi_1'''. \quad 2$$ [6501]

$$\frac{d.(\gamma_1.\cos.\Pi_1)}{dt} = \left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.\gamma_1.\sin.\Pi_1 \quad 1$$

$$-(0).\theta.\sin.\Upsilon-\boxed{0}.\gamma.\sin.\Pi-(0,1).\gamma_1'.\sin.\Pi_1'-(0,2).\gamma_1''.\sin.\Pi_1''-(0,3).\gamma_1'''.\sin.\Pi_1'''. \quad 2$$ [6502]

Substituting these values of $d.(e.\sin.\varpi)$, $d.(e.\cos.\varpi)$, $d.(\gamma_1.\sin.\Pi_1)$, $d.(\gamma_1.\cos.\Pi_1)$, in the expression of $an.\text{d}R$ [6492], we find that it vanishes; [6502'] hence $an.\text{d}R = 0$.*

$$-\Sigma'.lp.\cos.(pt+A) = -\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.\Sigma'.l.\cos.(pt+A)$$

$$+(0).\Sigma'.L.\cos.(pt+A)+\boxed{0}.\Sigma'.L'.\cos.(pt+A) \quad [6501a']$$

$$+(0,1).\Sigma'.l'.\cos.(pt+A)+(0,2).\Sigma'.l''.\cos.(pt+A)+(0,3).\Sigma'.l'''.\cos.(pt+A).$$

The first member of this equation is equal to the differential of $\gamma_1.\sin.\Pi_1$, deduced from the second member of [6199], and then divided by dt, as in the first member of [6501]. Moreover, if we substitute, in the second member of [6501a'], the values [6498d—g, col. 2; 6327, 6331], it becomes as in the second member of [6501]. In like [6501b] manner, if we multiply [6308] by $\sin.(pt+A)$, we shall obtain an equation exactly similar to [6501a'], $\cos.(pt+A)$ being changed into $\sin.(pt+A)$, in both members. Then the [6501c] *first member of the product* becomes equal to the differential of the second member of [6500], divided by dt, as in [6502]; and *the second member of the product* is easily [6501d] reduced to the form of the second member of [6502], by the substitution of the values [6498d—g, col. 1; 6326, 6330].

* (3367) We find, by inspection, that the coefficients of $-d.(e.\cos.\varpi)$, $d.(e.\sin.\varpi)$, $-d.(\gamma_1.\cos.\Pi_1)$, $d.(\gamma_1.\sin.\Pi_1)$, in the second member of [6492], are [6502a] respectively equal to the second members of the equations [6495, 6496, 6501, 6502]; we may therefore substitute the first members of these equations in [6492], instead of their equivalent values, and we shall get, without any reduction,

$$an.\text{d}R = -d.(e.\cos.\varpi).\frac{d.(e.\sin.\varpi)}{dt} + d.(e.\sin.\varpi).\frac{d.(e.\cos.\varpi)}{dt}$$

$$-d.(\gamma_1.\cos.\Pi_1).\frac{d.(\gamma_1.\sin.\Pi_1)}{dt} + d.(\gamma_1.\sin.\Pi_1).\frac{d.(\gamma_1.\cos.\Pi_1)}{dt}. \quad [6502b]$$

Now the terms in each line of the second member of this equation, mutually destroy each [6502c] other; hence we have $an.\text{d}R = 0$, as in [6502'].

We shall now resume the equation [6060], or rather its differential, from which we have deduced, in [931], the following expression ;*

[6503]
$$\frac{d.\delta v_{,}}{dt}=\frac{\frac{d.(2r.d\delta r+dr.\delta r)}{a^2n.dt^2}+\frac{3an}{\mu}.\int dR+\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)}{\sqrt{1-e^2}}.$$

[6503′] *We may here neglect the divisor* $\sqrt{1-e^2}$, and suppose it equal to unity. It is however plain, that if the numerator contain a constant quantity g,

[6504] it will produce, in $\frac{d.\delta v_{,}}{dt}$, the term $\frac{1}{2}\text{g}.e^2$, by the development of this divisor in a series ; and it will therefore be necessary to retain this divisor.

[6504′] But the constant quantity g, will produce, in $\delta v_{,}$, the term gt,† and then nt will no longer express the mean motion of m, which is contrary to what we have supposed ; *therefore the constant quantity* g *must vanish.*

[6505] This condition may always be satisfied by adding an appropriate constant quantity to the integral $\int dR$.

If we notice only the secular inequalities of e and ϖ, we shall have,‡

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* (3368) The equation [6060] becomes as in [6503], by dividing its differential by

[6503a] dt, and substituting the expression [6060b] ; observing also that, as $a=n^{-\frac{2}{3}}$ [6110] is constant [6492b—b′], we may bring a, n, from under the sign of integration.

[6504a] † (3369) If the second member of [6503] contain the term $\frac{\text{g}}{\sqrt{1-e^2}}=\text{g}.(1+\frac{1}{2}e^2+, \&c.)$,

[6504b] it will produce, in its integral, or in the value of $\delta v_{,}$, the terms $\delta v_{,}=\text{g}t+\frac{1}{2}\text{g}\int e^2.dt$; neglecting e^4, &c. ; therefore the mean motion nt will be increased by the quantity gt ; which is contrary to the hypothesis in [6062′], where we have assumed its value from

[6504c] observation, to be nt ; so that we must put $\text{g}=0$, and then the term $\frac{1}{2}\text{g}.\int e^2.dt$ [6504b] vanishes also from the expression of $\delta v_{,}$; by taking the constant quantity connected with

[6504d] $\int dR$ [6503], so as to produce this result, as is observed in [6505] and in [6121, &c.].

‡ (3370) The expression of r [6506] is the same as in [6200a] ; and by taking its

[6506a] differential, supposing e, ϖ, to be the variable quantities, we get the value of $\frac{dr}{dt}$, depending on the secular inequalities [6506c], observing that a, n, are constant [6503a]. Again, if we develop r [6506] according to the powers of t, by Taylor's theorem [607, &c.],

[6506b] noticing only the secular inequalities, and supposing r and its differentials in the second member to correspond to the epoch, or $t=0$, we shall get the expression [6506d] ;

$$r = a.\{1 - e.\cos.(nt + \varepsilon - \varpi)\};$$ [6506]

$$\delta r = -at.\left\{\frac{de}{dt}.\cos.(nt+\varepsilon-\varpi) + e.\frac{d\varpi}{dt}.\sin.(nt+\varepsilon-\varpi)\right\}.$$ [6507]

From these we get, by retaining only the terms multiplied by t, without sines and cosines of nt, and neglecting the differentials $\frac{dde}{dt^2}$, $\frac{dd\varpi}{dt^2}$, which are incomparably* smaller than $\frac{de}{dt}$, $\frac{d\varpi}{dt}$; † [6508]

$$\left(\frac{dr}{dt}\right) = -a.\left\{\frac{de}{dt}.\cos.(nt+\varepsilon-\varpi) + e.\frac{d\varpi}{dt}.\sin.(nt+\varepsilon-\varpi)\right\};$$ [6506c]

$$r = \mathrm{r} + \left(\frac{d\mathrm{r}}{dt}\right).t +, \&c.$$ [6506d]

Substituting [6506c] in [6506d], we get the value of r, or $\mathrm{r} + \delta r$, whence we obtain δr, as in [6507].

* (3371) The smallness of these quantities is manifest in the perfectly analogous case of the planets in [1122, &c.]. For the first of the equations [1122] shows that $\frac{de}{dt}$ is [6508a] composed of terms of the order $\boxed{0,1}\,.e'$, or of the order $m'e'$ [1082]; and its differential, divided by dt, gives $\frac{dde}{dt^2}$, of the order $m'.\frac{de'}{dt}$, and this last expression is of the order m'^2, as is evident from [1122 line 2]. Hence it appears that $\frac{dde}{dt^2}$, is of the order m' in [6508b] comparison with $\frac{de}{dt}$, and must therefore be much smaller, as in [6508]. A similar result is obtained by the numerical calculation in [3853a].

† (3372) Using for brevity the symbol $W = nt + \varepsilon - \varpi$, we may put the equations [6509a] [6506, 6507] under the forms [6509b, c]; and their differentials under the forms [6509d, e], [6509a'] neglecting terms of the order dde, $dd\varpi$, $de \times d\varpi$, de^2, &c., as in [6508];

$$r = a - ae.\cos.W;$$ [6509b]

$$\delta r = -at.\frac{de}{dt}.\cos.W - at.e.\frac{d\varpi}{dt}.\sin.W;$$ [6509c]

$$\frac{dr}{dt} = -a.\frac{de}{dt}.\cos.W + ane.\sin.W - ae.\frac{d\varpi}{dt}.\sin.W;$$ [6509d]

$$\frac{d\delta r}{dt} = -a.\frac{de}{dt}.\cos.W + ant.\frac{de}{dt}.\sin.W - ae.\frac{d\varpi}{dt}.\sin.W - ant.e.\frac{d\varpi}{dt}.\cos.W.$$ [6509e]

In substituting these expressions in the first members of [6509g, h], and retaining only the terms which are independent of the configurations, we may neglect the quantities which

[6509] $$\frac{2r.d\delta r+dr.\delta r}{a^2.ndt}=\tfrac{1}{2}t.\left\{e.\cos.\varpi.\frac{d.(e.\sin.\varpi)}{dt}-e.\sin.\varpi.\frac{d.(e.\cos.\varpi)}{dt}\right\}.$$

[6509′] Taking its differential, neglecting the differentials and products of the quantities $\frac{de}{dt}$, $\frac{d\varpi}{dt}$, we shall have,

[6510] $$\frac{d.(2r.d\delta r+dr.\delta r)}{a^2.ndt^2}=\tfrac{1}{2}.\left\{e.\cos.\varpi.\frac{d.(e.\sin.\varpi)}{dt}-e.\sin.\varpi.\frac{d.(e.\cos.\varpi)}{dt}\right\}.$$

Substituting for $\frac{d.(e.\sin.\varpi)}{dt}$, $\frac{d.(e.\cos.\varpi)}{dt}$, their values [6495, 6496], we shall obtain,*

[6511] $$\frac{d.\delta v_{,}}{dt}=\frac{d.(2r.d\delta r+dr.\delta r)}{a^2.ndt^2}=\tfrac{1}{2}.\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.e^2$$
$$-\tfrac{1}{2}.\boxed{0,1}.ee'.\cos.(\varpi'-\varpi)-\tfrac{1}{2}.\boxed{0,2}.ee''.\cos.(\varpi''-\varpi)-\tfrac{1}{2}.\boxed{0,3}.ee'''.\cos.(\varpi'''-\varpi).$$

The term $\int dR$ vanishes from the expression [6503], as we have seen

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[6509f] contain the first power of $\sin.W$, $\cos.W$, as well as their product, $\sin.W.\cos.W$; putting also $\sin.^2W=\frac{1}{2}$, $\cos.^2W=\frac{1}{2}$ [1,6] Int. Hence we shall have,

[6509g] $$\frac{2r.d\delta r}{dt}=2a^2.e.\frac{de}{dt}.\cos.^2W+2a^2.nt.e^2.\frac{d\varpi}{dt}.\cos.^2W=a^2.e.\frac{de}{dt}+a^2.nt.\frac{e^2.d\varpi}{dt};$$

[6509h] $$\frac{dr.\delta r}{dt}=-a^2.nt.e^2.\frac{d\varpi}{dt}.\sin.^2W=-\tfrac{1}{2}a^2.nt.\frac{e^2.d\varpi}{dt}.$$

The sum of the two expressions [6509g, h], being divided by a^2n, gives,

[6509i] $$\frac{2r.d\delta r+dr.\delta r}{a^2.ndt}=\frac{ede}{ndt}+\tfrac{1}{2}t.\frac{e^2.d\varpi}{dt}.$$

[6509k] We may neglect the first term of the second member, because its differential produces, in [6510], only terms of the order $edde$, or de^2, which are neglected in [6509a′]. The remaining term of [6509i] is easily reduced to the form of the second member of [6509], by substituting the following value of $d\varpi$, which is easily deduced from [54] Int. by putting $z=\varpi$.

[6509l] $$d\varpi=\cos.^2\varpi.d.(\text{tang}.\varpi)=\cos.^2\varpi.d.\left(\frac{e.\sin.\varpi}{e.\cos.\varpi}\right)=\frac{(e.\cos.\varpi).d.(e.\sin.\varpi)-(e.\sin.\varpi).d.(e.\cos.\varpi)}{e^2}.$$

The differential of [6509] being divided by dt, neglecting the quantities mentioned in [6509′], gives [6510].

[6511a] * (3373) Multiplying [6495] by $\frac{1}{2}e.\cos.\varpi$, and [6196] by $-\frac{1}{2}e.\sin.\varpi$, then taking the sum of the products, we get the second member of [6510]. This sum, being reduced [6511b] by means of the expression of e^2 [6492e], and that of $ee'.\cos.(\varpi'-\varpi)$ [6492f], with the similar expressions of $ee''.\cos.(\varpi''-\varpi)$, becomes as in [6511].

in [6502'].* The remaining term of [6503] is $\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)$, and it follows, [6512]
from § 5, Book VI., that by noticing only the action of m' upon m, and putting $\mu=1$ [6110c], the constant part of $\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)$, which is multiplied [6513]
by the squares and products of the excentricities and inclinations of the orbits, gives, in [6503], the following terms;†

$$\begin{aligned}\frac{d.\delta v_{\prime}}{dt}&=\tfrac{1}{2}m'.n.(e^2+e'^2).\left\{a^2.\left(\frac{dA^{(0)}}{da}\right)+2a^3.\left(\frac{ddA^{(0)}}{da^2}\right)+\tfrac{1}{2}a^4.\left(\frac{d^3A^{(0)}}{da^3}\right)\right\} &1\\ &-m'.n.ee'.\cos.(\varpi'-\varpi).\left\{2a^3.\left(\frac{ddA^{(1)}}{da^2}\right)+\tfrac{1}{2}a^4.\left(\frac{d^3A^{(1)}}{da^3}\right)\right\} &2\\ &+\tfrac{1}{4}m'.n.\left\{a^2a'.B^{(1)}+a^3a'.\left(\frac{dB^{(1)}}{da}\right)\right\}.\left\{\gamma_1'^2-2\gamma_1'.\gamma_1.\cos.(\Pi_1'-\Pi_1)+\gamma_1^2\right\}. &3\end{aligned}$$ [6514]

The action of the satellites m'' and m''', produces terms similar to those in [6514]. The sun's action produces, in $\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)$ [6503], the term‡ [6514']

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* (3374) We have in [6502'] $an.\text{d}R=0$, whence $\text{d}R=0$ and $f\text{d}R=$ constant; so that it may be neglected in [6503], as in [6504d]. [6512a]

† (3375) Neglecting the divisor $\sqrt{1-e^2}$, as in [6503'], we shall have, for the last term of the second member of [6503], the expression $\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)$. Substituting in it the value [6514a]
$\mu=1$ [6513], and that in [6089e], it becomes $2an.a.\left(\frac{dR}{da}\right)=2na^2.\left(\frac{dR}{da}\right)$. The symbol [6514b]
R [6503] is equivalent to δR in [3757d]; so that to conform to the notation in [3757d, &c.], we must change R into δR, in the expression [6514b], and it will become $2na^2.\left(\frac{d\delta R}{da}\right)$. The secular part of δR, depending on the second power of the [6514c]
excentricities and inclinations [3763], is given in [3764]. The part of this expression [6514d]
depending on $A^{(0)}$, $A^{(1)}$ [3761 lines 1, 2, 3], produces in δR [3772b, &c.] some terms which give in $2ndt.a^2.\left(\frac{d\delta R}{da}\right)$, the terms in the second member of [3772]; and by dividing them by dt, and substituting [6488e], we get, in [6514c], the terms in [6514 lines 1, 2]. [6514e]
The terms of δR depending on $B^{(1)}$ [3764 line 4], being substituted in the first member of [6514f], produces in this last function the following expression;

$$2na^2.\left(\frac{d\delta R}{da}\right)=\tfrac{1}{4}m'.na^2.a'.\left\{B^{(1)}+a.\left(\frac{dB^{(1)}}{da}\right)\right\}.\left\{(p'-p)^2+(q'-q)^2\right\}. \qquad [6514f]$$

Substituting the value of $(p'-p)^2+(q'-q)^2$ [6488h], it becomes as in [6514 line 3].

‡ (3376) Using the value of R [6042], and $\mu=1$ [6513], we get, by taking its partial differential relative to r, and multiplying by $\frac{2an.r}{dr}$, [6515a]

[6515] $$\frac{d.\delta v_{\prime}}{dt}=-2.\boxed{0}.\left\{e^2+H^2-\gamma_1^2+2\gamma_1.\gamma.\cos.(\Pi_1-\Pi)-\gamma^2\right\}.$$

Lastly, the part of $\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)$, depending on Jupiter's ellipticity, produces the term,*

[6516] $$\frac{d.\delta v_{\prime}}{dt}=3.(0).\{e^2-\theta^2-2\theta.\gamma_1.\cos.(\varphi+\Pi_1)-\gamma_1^2\}.$$

Therefore we shall have, by adding the functions [6511, 6514, 6515, 6516],

[6516′] the expression of $\frac{d.\delta v_{\prime}}{dt}$. To obtain that of $\frac{d.\delta v}{dt}$, or, in other words, the projection of $d.\delta v_{\prime}$ upon the fixed plane, divided by dt; we must, as in [3782], add to the preceding term of $\frac{d.\delta v_{\prime}}{dt}$, the quantity,†

[6515a′] $$\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)=-an.\frac{Sr^2}{D^3}.\{1-3s^2-3S'^2+3(1-s^2-S'^2).\cos.(2U-2v)+12sS'.\cos.(U-v)\}.$$

[6515b] Supposing the term $\frac{S}{D}$, in the value of R [6042], to be neglected as in [6277′, 6490, &c.]; we shall find, that the expression, in the second member of [6515a′], is equal to the retained part of R [6042] multiplied by $4an$; so that we shall have,

[6515c] $$\frac{2an}{\mu}.r\left(\frac{dR}{dr}\right)=4an.R\,;$$

[6515d] and if we substitute the value of $an.R$, depending on this part of the sun's action [6490], it will become as in [6515].

* (3377) The part of R, depending on Jupiter's ellipticity, is given in [6052].

[6516a] Substituting it in the first member of [6516c], and putting $\mu=1$ [6513], we get the second member of [6516c]; and by substituting in it the value of R [6052], it becomes

[6516b] as in [6516d]. Lastly, if we substitute in [6516d] the part of $an.R$ [6491], depending on Jupiter's ellipticity it becomes as in [6516].

[6516c] $$\frac{2an}{\mu}.r.\left(\frac{dR}{dr}\right)=6an.(\rho-\tfrac{1}{2}\varphi).\{\tfrac{1}{3}-(s-s_1)^2\}.\frac{MB^2}{r^3}$$

[6516d] $$=-6an.R.$$

[6517a] † (3378) The expression to be added to $dv_{\prime}$, or to $d.\delta v_{\prime}$, in the present notation, is given in [3782], under the form $\frac{1}{2}(qdp-pdq)$. Now the quantities φ, θ [1030′] are

[6517b] changed into γ_1, Π_1 respectively, in the present notation [6315, 6316]; and if we put γ_1 for $\tang.\gamma_1$, on account of its smallness, we shall find that the expression of p, q, [1032]

[6517c] will become $p=\gamma_1.\sin.\Pi_1$; $q=\gamma_1.\cos.\Pi_1$. Substituting these in the reduction of $d.\delta v_{\prime}$

[6517d] or $\frac{1}{2}(qdp-pdq)$ [6517a], we obtain in $\frac{d.\delta v_{\prime}}{dt}$ the term [6517]. Now multiplying [6501]

$$\frac{d.\delta v_{,}}{dt} = \tfrac{1}{2}.\left\{ \gamma_1.\cos.\tau_1.\frac{d.(\gamma_1.\sin.\tau_1)}{dt} - \gamma_1.\sin.\tau_1.\frac{d.(\gamma_1.\cos.\tau_1)}{dt} \right\} \qquad [6517]$$

$$= -\tfrac{1}{2}.\left\{ (0) + \boxed{0} + (0,1) + (0,2) + (0,3) \right\}.\gamma_1^2 \qquad 1$$

$$-\tfrac{1}{2}.(0).\theta.\gamma_1.\cos.(\Psi + \tau_1) + \tfrac{1}{2}.\boxed{0}.\gamma.\gamma_1.\cos.(\tau_1 - \tau) + \tfrac{1}{2}.(0,1).\gamma_1.\gamma_1'.\cos.(\tau_1 - \tau_1') \qquad 2 \qquad [6517']$$

$$+\tfrac{1}{2}.(0,2).\gamma_1.\gamma_1''.\cos.(\tau_1 - \tau_1'') + \tfrac{1}{2}.(0,3).\gamma_1.\gamma_1'''.\cos.(\tau_1 - \tau_1''') . \qquad 3$$

Collecting together all the terms* of $\frac{d.\delta v}{dt}$ and integrating, we shall obtain the secular equation of the satellite m. We may observe that we have, as in [6492e—h],

$$e^2 = (e.\cos.\varpi)^2 + (e.\sin.\varpi)^2 \qquad [6518]$$

$$ee'.\cos.(\varpi' - \varpi) = e.\cos.\varpi.\,e'.\cos.\varpi' + e.\sin.\varpi.\,e'.\sin.\varpi'. \qquad [6519]$$

We have seen, in [6493, 6494], that $e.\sin.\varpi$, $e.\cos.\varpi$, are represented by a series of terms of the following forms;

$$e.\sin.\varpi = -h.\sin.(gt + \Gamma) - h_1.\sin.(g_1 t + \Gamma_1) -, \text{\&c.}; \qquad [6520]$$

$$e.\cos.\varpi = -h.\cos.(gt + \Gamma) - h_1.\cos.(g_1 t + \Gamma_1) -, \text{\&c.}; \qquad [6521]$$

and $e'.\sin.\varpi'$, $e'.\cos.\varpi'$, &c. [6235k, l, m], are the sums of similar terms. [6522]
Hence we shall have, in the expression of $\frac{d.\delta v}{dt}$; *First.* Some constant [6522']
terms; *Second.* Some terms multiplied by cosines of angles of the forms $(g_1 - g).t + \Gamma_1 - \Gamma$, &c.† *We may neglect the constant parts, because the* [6523]

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by $\tfrac{1}{2}\gamma_1.\cos.\tau_1$, and [6502] by $-\tfrac{1}{2}\gamma_1.\sin.\tau_1$; then taking the sum of the products, we get the value of the function [6517]. This sum can be reduced by substituting the values [6517e] of γ_1^2, $\gamma_1.\gamma.\cos.(\tau_1 - \tau)$, $\theta.\gamma_1.\cos.(\Psi + \tau_1)$, &c. [6492e, g, h, &c.]; and by this means it will become as in [6517'].

* (3379) We must add together the expressions [6511, 6514, 6515, 6516, 6517'], and also the terms depending on m'', m''' [6514], which are similar to those explicitly [6518a] given in [6514]. The sum will be the complete value of the terms of $\frac{d.\delta v}{dt}$, now under consideration.

† (3380) Several of the terms of $\frac{d.\delta v}{dt}$ [6511, 6514], contain e^2, or $(e.\cos.\varpi)^2 + (e.\sin.\varpi)^2$ [6523a] [6518]; and if we substitute the values of $e.\sin.\varpi$, $e.\cos.\varpi$, [6520, 6521], and reduce, by means of [17, 20] Int., we shall get,

$$e^2 = h^2 + h_1^2 + \text{\&c.} + 2hh_1.\cos.\{(g_1 - g).t + \Gamma_1 - \Gamma\} + \text{\&c.}; \qquad [6523b]$$

[6524] *terms which result from them, after integration, are proportional to the time, and are therefore confounded with the mean motion of* m. We may apply the same considerations to the terms depending on the inclinations of the orbits.

13. The most important terms of the expression of the secular equation of m, are those depending on the secular variations of the equator and orbit of Jupiter.* They are analogous to those from which the moon's
[6525] secular equation arises, which we have developed in the seventh book [6528x].† To obtain them we must substitute, in the preceding expression

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as is evident, by the calculation in the first note in page 601 of the first volume of this
[6523c] work. Similar results are obtained from the terms of [6511, 6514], depending upon $ee'.\cos.(\varpi'-\varpi)$, developed as in [6519]. The constant terms, spoken of in [6522'], are
[6523d] like $h^2+h'^2+\&c.$ [6523b]; and the variable terms, or those which contain t, in [6523b, &c.], are similar to those which depend on $2hh_1.\cos.\{(g_1-g).t+\Gamma_1-\Gamma\}$ [6523].
[6523e] The integral of these terms produces in δv, terms of the form $\frac{2hh_1}{g_1-g}.\sin.\{(g_1-g).t+\Gamma_1-\Gamma\}$. Developing this, according to the powers of t, by means of [60] Int., it produces terms depending on the first power of t, of the form $2hh_1.t.\cos.(\Gamma_1-\Gamma)$, which is of the same
[6523f] order as hh_1, or e^2, ee', &c., and may therefore be neglected on account of its smallness
[6523g] [6057e]. The author also investigates the remaining term in [6514 line 3], depending on
[6523h] $B^{(1)}$; and finds, in [6547], that it is too small to be noticed. The other terms of $\frac{d.\delta v}{dt}$,
[6523i] enumerated in [6518a], are contained in the functions [6515, 6516, 6517'], and they produce, by development, the expression [6528]; as we shall see in [6528u—w].

[6525a] * (3381) This is shown in [6536], and in the corresponding note [6536g—i].

[6528a] † (3382) We shall now develop, according to the powers of t, the parts of $\frac{d.\delta v}{dt}$ which are mentioned in [6523i]; namely, those in [6515, 6516, 6517'], being the only
[6528b] ones of importance. In these developments we shall *always neglect the constant parts*, for the reasons stated in [6524]; *also the terms depending on* t^2, and the higher powers
[6528c] of t; because they introduce into the integral δv, only terms of the order t^3, t^4, &c.,
[6528d] which are insensible. Moreover we shall neglect the terms depending on the excentricities e, e', &c., on account of their smallness [6057e]. Retaining therefore, as in [6528b, &c.],
[6528e] only the terms which are multiplied by the first power of t, we find that the sum of the squares of [6400, 6401] gives γ^2 [6528g]; the sum of the squares of [6413, 6414] gives $\gamma_1{}^2$ [6528g]. Moreover, the double of the product of the two equations [6400, 6413],
[6528f] being added to the similar product of [6401, 6414], gives the third of the equations

of $\frac{d.\delta v}{dt}$, the values of $\gamma.\sin.\eta$, $\gamma.\cos.\eta$, Ψ, θ, $\gamma_1.\sin.\eta_1$, &c., which are found in § 10. Therefore if we neglect the constant terms, as in [6523], and [6526]

[6528g], the terms in its first member being reduced by means of [21] Int.

$\gamma^2=0;\quad \gamma_1^2=2(\lambda-1).\lambda.{}^1L.bt;\quad 2\gamma_1.\gamma.\{\sin.\eta.\sin.\eta_1+\cos.\eta.\cos.\eta_1\}=2\gamma_1.\gamma.\cos.(\eta_1-\eta)=2(\lambda-1).{}^1L.bt.$ [6528g]

Substituting these values, together with $H^2=2H_1.ct$ [6527], and $e^2=0$, in [6515], it becomes, [6528h]

$$-2\boxed{0}.\{2H_1.ct-2(\lambda-1).\lambda.{}^1L.bt+2(\lambda-1).{}^1L.bt\}=-4\boxed{0}.H_1.ct+4\boxed{0}.(1-\lambda)^2.{}^1L.bt;\quad [6528i]$$

being the same as the two terms in [6528 line 1] depending upon $\boxed{0}$.

Again we have, as in [6404], by neglecting terms of the order t^2,

$$\theta={}^1L;\qquad \theta.\cos.\Psi={}^1L;\qquad -\theta.\sin.\Psi=-{}^1L.pt.\quad [6528k]$$

Multiplying the equations [6414, 6413] respectively, by these values of $\theta.\cos.\Psi$, $-\theta.\sin.\Psi$; taking the sum of the products, reducing by means of [23] Int., and retaining terms of the order t, we get, [6528l]

$$\theta.\gamma_1.\{\cos.\Psi.\cos.\eta_1-\sin.\Psi.\sin.\eta_1\}=\theta.\gamma_1.\cos.(\Psi+\eta_1)=\lambda.{}^1L.bt.\quad [6528m]$$

Substituting this and the values of γ_1^2, θ^2, e^2, [6528g, h, k], in [6516], it becomes, by retaining only the terms depending on t,

$$3(0).\{-2\lambda.{}^1L.bt-2(\lambda-1).\lambda.{}^1L.bt\}=-6(0).\lambda^2.{}^1L.bt;\quad [6528n]$$

which is the same as the term depending on (0), in [6528 line 1].

We shall now compute the terms of [6517']. Substituting, in the first line of this expression, the value of γ_1^2 [6528g], it produces the terms in [6528 line 2], as is evident by inspection. In like manner, by substituting the value of $\theta.\gamma_1.\cos.(\Psi+\eta_1)=\lambda.{}^1L.bt$, [6528o] [6528m], in the term depending on (0), [6517' line 2], it produces the term depending on (0), in [6528 line 3]; also by substituting $\gamma\gamma_1.\cos.(\eta_1-\eta)=(\lambda-1).{}^1L.bt$ [6528g], in [6528p] the term of [6517' line 2], depending on $\boxed{0}$, it produces the term depending on $\boxed{0}$, in [6528 line 3].

If we change γ, η, into γ_1', η_1', respectively, in the identical equation [6492g], we get,

$$\gamma_1.\gamma_1'.\cos.(\eta_1-\eta_1')=(\gamma_1.\cos.\eta_1).(\gamma_1'.\cos.\eta_1')+(\gamma_1.\sin.\eta_1).(\gamma_1'.\sin.\eta_1').\quad [6528q]$$

Again, if we change the symbols γ_1, η_1, λ, [6315, 6316, 6343], corresponding to the satellite m, into γ_1', η_1', λ', respectively, corresponding to the satellite m', [6489, 6344], we shall find that the equations [6413, 6414] become, [6528r]

$$\gamma_1'.\sin.\eta_1'=(1-\lambda').{}^1L.{}^1pt+\lambda'.at;\quad [6528s]$$

$$\gamma_1'.\cos.\eta_1'=(\lambda'-1).{}^1L+\lambda'.bt.\quad [6528t]$$

Substituting these values and those of [6413, 6114], in the second member of [6528q], and retaining only the terms of the order t, we obtain,

$$\gamma_1.\gamma_1'.\cos.(\eta_1-\eta_1')=\{(\lambda-1).\lambda'+(\lambda'-1).\lambda\}.{}^1L.bt.\quad [6528u]$$

Multiplying this by $\frac{1}{2}(0,1)$, we get the term of [6517' line 2] depending on $(0,1)$, and

[6526′] suppose the excentricity H of Jupiter's orbit to be developed in the following series,

[6527] $$H = H_{1} + ct + \&c.; \quad [4407 \text{ line } 1]$$

[6527′] H_{1} being the value of H at the origin of the time t; we shall have,

Symbol H_{1}

[6528]
$$\frac{d.\delta v}{dt} = -4.\boxed{0}.H_{1}.ct + 4.\boxed{0}.(1-\lambda)^2.{}^1L.bt - 6(0).\lambda^2.{}^1L.bt \qquad 1$$
$$+(1-\lambda).\lambda.\left\{(0)+\boxed{0}+(0,1)+(0,2)+(0,3)\right\}.{}^1L.bt \qquad 2$$
$$-\tfrac{1}{2}.\lambda.(0).{}^1L.bt - \tfrac{1}{2}.(1-\lambda).\boxed{0}.{}^1L.bt \qquad 3$$
$$+\tfrac{1}{2}.(0,1).\{(\lambda-1).\lambda'+(\lambda'-1).\lambda\}.{}^1L.bt \qquad 4$$
$$+\tfrac{1}{2}.(0,2).\{(\lambda-1).\lambda''+(\lambda''-1).\lambda\}.{}^1L.bt \qquad 5$$
$$+\tfrac{1}{2}.(0,3).\{(\lambda-1).\lambda'''+(\lambda'''-1).\lambda\}.{}^1L.bt. \qquad 6$$

Hence it is easy to find, by means of the equations between λ, λ', λ'', λ''' [6347—6350],*

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arising from the action of the satellite m' upon m, as in [6528 line 4]. We may deduce [6528v] from it the similar expression in [6528 line 5], arising from the action of m'' upon m, or from the term depending on (0,2) in [6517′ line 3], by merely changing (0,1) into (0,2), and λ into λ''. We may also obtain, in the same way, that in [6528 line 6], arising [6528w] from the action of m''' upon m, or from the term depending on (0,3) in [6517′ line 3], by changing, in [6528 line 4], (0,1) into (0,3), and λ' into λ'''. Finally we may observe, that if we multiply the expression [6528] by dt, and integrate it, we shall obtain [6528x] terms of δv, depending on t^2, which are similar to those in the lunar theory [5541, 5543] depending upon the moon's secular equation, as is observed in [6525].

* (3383) Transposing in [6347] the terms depending on (0,1), (0,2), (0,3), we get,

[6529a] $$\lambda.(0)+(\lambda-1).\boxed{0} = (0,1).(\lambda'-\lambda)+(0,2).(\lambda''-\lambda)+(0,3).(\lambda'''-\lambda);$$

multiplying this by $\frac{1}{2}-\lambda$, we obtain,

[6529b] $$(\tfrac{1}{2}-\lambda).\left\{\lambda.(0)+(\lambda-1).\boxed{0}\right\} = (\tfrac{1}{2}-\lambda).\{(0,1).(\lambda'-\lambda)+(0,2).(\lambda''-\lambda)+(0,3).(\lambda'''-\lambda)\}.$$

Now the terms in the first member of this expression are the same as those of [6528 lines 2,3] [6529c] which depend on the functions (0), $\boxed{0}$, multiplied by the common factor ${}^1L.bt$, as is easily seen by connecting the coefficients, and making some slight reductions. We may [6529c′] therefore neglect the terms depending on (0), $\boxed{0}$ [6528 lines 2, 3], and instead of them substitute the second member of [6529b], multiplied by the common factor ${}^1L.bt$, namely,

$$\frac{d.\delta v}{dt}=-4.\boxed{0}.H_1.ct+4.\boxed{0}.(1-\lambda)^2.{}^1L.bt-6(0).\lambda^2.{}^1L.bt;$$ [6529]

which produces in δv, or in the motion of the satellite m, the following secular equation;*

$$\delta v=-2.\boxed{0}.H_1.ct^2+2.\boxed{0}.(1-\lambda)^2.{}^1L.bt^2-3(0).\lambda^2.{}^1L.bt^2.$$ [6530]

We may observe, with respect to the three first satellites, that the ratios which obtain, between their mean motions, change considerably their secular [6531]
inequalities, as we shall see in [6663—6711, &c.].

When there is but one satellite, we shall obtain, from [6347],† Value of λ for the moon.

$$\lambda=\frac{\boxed{0}}{\boxed{0}+(0)}.$$ [6532]

In the lunar theory $\boxed{0}$ is incomparably greater than (0), and thus we [6533]

$$(\tfrac{1}{2}-\lambda).{}^1L.bt.\{(0,1).(\lambda'-\lambda)+(0,2).(\lambda''-\lambda)+(0,3).(\lambda'''-\lambda)\}.$$ [6529d]

Hence it appears that the coefficient of $(0,1).{}^1L.bt$, in the three lines [6528 lines 2, 4; 6529d], are respectively,

$$(1-\lambda).\lambda;\qquad \tfrac{1}{2}(\lambda-1).\lambda'+\tfrac{1}{2}(\lambda'-1).\lambda;\qquad (\tfrac{1}{2}-\lambda).(\lambda'-\lambda).$$ [6529e]

If we add these terms together, we find that they mutually destroy each other. For the coefficient of λ', in the sum, setting the terms in the order in which they occur, without any reduction, is $\frac{1}{2}(\lambda-1)+\frac{1}{2}\lambda+(\frac{1}{2}-\lambda)=0$; and by neglecting λ', in the sum, it becomes, without reduction, $(1-\lambda).\lambda-\frac{1}{2}\lambda-(\frac{1}{2}-\lambda).\lambda=0$. In like manner we find, [6529f]
that the coefficients of $(0,2).{}^1L.bt$, and $(0,3).{}^1L.bt$, in [6528 lines 2, 4; 6529d], vanish. Therefore the whole expression [6528 lines 2—6] vanishes; and the formula [6529g]
[6528] is reduced to its first line, as in [6529].

* (3384) Multiplying [6529] by dt, and integrating, we obtain in v, or δv, the expression [6530]. [6530a]

† (3385) If there be only one satellite m, we must neglect (0,1), (0,2), (0,3), in the equation [6347], and it will become,

$$0=\left\{(0)+\boxed{0}\right\}.\lambda-\boxed{0}.$$ [6532a]

Dividing this by $(0)+\boxed{0}$, we get [6532].

shall have very nearly,*

[6534] $$\lambda = 1 - \frac{(0)}{\boxed{0}};$$

[6535] so that λ will differ but very little from unity; which reduces the preceding

[6536] expression of the secular equation to the single term† $-2.\boxed{0}.H_1.ct^2$.

[6534a] * (3386) The values of (0), $\boxed{0}$, [6216] give $\frac{(0)}{\boxed{0}} = \frac{4}{3}.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.\frac{n^2}{M^2}$; and if we change the elements relative to Jupiter and its first satellite, into those of the earth and moon

[6534b] respectively, we shall have the ellipticity of the earth $\rho = \frac{1}{305}$ [6044, 5593]; the centrifugal force $\varphi = \frac{1}{289}$ [6044′, 1594a]; hence $\rho - \frac{1}{2}\varphi = \frac{1}{645}$. Moreover,

[6534c] $\frac{M}{n} = \frac{\text{earth's mean motion}}{\text{moon's mean motion}} = m = 0{,}0748$ nearly, [6101′, 6062′, 5117]; also,

$$\frac{1}{a} = \frac{\text{earth's radius}}{\text{moon's dist. from earth}} = 0{,}0165510, \quad [6061, 6082, 5329].$$

Substituting these values in the expression [6534a], it becomes,

[6534d] $$\frac{(0)}{\boxed{0}} = \frac{4}{3}.\frac{1}{645}.(0{,}0165510)^2.(0{,}0748)^{-2} = \frac{1}{9881} \text{ nearly.}$$

[6534e] This quantity being very small we may neglect its square and higher powers, in the development of λ in terms of $\frac{(0)}{\boxed{0}}$, and then it becomes as in [6534].

[6536a] † (3387) Substituting the value of λ [6534], in the second term of [6530], it becomes of the order $\frac{(0)}{\boxed{0}}.(0)$, which may be neglected on account of its smallness [6534e].

[6536b] The third term is also very small in comparison with the first. For the ratio of these two terms is represented by $\frac{3}{2}.\frac{(0)}{\boxed{0}}.\frac{{}^1L}{H_1}.\frac{b}{c} = \frac{3}{2}.\frac{1}{9881}.\frac{{}^1L}{H_1}.\frac{b}{c}$ [6534d]; and we have in

[6536c] [6937, 6938] $\frac{b}{c} = \frac{1}{14}$ nearly; moreover the value of ${}^1L$ [6937], expressed in parts of the radius, is nearly equal to $0{,}05$; and the excentricity H_1 [6527′, 4080 line 5], is $0{,}048..$, which may be considered as equal to ${}^1L$; hence the ratio [6536b] becomes

[6536d] $\frac{3}{2}.\frac{1}{9881}.\frac{1}{14}$ nearly; consequently the term depending on (0) may be neglected in

[6536e] comparison with the first term of [6530], and then this expression of δv will be reduced

Substituting $\boxed{0} = \frac{3}{4}.\frac{M^2}{n}$ [6216], it becomes, as in [6536f], $-\frac{3}{2}.\frac{M^2}{n}.H_1.ct^2$; [6537]
which agrees with what we have found in [5541, 6536i].

The terms, which we have just had under consideration, will finally become very sensible. Of the other terms, the greatest are those which depend upon the products* $\theta' l$, $\theta' l'$, &c.; and we shall see hereafter, that l, l', &c. are small quantities, whose squares may be neglected, without any sensible error. This being premised, we shall consider the following term of $\frac{d.\delta v}{dt}$ [6515]; [6537′]

$$\frac{d.\delta v}{dt} = 2.\boxed{0}.\{\gamma_1^2 - 2\gamma\gamma_1.\cos.(\Pi_1 - \Pi) + \gamma^2\}. \qquad [6538]$$

Now it is easy to prove that,†

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to its first term, $\delta v = -2.\boxed{0}.H_1.ct^2$, as in [6536]. Substituting the value of $\boxed{0}$ [6216], it becomes $\delta v = -\frac{3}{2}.\frac{M^2}{n}.H_1.ct^2$, as in [6537]. This is the same as that which is deduced from the expression [5541] $\delta v = -\frac{3}{2}.m^2.f(e'^2 - E'^2).ndt$, by changing m [5117] into $\frac{M}{n}$ [6534c], E' into H_1, and e' into H, to conform to the present notation. For by this means it becomes nearly $-\frac{3}{2}.\frac{M^2}{n}.f(H^2 - H_1^2).dt$; and by substituting the value of H [6527], we get the term, [6536f] [6536g] [6536h]

$$-\frac{3}{2}.\frac{M^2}{n}.f\,2H_1.ctdt = -\frac{3}{2}.\frac{M^2}{n}.H_1.ct^2, \qquad [6536i]$$

as in [6537].

* (3388) If we compare [6430 line 2] with [7352 line 2], we see that $\xi''' l = l'''$ [6422] is of the order $2771''$; moreover $\theta' = 34352''$ [7217]. The squares and products of these expressions being divided by the radius in seconds, give $\theta'^2 = 1854''$, $\theta' l''' = 150''$, $l'''^2 = 12''$. The other values of l''' [7352 lines 3, 4] are much less than the preceding, so that generally their squares may be neglected, as in [6537′]. The terms depending on $\theta' l$, $\theta' l'$, produce in $\frac{d.\delta v}{dt}$, some terms connected with $\cos.(pt + A - \varpi')$, &c., which do not contain v, as in [6540]; and by integration they introduce into the expression of δv, the divisor p, which is small in comparison with n [6025k, p]; and on this account it becomes necessary to notice them, as in [6537′, &c.]. [6537a] [6537b] [6537c] [6537d] [6537e]

† (3389) The function [6538] is part of [6515], the other part being neglected as in [6552d]; and we shall see, in [6540], that the retained part of the function [6538] [6539a]

[6539] $$\tfrac{1}{2}.\{\gamma_1^2-2\gamma\gamma_1.\cos.(\Pi_1-\Pi)+\gamma^2\},$$

is equal to the part, which is independent of v, in the square of the expression of the latitude of m, above the plane of the orbit of Jupiter. We have given the expression of the latitude, in [6427]; and by developing its square, in sines and cosines of v and its multiples; neglecting the
[6539'] squares and products of l and l'; we shall find, for the double of the
[6539''] part which is independent of these sines and cosines,* the following value;

[6540] $$(1-\lambda)^2.\theta'^2+2(\lambda-1).\theta'.\left\{\begin{array}{l} l.\cos.(pt+\text{A}-\Psi')+l_1.\cos.(p_1t+\text{A}_1-\Psi') \\ +l_2.\cos.(p_2t+\text{A}_2-\Psi')+l_3.\cos.(p_3t+\text{A}_3-\Psi') \end{array}\right\}.$$

Hence the preceding term of $\frac{d.\delta v}{dt}$ produces the following expression;

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contains $\theta' l$, $\theta' l'$, &c. Now the latitude of the satellite m, above the fixed plane, is represented by $\gamma_1.\sin.(v-\Pi_1)$ [6332b]; and this latitude would be $\gamma.\sin.(v-\Pi)$ [6391],
[6539b] if it move in the plane of Jupiter's orbit. *Hence the latitude of the satellite m, above the orbit of Jupiter*, is $\gamma_1.\sin.(v-\Pi_1)-\gamma.\sin.(v-\Pi)$. Squaring this, and reducing the product by [17] Int., neglecting also the terms containing $2v$, we may put,

[6539c] $$\sin.^2(v-\gamma_1)=\tfrac{1}{2}; \quad \sin.^2(v-\Pi)=\tfrac{1}{2}; \quad \sin.(v-\gamma_1).\sin.(v-\Pi)=\tfrac{1}{2}.\cos.(\Pi_1-\Pi);$$

and then by substitution we get, as in [6539],

[6539d] $$\{\gamma_1.\sin.(v-\Pi_1)-\gamma.\sin.(v-\Pi)\}^2=\tfrac{1}{2}\gamma_1^2-\gamma\gamma_1.\cos.(\Pi_1-\Pi)+\tfrac{1}{2}\gamma^2.$$

* (3390) The expression [6427] becomes, by using the symbol Σ' [6324'], of the following abridged form, which represents *the latitude s, of the satellite m, above the orbit of Jupiter*;

[6540a] $$s=(\lambda-1).\theta'.\sin.(v+\Psi')+\Sigma'.l.\sin.(v+pt+\text{A}).$$

Squaring this expression, and neglecting terms of the second order in l, l', &c., as in [6539'], we shall have,

[6540b] $$s^2=(1-\lambda)^2.\theta'^2.\sin.^2(v+\Psi')+2(\lambda-1).\theta'.\sin.(v+\Psi').\Sigma'.l.\sin.(v+pt+\text{A}).$$

Reducing these products by [17] Int., and then neglecting, as in the preceding note, the terms depending on $2v$, we get,

[6540c] $$\sin.^2(v+\Psi')=\tfrac{1}{2}; \quad \sin.(v+\Psi').\sin.(v+pt+\text{A})=\tfrac{1}{2}.\cos.(pt+\text{A}-\Psi'); \ \&c.$$

Substituting these in [6540b], and then doubling the product, as in [6539''], we obtain
[6540d] [6540]. This represents the part of $\gamma_1^2-2\gamma\gamma_1.\cos.(\Pi_1-\Pi)+\gamma^2$, which is independent
[6540e] of $2v$; and by multiplying it by $2.\boxed{0}$, it produces, for the expression of [6538], the function [6541]; omitting the quantity,

[6540f] $$2(1-\lambda)^2.\theta'^2.\boxed{0}=2(1-\lambda)^2.{}^1L^2.\boxed{0}+4(1-\lambda)^2.{}^1L.bt+\&c. \ [6405];$$

because the constant part is neglected, as in [6505]; and the part depending on t, has already been noticed in [6529], where it was found by a different method.

$$\frac{d.\delta v}{dt} = 4.(\lambda-1).\boxed{0}.\theta'.\left\{\begin{array}{l} l.\cos.(pt+\text{A}-\Psi')+l_1.\cos.(p_1t+\text{A}_1-\Psi') \\ +l_2.\cos.(p_2t+\text{A}_2-\Psi')+l_3.\cos.(p_3t+\text{A}_3-\Psi') \end{array}\right\}. \qquad [6541]$$

We shall now consider the following terms, contained in [6516];

$$-3.(0).\{\theta^2+2\theta.\gamma_1.\cos.(\Psi+\Pi_1)+\gamma_1^2\}. \qquad [6542]$$

The expression of *the latitude of the satellite* m, *above the plane of Jupiter's equator, is,**

$$\begin{array}{l} \lambda.\theta'.\sin.(v+\Psi')+l.\sin.(v+pt+\text{A})+l_1.\sin.(v+p\,t+\text{A}_1) \\ \qquad +l_2.\sin.(v+p_2t+\text{A}_2)+l_3.\sin.(v+p_3t+\text{A}_3)\,; \end{array} \qquad [6543]$$

hence we easily find, that this term of $\frac{d.\delta v}{dt}$ produces the following;†

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* (3391) The latitude of the satellite m, if it move in the plane of Jupiter's equator, is $-\theta'.\sin.(v+\Psi')$ [6392], above the orbit of Jupiter. Subtracting this from s [6427], [6543a] which denotes the actual latitude above the same orbit [6426'], we obtain the real latitude, [6543b] counted from the equator of Jupiter, as in [6543].

† (3392) The latitude of the satellite m, above the fixed plane, is $\gamma_1.\sin.(v-\Pi_1)$ [6544a] [6332b]; but if it move in the plane of Jupiter's equator, its latitude will be $-\theta.\sin.(v+\Psi)$ [6322]. Subtracting this last expression from the preceding, we get *the latitude counted* [6544b] *from the plane of Jupiter's equator*, $\gamma_1.\sin.(v-\Pi_1)+\theta.\sin.(v+\Psi)$. This may be derived [6544c] from the expression [6539b], by changing γ into $-\theta$, and Π into $-\Psi$; and by making the same changes in [6539], omitting also the terms depending on $2v$, we get,

$$\{\gamma_1.\sin.(v-\Pi_1)+\theta.\sin.(v+\Psi)\}^2 = \tfrac{1}{2}\gamma_1^2+\gamma_1.\theta.\cos.(\Psi+\Pi_1)+\tfrac{1}{2}\theta^2\,; \qquad [6544d]$$

which represents the square of the latitude of m *above the plane of Jupiter's equator* [6544e] [6544b, &c.]. Now this latitude may also be found, as in the last note, by subtracting $-\theta'.\sin.(v+\Psi')$, which represents the latitude of the satellite above the orbit of Jupiter, supposing the motion of the satellite to be in the plane of Jupiter's equator, from [6544f] the general expression of the latitude s [6427], above the orbit of Jupiter [6426]; the difference is the function [6543], or $\lambda.\theta'.\sin.(v+\Psi')+\Sigma'.l.\sin.(v+pt+\text{A})$. The square [6544g] of this expression being substituted in the first member of [6544d], then multiplying by 6(0), and transposing all the terms to the opposite members, we get [6544h]. Reducing the second member of this expression, as we have done that in [6540b], we obtain [6544i].

$$-3(0)\{\theta^2+2\theta.\gamma_1.\cos.(\Psi+\Pi_1)+\gamma_1^2\} = -6(0).\{\lambda.\theta'.\sin.(v+\Psi')+\Sigma'.l.\sin.(v+pt+\text{A})\}^2 \qquad [6544h]$$

$$= -6(0).\{\tfrac{1}{2}\lambda^2.\theta'^2+\lambda.\theta'.\Sigma'.l.\cos.(pt+\text{A}-\Psi')\}. \qquad [6544i]$$

The first member of [6544h] is the same as in [6542]; and the terms under the sign Σ', in the second member, are the same as those in [6544]. The other term, $-3(0).\lambda^2.\theta'^2$, [6544k] may be neglected, as in the calculation [6540e, f]; observing, that after substituting the value of θ' [6405], we may reject the constant term $-3(0).\lambda^2.{}^1L^2$, as in [6505]; [6544l] and the term $-6(0).\lambda^2.{}^1L.bt$, has already been noticed in [6528n].

[6544] $$\frac{d.\delta v}{dt} = -6.(0).\lambda.\theta'.\left\{\begin{array}{l} l.\cos.(pt+\Lambda-\Psi')+l_1.\cos.(p_1t+\Lambda_1-\Psi') \\ +l_2.\cos.(p_2t+\Lambda_2-\Psi')+l_3.\cos.(p_3t+\Lambda_3-\Psi') \end{array}\right\}.$$

We may also notice the term of $\frac{d.\delta v}{dt}$, contained in [6514 line 3], namely,

[6545] $$\tfrac{1}{4}m'.n.\left\{a^2a'.B^{(1)}+a^3a'.\left(\frac{dB^{(1)}}{da}\right)\right\}.\{\gamma_1'^2-2\gamma_1'.\gamma_1.\cos.(\Pi_1'-\Pi_1)+\gamma_1^2\}.$$

[6546] We shall observe that $\gamma_1'.\cos.\Pi_1'-\gamma_1.\cos.\Pi_1$, and $\gamma_1'.\sin.\Pi_1'-\gamma_1.\sin.\Pi_1$, [6547c, d], are of the order λ, which is a very small fraction in the theory of the satellites of Jupiter. The sum of the squares of these two quantities gives,*

[6547] $$(\gamma_1'.\cos.\Pi_1'-\gamma_1.\cos.\Pi_1)^2+(\gamma_1'.\sin.\Pi_1'-\gamma_1.\sin.\Pi_1)^2=\gamma_1'^2-2\gamma_1'.\gamma_1.\cos.(\Pi_1'-\Pi_1)+\gamma_1^2\,;$$

which is of the order λ^2, we may therefore neglect it, without any sensible error.

Lastly, we shall consider the term of $\frac{d.\delta v}{dt}$, contained in [6517], namely,

[6548] $$\tfrac{1}{2}.\left\{\gamma_1.\cos.\Pi_1.\frac{d.(\gamma_1.\sin.\Pi_1)}{dt}-\gamma_1.\sin.\Pi_1.\frac{d.(\gamma_1.\cos.\Pi_1)}{dt}\right\}.$$

It is easy to prove that,†

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* (3393) Developing the first member of [6547], it becomes, without reduction,

[6547a] $$\gamma_1'^2.\{\cos.^2\Pi_1'+\sin.^2\Pi_1'\}-2\gamma_1'.\gamma_1.\{\cos.\Pi_1'.\cos.\Pi_1+\sin.\Pi_1'.\sin.\Pi_1\}+\gamma_1^2.\{\cos.^2\Pi_1+\sin.^2\Pi_1\}\,;$$

and this is easily reduced to the form in the second member of [6547], by using [24, &c.]

[6547b] Int. Subtracting the expression [6413] from [6413d], we get [6547c]; and by subtracting [6414] from [6413e], we get [6547d].

[6547c] $$\gamma_1'.\sin.\Pi_1'-\gamma_1.\sin.\Pi_1=(\lambda-\lambda').{}^1L.{}^1pt+(\lambda'-\lambda).at.$$

[6547d] $$\gamma_1'.\cos.\Pi_1'-\gamma_1.\cos.\Pi_1=(\lambda'-\lambda).{}^1L+(\lambda'-\lambda).bt$$

The second members of these expressions being of the order $\lambda'-\lambda$, the sum of their

[6547e] squares must be of the order $(\lambda'-\lambda)^2$; which produces, in [6545], only terms of the order $m'.(\lambda'-\lambda)^2$; and these may be neglected on account of their smallness [6829, 7206, &c.].

† (3394) The latitude of the satellite m, *above the plane of Jupiter's orbit*, is given in [6539b], and in another form in [6540a]. Now if we put these two expressions equal to each other, and transpose $\gamma.\sin.(v-\Pi)$, we shall get,

[6548a] $$\gamma_1.\sin.(v-\Pi_1)=\Sigma'.l.\sin.(v+pt+\Lambda)+(\lambda-1).\theta'.\sin.(v+\Psi')+\gamma.\sin.(v-\Pi).$$

[6548b] But from [21] Int. we have $\sin.(v+b)=\sin.v.\cos.b+\cos.v.\sin.b$; and if we develop the four terms of the equation [6548a], by means of this formula, we shall obtain an

[6548c] equation which must be satisfied for all values of v; so that we may put the terms depending on $\sin.v$ separately equal to each other, and we shall get [6550]; in like

$$\gamma_1.\sin.\Pi_1 = -l.\sin.(pt+\text{A}) - l_1.\sin.(p_1t+\text{A}_1) - \&\text{c.} - (\lambda-1).\theta'.\sin.\Psi' + \gamma.\sin.\Pi; \qquad [6549]$$

$$\gamma_1.\cos.\Pi_1 = l.\cos.(pt+\text{A}) + l_1.\cos.(p_1t+\text{A}_1) + \&\text{c.} + (\lambda-1).\theta'.\cos.\Psi' + \gamma.\cos.\Pi; \qquad [6550]$$

hence the term [6548] produces the following expression ;*

$$\frac{d.\delta v}{dt} = -\tfrac{1}{2}.(\lambda-1).\theta'.\left\{\begin{array}{l} pl.\cos.(pt+\text{A}-\Psi') + p_1l_1.\cos.(p_1t+\text{A}_1-\Psi') \\ +p_2l_2.\cos.(p_2t+\text{A}_2-\Psi') + p_3l_3.\cos.(p_3t+\text{A}_3-\Psi') \end{array}\right\}. \begin{array}{r} 1 \\ 2 \end{array} \qquad [6551]$$

Now if we add together the terms [6541, 6544, 6551], and integrate the sum, we shall have, for the corresponding part of δv, the following expression ;†

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manner the terms depending on $\cos.v$, give [6549], by changing the signs of all the terms. [6548d]
We may also observe that each of the expressions of $\gamma_1.\sin.\Pi_1$, $\gamma_1.\cos.\Pi_1$ [6549, 6550], contain two terms, depending on θ', γ, which are not inserted in the values of $\gamma_1.\sin.\Pi_1$, $\gamma_1.\cos.\Pi_1$, [6499, 6500]. [6548e] The reason for this omission is, that their differentials produce only insensible quantities in [6501, 6502], on account of the smallness of the coefficients of t, in the values θ', Ψ', γ, Π, [6927—6929, 4246], in comparison with the values of p [6025p]. [6548f] In fact the smallness of these coefficients enables us to consider θ', ϖ', γ, Π, as constant quantities, as in [7220]; and then their differentials vanish from [6501, 6502]; so that we may neglect them in using the differentials of [6499, 6500]. [6548g] We shall then have, by using the sign Σ', in like manner as in [6324'], so as to include the terms depending on p, p_1, p_2, p_3;

$$\frac{d.(\gamma_1.\sin.\Pi_1)}{dt} = -\Sigma'.pl.\cos.(pt+\text{A}); \qquad \frac{d.(\gamma_1.\cos.\Pi_1)}{dt} = -\Sigma'.pl.\sin.(pt+\text{A}). \qquad [6548h]$$

* (3395) Substituting the values [6549, 6550, 6548h] in [6548], and neglecting terms of the order l^2p, $l\gamma p$, &c. on account of their smallness, we find that it is only necessary [6551a] to retain the terms containing θ', in [6549, 6550]; hence we obtain, for the value of the function [6548], the first member of the following expression [6551c]. [6551b] This is reduced to the form in the second member, by using [24] Int.

$$-\tfrac{1}{2}.(\lambda-1).\theta'.\Sigma'.pl.[\cos.\Psi'.\cos.(pt+\text{A}) + \sin.\Psi'.\sin.(pt+\text{A})] = -\tfrac{1}{2}.(\lambda-1).\theta'.\Sigma'.pl.\cos.(pt+\text{A}-\Psi') \qquad [6551c]$$

This last expression agrees with that in [6551].

† (3396) The expression [6552] is the sum of the functions [6541, 6544, 6551], multiplied by dt, and then integrated. [6552a] For the terms multiplied by $\boxed{0}$, in [6552 lines 1, 2], are derived from [6541]; those multiplied by (0), from [6544]; and those in [6552 lines 3, 4], from [6551]. It will be seen in [6944] that there is but one term of the expression δv, or rather $\delta v'''$, [6552], which requires notice from its magnitude, [6552b] and this produces only $-49'',51$, in the motion of the fourth satellite; the values for the other satellites being insensible. We may remark that in finding the [6552c]

$$\delta v = -\left\{6(0).\lambda+4.(1-\lambda).\boxed{0}\right\}.\theta'.\left\{\begin{array}{l}\frac{l}{p}.\sin.(pt+\text{A}-\Psi')+\frac{l_1}{p_1}.\sin.(p_1t+\text{A}_1-\Psi') \\ +\frac{l_2}{p_2}.\sin.(p_2t+\text{A}_2-\Psi')+\frac{l_3}{p_3}.\sin.(p_3t+\text{A}_3-\Psi')\end{array}\right\}\begin{array}{l}1\\2\end{array}$$

[6552] $$+\tfrac{1}{2}.(1-\lambda).\theta'.\left\{\begin{array}{l}l.\sin.(pt+\text{A}-\Psi')+l_1.\sin.(p_1t+\text{A}_1-\Psi') \\ +l_2.\sin.(p_2t+\text{A}_2-\Psi')+l_3.\sin.(p_3t+\text{A}_3-\Psi')\end{array}\right\}. \qquad \begin{array}{l}3\\4\end{array}$$

[6553] This part of δv is hardly sensible, and we need not notice it, except in the fourth satellite [6944]. It must be modified relative to the other satellites, in consequence of the terms which depend upon the square of the disturbing force [6699, &c.].

[6553′] If we apply this expression to the moon,* we shall have $\lambda = 1 - \frac{(0)}{\boxed{0}}$

[6554] [6534], and $p = \boxed{0}$, very nearly [6553*d*]; hence we obtain,†

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functions [6538, 6542, 6545], from [6515, 6516, 6514] respectively, the terms depending

[6552*d*] on the excentricities e^2, e'^2, &c. are neglected; and it is evident that this can be safely done, because the excentricities [6057*e*] are very small in comparison with the factor θ'

[6552*e*] [7217], which occurs in the terms [6541, 6544, 6551]; and these last terms, which have the factor θ', are hardly sensible, as we have already observed in [6552*c*, 6553].

[6553*a*] * (3397) The retrograde motion of the moon's node is $\frac{3}{4}m^2.v$ nearly [4800].

[6553*b*] Substituting $m=\frac{M}{n}$, [6534*c*], and $v=nt$ [6439], it becomes, according to the present

[6553*c*] notation, $\frac{3}{4}.\frac{M^2}{n}.t=\boxed{0}.t$ [6216]; so that if we wish to investigate the value of δv

[6553*d*] [6552], corresponding to the retrograde motion of the moon's node, supposing it to be represented by pt [7133*c*], we shall have $pt=\boxed{0}.t$, or $p=\boxed{0}$, as in [6554].

[6554*a*] † (3398) Neglecting l_1, l_2, l_3, in [6552], it becomes, for the moon, as in [6554*b*]. Substituting in this the values $1-\lambda=\frac{(0)}{\boxed{0}}$; $p=\boxed{0}$, [6553′, 6554], it becomes as in [6554*c*]; observing that the preceding value of λ for the moon, gives very nearly $\lambda=1$ [6534*d*], which is used in the first term of [6554*c*];

[6554*b*] $$\delta v=\left\{-6\,(0).\frac{\lambda}{p}-4\,(1-\lambda).\boxed{0}.\frac{1}{p}+\tfrac{1}{2}(1-\lambda)\right\}.\theta'.l.\sin.(pt+\text{A}-\Psi')$$

[6554*c*] $$=\left\{-6-4+\tfrac{1}{2}\right\}.\frac{(0)}{\boxed{0}}.\theta'.l.\sin.(pt+\text{A}-\Psi')=-\tfrac{19}{2}.\frac{(0)}{\boxed{0}}.\theta'.l.\sin.(pt+\text{A}-\Psi').$$

This last expression is the same as in [6555], using the value of p [6554].

$$\delta v = -\tfrac{19}{2}.(0).\frac{\theta'.l}{p}.\sin.(pt+\text{A}-\Upsilon').$$ [6555]

Lunar inequality depending on the earth's ellipticity.

This expression agrees with that which we have found in [5389]; supposing, in this case, the obliquity of the ecliptic to be very small.* [6555']

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* (3399) Instead of using the obliquity of the ecliptic $\lambda = 23^d\ 28^m\ 17^s,9$ [5355], if [6555a] we suppose it to be small and equal to θ' [6360], we may change $\sin.\lambda.\cos.\lambda$ into $\sin.\theta'$ [6555b] or θ'; then the expression of δv [5389] will become,

$$\delta v = \tfrac{19}{2}.\frac{(\alpha\rho - \frac{1}{2}\alpha\varphi)}{g-1}.\frac{D^2}{a^2}.\gamma.\theta'.\sin.(\text{long. of the ascending node}).$$ [6555c]

Now we have $D = 1$ [5334, 6082'], $g-1 = \frac{3}{4}m^2$ [5347q]; or, in the present notation, by successive reductions, using [6534c, 6216, 6554], [6555d]

$$g - 1 = \tfrac{3}{4}.\frac{M^2}{n^2} = \frac{\boxed{0}}{n} = \frac{p}{n}.$$ [6555d']

Substituting these values in [6555c], and changing $\alpha\varphi$ into φ, also $\alpha\rho$ into ρ, [5333, 5333', 6044, 6044'], it becomes as in [6555e]; and by using (0) [6216], it changes into [6555f];

$$\delta v = \tfrac{19}{2}.\frac{(\rho - \frac{1}{2}\varphi)n}{a^2}.\frac{\gamma.\theta'}{p}.\sin.(\text{long. of the ascending node})$$ [6555e]

$$= \tfrac{19}{2}.(0).\frac{\gamma.\theta'}{p}.\sin.(\text{long. of the ascending node}).$$ [6555f]

If we subtract the longitude of the descending node of Jupiter's equator, or, as we may call [6555g] it, the longitude of the vernal equinox of Jupiter, $-\Upsilon'$ [6361], from the longitude v of the first satellite [6023c], both longitudes being counted from the same axis x; we shall [6555h] get $v+\Upsilon'$ for the distance of the satellite from that node, or equinox, counted according [6555i] to the order of the signs. Again $v+pt+\text{A}$ represents, as in [6300 line 2, 6298e], the distance of the satellite from its ascending node on Jupiter's orbit. Subtracting this last [6555k] expression from that in [6555i], we get the distance of the ascending node of the orbit of the satellite from Jupiter's vernal equinox, $(v+\Upsilon')-(v+pt+\text{A}) = -(pt+\text{A}-\Upsilon')$; [6555l] which is to be substituted, for the longitude of the node, in [6555f], and then it becomes,

$$\delta v = -\tfrac{19}{2}.(0).\frac{\gamma.\theta'}{p}.\sin.(pt+\text{A}-\Upsilon').$$ [6555m]

The greatest value of s [6300 line 1; 6298d, &c.] is l, which may therefore be taken for the inclination of the orbit of the satellite, instead of γ [4813] nearly. Substituting this [6555n] value of γ, in [6555m], it becomes as in [6555].

CHAPTER VI.

ON THE INEQUALITIES DEPENDING ON THE SQUARE OF THE DISTURBING FORCE.

14. We have already considered, in [1215—1242v], one of the most remarkable of these inequalities. *It depends, as we have seen, upon the circumstance, that, at the origin of the motion, the mean longitude of the first satellite, minus three times that of the second, plus twice that of the third,*
[6556] *differed but very little from the semi-circumference; and then the mutual attraction of these three satellites has caused this difference to vanish. We shall now resume, by another method, the abstruse theory of this inequality,* in order to give it a further development, and to determine its influence upon the various inequalities of these satellites.

[6557] *If we consider the orbits as variable ellipses, ζ representing the mean longitude of the satellite m,* we shall have, as in [1195, 6110c],

[6558] $$dd\zeta = 3andt.\text{d}R.$$

We shall notice, in the expression of the motion of the satellites, only those
[6559] terms which depend on the angle $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$, and have also the divisor $(n-3n'+2n'')^2$; since these terms may become sensible, on account of the extreme smallness of this divisor. It is plain that
[6560] $3andt.\text{d}R$ contains terms depending on the proposed angle, which may acquire this divisor by the double integrations. Moreover such terms cannot be introduced into the value of v, except through the expression of ζ; for it is evident, by the inspection of the values of de, $d\varpi$, $d\varepsilon$, de', &c., given
[6561] in [1258, 1258a, &c.], that they cannot produce such terms in v, at least when we carry on the approximation no further than to include the terms depending on the square of the disturbing force.* Therefore, by noticing

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[6561a] * (3400) We shall see, in [6567—6569], that R, R', R'', $\text{d}R$, $\text{d}R'$, $\text{d}R''$, do not contain terms depending on the angle $v-3v'+2v''$, connected with coefficients having the

only such of these terms as can acquire this divisor, by integration, we shall have,

$$ddv = 3andt.\text{d}R. \qquad [6562]$$

In like manner we shall have,

$$ddv' = 3a'n'.dt.\text{d}'R'; \qquad [6563]$$

$$ddv'' = 3a''n''.dt.\text{d}''R''; \qquad [6564]$$

R' and R'' being what R becomes relative to the satellites m' and m''; [6565]
and the characteristics d, d′, d″, correspond respectively to the co-ordinates [6566]
of the bodies m, m', m''. We must now determine the terms of $\text{d}R$, $\text{d}'R'$, Symbols d,d′,d″, R', R''.
$\text{d}''R''$, which depend on the angle $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$.

The expressions of R, R', R'', do not contain angles depending on [6567]
$v-3v'+2v''$; they give, by development, only terms depending on the radii vectores, the latitudes, the elongations, $v-v'$, $v-v''$, $v'-v''$, and the multiples of these elongations.* But by the substitution of the parts of [6568]
r, v, r', v', &c., which depend upon the disturbing forces, there may arise, [6569]
in $\text{d}R$, $\text{d}'R'$, $\text{d}''R''$, some terms of the order of the square of the disturbing forces, depending on the angle $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$. We have determined, in [6116, 6119], the perturbations of r, v, r', v', r'', v'', and we have seen in [6131, 6132], that *the principal inequalities of* r, v, *arising* [6570]
from the disturbing forces, depend upon the angle $2nt-2n't$; *that those of* r' *and* v' [6139, 6140, 6148, 6149], *depend upon the angles* $nt-n't$ *and*

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divisor $n-3n'+2n''$, or its square. For similar reasons it will follow, that the terms [6561b]
$r.\left(\frac{dR}{dr}\right)$, $(r\delta r)$, dr, δr [6094, 6057, &c.; 6049k, l, &c.], in δv [6060], do not contain
terms depending on that angle, and having that divisor, even when we notice quantities [6561c]
of the order of the square of the disturbing force. Moreover if this angle be found in
$r.\left(\frac{dR}{dr}\right)$, it will only produce in $\int ndt.r.\left(\frac{dR}{dr}\right)$, which occurs in δv [6060], a term [6561d]
having the first power of the divisor $n-3n'+2n''$, noticing the same order of terms. Hence it appears that the only term of δv [6060], which can produce this divisor, is that
depending on $\int\int 3andt.\text{d}R$, corresponding to $dd\zeta$ [6558]; or, as it may be written, [6561e]
$ddv = 3andt.\text{d}R$, corresponding to [6562], or to the similar values of ddv', ddv'' [6561f]
[6563, 6564].

* (3401) This is evident by comparing [949 or 951] with its development [6568a]
[957 or 1011], or with the expressions in [1226, 1227], which produce, as in [1230], some terms of $\text{d}R$, depending on the angle $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$.

$2n't-2n''t$; *lastly that those of* r'', v'' [6164, 6165], *depend on the angle*
[6571] $n't-n''t$. *These inequalities acquire, by integration, very small divisors, which render them much greater than the other inequalities, so that we need only notice them in the present question.* Some of the arguments of these inequalities, when combined with the elongations of the satellites and their
[6572] multiples, by addition or subtraction, may produce the angle $nt-3n't+2n''t$. The first of these arguments $2nt-2n't$, and the last, $n't-n''t$, cannot form it* by their combination with the angles $v-v'$, $v-v''$, $v'-v''$, and their multiples, as is evident by using the following mean values of v, v', v'' [6091, &c.], namely,

[6572'] $$v=nt+\varepsilon; \qquad v'=n't+\varepsilon'; \qquad v''=n''t+\varepsilon''.$$

Therefore in the expressions of $\mathrm{d}R$, $\mathrm{d}'R'$, $\mathrm{d}''R''$, *we may dispense with the consideration of the perturbations of the satellites* m, m''; *it will suffice to*
[6573] *notice those of the satellite* m'. The inequality of the satellite m', relative
[6574] to the angle $2n't-2n''t$, being combined by subtraction with the angle $v-v'$, and its inequality relative to the angle $nt-n't$ being combined, in like manner, with the angle $2v'-2v''$, will produce some terms depending
[6574'] on the angle† $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$.

[6572a] * (3402) Thus the angle $2nt-2n't$, being combined with a multiple of $v-v'=nt-n't+\varepsilon-\varepsilon'$, will not contain $n''t$, and will not therefore be of the form [6572].
[6572a'] The same angle $2nt-2n't$, when combined with a multiple of $v-v''=nt-n''t+\varepsilon-\varepsilon''$, will form an angle containing $-2n't$, instead of $-3n't$ [6572]. Lastly, if the angle
[6572b] $2nt-2n't$ be combined with a multiple of $v'-v''$, it will contain $2nt$ instead of nt, as is required in [6572]. In like manner $n't-n''t$, being combined with a multiple of $v-v'$, contains $-n''t$, instead of $+2n''t$ [6572]; if it be combined with a multiple of
[6572c] $v-v''$, it will contain $n't$ instead of $-3n't$; and if it be combined with a multiple of $v'-v''$, it will not contain nt [6572]. Hence it appears *that in noticing only the terms*
[6572d] *depending on the angle* $nt-3n't+2n''t$ [6572], *we may neglect the inequalities depending*
[6572e] *on the angles* $2nt-2n't$, $n't-n''t$; *or, in other words, we may neglect the terms* δr, δv
[6572f] [6131, 6132], *also* $\delta r''$, $\delta v''$ [6164, 6165], *with their differentials* $d\delta r$, $d\delta v$, $d\delta r''$, $d\delta v''$; and notice $\delta r'$, $\delta v'$, $d\delta r'$, $d\delta v'$ [6148, 6149, 6158, 6159], depending on $2n't-2n''t$, $nt-n't$.

† (3403) The angle $-2(n't-n''t)$, which occurs in [6148, 6149], being combined with
[6574a] $(v-v')$, or $nt-n't$, produces the angle $nt-3n't+2n''t$. In like manner, if we combine the angle $(nt-n't)$ [6158, 6159] with $-2(v'-v'')$, it produces $(nt-3n't+2n''t)$.
[6574b] The term of R depending on $v-v'$, is noticed in [6575, &c.]; and the term of R depending on $(2v'-2v'')$ is noticed in [6595, &c.].

We shall now consider the second term of R [6090], which we shall represent by $R=m'.A^{(1)}.\cos.(v-v')$. If we notice only this term we shall have,* [6575]

$$\mathrm{d}R=-m'.A^{(1)}.dv.\sin.(v-v')+m'.\left(\frac{dA^{(1)}}{dr}\right).dr.\cos.(v-v'). \qquad [6576]$$

If we neglect the perturbations of m, and the excentricities of the orbits, we shall have $dr=0$, and $dv=ndt$ [6572′]; therefore, [6577]

$$\mathrm{d}R=-m'.A^{(1)}.ndt.\sin.(v-v'); \qquad [6578]$$

which gives, in $\mathrm{d}R$, the following terms of the order of the square of the disturbing forces;†

$$\mathrm{d}R=m'.A^{(1)}.ndt.\delta v'.\cos.(v-v')-m'.\left(\frac{dA^{(1)}}{dr'}\right).ndt.\delta r'.\sin.(v-v'). \qquad [6579]$$

We have, in [6148, 6149], by noticing only the perturbations depending on the angle $2n't-2n''t$,

$$\frac{\delta r'}{a'}=-\frac{m''.n'.F'}{2\,(2n'-2n''-N')}.\cos.(2n't-2n''t+2\varepsilon'-2\varepsilon''); \qquad [6580]$$

$$\delta v'=\frac{m''.n'.F'}{2n'-2n''-N'}.\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon''). \qquad [6581]$$

Substituting these values in [6579], retaining only the terms depending on $nt-3n't+2n''t$, and observing that n is very nearly equal to $2n'$ [6151], [6582]

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* (3404) We have already remarked, in [6089b], that $A^{(1)}$ is a function of r, r', and does not contain v. Then taking the differential of R [6575], relative to the characteristic d, which affects only r, v, we get the expression of $\mathrm{d}R$ [6576]. Substituting in it the values of dr, dv [6577], we get [6578]; observing that the variations relative to the characteristic δ, of the second term of [6576], which contains dr, produces nothing of the required form or order in [6583]; $d\delta r$, $d\delta v$, being neglected, as in [6572f]. [6576a] [6576b] [6576c]

† (3405) Supposing r' to be increased by the quantity $\delta r'$ [6148], and v' by $\delta v'$ [6149], we shall find, by Taylor's theorem [617, &c.], that $A^{(1)}$ will become $A^{(1)}+\left(\frac{dA^{(1)}}{dr}\right).\delta r$, nearly; and $\sin.(v-v')$ will become $\sin.(v-v')-\delta v'.\cos.(v-v')$. Substituting these in [6578], we obtain the additional terms of $\mathrm{d}R$ [6579], depending on $\delta r'$, $\delta v'$. The value of $\delta v'$ is given in [6149], being the same as in [6581]; that of $\frac{\delta r'}{a}$ is deduced from [6148], by putting, in the first member, $\frac{r'}{a'}=1$ [6297c]. The terms of $\delta r'$, $\delta v'$ [6139, 6140], are neglected, because they do not produce, by combination, the angle [6574′]; as we have seen in [6572a, &c.]. [6579a] [6579b] [6579c] [6579d]

we shall have,*

[6583] $$3andt.\mathrm{d}R = -\frac{3m'.m''.n^3.\frac{a}{a'}.F'.\left\{2a'.A^{(1)}-a'^2.\left(\frac{dA^{(1)}}{da'}\right)\right\}}{8.(2n'-2n''-N')}.dt^2.\sin.(nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon'').$$

We have very nearly, as in [6145],

[6584] $$G = 2a'.A^{(1)} - a'^2.\left(\frac{dA^{(1)}}{da'}\right).$$

[6584′] Moreover $2n'-2n''$ is equal to $n-n'$ [6154]; or at least, their difference is so small that it has hitherto been insensible by observation. Therefore if we substitute, in [6583], the values [6572′],

[6585] $$nt+\varepsilon = v; \qquad n't+\varepsilon' = v'; \qquad n''t+\varepsilon'' = v'',$$

which can be done in the present instance; and also the value $ddv = 3andt.\mathrm{d}R$ [6562], we shall obtain,

[6586] $$\frac{ddv}{dt^2} = -\frac{3m'.m''.n^3.\frac{a}{a'}.F'G}{8.(n-n'-N')}.\sin.(v-3v'+2v'').$$

[6587] The part of R, relative to the action of m'' upon m, contains only terms depending on the angle $v-v''$ and its multiples; therefore it adds no term to this value of $\frac{ddv}{dt^2}$ [6572a', &c.].

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* (3406) Substituting $n' = \frac{1}{2}n$ [6151] in the numerators of the coefficients of [6580, 6581], and then their results in [6579], we get,

[6583a] $$\mathrm{d}R = -\frac{m'.m''.n^2.F'}{8.(2n'-2n''-N')}.\left\{\begin{array}{l} -4A^{(1)}.\cos.(v-v').\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon'') \\ -2a'.\left(\frac{dA^{(1)}}{dr'}\right).\sin.(v-v').\cos.(2n't-2n''t+2\varepsilon'-2\varepsilon'') \end{array}\right\}.dt.$$

[6583b] Putting now $v-v' = nt-n't+\varepsilon-\varepsilon'$ [6572′], and reducing the expressions by means of [18, 19] Int., retaining only the terms depending on the angle [6574′], we get,

[6583c] $$\mathrm{d}R = -\frac{m'.m''.n^2.F'}{8.(2n'-2n''-N')}.\left\{2A^{(1)}-a'.\left(\frac{dA^{(1)}}{dr'}\right)\right\}.dt.\sin.(nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon'').$$

[6583d] Multiplying this by $3andt$, it produces the expression [6583]; observing, that by neglecting the excentricity we may change $\left(\frac{dR}{dr'}\right)$ into $\left(\frac{dR}{da'}\right)$, as in [6202$b$, &c.].

[6583e] Substituting in [6583] the values G [6584], $2n'-2n''$ [6584′], and ddv [6562], also the expressions [6585], we obtain the equation [6586]. Multiplying this by dt^2, and substituting [6562], it becomes,

[6583f] $$3a.ndt.\mathrm{d}R = -\frac{3m'.m''.n^3.\frac{a}{a'}.F'G.dt^2}{8.(n-n'-N')}.\sin.(v-3v'+2v'').$$

We shall now consider the term $R'=m.A_{,}^{(1)}.\cos.(v-v')$, being a part of the expression of R', arising from the action of m upon m', as we have seen in [6134'].* If we notice only this term, we shall have, [6588]

$$\mathrm{d}'R'=m.A_{,}^{(1)}.dv'.\sin.(v-v')+m.dr'.\left(\frac{dA_{,}^{(1)}}{dr'}\right).\cos.(v-v'). \qquad [6589]$$

This function, being developed, contains the following terms;

$$\mathrm{d}'R'=m.\left(\frac{dA_{,}^{(1)}}{da'}\right).n'dt.\delta r'.\sin.(v-v')+m.A_{,}^{(1)}.d\delta v'.\sin.(v-v') \qquad 1$$

$$-m.A_{,}^{(1)}.n'dt.\delta v'.\cos.(v-v')+m.\left(\frac{dA_{,}^{(1)}}{da'}\right).d\delta r'.\cos.(v-v'). \qquad 2 \qquad [6590]$$

Substituting, in [6590], the preceding values of $\delta r'$, $\delta v'$ [6580, 6581]; and observing that $n''=\frac{1}{2}n'$ [6151] nearly, also that we have very nearly, as in [6137, 6145], [6591]

$$G=2a'.A_{,}^{(1)}-a'^2.\left(\frac{dA_{,}^{(1)}}{da'}\right); \qquad [6592]$$

we shall obtain, †

$$3a'.n'dt.\mathrm{d}'R'.=\frac{3m.m''.n^3.F'G.dt^2}{16.(n-n'-N')}.\sin.(v-3v'+2v''). \qquad [6593]$$

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* (3407) Comparing [6090, 6134'] with the definition of R' [6565], we obtain the term of R' [6588]. Its differential relative to the characteristic d', which affects only r', v' [6566], gives [6589]. Supposing now r' to be augmented by $\delta r'$, and v' by $\delta v'$; then developing the second member of [6589], according to the powers of $\delta r'$, $\delta v'$ [617], retaining only the terms depending on the first power, we shall get [6590]. For the variations of the three quantities $A^{(1)}$, dv', $\sin.(v-v')$, which enter as factors into the first term of the second member of [6589], produce respectively the three terms of [6590], containing $\delta r'$, $d\delta v'$, $\delta v'$; observing that we may substitute $n'dt$ for dv' [6585]. Moreover the variation of dr', in the last term of [6589], produces the last term of [6590]. The variations of $dA_{,}^{(1)}$, $\cos.(v-v')$, in the last term of [6589], are neglected, because they are multiplied by dr', which is of the order e', and may therefore be neglected, as in [6577], since they produce nothing of the required form and order. [6588a] [6588b] [6588b'] [6588c] [6588d] [6588e] [6588f]

† (3408) Taking the differential of [6580], and comparing the result with [6581], after substituting in the coefficient $2n'-2n''=n-n'=n'$ [6154, 6151], we get the first of the expressions [6593c]. Then taking the differential of [6581] and comparing it with [6580], after making similar substitutions, we get the second of the expressions [6593c], [6593a] [6593b]

$$\frac{d.\delta r'}{a'}=\tfrac{1}{2}n'dt.\delta v'; \qquad d.\delta v'=-2n'dt.\frac{\delta r'}{a'}. \qquad [6593c]$$

Substituting these expressions in the two lines of the second member of [6590], they

We may here observe, that, by comparing the two expressions of $3andt.\mathrm{d}R$, $3a'n'dt.\mathrm{d}'R'$ [6583, 6593], we shall obtain,*

[6594] $$m.\mathrm{d}R+m'.\mathrm{d}'R'=0;$$

which is conformable to what we have found in [1202].

[6595] *The part of* R' *relative to the action of* m'' *upon* m', *contains the term* $R'=m''.A'^{(2)}.\cos.(2v'-2v'')$ [6090, 6146]. Noticing only this term, and

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[6593d] produce the two lines of the value of $\mathrm{d}'R'$ [6593e] respectively; and by using the value of G [6592], it becomes as in [6593f].

[6593e] $$\mathrm{d}'R'=-\frac{m.n'dt}{a'}\cdot\frac{\delta r'}{a'}\cdot\sin.(v-v').\left\{-a'^2.\left(\frac{dA_{\prime}^{(1)}}{da'}\right)+2a'.A_{\prime}^{(1)}\right\}$$
$$-\frac{m.n'dt}{a'}\cdot\tfrac{1}{2}\,\delta v'.\cos.(v-v').\left\{2a'.A_{\prime}^{(1)}-a'^2.\left(\frac{dA_{\prime}^{(1)}}{da'}\right)\right\}$$

[6593f] $$=-\frac{m.n'dt}{a'}\cdot G.\left\{\frac{\delta r'}{a'}\cdot\sin.(v-v')+\tfrac{1}{2}\,\delta v'.\cos.(v-v')\right\}.$$

[6593g] Now if we put for brevity $B=\frac{m''.n'.F'}{2n'-2n''-\mathcal{N}'}$, and use the values of $v,\ v',\ v''$ [6572'], we shall find that the expressions [6580, 6581] may be put under the following forms;

[6593h] $$\frac{\delta r'}{a'}=-\tfrac{1}{2}B.\cos.(2v'-2v'')\ ;\qquad \delta v'=B.\sin.(2v'-2v'').$$

[6593i] Multiplying the first of these equations by $\sin.(v-v')$, and the second by $\frac{1}{2}\cos.(v-v')$; then taking the sum of the products, and reducing successively, by means of [22] Int., we get,

[6593k] $$\frac{\delta r'}{a'}\cdot\sin.(v-v')+\tfrac{1}{2}\delta v'.\cos.(v-v')=-\tfrac{1}{2}B.\left\{\sin.(v-v').\cos.(2v'-2v'')-\cos.(v-v').\sin.(2v'-2v'')\right\}$$

[6593l] $$=-\tfrac{1}{2}B.\sin.(v-3v'+2v'').$$

[6593m] Substituting this in [6593f], and then multiplying by $3a'.n'dt$, we get the first of the expressions [6593n]. Re-substituting B [6593g], $n'=\frac{1}{2}n$ [6151], and $2n'-2n''=n-n'$ [6593a], we finally get [6593o], which is the same as [6593].

[6593n] $$3a'.n'dt.\mathrm{d}'R'=\tfrac{3}{2}.m.n'^2.BG.dt^2.\sin.(v-3v'+2v'')=\frac{3m.m''.n'^3.F'G.dt^2}{2.(2n'-2n''-\mathcal{N}')}\cdot\sin.(v-3v'+2v'')$$

[6593o] $$=\frac{3m.m''.n^3.F'G.dt^2}{16.(n-n'-\mathcal{N}')}\cdot\sin.(v-3v'+2v'').$$

[6594a] * (3409) Multiplying [6583f] by $\frac{m}{3a.ndt}$, and [6593] by $\frac{m'}{3a'.n'dt}=\frac{2m'}{3a'.ndt}$ nearly [6151]; we find that the second members of these products are,

[6594b] $$\mp\frac{m.m'.m''.n^2.F'G.dt}{8.(n-n'-\mathcal{N}').a'}\cdot\sin.(v-3v'+2v'');$$

[6594c] and as they have different signs, their sum vanishes; hence the sum of the terms in the first members of the same products, becomes $m.\mathrm{d}R+m'.\mathrm{d}'R'=0$, as in [6594].

taking its differential relative to d', which only affects r', v' [6566], we shall have,

$$\mathrm{d}'R' = -2m''.A'^{(2)}.dv'.\sin.(2v'-2v'')+m''.dr'.\left(\frac{dA'^{(2)}}{da'}\right).\cos.(2v'-2v''). \qquad [6596]$$

This function, being developed, contains the following terms;*

$$\begin{aligned}\mathrm{d}'R' = &-2m''.\left(\frac{dA'^{(2)}}{da'}\right).n'dt.\delta r'.\sin.(2v'-2v'')-2m''.A'^{(2)}.d\delta v'.\sin.(2v'-2v'')\\ &-4m''.n'dt.A'^{(2)}.\delta v'.\cos.(2v'-2v'')+m''.\left(\frac{dA'^{(2)}}{da'}\right).d\delta r'.\cos.(2v'-2v'').\end{aligned} \qquad [6597]$$

We have, by noticing only the action of the satellite m upon m', [6139, 6140],†

$$\frac{\delta r'}{a'} = -\frac{m.n'.G}{2.(n-n'-N')}.\cos.(nt-n't+\varepsilon-\varepsilon'); \qquad [6598]$$

$$\delta v' = \frac{m.n'.G}{n-n'-N'}.\sin.(nt-n't+\varepsilon-\varepsilon'). \qquad [6599]$$

Then, by observing that $n'=2n''$ nearly [6151], we shall have, with a considerable degree of accuracy, by [6147],

$$F' = -4a'.A'^{(2)}-a'^2.\left(\frac{dA'^{(2)}}{da'}\right); \qquad [6600]$$

hence we obtain,‡

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* (3410) The expression [6597] is deduced from [6596], in the same manner as [6590] is obtained from [6589], in [6588a—f]; namely, by finding the increment of [6596] from the change of r', v', into $r'+\delta r'$, $v'+\delta v'$, respectively; neglecting, as in [6588f], the two terms having the factor dr'. [6597a] [6597b]

† (3411) Substituting $\frac{r'}{a'}=1$ [6579c] in the first member of [6139], it becomes as in [6598]. The value of $\delta v'$ [6140] is the same as in [6599]. Substituting $n''=\frac{1}{2}n'$ [6151] in [6147], it becomes as in [6600]. [6600a]

‡ (3412) The computation of [6601] from [6597] is similar to that of finding [6593] from [6590], as in [6593a—o]. In the first place, the differentials of $\delta r'$, $\delta v'$.[6598, 6599], being compared with [6599, 6598] respectively, give the same results as in [6593c]. Substituting these values in the two lines of the second member of [6597], they produce respectively the two lines in the second member of the following expression; [6601a] [6601b]

$$\begin{aligned}\mathrm{d}'R' = &\,m''.n'dt.\frac{\delta r'}{a'}.\sin.(2v'-2v'').\left\{-2a'.\left(\frac{dA'^{(2)}}{da'}\right)+4A'^{(2)}\right\}\\ &+m''.n'dt.\delta v'.\cos.(2v'-2v'').\left\{-4A'^{(2)}+\tfrac{1}{2}a'.\left(\frac{dA'^{(2)}}{da'}\right)\right\}.\end{aligned} \qquad [6601c]$$

[6601] $$3a'.n'dt.\mathrm{d}'R' = \frac{3m.m''.n^3.F'G.dt^2}{32.(n-n'-N')}.\sin.(v-3v'+2v'').$$

Connecting these two terms of $3a'.n'dt.\mathrm{d}'R'$ [6593, 6601], we get, as in [6601m],

[6602] $$\frac{ddv'}{dt^2} = \frac{9m.m''.n^3.F'G}{32.(n-n'-N')}.\sin.(v-3v'+2v'').$$

It now remains to consider the value of $\mathrm{d}''R''$. The part of R'', depending

[6603] on $2v'-2v''$ [6574], is as in [6595, &c.], $R''=m'.A'^{(2)}.\cos.(2v'-2v'')$; and by noticing only this term, we shall have,

[6604] $$\mathrm{d}''R'' = 2m'.A'^{(2)}.dv''.\sin.(2v'-2v'')+m'.dr''.\left(\frac{dA'^{(2)}}{dr''}\right).\cos.(2v'-2v'').$$

This function contains the following terms ;*

[6601d] If we now put, for brevity, $B'=\frac{m.n'.G}{n-n'-N'}$, and use the values of v, v', v'' [6572'], we shall find that the expressions [6598, 6599], may be put under the following forms ;

[6601e] $$\frac{\delta r'}{a'}=-\tfrac{1}{2}B'.\cos.(v-v')\,; \qquad \delta v'=B'.\sin.(v-v').$$

[6601f] Multiplying these values by $\sin.(2v'-2v'')$, $\cos.(2v'-2v'')$, respectively; reducing the products by [18, 19] Int., and retaining only the terms depending on the angle $v-3v'+2v''$, we get,

[6601g] $$\frac{\delta r'}{a'}.\sin.(2v'-2v'')=\tfrac{1}{4}B'.\sin.(v-3v'+2v'')\,; \qquad \delta v'.\cos.(2v'-2v'')=\tfrac{1}{2}B'.\sin.(v-3v'+2v'').$$

Substituting these last expressions in [6601c], and making successive reductions, using F' [6600], we get,

[6601h] $$\mathrm{d}'R'=m''.n'.dt.\left\{\tfrac{1}{4}.\left[-2a'.\left(\frac{dA'^{(2)}}{da'}\right)+4A'^{(2)}\right]+\tfrac{1}{2}.\left[-4A'^{(2)}+\tfrac{1}{2}a'.\left(\frac{dA'^{(2)}}{da'}\right)\right]\right\}.B'.\sin.(v-3v'+2v'')$$

[6601i] $$=m''.n'.dt.\left\{-A'^{(2)}-\tfrac{1}{4}a'.\left(\frac{dA'^{(2)}}{da'}\right)\right\}.B'.\sin.(v-3v'+2v'')$$

[6601k] $$=m''.n'.dt.\frac{F'}{4a'}.B'.\sin.(v-3v'+2v'').$$

[6601l] Multiplying this last expression by $3a'n'.dt$, and re-substituting the value of B' [6601d], also $n'^3=\frac{1}{8}.n^3$ nearly [6151], we get [6601]. The sum of the two parts of $3a'.n'dt.\mathrm{d}'R'$ [6593, 6601], being taken and substituted in [6563], gives,

[6601m] $$ddv'=3a'n'.dt.\mathrm{d}'R'=\frac{9m.m''.n^3.F'G.dt^2}{32.(n-n'-N')}.\sin.(v-3v'+2v'').$$

Dividing this by dt^2, we get [6602].

[6605a] * (3413) We have, in [6595], $R'=m''.A'^{(2)}.\cos.(2v'-2v'')$; and as $A'^{(2)}$ is the

[6605b] same for both satellites m', m'' [6466d], we shall have $R''=m'.A'^{(2)}.\cos.(2v'-2v'')$, as in [6603]. The differential of this expression, relative to the characteristic d'', which

$$d''R''=2m'.n''dt.\delta r'.\left(\frac{dA'^{(2)}}{da'}\right).\sin.(2v'-2v'')+4m'.A'^{(2)}.n''dt.\delta v'.\cos.(2v'-2v'').$$ [6605]

Substituting the parts of $\delta v'$ and $\delta r'$, depending on the angle $nt-n't$ [6598, 6599], observing also that we have very nearly $n'=\frac{1}{2}n$, $n''=\frac{1}{4}n$ [6151], we obtain, [6605']

$$3a''.n''dt.d''R''=-\frac{3m.m'.n^3.F'G.dt^2}{64.(n-n'-N')}.\frac{a''}{a'}.\sin.(v-3v'+2v'').$$ [6606]

Hence we easily find, that, by noticing only the reciprocal action of m' upon m'', we shall have,*

$$m'.d'R'+m''.d''R''=0;$$ [6607]

which agrees with [1202]. Therefore we have,†

$$\frac{ddv''}{dt^2}=-\frac{3m.m'.n^3.F'G}{64.(n-n'-N')}.\frac{a''}{a'}.\sin.(v-3v'+2v'').$$ [6608]

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affects only r'', v'', gives [6604]. Supposing r' to be increased by $\delta r'$, and v' by $\delta v'$, we find that the expression [6604] will be increased, by the terms in the second member of [6605], as is evident by proceeding as in [6588a, &c.], and neglecting the terms multiplied by dr'', as in [6588f]; also those depending on $d\delta r''$, $d\delta v''$, as in [6572f]; and finally putting $dv''=n''dt$ [6572']. Now if we substitute, in [6605], the values [6601e], we shall get [6605f]. Reducing the products by [18, 19] Int., and retaining only the terms depending on the angle $v-3v'+2v''$, we get [6605g]; and by using F', B' [6600, 6601d], we obtain [6605h]. [6605c] [6605d] [6605e]

$$d''R''=m'.n''dt.B'.\left\{-a'.\left(\frac{dA'^{(2)}}{da'}\right).\cos.(v-v').\sin.(2v'-2v'')+4A'^{(2)}.\sin.(v-v').\cos.(2v'-2v'')\right\}$$ [6605f]

$$=m'.n''dt.B'.\left\{\tfrac{1}{2}a'.\left(\frac{dA'^{(2)}}{da'}\right)+2A'^{(2)}.\right\}.\sin.(v-3v'+2v'')$$ [6605g]

$$=-m'.n''dt.B'.\frac{F'}{2a'}.\sin.(v-3v'+2v'')=-\frac{m.m'.n'n''.F'G.dt}{2.(n-n'-N').a'}.\sin.(v-3v'+2v'').$$ [6605h]

Multiplying this by $3a''.n''dt$, and substituting, in the second member of the product, the values $n'=\frac{1}{2}n$, $n''=\frac{1}{4}n$ [6605'], we obtain [6606]. [6605i]

* (3414) Multiplying [6601] by $\frac{m'}{3a'.n'dt}=\frac{2m'}{3a'.ndt}$; and [6606] by $\frac{m''}{3a''.n''dt}=\frac{4m''}{3a''.ndt}$; [6607$a$]
we find that the second members of the products become,

$$\mp\frac{m.m'.m''.n^2.F'G.dt}{16.(n-n'-N').a'}.\sin.(v-3v'+2v'');$$ [6607b]

and being of different signs, their sum vanishes in the second member, and the first member of the sum becomes as in [6607].

† (3415) Substituting [6606] in [6564], and dividing by dt^2, we get [6608]. [6608a]

Now if we use the following values of k, φ,

[6609] $$\mathrm{k} = -\frac{3n.F'G}{8.(n-n'-N')} \cdot \left\{ \frac{a}{a'} \cdot m'.m'' + \tfrac{9}{4}.m.m'' + \frac{a''}{4a'} \cdot m.m' \right\};$$

[6610] $$\varphi = v - 3v' + 2v'';$$

we shall obtain, by connecting together the expressions of $\frac{ddv}{dt^2}$, $-\frac{3ddv'}{dt^2}$, $\frac{2ddv''}{dt^2}$ [6586, 6602, 6608], the following equation;*

[6611] $$\frac{dd\varphi}{dt^2} = \mathrm{k}.n^2.\sin.\varphi.$$

15. We may suppose k and n^2 to be constant in this equation, because their variations are very small; then, by integration, we obtain,†

[6612] $$dt = \frac{\pm d\varphi}{\sqrt{\mathrm{c} - 2\mathrm{k}.n^2.\cos.\varphi}};$$

[6612′] c being an arbitrary constant quantity. The different values which c may have, gives rise to the three following cases;

Cases of the value of φ

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* (3416) Multiplying [6602] by -3, also [6608] by 2, then adding the products

[6611a] to the equation [6586], we find that the first member of the sum is $\frac{ddv-3ddv'+2ddv''}{dt^2}$,

[6611b] which is easily reduced to the form $\frac{dd\varphi}{dt^2}$, by using φ [6610]; being the same as the first member of [6611]. Moreover the second member of this sum, by using the value of k [6609], becomes equal to $\mathrm{k}n^2.\sin.(v-3v'+2v'') = \mathrm{k}n^2.\sin.\varphi$, as in [6611].

† (3417) The equation [6611] is the same as [1235], changing V into φ, and β

[6612a] into k. Its integral, found as in [1236a], is the same as in [6612]. From this equation

[6612b] we may deduce the same results as in [1236′, &c.]; and the three cases relative to the value of c, given in [6613, 6615, 6617], correspond respectively to [1236′, 1237^vii, 1237^vi].

[6612c] We may observe that the symbols c, k, are printed in the Roman, instead of the Italic type used by the author, to distinguish them from c, k [6021i, x]. For the same reasons we have placed a mark below his symbol ϖ, in [6620, &c.], to distinguish it from ϖ

[6612d] [6024g]. We may also remark that the equation [6611] will not be altered, if we change,

[6612e] in [6610], φ into $-\varphi$, putting $\varphi = -(v-3v'+2v'')$; because this will merely change

[6612f] the signs of both members of [6611], without altering the form or value of the equation; so that instead of the expression [6610], we may put, more generally,

[6612g] $$\varphi = \pm(v-3v'+2v''), \quad \text{or} \quad \pm\varphi = v-3v'+2v''.$$

[6612h] Hence we see that the sign of φ is indeterminate in the differential equation [6611], and the same occurs in its integral [6612, &c.]

First. If the *constant quantity* c, *neglecting its sign, exceed* $2kn^2$, it must necessarily be *positive.* Then the angle $\pm\varphi$ *will increase indefinitely,* and may become equal to one, two, three, &c. semi-circumferences. [6613] First case. [6613']

Second. If the *constant quantity* c, *neglecting its sign, be less than* $2kn^2$, k *being positive,* the radical $\sqrt{c-2kn^2.\cos.\varphi}$ will become imaginary, when $\pm\varphi$ is equal to nothing, or to one, two, &c. circumferences. In this case the angle φ *will merely oscillate about the semi-circumference,* which will represent its mean value. [6614] Second case $\varphi=200^\circ$ [6615]

Third. If the *constant quantity* c, *neglecting its sign, be less than* $2kn^2$, k *being negative,* the radical $\sqrt{c-2kn^2.\cos.\varphi}$ will become imaginary, when $\pm\varphi$ is equal to an uneven number of semi-circumferences. In this case *the angle* φ *will oscillate about zero,* so that its mean value is nothing. [6616] Third case $\varphi=0^\circ$ [6617]

The case of $c=\pm 2kn^2$, may be supposed to be included in the preceding forms. We may, moreover, consider the probability of this case as infinitely small. We shall now see which of these cases really obtains, in the system of these satellites. [6618]

We shall find, in [7270, 7272], that k *is positive; therefore the third case* [6616] *must be excluded;* and the angle φ must either increase indefinitely [6613'], or oscillate about the semi-circumference [6615]. If we put, [6619] Symbol $\varpi_,$

$$\varphi=\pi\pm\varpi_, ; \qquad [6620]$$

π *being the semi-circumference whose radius is* 1; we shall have,*

$$dt=\frac{d\varpi_,}{\sqrt{c+2kn^2.\cos.\varpi_,}}. \qquad [6621]$$

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* (3418) The equation [6621] is easily deduced from [6612], by the substitution of [6620]. Now if the angle $\varpi_,$ increase indefinitely, the limits of $c+2kn^2.\cos.\varpi_,$ will be $c+2kn^2$, $c-2kn^2$; and it is evident that $c+2kn^2.\cos.\varpi_,$, will sometimes become negative, unless c be positive and greater than $2kn^2$; but if $c-2kn^2.\cos.\varpi_,$ be negative, the denominator of the expression of dt [6621], will become imaginary, and $\varpi_,$ will not increase with t. Hence we evidently see that, if $\varpi_,$ increase indefinitely with t, we shall have c positive and greater than $2kn^2$, as in [6621']; consequently the radical $\sqrt{c+2kn^2.\cos.\varpi_,}$, will exceed $\sqrt{2kn^2}$ or $n.\sqrt{2k}$, while $\varpi_,$ varies from 0° to 100°; and within these limits we shall always have, from [6621], $dt<\frac{d\varpi_,}{n.\sqrt{2k}}$, whose integral gives $t<\frac{\varpi_,}{n.\sqrt{2k}}$ [6622]. Putting $\varpi_,=\frac{1}{2}\pi$, we get $t<\frac{\pi}{2n.\sqrt{2k}}$, as in [6623]; hence we infer, as in [6624], that $\varpi_,$ or φ [6620], does not indefinitely increase, so that [6622a] [6622b] [6622c] [6622d] [6622e]

[6621′] *If the angles* $\pm\varphi$ *and* $\varpi_{,}$ *increase indefinitely,* c *will be positive, and greater than* $2\mathrm{k}n^2$; hence we shall have, in the interval comprised between $\varpi_{,}=0$, and $\varpi_{,}$ equal to a quarter of the circumference, $dt<\frac{d\varpi_{,}}{n.\sqrt{2\mathrm{k}}}$,

[6622] consequently $t<\frac{\varpi_{,}}{n.\sqrt{2\mathrm{k}}}$. Therefore the time t, which is required for the angle $\varpi_{,}$ to increase to a quarter of the circumference, will be less than

[6623] $\frac{\pi}{2n.\sqrt{2k}}$. We shall see, in [7274], that this time is less than two years.

[6624] *Now since the discovery of the satellites, the angle* $\varpi_{,}$ *has always appeared to be insensible, or extremely small; so that it does not increase indefinitely* [6613′]. *Therefore it must oscillate about zero, making its mean value equal*

[6625] *to nothing* [6622g]. *This is confirmed by observation, and furnishes a new and remarkable proof of the mutual attraction of Jupiter's satellites.*

Remarkable proof of the mutual attraction of Jupiter's satellites.

Hence we may deduce several important consequences. The following equation, which is deduced from [6610, 6620],

[6626] $$v-3v'+2v''=\pi\pm\varpi_{,},$$

gives, by putting the quantities, which are not periodical, separately equal to nothing,*

[6627] $$nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=\pi.$$

[6622f] it does not come under the form in case 1, [6613′]; and we have seen that it does not come under case 3, [6619]; therefore we are restricted to case 2, [6615], in which the

[6622g] mean value of φ is equal to π; or, in other words, $\varpi_{,}$ [6620] is equal to *zero*, as in

[6622h] [6624], and $\mathrm{c}<2\mathrm{k}n^2$ [6614].

[6628a] * (3419) The longitudes v, v', v'', are composed of the mean values [6572′], the terms depending on the libration [6652—6654], and the periodical terms. Now if we

[6628b] neglect these periodical terms, and substitute the others, in the first member of [6626]; then put the terms depending on the libration in the first member equal $\pm\varpi_{,}$, as in [6620], we shall get, from the remaining terms, the equation [6627]. This equation may

[6628b′] be put under the form $(nt-n't+\varepsilon-\varepsilon')-2.(n't-n''t+\varepsilon'-\varepsilon'')=\pi$, which is used hereafter. This equation holds good for all values of t; and if we put $t=0$, it becomes

[6628c] $\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, as in [6639]. Subtracting this from [6627], we get, for all values of

[6628d] t, $nt-3n't+2n''t=0$; dividing this by t, we obtain the equation [6628]. The

[6628e] equation [6628d] may be put under the form $nt+2n''t=3n't$, as in [6629]; and the equation [6627] corresponds to the theorem in [6630]. The first member of [6627] may

[6628f] be put under the form $2.(n''t-nt+\varepsilon''-\varepsilon)-3.(n't-nt+\varepsilon'-\varepsilon)$; and by substituting the values [6240p], it becomes $2.(\Theta''-\Theta)-3.(\Theta'-\Theta)=\Theta-3\Theta'+2\Theta''$; hence the equation

Hence we deduce,

$$n-3n'+2n''=0.$$ [6628]

From these equations we obtain the following important results. FIRST. *The mean motion of the first satellite, plus twice that of the third, is exactly equal to three times that of the second.* SECOND. *The mean longitude of the first, minus three times that of the second, plus twice that of the third, is always exactly equal to two right angles. The same result holds good relatively to the mean synodical longitudes. For in the equation,*

Laws of La Place. [6629] [6630] [6630']

$$nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=\pi \text{ [6627]},$$ [6631]

we may refer the angles to an axis moving according to any law, since the position of that axis vanishes from this equation; *therefore we may suppose that* $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$, *denote the mean synodical longitudes.** [6632]

Hence it follows that *the three first satellites cannot all be eclipsed at the same time.* For $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$, being supposed to express the

The three first satellites cannot be eclipsed at the same time.

[6627] may be put under the following form, which is used hereafter, in [7391, &c.];

$$\Theta-3\Theta'+2\Theta''=\pi=200^{\circ}.$$ [6628g]

We may remark that, in substituting the numerical values of Θ, Θ', Θ'', in the first member of [6628g], it may be necessary to add or subtract a circumference 400°, or a multiple of 400°, to make it equal to the second member 200°. This will be evident by using the values of Θ, Θ', Θ'', given in [7496, 7440, 7386], which, by neglecting the minutes and seconds, are nearly expressed by [6628h] [6628i]

$$\Theta=16^{\circ}+t.82583^{\circ}; \qquad \Theta'=346^{\circ}+t.41141^{\circ}; \qquad \Theta''=11^{\circ}+t.20420^{\circ};$$ [6628k]

t being the number of Julian years elapsed since the epoch of 1750 [7284]. Now if $t=0$, we shall have $\Theta=16^{\circ}$, $\Theta'=346^{\circ}$, $\Theta''=11^{\circ}$; substituting these in the first member of [6628g], it becomes $16^{\circ}-1038^{\circ}+22^{\circ}=-1000^{\circ}$; which, by adding three circumferences, or 1200°, becomes equal to the second member of [6628g]. Again, if we suppose $t=0{,}002$, the preceding values [6628k] become $\Theta=181^{\circ}{,}16$; $\Theta'=28^{\circ}{,}28$; $\Theta''=51^{\circ}{,}84$; hence $\Theta-3\Theta'+2\Theta''=181^{\circ}{,}16-84^{\circ}{,}84+103^{\circ}{,}68=200^{\circ}$, as in [6628g]. Lastly, if $t=0{,}005$, the expressions [6628k] are very nearly represented by $\Theta=28^{\circ}{,}9$; $\Theta'=151^{\circ}{,}7$; $\Theta''=113^{\circ}{,}1$; consequently, [6628l] [6628m] [6628n]

$$\Theta-3\Theta'+2\Theta''=28^{\circ}{,}9-455^{\circ}{,}1+226^{\circ}{,}2=-200^{\circ}.$$ [6628o]

Hence it is evident that the second member of [6628g] may have the sign $\pm$; or if we put i for an integral number, positive or negative, including zero, the equation [6628g] may be more generally expressed in the following manner; [6628p]

$$\Theta-3\Theta'+2\Theta''=200^{\circ}+400^{\circ}.i.$$ [6628q]

* (3420) This is fully demonstrated in [1240'', 1240a, b], and it is unnecessary to make any further remarks on this subject. [6632a]

[6633] mean synodical motions, we shall have, in the simultaneous eclipses of the first and second satellites, $nt+\varepsilon$ and $n't+\varepsilon'$ equal to π; so that the equation [6631] becomes,

[6634] $$2n''t+2\varepsilon''=3\pi;$$

and then the mean longitude of the third satellite will be equal to $\frac{3}{2}\pi$.

In the simultaneous eclipses of the first and third satellites, $nt+\varepsilon$ and $n''t+\varepsilon''$ are equal to π; and [6631] becomes,

[6635] $$3n't+3\varepsilon'=2\pi;$$

so that the mean synodical longitude of the second satellite is $\frac{2}{3}\pi$. Lastly, in the simultaneous eclipses of the second and third satellites, $n't+\varepsilon'$ and $n''t+\varepsilon''$ are equal to π; hence [6631] gives,

[6636] $$nt+\varepsilon=2\pi.$$

The mean synodical longitude of the first satellite is then nothing, and
[6636'] instead of being eclipsed, it may produce upon Jupiter an eclipse of the sun.

We have seen, in [6160, 6161], that the two principal inequalities of the second satellite, which are produced by the action of the first and third
[6637] satellites, are, by means of the preceding theorems, reduced to one single term, producing the great inequality, discovered by observation, in the motion
[6637'] of the second satellite. *Therefore these two inequalities will always be united*, and there is no fear that they will be separated after the lapse of many ages.

Were it not for the mutual action of the satellites, the two equations [6628, 6628c],

[6638] $$n-3n'+2n''=0;$$

[6639] $$\varepsilon-3\varepsilon'+2\varepsilon''=\pi;$$

would have no connection with each other. We must therefore suppose, at the origin of the motion, that the epochs and mean motions of the satellites have been so arranged as to satisfy these equations, which is extremely improbable;
[6639'] *and even in this case, the smallest force, such as the attraction of the planets and comets, would finally produce a change in these relations. But the reciprocal action of the satellites removes this objection, and gives stability to the preceding relations.* For, by what has been said, we have, at the origin of the motion,*

[6640a] * (3421) The differentials of [6620, 6610] give $\pm d\varpi_{,}=d\varphi=dv-3dv'+2dv''$; and from [6621, 6626], we have,

$$\frac{dv}{ndt}-\frac{3dv'}{ndt}+\frac{2dv''}{ndt}=\pm\sqrt{\frac{\text{c}}{n^2}-2\text{k}.\cos.(\varepsilon-3\varepsilon'+2\varepsilon'')}\,; \qquad [6640]$$

c being less than $2\text{k}n^2$ [6622h]; therefore, to render the preceding [6641]
theorems accurate, it is only necessary to have, at the origin of the motion,
the function $\frac{dv}{ndt}-\frac{3dv'}{ndt}+\frac{2dv''}{ndt}$, comprised between the limits,* [6641′]

$$+2\text{k}^{\frac{1}{2}}.\sin.(\tfrac{1}{2}\varepsilon-\tfrac{3}{2}\varepsilon'+\varepsilon'') \qquad [6642]$$

$$-2\text{k}^{\frac{1}{2}}.\sin.(\tfrac{1}{2}\varepsilon-\tfrac{3}{2}\varepsilon'+\varepsilon'')\,; \qquad [6643]$$

and it will suffice to ensure the stability of the system, if the foreign attractions should always leave the preceding function within these limits.

We know, by observation, that the angle $\varpi_{\prime}$ is very small; therefore we may suppose $\cos.\varpi_{\prime}=1-\frac{1}{2}\varpi_{\prime}^2$ [44] Int. We shall now put, [6644]

$$\frac{\text{c}+2\text{k}n^2}{n^2.\text{k}}=\beta_{\prime}^2\,; \qquad [6645]$$

$\beta_{\prime}$ being arbitrary, because it contains the arbitrary constant quantity c; [6645′]
then the differential equation [6621] gives,†

$\pm d\varpi_{\prime}=\pm dt.\sqrt{\{\text{c}+2\text{k}n^2.\cos.(v-3v'+2v''-\pi)\}}=\pm dt.\sqrt{\{\text{c}-2\text{k}n^2.\cos.(v-3v'+2v'')\}}$. [6640$b$]
Putting these two expressions of $\pm d\varpi_{\prime}$ equal to each other, and then dividing by ndt, we get,

$$\frac{dv}{ndt}-\frac{3dv'}{ndt}+\frac{2dv''}{ndt}=\pm\sqrt{\left\{\frac{\text{c}}{n^2}-2\text{k}.\cos.(v-3v'+2v'')\right\}}\,; \qquad [6640c]$$

and at the origin of the motion, when $t=0$, we have $v=\varepsilon$, $v'=\varepsilon'$, $v''=\varepsilon''$ [6572′]; [6640d]
hence it becomes as in [6640].

* (3122) Putting, for brevity, the function [6641′], or the first member of [6640] equal to φ', we shall obtain [6641c], by taking the square of the equation [6640]. Now [6641a]
we have $\cos.(\varepsilon-3\varepsilon'+2\varepsilon'')=1-2.\sin.^2(\frac{1}{2}\varepsilon-\frac{3}{2}\varepsilon'+\varepsilon'')$, from [1] Int. Substituting this in [6641b]
[6641c], we obtain [6641d].

$$\varphi'^2=\frac{\text{c}}{n^2}-2\text{k}.\cos.(\varepsilon-3\varepsilon'+2\varepsilon'') \qquad [6641c]$$

$$=\frac{\text{c}-2\text{k}n^2}{n^2}+4\text{k}.\sin.^2(\tfrac{1}{2}\varepsilon-\tfrac{3}{2}\varepsilon'+\varepsilon'')\,; \qquad [6641d]$$

and since $\text{c}-2\text{k}n^2$ is negative [6641], we shall have $\varphi'^2<4\text{k}.\sin.^2(\frac{1}{2}\varepsilon-\frac{3}{2}\varepsilon'+\varepsilon'')$, or [6641$e$]
$\varphi'<2\text{k}^{\frac{1}{2}}.\sin.(\frac{1}{2}\varepsilon-\frac{3}{2}\varepsilon'+\varepsilon'')$, as in [6642, 6643].

† (3123) The equation [6646] is deduced from [6621], in the same manner as [1239] [6646a]
is derived from [1238]; observing, that by changing β into k, and ϖ into $\varpi_{\prime}$, in [6646b]
[1238], it becomes as in [6621]; and by making the same changes in [1239], we get

[6646] $$\varpi_{,} = \beta_{,}.\sin.(nt.\sqrt{\mathrm{k}} + A);$$

A being another arbitrary constant quantity.

The motions of the four satellites of Jupiter being determined by twelve [6647] differential equations of the second order,* their theory ought to contain twenty-four arbitrary constant quantities. Four of these correspond to the mean motions of the satellites, or, in other words, to their mean distances; [6647′] four correspond to the epochs of the mean longitudes; eight depend on the excentricities and the aphelia; and eight others upon the inclinations and nodes of the orbits. The preceding theorems [6638, 6639] establish two relations between the mean motions and the epochs of the mean longitudes [6648] of the three first satellites, which reduces these twenty-four arbitrary quantities to twenty-two. To supply this deficiency the two new arbitrary quantities $\beta_{,}$, A, are introduced into the expression of $\varpi_{,}$ [6646].

If we resume the equation [6586] and substitute in it the following expression, which is deduced from [6610, 6620, 6646],†

[6649] $$v - 3v' + 2v'' = \pi + \beta_{,}.\sin.(nt.\sqrt{\mathrm{k}} + A);$$

[6646c] $\varpi_{,} = \lambda.\sin.(nt.\sqrt{\mathrm{k}} + \gamma)$; which is of the same form as [6646]; the constant quantities λ, γ, being changed into $\beta_{,}$, A, respectively. We may incidentally remark, that the symbols $\beta_{,}$, $\varpi_{,}$, [6646] are not accented in the original work; we have accented them to distinguish them from β, ϖ [6023n, 6024g].

[6647a] * (3424) Each of the *four* satellites furnishes *three* equations of the second order, similar to those in [6057, 6060, 6077]; making in all *twelve* equations of the *second* order; [6647b] and as the complete integral of an equation of the second order, introduces two arbitrary constant quantities, we shall have in all twenty-four arbitrary constant quantities, as in [6647].

† (3425) Substituting the computed value of $\varpi_{,}$ [6646], in the expression of φ [6610, 6620], using also $\beta_{,}$ for $\pm\beta_{,}$, we get, as in [6649],

[6650a] $$\varphi = v - 3v' + 2v'' = \pi + \beta_{,}.\sin.(nt.\sqrt{\mathrm{k}} + A).$$

The sine of this expression gives,

[6650b] $$\sin.(v - 3v' + 2v'') = -\sin.\{\beta_{,}.\sin.(nt.\sqrt{\mathrm{k}} + A)\} = -\beta_{,}.\sin.(nt.\sqrt{\mathrm{k}} + A) \quad [43] \text{ Int.},$$

neglecting terms of the order $\beta_{,}^3$, on account of their smallness [6658]. Substituting this in [6586], we get [6650]; multiplying this by dt, integrating and then adding a [6650c] constant quantity $n_{,}$, we obtain the value of $\frac{dv}{dt}$. Again, multiplying by dt, and integrating, adding the constant quantity $\varepsilon_{,}$, we obtain $v = n_{,}t + \varepsilon_{,} +$ function [6651]. [6650d] Now we may suppose $\varepsilon_{,}$ to be included in the epoch ε, and $n_{,}t$ in the mean motion

we shall obtain,

$$\frac{ddv}{dt^2} = \frac{3m'.m''.n^3.F'G}{8.(n-n'-N')} \cdot \frac{a}{a'} \cdot \beta_{,} \cdot \sin.(nt.\sqrt{k}+A). \qquad [6650]$$

Integrating this expression, and neglecting the arbitrary constant quantities which form part of the epoch and the mean longitude, we get,

$$v = -\frac{3m'.m''.n.F'G}{8k.(n-n'-N')} \cdot \frac{a}{a'} \cdot \beta_{,} \cdot \sin.(nt.\sqrt{k}+A); \qquad [6651]$$

and by substituting the value of k, we shall have,

$$v = \frac{\beta_{,} \cdot \sin.(nt.\sqrt{k}+A)}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}. \qquad [6652]$$

In like manner we shall find,*

$$v' = \frac{-\frac{3a'.m}{4a.m'} \cdot \beta_{,} \cdot \sin.(nt.\sqrt{k}+A)}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}; \qquad [6653]$$

$$v'' = \frac{\frac{a''.m}{8a.m''} \cdot \beta_{,} \cdot \sin.(nt.\sqrt{k}+A)}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}. \qquad [6654]$$

Therefore the three first satellites are subjected to an inequality depending on the angle $nt.\sqrt{k}+A$. It is by observation alone that we can determine the limit of the arbitrary quantity $\beta_{,}$, and the time when this inequality [6655]

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nt; and we may therefore neglect the constant quantities $n_{,}$, $\varepsilon_{,}$, and we shall have $v =$ function [6651]. Now from [6609] we have, [6650e]

$$k = -\frac{3m'.m''.n.F'G}{8.(n-n'-N')} \cdot \frac{a}{a'} \cdot \left\{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right\}; \qquad [6650f]$$

substituting this in [6651], we get [6652].

* (3426) Multiplying the second member of [6586] by $-\frac{3a'.m}{4a.m'}$, it becomes equal to the second member of [6602]; hence we get $\frac{ddv'}{dt^2} = -\frac{3a'.m}{4a.m'} \cdot \frac{ddv}{dt^2}$. Integrating and neglecting the constant quantities as in [6650e], we get $v' = -\frac{3a'.m}{4a.m'}.v$; and by substituting the value of v [6652], we obtain [6653]. In like manner, by multiplying [6586] by $\frac{a''.m}{8a.m''}$, it becomes equal to [6608]; hence $\frac{ddv''}{dt^2} = \frac{a''.m}{8a.m''} \cdot \frac{ddv}{dt^2}$; whose integral gives $v'' = \frac{a''.m}{8a.m''}.v$; and by substituting [6652], we get [6654]. [6654a] [6654b] [6654c]

vanishes, which depends upon the value of A. This inequality deserves the particular attention of astronomers. *We may consider it as a libration of the mean motions of the three first satellites. By means of this libration the*
[6656] *difference of the mean motions of the first and second satellites, minus twice*
Libration of Jupiter's satellites. *the difference of the mean motions of the second and third satellites, oscillates always about two right angles* [6628b']. *For this reason we shall designate*
[6657] *this inequality by the name of the libration of Jupiter's satellites.* It has considerable analogy with the libration of the lunar spheroid, whose analysis we have given in [3460—3470]; and in like manner as in the lunar theory it replaces two arbitrary constant terms of the mean longitudes. Moreover,
[6658] *this libration, as in the lunar theory, is insensible*; and this arises from circumstances depending upon the primitive motions of the satellites.

The radii vectores of the three satellites are subjected to the same
[6658'] inequality; for they produce, in the expression of the *mean motion* $\int ndt$, the inequality,

[6659]
$$\frac{\beta_{,}.\sin.(nt.\sqrt{\mathrm{k}}+A)}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}; \; [6652]$$

[6659'] which gives in n, considered as variable, the following inequality, which we shall represent by δn;*

[6660]
$$\delta n=\frac{n.\beta_{,}.\sqrt{\mathrm{k}}.\cos.(nt.\sqrt{\mathrm{k}}+A)}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}};$$

[6661] and since $a=n^{-\frac{2}{3}}$ [6110], we shall have,

[6662a] * (3427) Putting $n_{,}t$ for the mean value of $\int ndt$, and v for the quantity [6652], we shall have, for the mean motion [6658'], including the libration, the expression
[6662b] $\int ndt=n_{,}t+v$. Its differential being divided by dt, gives $n=n_{,}+\frac{dv}{dt}$; the quantity $\frac{dv}{dt}$ [6652], being the same as the second member of [6660], or the value of δn; hence
[6662c] $n=n_{,}+\delta n$. Substituting this in [6661], we get $a=(n_{,}+\delta n)^{-\frac{2}{3}}=n_{,}^{-\frac{2}{3}}-\frac{2}{3}.n_{,}^{-\frac{5}{3}}.\delta n$, neglecting the higher powers of δn, so that a contains the inequality $-\frac{2}{3}.n^{-\frac{5}{3}}.\delta n$, which we shall represent by δa; and we shall have very nearly,

[6662d]
$$\delta a=-\tfrac{2}{3}.n^{-\frac{5}{3}}.\delta n=-\tfrac{2}{3}.n^{-\frac{2}{3}}.\frac{\delta n}{n}=-\tfrac{2}{3}.a.\frac{\delta n}{n},$$

as in [6662]. In like manner we shall have, for the satellites m', m'',

[6662e]
$$\delta a'=-\tfrac{2}{3}.a'.\frac{\delta n'}{n'}; \qquad \delta a''=-\tfrac{2}{3}.a''.\frac{\delta n''}{n''}.$$

$$\delta u = -\tfrac{2}{3}.a.\frac{\delta n}{n};$$ [6662]

Libration of the radius vector.

which is the variation of the radius vector r, depending on the expression [6659]. We may obtain, in the same way, the corresponding variations of r' and r'' [6662e].

16. *The libration of the three first satellites of Jupiter, modifies all their inequalities of a long period. It gives to their expressions a particular form, which connects them together, and is a singular case of the analysis of perturbations.* If we suppose that $\lambda_{,}.\sin.(\mathrm{i}t+\mathrm{o})$, is an inequality of a long period of the satellite m, which would take place if it were not modified by the action of the two satellites m', m'', &c.; and that $\lambda_{,}'.\sin.(\mathrm{i}t+\mathrm{o})$ and $\lambda_{,}''.\sin.(\mathrm{i}t+\mathrm{o})$, are the corresponding inequalities of the satellites m' and m''; we shall have, by noticing only these inequalities,* [6663] [6664] [6665]

Singular case in the analysis of the perturbations.

$$\frac{ddv}{dt^2} = -\mathrm{i}^2.\lambda_{,}.\sin.(\mathrm{i}t+\mathrm{o});$$ [6666]

$$\frac{ddv'}{dt^2} = -\mathrm{i}^2.\lambda_{,}'.\sin.(\mathrm{i}t+\mathrm{o});$$ [6667]

$$\frac{ddv''}{dt^2} = -\mathrm{i}^2.\lambda_{,}''.\sin.(\mathrm{i}t+\mathrm{o}).$$ [6668]

But by noticing only the inequality of libration, we have as in [6586, 6650f],†

$$\frac{ddv}{dt^2} = \frac{\mathrm{k}.n^2.\sin.(v-3v'+2v'')}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}.$$ [6669]

Connecting together the two expressions [6666, 6669], we shall have,

$$\frac{ddv}{dt^2} = \frac{\mathrm{k}n^2.\sin.(v-3v'+2v'')}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}} - \mathrm{i}^2.\lambda_{,}.\sin.(\mathrm{i}t+\mathrm{o}).$$ [6670]

We shall suppose that

$$v = Q.\sin.(\mathrm{i}t+\mathrm{o}); \qquad v' = Q'.\sin.(\mathrm{i}t+\mathrm{o}); \qquad v'' = Q''.\sin.(\mathrm{i}t+\mathrm{o});$$ [6671]

represent the inequalities of v, v', v'', depending on $\mathrm{i}t+\mathrm{o}$, and modified

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* (3428) Taking the second differentials of the assumed values of the terms of v, v', v'', [6664, 6665], and dividing them by dt^2, we obtain the expressions [6666—6668]. [6666a]

† (3429) Dividing the expression [6586] by that in [6650f], and multiplying the result by k, we get [6669]. Adding this to [6666], we obtain [6670]. [6670a]

by the reciprocal action of the satellites; then putting,*

[6672] $$v-3v'+2v''=\varpi+(Q-3Q'+2Q'').\sin.(\mathrm{i}t+\mathrm{o});$$

we shall have,

[6673] $$\frac{ddv}{dt^2}=-\left\{\mathrm{i}^2.\lambda'+\frac{\mathrm{k}n^2.(Q-3Q'+2Q'')}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}\right\}.\sin.(\mathrm{i}t+\mathrm{o}).$$

Substituting, in the first member, the value of v [6671], we get,†

[6674] $$Q=\lambda_{,}+\frac{\mathrm{k}n^2.(Q-3Q'+2Q'')}{\mathrm{i}^2.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}.$$

In like manner we shall find,‡

6675] $$Q'=\lambda_{,}'-\frac{\frac{3a'.m}{4a.m'}.\mathrm{k}n^2.(Q-3Q'+2Q'')}{\mathrm{i}^2.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)};$$

[6676] $$Q''=\lambda_{,}''+\frac{\frac{a''.m}{8a.m''}.\mathrm{k}n^2.(Q-3Q'+2Q'')}{\mathrm{i}^2.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}.$$

[6672a] * (3430) The object of the present calculation is to notice only angles of the form $\mathrm{i}t+\mathrm{o}$, where i is very small, as in [6664, 6672]; and this expression [6672] is found, by substituting in $v-3v'+2v''$ the values [6671], and adding to the result the mean value ϖ of the function $v-3v'+2v''$ [6626]. The sine of this expression of $v-3v'+2v''$ gives,

[6672b] $\sin.(v-3v'+2v'')=-\sin.\{(Q-3Q'+2Q'').\sin.(\mathrm{i}t+\mathrm{o})\}=-(Q-3Q'+2Q'').\sin.(\mathrm{i}t+\mathrm{o})$;

[6672c] neglecting the higher powers of $(Q-3Q'+2Q'')$ on account of its smallness. Substituting this in [6670], it becomes as in [6673].

† (3431) The assumed value of v [6671] gives, by taking its second differential,

[6674a] $\frac{ddv}{dt^2}=-\mathrm{i}^2Q.\sin.(\mathrm{i}t+\mathrm{o})$; substituting this in the first member of [6673], then dividing by $-\mathrm{i}^2.\sin.(\mathrm{i}t+\mathrm{o})$, we get [6674].

[6675a] ‡ (3432) We see in [6654a, c] that the parts of $\frac{ddv'}{dt^2}$, $\frac{ddv''}{dt^2}$, containing k, are found

[6675b] by changing successively, in the similar expression of $\frac{ddv}{dt^2}$ [6669], k into $-\frac{3a'.m.\mathrm{k}}{4a.m'}$, and

[6675c] $\frac{a''.m.\mathrm{k}}{8a.m''}$. To these we must add the corresponding terms in [6667, 6668], and we shall

[6675d] obtain the expression of $\frac{ddv'}{dt^2}$, $\frac{ddv''}{dt^2}$, in forms like that of $\frac{ddv}{dt^2}$ [6673]; and which may

These three equations give,*

$$Q-3Q'+2Q''=\frac{\mathrm{i}^2.(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{\mathrm{i}^2-\mathrm{k}n^2}\ ; \qquad [6677]$$

consequently,

$$v=\left\{\lambda_{,}+\frac{\mathrm{k}n^2.(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{(\mathrm{i}^2-\mathrm{k}n^2).\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.\sin.(\mathrm{i}t+\mathrm{o})\ ; \qquad [6678]$$

$$v'=\left\{\lambda_{,}'-\frac{\frac{3a'.m}{4a.m'}.\mathrm{k}n^2.(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{(\mathrm{i}^2-\mathrm{k}n^2).\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.\sin.(\mathrm{i}t+\mathrm{o})\ ; \qquad [6679]$$

$$v''=\left\{\lambda_{,}''+\frac{\frac{a''.m}{8a.m''}.\mathrm{k}n^2.(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{(\mathrm{i}^2-\mathrm{k}n^2).\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.\sin.(\mathrm{i}t+\mathrm{o}). \qquad [6680]$$

It is important here to remark, that in the preceding analysis we have supposed [6680′]
i *to be much less than* $n-2n'$, *or* $n'-2n''$. *For in changing, as we have done in* [6585], $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$ into v, v', v'', respectively, in [6681]
the angle $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$, [6583], it is necessary that the same changes may be permitted in the values of $\frac{r'\delta r'}{a'^2}$ and $\delta v'$ [6580, 6581, 6598, 6599], which we have used in computing [6586, &c.]; [6681′]
observing that these expressions depend upon the angles $2n't-2n''t+2\varepsilon'-2\varepsilon''$, and $n't-nt+\varepsilon'-\varepsilon$. We shall first consider the part of $\frac{r'\delta r'}{a'^2}$ depending

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be derived from it, by changing, as in [6666—6668], $\lambda_{,}$ into $\lambda_{,}'$ or $\lambda_{,}''$ respectively, and k as in [6675*b*, *c*]; moreover, we must change, as in [6671], Q into Q', or Q'', respectively. Now making these changes in the values of k, $\lambda_{,}$, Q, we get successively, from Q [6674], the expressions of Q', Q'' [6675, 6676]. [6675*e*]

* (3433) Multiplying [6675] by -3, also [6676] by 2, and adding the products to [6674], we find that the factor $1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}$ occurs in the numerator and denominator of the term of the sum which is connected with k, and by rejecting it, we get, [6677*a*]

$$Q-3Q'+2Q''=(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')+\frac{\mathrm{k}n^2}{\mathrm{i}^2}.(Q-3Q'+2Q''). \qquad [6677b]$$

Hence we easily deduce the value of $Q-3Q'+2Q''$ [6677]; and by substituting it, in the values of Q, Q', Q'' [6674—6676], then these values in the expressions of v, v', v'' [6671], we get the final results in [6678—6680].

[6682] upon the angle $nt-n't+\varepsilon-\varepsilon'$. We have in § 3, by noticing only the terms depending on $\cos.(v-v')$,*

[6683] $$0=\frac{d^2.(r'\delta r')}{a'^2.dt^2}+N'^2.\frac{(r'\delta r')}{a'^2}+P.\cos.(v-v');$$

P *being a constant coefficient.* If we substitute the values,

[6684] $$v=nt+\varepsilon+Q.\sin.(it+\text{o})\,; \qquad v'=n't+\varepsilon'+Q'.\sin.(it+\text{o}),$$

corresponding to [6585, 6671], we shall get, by developing $\cos.(v-v')$, the following expression ;†

[6685]
$$\begin{aligned}0=\frac{d^2.(r'\delta r')}{a'^2.dt^2}+N'^2.\frac{(r'\delta r')}{a'^2} & \qquad 1\\ +P.\cos.(nt-n't+\varepsilon-\varepsilon') & \qquad 2\\ +\tfrac{1}{2}P.(Q'-Q).\cos.(nt-n't+\varepsilon-\varepsilon'-it-\text{o}) & \qquad 3\\ -\tfrac{1}{2}P.(Q'-Q).\cos.(nt-n't+\varepsilon-\varepsilon'+it+\text{o}). & \qquad 4\end{aligned}$$

Hence we deduce by integration, observing that i [6680'] and $n-2n'$
[6686] [6151a] are very small relative to n; also N' differs but very little from n' [6025f],‡

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* (3434) The terms of [6114 line 4], depending on the angle $n't-nt+\varepsilon'-\varepsilon$, are
[6683a] computed in [6092 line 1, 6094 line 2], from the expression of R [6090], and depend wholly on the term $A^{(1)}.\cos.(v'-v)$ of [6090]. Hence it is evident that it will be more correct to retain $v'-v$, instead of $n't-nt+\varepsilon'-\varepsilon$; and then the expression of [6114], corresponding to this angle, will be of the form,

[6683b] $$\frac{d^2.(r\delta r)}{a^2.dt^2}+N^2.\frac{(r\delta r)}{a^2}+P_{,}.\cos.(v'-v)\,;$$

$P_{,}$ being a constant quantity depending on $A^{(1)}$ and on its differential relative to a.
[6683c] Changing the elements of m into those of m', [6024], and the contrary, also supposing $P_{,}$ to become P, we get the equation [6683], corresponding to the satellite m'.

[6685a] † (3435) The difference of the values of v, v' [6684] gives, by using for brevity the symbol $T'=nt-n't+\varepsilon-\varepsilon'$,

[6685b] $$v-v'=T'-(Q'-Q).\sin.(it+\text{o}).$$

The cosine of this expression being found, by means of [61] Int., becomes as in [6685c], and this expression is easily reduced to the form [6685d], by reducing the last term of its second member, by means of [17] Int.,

[6685c] $$\cos.(v-v')=\cos.T'+(Q'-Q).\sin.T'.\sin.(it+\text{o})$$

[6685d] $$=\cos.T'+\tfrac{1}{2}(Q'-Q).\cos.(T'-it-\text{o})-\tfrac{1}{2}(Q'-Q).\cos.(T'+it+\text{o}).$$

Substituting this last expression in [6683], it becomes as in [6685].

[6687a] ‡ (3436) From the equation [6685] we obtain, by integration, as in [6049k, l], the values of $\frac{r'\delta r'}{a'^2}$. This is done by dividing the terms in [6685 lines 2, 3, 4], by

$$\begin{aligned}\frac{r'\delta r'}{a'^2} = & \frac{P}{n.(n-n'-N')}\cdot\cos.(nt-n't+\varepsilon-\varepsilon') & 1\\ & +\frac{P.(Q'-Q)}{2n.(n-n'-\mathrm{i}-N')}\cdot\cos.(nt-n't+\varepsilon-\varepsilon'-\mathrm{i}t-\mathrm{o}) & 2\\ & -\frac{P.(Q'-Q)}{2n.(n-n'+\mathrm{i}-N')}\cdot\cos.(nt-n't+\varepsilon-\varepsilon'+\mathrm{i}t+\mathrm{o}). & 3\end{aligned}$$ [6687]

If we suppose $\frac{\mathrm{i}}{n-n'-N'}$ to be so small that we may neglect it without any sensible error, we shall have, [6687′]

$$\frac{r'\delta r'}{a'^2} = \frac{P}{n.(n-n'-N')}\cdot\left\{\begin{matrix}\cos.(nt-n't+\varepsilon-\varepsilon') \\ +\tfrac{1}{2}(Q'-Q).\left\{\begin{matrix}\cos.(nt-n't+\varepsilon-\varepsilon'-\mathrm{i}t-\mathrm{o}) \\ -\cos.(nt-n't+\varepsilon-\varepsilon'+\mathrm{i}t+\mathrm{o})\end{matrix}\right\}\end{matrix}\right\}. \quad \begin{matrix}1\\2\\3\end{matrix}$$ [6688]

Substituting $v=nt+\varepsilon+Q.\sin.(\mathrm{i}t+\mathrm{o})$; $v'=n't+\varepsilon'+Q'.\sin.(\mathrm{i}t+\mathrm{o})$ [6684], we obtain,* [6689]

$$\frac{r'\delta r'}{a'^2} = \frac{P}{n.(n-n'-N')}\cdot\cos.(v-v').$$ [6690]

It is evident that the corresponding inequality of $\delta v'$ will be,†

$$\delta v' = -\frac{2P}{n.(n-n'-N')}\cdot\sin.(v-v').$$ [6691]

$(n-n')^2-N'^2$; $(n-n'-\mathrm{i})^2-N'^2$; $(n-n'+\mathrm{i})^2-N'^2$, respectively. These divisors are easily reduced to the forms $(n-n'+N').(n-n'-N')$; $(n\text{-}n'\text{-}\mathrm{i}+N').(n\text{-}n'\text{-}\mathrm{i}\text{-}N')$; $(n-n'+\mathrm{i}+N').(n-n'+\mathrm{i}-N')$. But we find from [6025f, 6680′], that $-n'+N'$, $-n'\mp\mathrm{i}+N'$, are extremely small; therefore these divisors are nearly represented by $n.(n-n'-N')$; $n.(n-n'-\mathrm{i}-N')$; $n.(n-n'+\mathrm{i}-N')$. Dividing the terms of [6685 lines 2, 3, 4], by these divisors respectively, we obtain the corresponding terms [6687]. Again the three divisors [6687d] may evidently be represented by the quantity $n.(n-n'-N')$, multiplied respectively by 1, $1-\frac{\mathrm{i}}{n-n'-N'}$, $1+\frac{\mathrm{i}}{n-n'-N'}$; and if we suppose $\mp\frac{\mathrm{i}}{n-n'-N'}$ to be so small that it may be neglected, the three divisors will all be equal to $n.(n-n'-N')$; and then the expression [6687] becomes of the form [6688]. [6687b] [6687c] [6687d] [6687e] [6687f]

* (3437) The expression [6685d], which represents the development of $\cos.(v-v')$, is the same as the terms between the braces, in the second member of [6688], using T' [6685a]. Substituting this in [6688], we get [6690]. [6690a]

† (3438) The terms of δv [6119 line 3], which contain the divisor $(n-n')^2-N^2$, are introduced into [6118b] by means of the term $\frac{2.d.\left(\frac{r\delta r}{a^2}\right)}{ndt}$ [6118b, x]; so that by [6691a]

We may apply the same reasoning to the parts of $\frac{r'\delta r'}{a'^2}$ and $\delta v'$, [6580, 6581], depending on the angle $2n't-2n''t+2\varepsilon'-2\varepsilon''$, and we shall find that we may, in like manner, change the angles $n't+\varepsilon'$ and $n''t+\varepsilon''$,
[6692] respectively, into v' and v''; provided we consider only the inequalities
[6692'] of v' and v'' depending upon any angle $\mathrm{i}t+\mathrm{o}$, in which i is much less than $n-2n'$ or $n'-2n''$.*

The inequality depending on $Mt+E-I$, which we have found in
[6693] [6285 line 2], is of this kind; its period being about ten times greater than that of the angle $nt-2n't$.† The expression of this inequality [6285 line 2],
[6693'] being compared with the assumed value $\lambda_{,}.\sin.(\mathrm{i}t+\mathrm{o})$ [6664], gives,‡

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noticing only the terms with this divisor we shall have $\delta v=\frac{2.d.\left(\frac{r\delta r}{a^2}\right)}{ndt}$; and in like manner
[6691b] $\delta v'=\frac{2.d.\left(\frac{r'\delta r'}{a'^2}\right)}{n'dt}$. Substituting in this last expression the value of $\frac{r'\delta r'}{a'^2}$ [6690], and observing that we have very nearly, by using the values of v, v' [6572'], and $n=2n'$ [6151],
[6691c] $d.\cos.(v-v')=-(dv-dv').\sin.(v-v')=-(ndt-n'dt).\sin.(v-v')=-n'dt.\sin.(v-v')$, we get the expression of $\delta v'$ [6691].

[6692a] * (3439) If we change v, n, ε, Q, into v'', n'', ε'', Q'', in [6683, &c.], and then write $P.\cos.(2v''-2v')$ instead of $P.\cos.(v-v')$, we shall obtain an equation similar to
[6692b] [6683], but depending on the angle $2n't-2n''t+2\varepsilon'-2\varepsilon''$; and if we follow the calculation from [6683] to [6690], we shall obtain a similar result to that in [6690]; supposing as in
[6692c] [6687f] that $\frac{\mathrm{i}}{2n'-2n''-N'}$ or $\frac{\mathrm{i}}{n-n'-N'}$ [6154] is a very small quantity, which may be neglected.

[6694a] † (3440) The values of n, n', M [6025k, m] give $n-2n'=3001156''$, $M=337211''$; of which the former is nearly ten times greater than the latter. We may
[6694b] also observe that the values of n [6025k], and $k=0,000000607302$ [7272], give
[6694c] $n.\sqrt{\mathrm{k}}=643563''$, which is about $\frac{1}{5}$ part of $n-2n'$ [6694a]. Hence it is evident that the angles Mt, $n.\sqrt{\mathrm{k}}.t$, which occur in δv [6285, 6652, &c.], may be considered as
[6694d] coming under the form of the terms spoken of in [6680', &c.]. The other terms of this kind are those which depend on the secular equations of the motions of the satellites, and on the motions of Jupiter's equator and orbit.

[6695a] ‡ (3441) Putting the term [6285 line 2] equal to the assumed value of δv [6693'], we get the expression of $\lambda_{,}$ [6694]. If we put the corresponding terms of $\delta v'$, $\delta v''$ equal

$$\lambda_{\prime} = -\frac{3M}{n}.H; \qquad \lambda_{\prime}' = -\frac{6M}{n}.H; \qquad \lambda_{\prime}'' = -\frac{12M}{n}.H.$$ [6694]

Hence we deduce,

$$\lambda_{\prime}-3\lambda_{\prime}'+2\lambda_{\prime}'' = -\frac{9M}{n}.H;$$ [6695]

therefore, by noticing only this inequality, we shall have,*

$$\delta v = -\frac{3M}{n}.\left\{1+\frac{3\mathrm{k}n^2}{(M^2-\mathrm{k}n^2).\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.H.\sin.(Mt+E-I);$$ [6696]

$$\delta v' = -\frac{6M}{n}.\left\{1-\frac{9a'.m.\mathrm{k}n^2}{8a.m'.(M^2-\mathrm{k}n^2).\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.H.\sin.(Mt+E-I);$$ [6697]

$$\delta v'' = -\frac{12M}{n}.\left\{1+\frac{3a''.m.\mathrm{k}n^2}{32a.m''.(M^2-\mathrm{k}n^2).\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.H.\sin.(Mt+E-I).$$ [6698]

We have determined, in § 12, [6487—6552], the secular inequalities of the satellites. The most sensible parts of these inequalities are those which depend on the secular variations of the orbit of Jupiter, and on the position of its equator [6525]. The values of i relative to them are very small,† [6699]

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to the similar values in [6665], we shall get $\lambda_{\prime}' = -\frac{3M}{n'}.H$; $\lambda_{\prime}'' = -\frac{3M}{n''}.H$; which [6695b]
are easily deduced from $\lambda_{\prime}$ by accenting the letters $\lambda_{\prime}$, n; and by using $n'=\frac{1}{2}n$,
$n''=\frac{1}{4}n$ [6151], they become as in [6694]. Substituting the values of $\lambda_{\prime}$, $\lambda_{\prime}'$, $\lambda_{\prime}''$ [6695c]
[6694], in the first member of [6695], it becomes as in its second member.

* (3442) Substituting the values [6694, 6695], and $\mathrm{i}=M$, $\mathrm{o}=E-I$, in [6696a]
[6678—6680], we get the expressions [6696—6698] respectively; observing that the
expression of $\lambda_{\prime}-3\lambda_{\prime}'+2\lambda_{\prime}''$ [6695], may be put under either of the three forms $3\lambda_{\prime}$, [6696b]
$\frac{3}{2}\lambda_{\prime}'$, $\frac{3}{4}\lambda_{\prime}''$, as is evident by substituting the values [6694], and comparing with [6695].

† (3443) The coefficients i corresponding to these secular variations, are of the order [6699a]
b, c, which occur in the expression of δv [6530]; and these quantities are very small,
being less than a second [6937, 6938], so that bt or ct does not vary a second in a year.
Comparing this with $n.\sqrt{\mathrm{k}}=643563''$ [6694c], we find that the ratio $\frac{\mathrm{i}}{n.\sqrt{\mathrm{k}}}$ is of the [6699b]
order $\frac{1}{643563}$, and its square $\frac{\mathrm{i}^2}{\mathrm{k}n^2}$ is very small, so that i^2 is very small in comparison [6699c]
with $\mathrm{k}n^2$, and may therefore be neglected; and then the formulas [6678—6680] become
as in [6700—6702]; because the factor $\frac{\mathrm{k}n^2}{\mathrm{i}^2-\mathrm{k}n^2}$ becomes equal to -1. [6699d]

[6699′] so that we may neglect i^2 in comparison with $\mathrm{k}n^2$; therefore we shall have, by noticing only these inequalities,

[6700] $$v = \Sigma'.\left\{\lambda_{,} - \frac{(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}\right\}.\sin.(\mathrm{i}t+\mathrm{o});$$

[6701] $$v' = \Sigma'.\left\{\lambda_{,}' + \frac{3a'.m.(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{4a.m'.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.\sin.(\mathrm{i}t+\mathrm{o});$$

[6702] $$v'' = \Sigma'.\left\{\lambda_{,}'' - \frac{a''.m.(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')}{8a.m''.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)}\right\}.\sin.(\mathrm{i}t+\mathrm{o}).$$

[6703] the characteristic Σ' refers, as in [6324′], to all the inequalities of the form $\sin.(\mathrm{i}t+\mathrm{o})$. From these equations we deduce,*

[6704] $$v-3v'+2v''=0.$$

Hence the inequalities with a long period, in which i^2 is considerably less than $\mathrm{k}n^2$, do not trouble the relations we have established in
[6705] [6627, 6628, 6631, 6639], between the three first satellites, by their mutual action; these inequalities being so modified as to satisfy these relations.
[6705′] In this way we have seen, in [3461], that the action of the earth upon the lunar spheroid, makes the rotatory motion of that spheroid participate in the motion of revolution, and by this means it maintains the equality between these mean motions.

[6706] We shall represent by Ct^2, $C't^2$, $C''t^2$, the secular equations of the three first satellites depending on the secular variations of the orbit and equator

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[6703a] * (3444) Multiplying [6701] by -3, and [6702] by $+2$, then adding these products to [6700], we find that the first member of the sum becomes $v-3v'+2v''$, as in [6704]. Moreover the terms of the second member of the sum, which have explicitly

[6703b] the factor $\dfrac{\lambda_{,}-3\lambda_{,}'+2\lambda_{,}''}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}}$, will be equal to this quantity, multiplied by

[6703b′] $$-\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right);$$

and by reduction it becomes $-(\lambda_{,}-3\lambda_{,}'+2\lambda_{,}'')$. This is destroyed by the other terms, $\lambda_{,}-3\lambda_{,}'+2\lambda_{,}''$, in that sum; so that we shall have $v-3v'+2v''=0$, as in [6704]. In this calculation we neglect terms in the second member of the order $\frac{\mathrm{i}^2}{\mathrm{k}n^2}.\lambda_{,}.\sin.(\mathrm{i}t+\mathrm{o})$,
[6703c] which are insensible from their smallness; but it is evident that if $\frac{\mathrm{i}^2}{\mathrm{k}n^2}$ be large, as in [6696—6698], the equation [6704] will not be satisfied.

of Jupiter, and also upon the resistance of the ether; these equations being independent of the mutual action of these satellites. We shall obtain these secular equations, by developing [6664, &c.], $\Sigma'.\lambda_{,}.\sin.(it+o)$, $\Sigma'.\lambda_{,}'.\sin.(it+o)$, $\Sigma'.\lambda_{,}''.\sin.(it+o)$, as far as the second power of t;* observing that the terms of the series which are independent of t, are confounded with the epochs of the longitudes; and those which depend on the first power of t, are confounded with the mean motions. The preceding expressions of v, v', v'', will thus give, for the secular equations, modified by the reciprocal action of the satellites,† [6707] [6708]

$$v = \left\{ C - \frac{(C-3C'+2C'')}{1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}} \right\}.t^2; \qquad [6709]$$

$$v' = \left\{ C' + \frac{3a'.m.(C-3C'+2C'')}{4a.m'.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)} \right\}.t^2; \qquad [6710]$$

$$v'' = \left\{ C'' - \frac{a''.m.(C-3C'+2C'')}{8a.m''.\left(1+\frac{9a'.m}{4a.m'}+\frac{a''.m}{4a.m''}\right)} \right\}.t^2. \qquad [6711]$$

These values may be used, without any sensible error, during several centuries, and they will suffice for a long period for astronomical purposes. [6711′]

17. *The nearly commensurable ratios of the mean motions of the three first satellites introduce some sensible terms into the equations which are given in* [6217, 6220, 6221], *for the determination of the variations of the excentricities and perijoves of the orbits.* To prove this, we shall resume the values of df, df' [1257]; which give, by observing that μ is very nearly [6712] [6713]

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* (3445) We have from [21] Int. $\sin.(it+o)=\sin.it.\cos.o+\cos.it.\sin.o$; and from [43, 44] Int. we have $\sin.it=it-$&c.; $\cos.it=1-\frac{1}{2}i^2t^2+$&c. Substituting these and neglecting terms of the order t^3, we shall have $v=\Sigma'.\lambda_{,}.\sin.(it+o)=A+Bt+Ct^2$; A, B, C, being constant quantities. Now supposing A to be connected with the epoch, and Bt with the mean motion, as in [6708], it will become $v=Ct^2$; and in like manner we have, for the second and third satellites, $v'=C't^2$, $v''=C''t^2$, as in [6706]; or as they may be written, [6707a] [6707b] [6707c]

$$\Sigma'.\lambda_{,}.\sin.(it+o)=Ct^2; \qquad \Sigma'.\lambda_{,}'.\sin.(it+o)=C't^2; \qquad \Sigma'.\lambda_{,}''.\sin.(it+o)=C''t^2. \qquad [6707d]$$

† (3446) Substituting the values [6707d] in [6700—6702], we get [6709—6711] respectively. [6709a]

equal to 1 [6110c],*

[6714]
$$d.(e.\cos.\varpi)=-\frac{a.ndt}{\sqrt{1-e^2}}.\left\{2.\cos.v+\tfrac{3}{2}.e.\cos.\varpi+\tfrac{1}{2}e.\cos.(2v-\varpi)\right\}.\left(\frac{dR}{dv}\right) \qquad 1$$
$$-a^2.ndt.\sqrt{1-e^2}.\left(\frac{dR}{dr}\right).\sin.v\,; \qquad 2$$

[6715]
$$d.(e.\sin.\varpi)=-\frac{a.ndt}{\sqrt{1-e^2}}.\left\{2.\sin.v+\tfrac{3}{2}e.\sin.\varpi+\tfrac{1}{2}e.\sin.(2v-\varpi)\right\}.\left(\frac{dR}{dv}\right) \qquad 1$$
$$+a^2.ndt.\sqrt{1-e^2}.\left(\frac{dR}{dr}\right).\cos.v. \qquad 2$$

[6716] If we neglect the excentricity e, in the second member of [6714], and retain in R [6052, 6090], only the terms,†

[6717]
$$R=-\frac{(\rho-\frac{1}{2}\varphi)}{3r^3}+m'.A^{(2)}.\cos.(2v-2v'),$$

[6718] we find, by substituting $r^2=a^2+2r\delta r$ [6057b], and $v=nt+\varepsilon+\delta v$ [6572′, &c.] in [6714], that it will become‡,

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* (3447) Putting $\mu=1$, as in [6110c], we get, from [1254], $f=e.\cos.\varpi$,

[6714a] $f'=e.\sin.\varpi$. Substituting their differentials in the first members of the equations [1257], we obtain [6714, 6715].

† (3448) The first term of R [6052] is $-\frac{1}{3}.(\rho-\frac{1}{2}\varphi).\frac{MB^2}{r^3}$; and since $M=1$,

[6717a] $B=1$ [6297d], it becomes as in the first term of [6717]. Its second term is the same as the third term of [6090]. We shall see that these produce, in $d.(e.\cos.\varpi)$ [6724], terms

[6717b] depending on the angle $nt-2n't+\varepsilon-2\varepsilon'$; and when they are combined with the terms of δv, $\delta v'$ [6725, 6726] depending on $nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma$, they produce, as in

[6717c] [6728], terms depending on the slowly varying angle $gt+\Gamma$; and finally introduce, in [6729′], a change in the value of hg, which must be noticed, as in [6731′, &c.], although it is of the order of the cube of the disturbing force.

‡ (3449) If we neglect terms of the order e, in the second member of [6714], we shall get,

[6718a]
$$d.(e.\cos.\varpi)=a.ndt.\left\{-2.\left(\frac{dR}{dv}\right).\cos.v-a.\left(\frac{dR}{dr}\right).\sin.v\right\}.$$

Now taking the partial differentials of R [6717] relative to v, r, we obtain,

[6718b]
$$\left(\frac{dR}{dv}\right)=-2m'.A^{(2)}.\sin.(2v-2v')\,; \qquad \left(\frac{dR}{dr}\right)=\frac{(\rho-\frac{1}{2}\varphi)}{r^4}+m'.\left(\frac{dA^{(2)}}{dr}\right).\cos.(2v-2v').$$

[6718c] But from [6089e, 6091] we have very nearly $\left(\frac{dR}{dr}\right)=\left(\frac{dR}{da}\right)$. This may be substituted in the last term of [6718b], which is not much affected by the excentricity on account of its smallness. Substituting these values in [6718a], we get,

$$d.(e.\cos.\varpi) = 4m'.ndt.aA^{(2)}.\sin.(2v-2v').\cos.v \qquad 1$$
$$-m'.ndt.a^2.\left(\frac{dA^{(2)}}{da}\right).\cos.(2v-2v').\sin.v \qquad 2$$
$$-ndt.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.\sin.(nt+\varepsilon) \qquad 3$$
$$+ndt.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.\left\{-\delta v.\cos.(nt+\varepsilon)+4.\frac{r\delta r}{a^2}.\sin.(nt+\varepsilon).\right\} \qquad 4$$
[6719]

We shall only notice, in the second member of this equation, the terms depending on the angle $nt-2n't+\varepsilon-2\varepsilon'$; observing that we have very nearly, as in [6131, 6132], [6720]

$$\frac{r\delta r}{a^2} = -\frac{m'.nF}{2.(2n-2n'-N)}.\cos.(2nt-2n't+2\varepsilon-2\varepsilon'); \qquad [6721]$$

$$\delta v = \frac{m'.nF}{2n-2n'-N}.\sin.(2nt-2n't+2\varepsilon-2\varepsilon'); \qquad [6722]$$

also as in [6130, 6151],

$$F = -4aA^{(2)} - a^2.\left(\frac{dA^{(2)}}{da}\right). \qquad [6723]$$

Then we shall have,*

$$d.(e.\cos.\varpi) = -\tfrac{1}{2}.m'.F.ndt.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon'). \qquad [6724]$$

$$d.(e.\cos.\varpi) = 4m'.ndt.aA^{(2)}.\sin.(2v-2v').\cos.v - m'.ndt.a^2.\left(\frac{dA^{(2)}}{da}\right).\cos.(2v-2v').\sin.v \qquad 1$$
$$-a^2.ndt.(\rho-\tfrac{1}{2}\varphi).\frac{\sin.v}{r^4}. \qquad 2$$
[6718d]

The first and second terms of the second member of this expression are the same as those in [6719 lines 1, 2]. The last term of [6718d] produces the terms in [6719 lines 3, 4]; as appears by substituting the values of r^2, v [6718]; which give by development, using [60] Int., the expressions [6718f]; also their product in [6718g], neglecting terms of the second order in δr, δv, [6718e]

$$r^{-4} = a^{-4}.\left(1-4.\frac{r\delta r}{a^2}\right); \qquad \sin.v = \sin.(nt+\varepsilon+\delta v) = \sin.(nt+\varepsilon)+\delta v.\cos.(nt+\varepsilon) \qquad [6718f]$$

$$\frac{\sin.v}{r^4} = \frac{1}{a^4}.\left\{\sin.(nt+\varepsilon)+\delta v.\cos.(nt+\varepsilon)-4.\frac{r\delta r}{a^2}.\sin.(nt+\varepsilon)\right\}. \qquad [6718g]$$

Multiplying [6718g] by $-a^2.ndt.(\rho-\frac{1}{2}\varphi)$, we get the value of the term [6718d line 2], as in [6719 lines 3, 4].

* (3450) If we substitute the values $\frac{r\delta r}{a^2}$, δv [6721, 6722] in [6719], and reduce the products by means of [17, 20] Int., retaining only the angle $v-2v'$, or $nt-2n't+\varepsilon-2\varepsilon'$, we shall get [6724]. For if we retain only this angle we have, [6724a]

[6724'] The mean longitude in the variable ellipsis is increased, in [6261], by terms depending on the angle $nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma$, which become sensible on account of the small divisors affecting them. We shall put,

[6725] $$\delta v = Q.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma);$$

[6726] $$\delta v' = Q'.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma);$$

for the terms of v, v' [6261, 6268], depending on this angle. We must increase $nt+\varepsilon$, $n't+\varepsilon'$, in the second member of [6724], by the [6727] quantities δv, $\delta v'$ [6725, 6726], respectively;* and we shall obtain, in the expression of $d.(e.\cos.\varpi)$ [6724], the following term;

[6728] $$d.(e.\cos.\varpi) = \tfrac{1}{4}m'.F.ndt.\left\{1-\frac{(0)}{2n-2n'-N)}\right\}.(2Q'-Q).\sin.(gt+\Gamma).$$

[6724b] $$\sin.(2v-2v').\cos.v = \tfrac{1}{2}.\sin.(v-2v') = \tfrac{1}{2}.\sin.(nt-2n't+\varepsilon-2\varepsilon');$$

[6724c] $$\cos.(2v-2v').\sin.v = -\tfrac{1}{2}.\sin.(v-2v') = -\tfrac{1}{2}.\sin.(nt-2n't+\varepsilon-2\varepsilon');$$

[6724d] $$-\delta v.\cos.(nt+\varepsilon) = -\tfrac{1}{2}.\frac{m'.nF}{(2n-2n'-N)}.\sin.(nt-2n't+\varepsilon-2\varepsilon');$$

[6724e] $$4.\frac{r\delta r}{a^2}.\sin.(nt+\varepsilon) = \tfrac{1}{2}.\frac{2m'.nF}{(2n-2n'-N)}.\sin.(nt-2n't+\varepsilon-2\varepsilon').$$

Substituting the values [6724b—e] in [6719], we obtain,

[6724f] $$d.(e.\cos\varpi) = -\tfrac{1}{2}m'.ndt.\left\{-4aA^{(2)}-a^2.\left(\frac{dA^{(2)}}{da}\right)\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon')$$
$$+\tfrac{1}{2}ndt.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}.\frac{m'.nF}{2n-2n'-N}.\sin.(nt-2n't+\varepsilon-2\varepsilon');$$

and if we use the value of F [6723], and (0) [6216], we get [6724].

[6728a] * (3451) This process of increasing the mean longitudes $nt+\varepsilon$, $n't+\varepsilon'$, by the inequalities δv, $\delta v'$ [6725, 6726] of a long period, is frequently used in this work; as, for [6728b] example, in [1232a]. Now if we put for brevity $q=(2Q'-Q).\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma)$, we shall find that the angle $nt-2n't+\varepsilon-2\varepsilon'$ will be *decreased*, by means of [6725, 6726], [6728c] by the quantity q; so that if we neglect quantities of the order q^2, and find the sine of the angle, in the first member of [6728d], by means of [60] Int., we shall get,

[6728d] $$\sin.(nt-2n't+\varepsilon-2\varepsilon'-q) = \sin.(nt-2n't+\varepsilon-2\varepsilon')-q.\cos.(nt-2n't+\varepsilon-2\varepsilon').$$

Substituting this in the second member of [6724], we find that the part, depending on q, produces in $d.(e.\cos.\varpi)$, the following term;

[6728e] $$d.(e.\cos.\varpi) = \tfrac{1}{2}m'.F.ndt.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.q.\cos.(nt-2n't+\varepsilon-2\varepsilon').$$

If we now re-substitute the value of q [6728b], and reduce the product by means of [18] [6728f] Int., retaining only the term depending on the slowly varying angle $gt+\Gamma$, it will become as in [6728]. The other part is neglected, because it does not increase so essentially by the integration.

We have, in [6236], by retaining only the term depending on the angle $gt+\Gamma$ [6728],

$$e.\cos.\varpi = -h.\cos.(gt+\Gamma).$$ [6729]

We must therefore increase the value of hg, which is given by the equation [6217], by the quantity,* [6729']

$$\tfrac{1}{4}m'.Fn.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.(2Q'-Q).$$ [6730]

This is equivalent to the subtraction of the expression [6730], *from the second member of the equation* [6217]. The term [6730']

$$\tfrac{1}{4}m'.Fn.\frac{(0)}{2n-2n'-N}.(2Q'-Q),$$ [6731]

in [6730], *is of the order of the cube of the disturbing force* [6731a]; *but on account of the great ellipticity of Jupiter and the nearly commensurable ratio of the mean motions of the two first satellites, the fraction* $\frac{(0)}{2n-2n'-N}$. *is very nearly equal to* $\frac{1}{6}$; *so that it becomes necessary to notice it.*† [6731'] [6732]

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* (3452) If we retain in [6236] only the term depending on the angle $gt+\Gamma$, which occurs in [6728], we shall have, as in [6729], $e.\cos.\varpi=-h.\cos.(gt+\Gamma)$; whose differential gives $d.(e.\cos.\varpi)=hg.dt.\sin.(gt+\Gamma)$. Now by noticing only the *first power* of the disturbing force, and the first power of the excentricity, we have obtained the equation [6217], which may be put under the form $hg=A$; denoting for brevity, by $-A$, all the terms of the second member of the equation [6217], excepting the first, hg. Substituting this in [6730b], we get $d.(e.\cos.\varpi)=A.dt.\sin.(gt+\Gamma)$. This expression of $d.(e.\cos.\varpi)$ is to be increased by the terms of a higher order, computed in [6728], which may be represented by $\delta A.dt.\sin.(gt+\Gamma)$; supposing the function [6730] to be represented by δA. Adding together the terms of $d.(e.\cos.\varpi)$ [6730b,f], we get for its corrected value, [6730a] [6730b] [6730c] [6730d] [6730e] [6730f] [6730g]

$$d.(e.\cos.\varpi)=(A+\delta A).dt.\sin.(gt+\Gamma).$$ [6730h]

Hence we see that the effect of these additional terms, depending on δA, is to change the coefficient A [6730f] into $A+\delta A$ [6730h]; or $-A$ into $-A-\delta A$; therefore the second member of the equation [6217], which is represented by $hg-A$ [6730e], must be augmented by the quantity $-\delta A$ [6730g]; or, in other words, the function [6730] must be *subtracted* from the second member of [6217], according to the direction in [6730']. [6730i] [6730k]

† (3453) The quantities m', (0), $2Q'-Q$, are each of the same order as the disturbing force, as is evident from [6216, 6261, 6268, 6725, 6726]; hence the function [6731] is of the order of the *cube* of the disturbing force, as in [6731']. The values of [6731a]

In like manner, by noticing in R' only the terms which are contained in the following expression [6733], we obtain in $d.(e'.\cos.\varpi')$, the terms in [6734];*

[6733] $$R' = -\frac{(\rho-\frac{1}{2}\varphi)}{3r'^3} + m''.A'^{(2)}.\cos.(2v'-2v'')+m.A_{\prime}^{(1)}.\cos.(v-v');$$

[6734]
$$d.(e'.\cos.\varpi') = -2m.n'dt.a'A_{\prime}^{(1)}.\sin.(v-v').\cos.v' \dagger \qquad 1$$
$$-m.n'dt.a'^2.\left(\frac{dA_{\prime}^{(1)}}{da'}\right).\cos.(v-v').\sin.v' \qquad 2$$
$$+4m''.n'dt.a'.A'^{(2)}.\sin.(2v'-2v'').\cos.v' \qquad 3$$
$$-m''.n'dt.a'^2.\left(\frac{dA'^{(2)}}{da'}\right).\cos.(2v'-2v'').\sin.v' \qquad 4$$
$$-n'dt.\frac{(\rho-\frac{1}{2}\varphi)}{a'^2}.\sin.(n't+\varepsilon') \qquad 5$$
$$+n'dt.\frac{(\rho-\frac{1}{2}\varphi)}{a'^2}.\left\{-\delta v'.\cos.(n't+\varepsilon')+4.\frac{r'\delta r'}{a'^2}.\sin.(n't+\varepsilon')\right\}. \qquad 6$$

[6731b] n, n', N [6025b, e] give $2n-2n'-N=n.0{,}004323$. Dividing the value of (0) [6216]
[6731c] by this last expression, we get $\frac{(0)}{2n-2n'-N}=\frac{(\rho-\frac{1}{2}\varphi)}{0{,}004323.a^2}$; substituting $a=5{,}698491$
[6731d] [6797], and $\rho-\frac{1}{2}\varphi=0{,}0219013$ [7152], it becomes nearly equal to $\frac{1}{6}$, as in [6732]; so that, on account of its magnitude, the author has noticed it.

* (3454) The terms of R' [6733, 6565], depending on the angles $v-v'$, $2v'-2v''$,
[6733a] correspond, according to the notation in [6134, 6146], to the terms of [6090] depending on the angles $v-v'$, $2v-2v'$; and from a little consideration it will be evident, that it is
[6733b] not necessary to notice any other angles, when computing the terms corresponding to the angle mentioned in [6724']. Again, the term of [6733], depending on the ellipticity of
[6733c] Jupiter, is the same as that in [6717], changing r into r', to correspond to the present case.

† (3455) The expression of $d.(e.\cos.\varpi)$ [6718a], gives that of $d.(e'.\cos.\varpi')$, by
[6734a] changing the elements of m into those of m', and R [6717] into R' [6733]; hence we get,

[6734b] $$d.(e'.\cos.\varpi')=a'.n'dt.\left\{-2.\left(\frac{dR'}{dv'}\right).\cos.v'-a'.\left(\frac{dR'}{dr'}\right).\sin.v'\right\};$$

[6734c] and by substituting the value of R' [6733], we obtain [6734]. This process of substitution may be considerably shortened, by derivation from [6719]; observing that the first and second terms of R [6717], produce the first and second terms of R' [6733];
[6734d] by changing the elements of m into those of m'; and the elements of m' into those of m''. Now making these changes in [6719], we get the terms in [6734 lines 3, 4, 5, 6].
[6734e] The remaining term of R' [6733], is $m.A_{\prime}^{(1)}.\cos.(v-v')$; and by substituting it for R', in [6734b], we get the terms in [6734 lines 1, 2].

Now we have, as in [6139, 6148; 6140, 6149], *

$$\frac{r'\delta r'}{a'^2}=-\frac{m.n'.G}{2.(n-n'-N')}.\cos.(nt-n't+\varepsilon-\varepsilon')-\frac{m''.n'.F'}{2.(n-n'-N')}.\cos.(2n't-2n''t+2\varepsilon'-2\varepsilon'');\quad [6735]$$

$$\delta v'=\frac{m.n'.G}{n-n'-N'}.\sin.(nt-n't+\varepsilon-\varepsilon')+\frac{m''.n'.F'}{n-n'-N'}.\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon'').\quad [6736]$$

We have in [6137, 6138], very nearly,†

$$G=2a'.A_{,}^{(1)}-a'^2.\left(\frac{dA_{,}^{(1)}}{da'}\right);\quad [6737]$$

hence we find, by retaining only the terms which depend on the angles $nt-2n't+\varepsilon-2\varepsilon'$, $n't-2n''t+\varepsilon'-2\varepsilon''$, ‡

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* (3456) The sum of the terms [6139, 6148], gives [6735], by changing the divisor $2n'-2n''-N'$ into $n-n'-N'$ [6154]. In like manner, the sum of [6140, 6149] gives [6736]. [6735a]

† (3457) Changing the divisor $n-n'$ into n' [6138], we find that the expression of G [6137] becomes as in [6737]. [6737a]

‡ (3458) If we retain only the angles $v-2v'$, $v'-2v''$, or the corresponding mean values $nt-2n't+\varepsilon-2\varepsilon'$, $n't-2n''t+\varepsilon'-2\varepsilon''$, we may substitute, in [6734], the following expressions, which are easily deduced from [18, 19] Int.; [6738a]

$$\sin.(v-v').\cos.v' = \tfrac{1}{2}.\sin.(v-2v')\quad [6738b]$$

$$\cos.(v-v').\sin.v' = -\tfrac{1}{2}.\sin.(v-2v')\quad [6738c]$$

$$\sin.(2v'-2v'').\cos.v' = \tfrac{1}{2}.\sin.(v'-2v'')\quad [6738d]$$

$$\cos.(2v'-2v'').\sin.v' = -\tfrac{1}{2}.\sin.(v'-2v'').\quad [6738e]$$

Substituting these values in [6734 lines 1, 2, 3, 4] respectively, we obtain the following terms;

$$-\tfrac{1}{2}m.n'dt.\left\{2a'A_{,}^{(1)}-a'^2.\left(\frac{dA_{,}^{(1)}}{da'}\right)\right\}.\sin.(v-2v')-\tfrac{1}{2}m''.n'dt.\left\{-4a'A'^{(2)}-a'^2.\left(\frac{dA'^{(2)}}{da'}\right)\right\}.\sin.(v'-2v'').\quad [6738f]$$

If we substitute in this expression the values of G, F' [6737, 6600], and the mean values of the angles [6738a], we shall obtain respectively the terms in [6738 lines 1, 2], which are independent of the quantity (1). We shall now compute the remaining terms of [6734 lines 5, 6], depending on $(\rho-\frac{1}{2}\varphi)$; and if we substitute $(1)=\frac{(\rho-\frac{1}{2}\varphi)}{a'^2}.n'$ [6217c], neglecting $-(1).dt.\sin.(n't+\varepsilon')$ [6734 line 5], which produces no angle of the form $v-2v'$ or $v'-2v''$, we shall find that the remaining terms [6734 line 6] become, [6738g] [6738h]

$$(1).dt.\left\{-\delta v'.\cos.(n't+\varepsilon')+4.\frac{r'\delta r'}{a'^2}.\sin.(n't+\varepsilon')\right\}.\quad [6738i]$$

Substituting the values of $\delta v'$, $\frac{r'\delta r'}{a'^2}$ [6735, 6736], we get the following expression of the terms of [6734], depending on (1);

[6738]
$$d.(e'.\cos.\varpi') = -\tfrac{1}{2} m.n'dt.G.\left\{1-\frac{(1)}{n-n'-N'}\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon') \qquad 1$$
$$-\tfrac{1}{2} m''.n'dt.F'.\left\{1-\frac{(1)}{n-n'-N'}\right\}.\sin.(n't-2n''t+\varepsilon'-2\varepsilon''). \qquad 2$$

We shall put the term of $\delta v''$, which we have found in [6269], under the following form;

[6739]
$$\delta v'' = Q''.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

Now we have, in [6627],

[6740]
$$nt-2n't+\varepsilon-2\varepsilon' = \pi + n't-2n''t+\varepsilon'-2\varepsilon'';$$

hence we obtain, in $d.(e'.\cos.\varpi')$, the following terms;*

[6741]
$$d.(e'.\cos.\varpi') = \tfrac{1}{4} m.n'dt.\left\{1-\frac{(1)}{n-n'-N'}\right\}.G.(2Q'-Q).\sin.(gt+\Gamma) \qquad 1$$
$$-\tfrac{1}{4} m''.n'dt.\left\{1-\frac{(1)}{n-n'-N'}\right\}.F'.(2Q''-Q').\sin.(gt+\Gamma); \qquad 2$$

[6738k] $\frac{-m.n'dt.(1)}{2.(n-n'-N')}.G.\{2.\cos.(n't+\varepsilon').\sin.(nt-n't+\varepsilon-\varepsilon')+4.\sin.(n't+\varepsilon').\cos.(nt-n't+\varepsilon-\varepsilon')\}$

[6738l] $\frac{-m''.n'dt.(1)}{2.(n-n'-N')}.F'.\{2.\cos.(n't+\varepsilon').\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon'')+4.\sin.(n't+\varepsilon').\cos.(2n't-2n''t+2\varepsilon'-2\varepsilon'')\}.$

Now if we retain only the angles mentioned in [6738a], we shall have, by means of [18, 19] Int.,

[6738m] $2.\cos.(n't+\varepsilon').\sin.(nt-n't+\varepsilon-\varepsilon') = \sin.(nt-2n't+\varepsilon-2\varepsilon')$;

[6738n] $4.\sin.(n't+\varepsilon').\cos.(nt-n't+\varepsilon-\varepsilon') = -2.\sin.(nt-2n't+\varepsilon-2\varepsilon')$;

[6738o] $2.\cos.(n't+\varepsilon').\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon'') = \sin.(n't-2n''t+\varepsilon'-2\varepsilon'')$;

[6738p] $4.\sin.(n't+\varepsilon').\cos.(2n't-2n''t+2\varepsilon'-2\varepsilon'') = -2.\sin.(n't-2n''t+\varepsilon'-2\varepsilon'')$.

Substituting [6738m—p] in [6738k, l], and making some slight reductions, it becomes,

[6738q] $-\frac{m.n'dt.(1)}{2.(n-n'-N')}.G.\{-\sin.(nt-2n't+\varepsilon-2\varepsilon')\}-\frac{m''.n'dt.(1)}{2.(n-n'-N')}.F'.\{-\sin.(n't-2n''t+\varepsilon'-2\varepsilon'')\}$;

which is the same as the part of [6738], depending on (1).

[6741a] * (3459) We shall suppose, as in [6728a, &c.], $nt+\varepsilon$, $n't+\varepsilon'$, $n''t+\varepsilon''$, to be increased by the quantities δv, $\delta v'$, $\delta v''$ [6725, 6726, 6739] respectively; and shall use,
[6741b] for brevity, the value of q [6728b], and the similar value of q' [6741c], which is easily reduced to the form [6741d], by the substitution of [6740];

[6741c] $q' = \quad (2Q''-Q').\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma)$;

[6741d] $q' = -(2Q''-Q').\sin.(n't-2n''t+\varepsilon'-2\varepsilon''+gt+\Gamma)$.

[6741e] Then we see, as in [6728c], that the angle $nt-2n't+\varepsilon-2\varepsilon'$ is *increased* by the quantity
[6741f] $-q$; and its sine is increased by the quantity $-q.\cos.(nt-2n't+\varepsilon-2\varepsilon')$ [6728d]. In like manner, the angle $n't-2n''t+\varepsilon'-2\varepsilon''$ is *increased* by the quantity $-q'$ [6741c],
[6741g] and its sine is *increased* by the quantity $-q'.\cos.(n't-2n''t+\varepsilon'-2\varepsilon'')$, as is evident by the

we must therefore *add*, to the second member of the equation [6220], the quantity,* [6741']

$$-\tfrac{1}{4}m.n'.\left\{1-\frac{(1)}{n-n'-N'}\right\}.G.(2Q'-Q) \qquad 1$$
$$+\tfrac{1}{4}m''.n'.\left\{1-\frac{(1)}{n-n'-N'}\right\}.F''.(2Q''-Q'). \qquad 2 \qquad [6742]$$

In like manner we shall find that we must *add* to the second member of the equation [6221], the term,† [6742']

$$\tfrac{1}{4}m'.n''.\left\{1-\frac{(2)}{n'-n''-N''}\right\}.G'.(2Q''-Q'). \qquad [6743]$$

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method of reduction in [6728d]. Now if we substitute the values of $q,\ q'$ [6728b, 6741d] in the first members of [6741i,k], and reduce by [18, 19] Int., retaining only the terms depending on the angle $gt+\Gamma$, we shall get the second members of the expressions [6741i,k], [6741h]

$$-q.\cos.(nt-2n't+\varepsilon-2\varepsilon')=-\tfrac{1}{2}.(2Q'-Q).\sin.(gt+\Gamma)\,; \qquad [6741i]$$
$$-q'.\cos.(n't-2n''t+\varepsilon'-2\varepsilon'')=\ \tfrac{1}{2}.(2Q''-Q').\sin.(gt+\Gamma). \qquad [6741k]$$

Substituting in [6738 line 1] the increment [6741f,i], and in [6738 line 2] the increment [6741g,k], we get the terms in the second member of [6741 lines 1, 2] respectively.

* (3460) We have found that the second member of [6728], being divided by $dt.\sin.(gt+\Gamma)$, gives the function [6730], which is to be subtracted from the second member of the equation [6217], as in [6730']. In like manner, if we divide the second member of [6741] by $dt.\sin.(gt+\Gamma)$, we shall get the quantity which is to be subtracted from the second member of [6220]; or, by changing the signs, we get the quantities to be added to the second member of [6220], as in [6742]. [6742a] [6742b]

† (3461) We may deduce $d.(e''.\cos.\varpi'')$ from $d.(e'.\cos.\varpi')$ [6738], by changing (1) into (2), as in [6217c], and increasing the accents of the elements of $m,\ m',\ m''$, by unity; it being evident that m' is acted upon by $m,\ m''$; as m'' is by $m',\ m'''$; and as the term depending on the angle $n''t-2n'''t+\varepsilon''-2\varepsilon'''$, is not increased by integration, it may be neglected, and then we shall get from [6738], [6743a] [6743b]

$$d.(e''.\cos.\varpi'')=-\tfrac{1}{2}m'.n''dt.G'.\left\{1-\frac{(2)}{n'-n''-N''}\right\}.\sin.(n't-2n''t+\varepsilon'-2\varepsilon'')\,; \qquad [6743c]$$

observing that this change makes G [6137] become G' [6163]. Now by following the calculation in [6738—6742], we get the same result as in [6743]; but this is more easily obtained by derivation. For if we put in [6738], $G=0$, $F'=\frac{m'.n''}{m''.n'}.G'$, and change $\frac{(1)}{n-n'-N'}$ into $\frac{(2)}{n'-n''-N''}$, it becomes identical with [6743c]; and the same changes being made in [6742], which is deduced from [6738], it becomes as in [6743]. [6743d] [6743e]

Hence the equations [6217, 6220, 6221, 6222] become respectively,*

[6744]
$$0=h.\left\{g-(0)-\boxed{0}-(0,1)-(0,2)-(0,3)\right\} \quad 1$$
$$+\boxed{0,1}.h'+\boxed{0,2}.h''+\boxed{0,3}.h'''-\tfrac{1}{4}m'.n.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.F.(2Q'-Q); \quad 2$$

[6745]
$$0=h'.\left\{g-(1)-\boxed{1}-(1,0)-(1,2)-(1,3)\right\} \quad 1$$
$$+\boxed{1,0}.h+\boxed{1,2}.h''+\boxed{1,3}.h'''-\tfrac{1}{4}m.n'.\left\{1-\frac{(1)}{n-n'-N'}\right\}.G.(2Q'-Q) \quad 2$$
$$+\tfrac{1}{4}m''.n'.\left\{1-\frac{(1)}{n-n'-N'}\right\}.F'.(2Q''-Q'); \quad (V)\ 3$$

[6746]
$$0=h''.\left\{g-(2)-\boxed{2}-(2,0)-(2,1)-(2,3)\right\} \quad 1$$
$$+\boxed{2,0}.h+\boxed{2,1}.h'+\boxed{2,3}.h'''+\tfrac{1}{4}m'.n''.\left\{1-\frac{(2)}{n'-n''-N''}\right\}.G'.(2Q''-Q'); \quad 2$$

[6747]
$$0=h'''.\left\{g-(3)-\boxed{3}-(3,0)-(3,1)-(3,2)\right\} \quad 1$$
$$+\boxed{3,0}.h+\boxed{3,1}.h'+\boxed{3,2}.h''. \quad 2$$

It may be thought that the equations [6337—6341], which correspond to the inclinations and nodes of the orbits, can acquire, in like manner, some sensible terms depending on the square of the disturbing force; but it is
[6748] easy to satisfy ourselves that this is not the case, by considering the differential equations of these motions, which are given in [1331].†

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* (3462) Subtracting, from the equation [6217], the expression [6730], according to
[6744a] the directions in [6730′], it becomes as in [6744]. Adding, as in [6741′], the function [6742] to the second member of [6220], gives [6745]. The function [6743] being added to the second member of [6221], as in [6742′], gives [6746]. Lastly, [6757] is the same as [6222].

† (3463) If we neglect the terms of the equations [1331], depending on s^2, and substitute, as in [1, 6, 31] Int.,

[6748a] $\sin.(v-\theta).\cos.(v-\theta)=\tfrac{1}{2}.\sin.(2v-2\theta)$; $\sin.^2(v-\theta)=\tfrac{1}{2}-\tfrac{1}{2}.\cos.(2v-2\theta)$; $\cos.^2(v-\theta)=\tfrac{1}{2}+\tfrac{1}{2}.\cos.(2v-2\theta)$;

we shall get,

[6748b]
$$d.\text{tang}.\varphi=\frac{dt.\text{tang}.\varphi}{2c}.\left\{r.\left(\frac{dR}{dr}\right).\sin.(2v-2\theta)+\left(\frac{dR}{dv}\right)+\left(\frac{dR}{dv}\right).\cos.(2v-2\theta)\right\}-\frac{dt}{c}.\cos.(v-\theta).\left(\frac{dR}{ds}\right);$$

[6748c]
$$d\theta.\text{tang}.\varphi=\frac{dt.\text{tang}.\varphi}{2c}.\left\{r.\left(\frac{dR}{dr}\right)-r.\left(\frac{dR}{dr}\right).\cos.(2v-2\theta)+\left(\frac{dR}{dv}\right).\sin.(2v-2\theta)\right\}-\frac{dt}{c}.\sin.(v-\theta).\left(\frac{dR}{ds}\right).$$

18. *The square of the principal inequality of the satellites, which we have developed in* [6126—6193], *can produce a sensible term, which we shall now investigate. This inequality may be supposed to correspond to a variable ellipsis, and in this case it affects the excentricity and the perijove of the orbit.* [6748′]
Now since $\delta v = (\text{I}).\sin.(2nt-2n't+2\varepsilon-2\varepsilon')$ [6172], denotes this inequality [6749]
in the motion of the first satellite, and $2e.\sin.(nt+\varepsilon-\varpi)$ [6240a], the first [6750]
term of the elliptical part of v; *if we represent by* $\delta.(e.\sin.\varpi)$, *and* $\delta.(e.\cos.\varpi)$, *the variations of* $e.\sin.\varpi$ *and* $e.\cos.\varpi$, *depending on the disturbing force*, we shall obtain in v the inequality,* [6751]

$$\delta v = 2\delta.(e.\cos.\varpi).\sin.(nt+\varepsilon)-2\delta.(e.\sin.\varpi).\cos.(nt+\varepsilon). \qquad [6752]$$

We shall put the inequality δv [6749], under the following form;

$$\begin{aligned}\delta v = (\text{I}).\cos.(nt-2n't+\varepsilon-2\varepsilon').\sin.(nt+\varepsilon) \quad & 1\\ +(\text{I}).\sin.(nt-2n't+\varepsilon-2\varepsilon').\cos.(nt+\varepsilon). \quad & 2\end{aligned} \qquad [6753]$$

If we substitute, in these expressions, the values of $\left(\frac{dR}{dr}\right)$, $\left(\frac{dR}{dv}\right)$ [6718b], and reduce the products by means of the formulas [17—20] Int., we shall find that the resulting terms are connected with angles of the form $2v-2v'\pm(2v-2\theta)$, or $2v-2\theta$, and that they do [6748d]
not increase by the integrations like those of the forms $2v-2v'\pm v$, or rather $v-2v'$, which are retained in [6724b—e]; therefore we may neglect them; and for similar reasons we may neglect other terms arising from the partial differentials of R, relative to r and v. The terms of [6748b, c], depending on $\left(\frac{dR}{ds}\right)$, may also be neglected; for the [6748e]
expression of $\left(\frac{dR}{ds}\right)$, deduced from [6297], is of the *small* order s, s'; and if we substitute the values of $s=\text{tang}.\varphi.\sin.(v-\theta)$, $s'=\text{tang}.\varphi'.\sin.(v'-\theta')$ [1330″], we shall [6748f]
easily perceive that there are no terms produced, of the proposed order, in [6748b, c], which can become sensible by the integrations. Similar remarks may be made relative to the [6748g]
terms of the order of the square of the disturbing force for the other satellites m', m'', m'''. [6748h]

* (3464) From [22] Int. we get, $\sin.(nt+\varepsilon-\varpi)=\cos.\varpi.\sin.(nt+\varepsilon)-\sin.\varpi.\cos.(nt+\varepsilon)$; multiplying this by $2e$, we obtain the term of v [6750], under the following form; [6752a]

$$2.(e.\cos.\varpi).\sin.(nt+\varepsilon)-2.(e.\sin.\varpi).\cos.(nt+\varepsilon). \qquad [6752b]$$

Substituting the parts of $\delta.(e.\cos.\varpi)$, $\delta.(e.\sin.\varpi)$, depending on the disturbing force, it becomes as in [6752]. Putting in the formula [21] Int., $a=nt+\varepsilon$, $b=nt-2n't+\varepsilon-2\varepsilon'$, [6752$c$]
we get an expression of $\sin.(2nt-2n't+2\varepsilon-2\varepsilon')$; and by multiplying it by (I), we get [6752d]
the value of δv [6749], as in [6753]. Comparing together the coefficients of $\cos.(nt+\varepsilon)$, in the equivalent expressions [6752, 6753], we get [6754]. In like manner the coefficients [6752e]
of $\sin.(nt+\varepsilon)$, being put equal to each other, give [6755].

Comparing together the two expressions [6752, 6753], we get,

[6754] $$2\delta.(e.\sin.\varpi) = -(1).\sin.(nt-2n't+\varepsilon-2\varepsilon');$$

[6755] $$2\delta.(e.\cos.\varpi) = \quad (1).\cos.(nt-2n't+\varepsilon-2\varepsilon').$$

These values are the same as those which we have obtained in the preceding article. For we have found in [6724],

[6756] $$d.(e.\cos.\varpi) = -\tfrac{1}{2}m'.F.ndt.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.\sin.(nt-2n't+\varepsilon-2\varepsilon').$$

Hence we obtain, by integration, the corresponding expression of $\delta.(e.\cos.\varpi)$ [6751], or,

[6757] $$2\delta.(e.\cos.\varpi) = \frac{m'.F.n}{n-2n'}.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon').$$

[6758] If we now observe that we have very nearly* $N=n-(0)$ [6758b], and by [6132, 6172], we have,

[6759] $$(1) = \frac{m'.F.n}{2n-2n'-N};$$

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* (3465) We may neglect the terms of N^2 [6124] depending on M, $A^{(0)}$, in comparison with that depending on $(\rho-\frac{1}{2}\varphi)$; as is evident from the estimated values

[6758a] [6485a, &c.]; hence [6124] becomes $N^2=n^2.\left\{1-2.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}\right\}$. Its square root gives

[6758b] very nearly, by using (0) [6216], $N=n-\frac{(\rho-\frac{1}{2}\varphi).n}{a^2}=n-(0)$, as in [6758]. Moreover

[6758c] by comparing the expressions of δv [6132, 6172], we obtain the value of (1) [6759]. Substituting this in [6755], we get [6758d]; making successive reductions, by changing the forms of the factors, and using $n-N=(0)$ [6758], we finally obtain [6758f], which is the same as [6757].

[6758d] $$2.\delta.(e.\cos.\varpi) = \frac{m'.F.n}{2n-2n'-N}.\cos.(nt-2n't+\varepsilon-2\varepsilon')$$

[6758e] $$=\frac{m'.F.n}{n-2n'}.\left\{1-\frac{(n-N)}{2n-2n'-N}\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon')$$

[6758f] $$=\frac{m'.F.n}{n-2n'}.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon').$$

We may, in a similar manner, prove the identity of the expressions of $2.\delta.(e.\sin.\varpi)$, deduced from [6715, 6754]. For by inspection we see, that [6715] may be derived from

[6758g] [6714], by changing v, ϖ, into $v-100^{\circ}$, $\varpi-100^{\circ}$, respectively, in the terms which are independent of R; and the same changes are to be made in [6719], leaving the angle $2v-2v'$ [6719 lines 1, 2], or $2nt-2n't+2\varepsilon-2\varepsilon'$, unaltered, because it is derived from

[6758h] R; so that if we put the angle $nt-2n't+\varepsilon-2\varepsilon'$, which occurs in [6724], under the form $(2nt-2n't+2\varepsilon-2\varepsilon')-(nt+\varepsilon)$, we must consider $(2nt-2n't+2\varepsilon-2\varepsilon')$ as invariable;

[6758i] but $-(nt+\varepsilon)$ is to be changed into $-(nt+\varepsilon-100^{\circ}) = -(nt+\varepsilon)+100^{\circ}$; or, in other

we shall find that this second value of $2.\delta.(e.\cos.\varpi)$ agrees with the preceding. Now the elliptical expression of v [669], contains the following term ;* [6760]

$$\tfrac{5}{4}e^2.\sin.(2nt+2\varepsilon-2\varpi) = \tfrac{5}{4}.(e^2.\cos.^2\varpi-e^2.\sin.^2\varpi).\sin.(2nt+2\varepsilon) \qquad 1$$
$$-\tfrac{5}{2}e^2.\sin.\varpi.\cos.\varpi.\cos.(2nt+2\varepsilon). \qquad 2 \qquad [6761]$$

Changing $e.\sin.\varpi$ into $e.\sin.\varpi+\delta.(e.\sin.\varpi)$, and $e.\cos.\varpi$ into $e.\cos.\varpi+\delta.(e.\cos.\varpi)$, we find that the expression of v contains the term, [6762]

$$\delta v = \tfrac{5}{4}.\{[\delta.(e.\cos.\varpi)]^2-[\delta.(e.\sin.\varpi)]^2\}.\sin.(2nt+2\varepsilon) \qquad 1$$
$$-\tfrac{5}{2}.\delta.(e.\cos.\varpi).\delta.(e.\sin.\varpi).\cos.(2nt+2\varepsilon). \qquad 2 \qquad [6763]$$

Substituting for $\delta.(e.\cos.\varpi)$, $\delta.(e.\sin.\varpi)$, their preceding values, there will arise in v the inequality,†

$$\delta v = \tfrac{5}{16}.(\,\text{I}\,)^2.\sin.(4nt-4n't+4\varepsilon-4\varepsilon'); \qquad [6764]$$

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words, we must increase the angle $nt-2n't+\varepsilon-2\varepsilon'$ [6724] by 100°; or change *sin.* into *cos.* Hence we get from [6724],

$$d.(e.\sin.\varpi) = -\tfrac{1}{2}m'.F.ndt.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.\cos.(nt-2n't+\varepsilon-2\varepsilon'). \qquad [6758k]$$

Now by comparing the second members of [6758d,f], we get $1-\frac{(0)}{2n-2n'-N}=\frac{n-2n'}{2n-2n'-N}$; [6758$l$]

substituting this in [6758k], and then taking the integral, we get,

$$\delta.(e.\sin.\varpi) = -\tfrac{1}{2}.\frac{m'.F.n}{2n-2n'-N}.\sin.(nt-2n't+\varepsilon-2\varepsilon') \qquad [6758m]$$
$$= -\tfrac{1}{2}.(\text{I}).\sin.(nt-2n't+\varepsilon-2\varepsilon'). \quad [6759] \qquad [6758n]$$

Multiplying this last expression by 2, it becomes as in [6754].

* (3466) We obtain by development, as in [22, 32, 31] Int.,

$$\sin.\{(2nt+2\varepsilon)-2\varpi\} = \cos.2\varpi.\sin.(2nt+2\varepsilon)-\sin.2\varpi.\cos.(2nt+2\varepsilon) \qquad [6760a]$$
$$= (\cos.^2\varpi-\sin.^2\varpi).\sin.(2nt+2\varepsilon)-2.\sin.\varpi.\cos.\varpi.\cos.(2nt+2\varepsilon). \qquad [6760b]$$

Multiplying this by $\frac{5}{4}e^2$, we get [6761]. Substituting the augmented values of $e.\sin.\varpi$, $e.\cos.\varpi$ [6762], and neglecting terms of the order $(e.\sin.\varpi).\delta.(e.\sin.\varpi)$, we get [6763]. [6760c] These neglected terms are much smaller than those which are retained, because $2e.\sin.\varpi$ is less than $2\delta.(e.\sin.\varpi)$, the former being of the order $40''$ [6057f], and the latter of [6760d] the order $5050''$, [7513 line 5, 6754, 6172]. Similar remarks may be made relative to the corresponding terms in [7467 lines 1, 4], in the value of v'. Those of v'' are [6760e] neglected from their smallness in [6766'].

† (3467) Putting for brevity $T = nt-2n't+\varepsilon-2\varepsilon'$, we obtain from [6754, 6755], by using [31, 32] Int., [6764a]

we shall find in the same manner, in v', the inequality [6765i]; *

[6765] $$\delta v'=\tfrac{5}{16}\cdot(\mathrm{II})^2.\sin.(2nt-2n't+2\varepsilon-2\varepsilon');$$

and in v'', the inequality,†

[6766] $$\delta v''=\tfrac{5}{16}.(\mathrm{III})^2.\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon'').$$

[6764b] $$[\delta.(e.\cos.\varpi)]^2-[\delta.(e.\sin.\varpi)]^2=\tfrac{1}{4}.(\mathrm{I})^2.\{\cos.^2T-\sin.^2T\}=\tfrac{1}{4}.(\mathrm{I})^2.\cos.2T;$$

[6764c] $$\delta.(e.\sin.\varpi).\delta.(e.\cos.\varpi)=-\tfrac{1}{4}.(\mathrm{I})^2.\sin.T.\cos.T=-\tfrac{1}{8}.(\mathrm{I})^2.\sin.2T.$$

Substituting these in [6763] and reducing by means of [21] Int., we get, by successive operations, the expression [6764e], being the same as in [6764];

[6764d] $$\delta v=\tfrac{5}{16}.(\mathrm{I})^2.\{\cos.2T.\sin.(2nt+2\varepsilon)+\sin.2T.\cos.(2nt+2\varepsilon)\}$$

[6764e] $$=\tfrac{5}{16}.(\mathrm{I})^2.\sin.(2nt+2\varepsilon+2T)=\tfrac{5}{16}.(\mathrm{I})^2.\sin.(4nt-4n't+4\varepsilon-4\varepsilon').$$

* (3468) Using the abridged symbol T [6764a], we get, from [6173], the first of the expressions of $\delta v'$ [6765a]. Developing this by means of [21] Int., we get [6765b].

[6765a] $$\delta v'=-(\mathrm{II}).\sin.(T+n't+\varepsilon')$$

[6765b] $$=-(\mathrm{II}).\sin.T.\cos.(n't+\varepsilon')-(\mathrm{II}).\cos.T.\sin.(n't+\varepsilon').$$

[6765c] Now the elliptical value of v', similar to that of v [6750], is $2e'.\sin.(n't+\varepsilon'-\varpi')$; and if we develop it, as in [6752b], it becomes,

[6765d] $$2.(e'.\cos.\varpi').\sin.(n't+\varepsilon')-2.(e'.\sin.\varpi').\cos.(n't+\varepsilon');$$

and the variations $\delta.(e'.\sin.\varpi')$, $\delta.(e'.\cos.\varpi')$, produce in this term of v', the following expression, which is similar to [6752];

[6765e] $$\delta v'=2.\delta.(e'.\cos.\varpi').\sin.(n't+\varepsilon')-2.\delta.(e'.\sin.\varpi').\cos.(n't+\varepsilon').$$

Putting the two expressions of R' [6765b, e] equal to each other, and then comparing separately the terms which are multiplied by $\sin.(n't+\varepsilon')$, $\cos.(n't+\varepsilon')$, we obtain the two following equations;

[6765f] $$2.\delta.(e'.\cos.\varpi')=-(\mathrm{II}).\cos.T; \qquad 2.\delta.(e'.\sin.\varpi')=(\mathrm{II}).\sin.T.$$

From these values we get, in like manner as in [6764b, c],

[6765g] $$[\delta.(e'.\cos.\varpi')]^2-[\delta.(e'.\sin.\varpi')]^2=\tfrac{1}{4}.(\mathrm{II})^2.\cos.2T;\quad \delta.(e'.\sin.\varpi').\delta.(e'.\cos.\varpi')=-\tfrac{1}{8}.(\mathrm{II})^2.\sin.2T.$$

Now accenting the letters v, n, ε, ϖ, in [6763], and then substituting their reduced values [6765g], we get the expression [6765h], which is similar to [6764d]; and by successive reductions, using [21] Int., we get [6765i], corresponding to [6765].

[6765h] $$\delta v'=\tfrac{5}{16}.(\mathrm{II})^2.\{\cos 2T.\sin.(2n't+2\varepsilon')+\sin.2T.\cos.(2n't+2\varepsilon')\}$$

[6765i] $$=\tfrac{5}{16}.(\mathrm{II})^2.\sin.(2n't+2\varepsilon'+2T)=\tfrac{5}{16}.(\mathrm{II})^2.\sin.(2nt-2n't+2\varepsilon-2\varepsilon').$$

† (3469) Putting for a moment for brevity $T'=n't-2n''t+\varepsilon'-2\varepsilon''$, we find that the value of $\delta v''$ [6174] becomes, as in [6766a]; and by developing, as in the preceding notes, it becomes as in [6766b].

[6766a] $$\delta v''=-(\mathrm{III}).\sin.(T'+n''t+\varepsilon'')$$

[6766b] $$=-(\mathrm{III}).\sin.T'.\cos.(n''t+\varepsilon'')-(\mathrm{III}).\cos.T'.\sin.(n''t+\varepsilon'').$$

Comparing this with the expression [6766c], which is similar to [6752], and arises from the greatest of the elliptical terms of v'',

These inequalities are very small; that which corresponds to v' [6765 or 6947], *is the only one deserving of notice.* [6766']

The square of the disturbing force introduces also, into the coefficients of the chief inequality of each of the three first satellites, some quantities, which increase considerably by the divisor of the order $(n-2n')^2$, which affects them. We have noticed, in § 4, the sensible part of those quantities which depend on the product of the masses of the satellites by the ellipticity of Jupiter's spheroid, in determining with precision the values of N, N', N'' [6128, &c.].* The other parts are so small, that they may be neglected without any sensible error. [6767] [6767']

19. We have determined in [6530, &c.] the secular equations of the motions of the satellites of Jupiter, and we have observed that the only part of these equations which can finally become sensible, is that which depends on the secular variations in the elements of Jupiter's orbit, and in the position of its equator [6525]. *If the part of* $\mathrm{d}R$, *depending on the square of the disturbing force, contain any terms of the form* $\mathrm{d}R = Q.H^2.dt$, Q *being* [6768] [6769]

$$\delta v'' = 2.\delta.(e''.\cos.\varpi'').\sin.(n''t+\varepsilon'') - 2.\delta.(e''.\sin.\varpi'').\cos.(n''t+\varepsilon'')\,; \qquad [6766c]$$

we get the two equations [6766d], which are similar to [6765f]; from these we easily deduce the expressions [6766e], corresponding to [6765g];

$$2.\delta.(e''.\cos.\varpi'') = -(\text{III}).\cos.T'\,; \qquad 2.\delta.(e''.\sin.\varpi'') = (\text{III}).\sin.T'\,; \qquad [6766d]$$

$$[\delta.(e''.\cos.\varpi'')]^2 - [\delta.(e''.\sin.\varpi'')]^2 = \tfrac{1}{4}.(\text{III})^2.\cos.2T'\,;$$
$$\delta.(e''.\sin.\varpi'').\delta.(e''.\cos.\varpi'') = -\tfrac{1}{8}.(\text{III})^2.\sin.2T'. \qquad [6766e]$$

Now putting two accents on v, n, ε, ϖ [6763], we obtain the corresponding term of $\delta v''$; and by substituting [6766e], we get, by successive reductions, the expression [6766g], which is similar to that in [6765i].

$$\delta v'' = \tfrac{5}{16}.(\text{III})^2.\{\cos.2T'.\sin.(2n''t+2\varepsilon'')+\sin.2T'.\cos.(2n''t+2\varepsilon'')\} \qquad [6766f]$$
$$= \tfrac{5}{16}.(\text{III})^2.\sin.(2n''t+2\varepsilon''+2T') = \tfrac{5}{16}.(\text{III})^2.\sin.(2n't-2n''t+2\varepsilon'-2\varepsilon''). \qquad [6766g]$$

This last expression corresponds to [6766].

* (3470) The terms here referred to are those in [6139, 6140; 6160, 6161; 6164, 6165]. For we have very nearly, as in [6758, &c.], $N = n-(0)$; $N' = n'-(1)$, &c.; hence we get, by developing in series the following expression of the divisor in [6139, &c.], [6767a]

$$\frac{1}{n-n'-N'} = \frac{1}{n-2n'+(1)} = \frac{1}{n-2n'} - \frac{(1)}{(n-2n')^2} + \text{\&c.}\,; \qquad [6767b]$$

and in this way the term depending on the ellipticity (1), is virtually introduced with the divisor $(n-2n')^2$, being of the same order as the terms which are mentioned in [6767]. Moreover we have seen in [6485a—e], that the part depending on the ellipticity of Jupiter is greater than those arising from the action of the sun and satellites, as in [6767']. [6767c]

[6770] *a constant coefficient, and* H [6527] *the excentricity of Jupiter's orbit*, it is [6771] evident that the corresponding part of the double integral $ff3andt.\mathrm{d}R$, which enters into the expression of δv [6771c], will acquire, by integration, [6771'] a divisor of the order of the square of the disturbing force,* which will render it sensible, and it will become of the same order as the quantities we [6772] have determined in [6530, &c.]; it is therefore important to notice such terms, or to prove that no such ones exist.

In the first place we may observe, that in the function R, the quantity [6773] H^2 cannot be multiplied by the $\sin.2I$, or $\cos.2I$; I being the longitude of the perihelion of Jupiter [6276]; because the value of R is independent of the arbitrary point, which is taken for the origin of the longitudes [949—950]. The non-periodical part of R, depending on the square of the disturbing force, and multiplied by H^2, can be produced only by the combination of two angles which mutually destroy each other, under the sign [6774] of *cosine*; for R evidently contains only cosines.† We shall put H^2 under the following form; ‡

[6775]
$$H^2.\cos.\left\{\begin{array}{c} i.(nt-Mt+\varepsilon-E)+2.(Mt+E-I) \\ -i.(nt-Mt+\varepsilon-E)-2.(Mt+E-I) \end{array}\right\}.$$

[6771a] * (3471) The square of the excentricity e^2 [1109], which is named H^2 in [6770], depends on angles of the form $gt+\beta$; g *being of the same order as the disturbing forces* [6771b] [1097c]; so that if we put $H^2=\Sigma'.N.\cos.(gt+\beta)$, we shall have, as in [6769], $\mathrm{d}R=Q.dt.\Sigma'.N.\cos.(gt+\beta)$. Now we see, in [6060 or 6561f], that v contains a term [6771c] of the form $\delta v=ff3andt.\mathrm{d}R$ [6771]; and by substituting the preceding value of $\mathrm{d}R$ [6771d] [6771c], it becomes of the form $\delta v=ff3an.Qdt^2.\Sigma'.N.\cos.(gt+\beta)$; then it is evident that the double integration introduces the divisor g^2, of the same order as the square of the disturbing force [6771a]; as is stated in [6771'].

[6774a] † (3472) This is evident, by comparing the expression of R [949] with its developments [957, &c.]; and it will also be manifest from the development of R in the following note.

‡ (3473) If we develop the value of R relative to the sun's action, as we have done [6775a] for that of the satellite, in [6090, &c.], without restricting it to its first terms, as in [6100, 6103, &c.]; but carrying on the approximation so as to contain multiples of the angle $nt-Mt+\varepsilon-E$, we shall find that R will contain cosines of angles of the form $\cos.i.(nt-Mt+\varepsilon-E)$; i being an integral number, including $i=0$ [954'']. Moreover the term $H.\cos.(Mt+E-I)$ is introduced, in [6275], by means of the excentricity of [6775b] Jupiter's orbit; and its square will produce $H^2.\cos.2.(Mt+E-I)$. Hence it follows

Part of the angle contained under the sign of *cosine* may appertain to the co-ordinates of m, and this part is the only one which must be varied in R to obtain $\mathrm{d}R$; now whatever this part may be, it is evident that the [6776]
differential of the preceding term is nothing; therefore $\mathrm{d}R$ contains no term of the form $Q.H^2.dt$ [6769], depending on the square of the disturbing force. It is evident that the same reasoning may be applied to the terms, depending on the change of the equator and orbit of Jupiter. [6777]
Thus the square of the disturbing force does not introduce any sensible quantity in the secular equations of the satellites of Jupiter, or in the secular equation of the moon.

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that the angles connected with the coefficient H^2, which mutually destroy each other, and produce H^2 independent of the *cosines*, must be of the forms assumed in [6775]. [6775c]
Now the differential of a term of the form [6775], being taken relative to the characteristic [6775d]
d, which affects only the co-ordinates of m, changes the *cosine* into *sine;* and as the angle under the sign of *cosine* or *sine* is identically nothing, *its sine must vanish*, so that [6775e]
$\mathrm{d}R = 0$, as in [6776, &c.].

CHAPTER VII.*

NUMERICAL VALUES OF THE PRECEDING INEQUALITIES.

20. To reduce the preceding inequalities to numbers, we must know the times of the sideral revolutions of the satellites, and their mean distances from Jupiter's centre. These durations are by the tables,

Sideral revolutions.

[6778]	First satellite	1^{days},769137787 ;	log. = 0,2477617 ;
[6779]	Second satellite	3 ,551181017 ;	log. = 0,5503728 ;
[6780]	Third satellite	7 ,154552808 ;	log. = 0,8545825 ;
[6781]	Fourth satellite	16 ,689019396 ;	log. = 1,2224308 .

Values of n, n', n'', n'''.

The values of n, n', n'', n''', are inversely proportional to these times; hence we have,†

[6782] $n = n'''. \, 9{,}433419$; log. coeff. = 0,9746691 ;

[6783] $n' = n'''. \, 4{,}699569$; log. coeff. = 0,6720580 ;

[6784] $n'' = n'''. \, 2{,}332643$; log. coeff. = 0,3678485 .

[6778a] * (3474) In the original work this is erroneously named Chapter VI, and a similar error is found in the names of all the succeeding chapters of this book. We have corrected these mistakes, by increasing each of the numbers by unity.

[6781a] † (3475) Dividing the number of days in [6781] by those in [6778, 6779, 6780], we get the coefficient of n''', in the second members of the expressions [6782, 6783, 6784]
[6781b] respectively. Instead of expressing these quantities in terms of n''', we may denote them by terms of n, n', n''; and as these values are frequently required in this work, we have
[6781c] inserted them, for convenience, at the beginning of this volume, in [6025a—p]. These expressions are easily deduced from those in [6782—6784]. For the same cause we have
[6781d] also reduced the values [6835—6838, 6840] to similar forms, as in [6025c—i]. The mean sideral motion of the first satellite, in the time 1^{day},769137787, [6778], is 4000000″;
[6781e] hence its motion in one Julian year is 825826010″; which represents the value of n
[6781f] [6062′, 6025k]. In like manner we get, from [6779, 6780, 6781], the values of n', n'', n''',

To determine the mean distances a, a', a'', a''', we shall observe that the greatest elongation of the fourth satellite from Jupiter, in its mean distance, and seen at the mean distance of Jupiter from the sun, has been observed by Pound,* to be $1530'',864 = 496^{s}$. He also observed, at the same [6785]

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as in [6025*k*]. Finally, in order to obtain, at one view, a comparison of the different angles, we have inserted in [6025*o*, *p*] the values of g, g_1, g_2, g_3 [7176, 7183, 7190, 7195]; [6781*g*] and those of p, p_1, p_2, p_3 [7226, 7233, 7238, 7245], M [7253*b*], $n-M$, &c. [6025*n*]. The angles given in [6025*m*—*p*], according to the theory, are used by Delambre in the edition of his tables, printed at Paris in 1817; but the values of n, n', &c. are varied a little. The times of the synodical motions, used in the construction of the tables of this last edition, are as follows, using sexagesimal hours, minutes and seconds; [6781*h*]

First satellite	1^d	18^h	28^m	$35^s,945374812 =$	1^{days}	$,769860479$;	[6781*i*]
Second satellite	3	13	17	53 ,730106062 =	3	,554094098;	[6781*k*]
Third satellite	7	03	59	35 ,82511281 =	7	,166386865;	[6781*l*]
Fourth satellite	16	18	05	07 ,0209844 =	16	,753553484.	[6781*m*]

Supposing, as in Bouvard's tables of Jupiter, that the motion of the planet, from the fixed equinox in a Julian year, is $33^\circ,7212094$ or $33^\circ,7366724$ from the earth's moveable equinox, we shall obtain the motions n, n', n'', n''', of the satellites, from the same moveable equinox, in 100 Julian years, according to these tables of Delambre, namely, [6781*n*]

First satellite	8258261°	,65128;	[6781*o*]
Second satellite	4114125	,82948;	[6781*p*]
Third satellite	2042057	,91292;	[6781*q*]
Fourth satellite	875427	,46850;	[6781*r*]

agreeing very nearly with the values adopted by the author [7253—7257]. [6781*s*]

* (3476) Pound's measure of the elongations $1530'',864$, being divided by the semi-diameter $\frac{1}{2}\times 120'',3704$, gives the value of a''', expressed in parts of this [6787*a*] semi-diameter, as in [6787]. The elongations of the fourth satellite were observed very carefully by Professor Airy, in the autumn of 1832, and the results of his observations are [6787*b*] given in the sixth volume of the Memoirs of the Astronomical Society of London. With these new observations, he has re-computed the mass of Jupiter, by the method given by [6787*c*] La Place in [4065]; and has increased it from $\frac{1}{1070.5}$ [4061*d*] to $\frac{1}{1048.7}$; which agrees [6787*d*] very nearly with Encke's estimate $\frac{1}{1050.117}$, from the perturbations of Vesta; and with that of Nicolai, $\frac{1}{1053.924}$ [4061*g*], from the perturbations of Juno; and by this means he [6787*e*] has nearly removed the discrepancy in the different estimates of the mass, mentioned in [4061*g*—*m*].

[6786] distance, the diameter of Jupiter's equator to be $120'',3704 = 39^{s}$. *Taking this semi-diameter for unity*, we shall have,

[6787] $$a''' = 25{,}43590\,; \qquad \log. a''' = 1{,}4054471.$$

There may be some inaccuracy in this value of the ratio of a''' to the semi-diameter of Jupiter's equator, arising from the uncertainty in the [6788] estimate of Jupiter's diameter, in consequence of the effect of the irradiation in its measure. *This cannot produce any sensible error in the following* [6788'] *results; but the unity we make use of, may not accurately represent the semi-diameter of Jupiter's equator.*

With respect to the distances a, a', a'', it is much more accurate to compute them, from the value of a''', by Kepler's law [387], than to deduce them from observation. According to this law, the mean distance a, of [6789] the first satellite from Jupiter's centre, is $a = a'''.\sqrt[3]{\frac{n'''^2}{n^2}}$.* But the accuracy of this expression is a little impaired, by the forces disturbing the motions of the satellites, which, as we have seen in [6079'', 6123], add to the mean distances a and a''' the quantities δa and $\delta a'''$; whose analytical values are given in [6123, 6125]. *The only sensible term in these* [6789'] *quantities is that depending on the oblateness of Jupiter*, which we may [6790] represent by $\delta a = a.\frac{(\rho-\frac{1}{2}\varphi)}{3a^2}$; † we must therefore add this quantity to the

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[6789a] * (3477) We have, as in [6110], $a^3n^2 = 1 = a'''^3 n'''^2$; hence we easily obtain [6789].

[6790a] † (3478) It is evident from the calculations in [6485a—e, &c.], that the terms of the second member of [6123], depending on M, m', are much smaller than those depending [6790b] on $\rho-\frac{1}{2}\varphi$; and by neglecting them we shall have $\frac{\delta a}{a} = \frac{(\rho-\frac{1}{2}\varphi)}{3a^2}$, as in [6790]. Now [6790c] from [6789a] we have $a = n^{-\frac{2}{3}}$ nearly; and by adding it to $\delta a = a.\frac{(\rho-\frac{1}{2}\varphi)}{3a^2} = n^{-\frac{2}{3}}.\frac{(\rho-\frac{1}{2}\varphi)}{3a^2}$, we get the expression [6792]. Putting three accents to the letters a, n, in [6792], we get the expression [6793], corresponding to the fourth satellite m'''. Dividing [6792] [6790d] by [6793], and multiplying the result by a''', we get [6794]. Substituting in [6794] the values of n, a''' [6782, 6787], we obtain [6797]; observing that we may use the [6790e] value of a [6795] in finding the divisor $\frac{1}{a^2}$; but if we wish merely to verify the value of a [6797], it will be rather more accurate to substitute, at once, in the term $\frac{1}{a^2}$ [6794], the [6790f] value of a [6797]; and we shall find that the result agrees with the calculation of the author. In like manner, by accenting successively the letters a, n, [6794], with one or

value of a, given by the equation $n^2 = \frac{1}{a^3}$ [6110c]. Hence we shall have, [6791]

$$a = n^{-\frac{2}{3}} . \left\{ 1 + \tfrac{1}{3} . \frac{(\rho - \frac{1}{2}\varphi)}{a^2} \right\} ;$$ [6792]

and for the same reason,

$$a''' = n'''^{-\frac{2}{3}} . \left\{ 1 + \tfrac{1}{3} . \frac{(\rho - \frac{1}{2}\varphi)}{a'''^2} \right\} .$$ [6793]

Therefore we shall have,

$$a = \left\{ 1 + \tfrac{1}{3} . (\rho - \tfrac{1}{2}\varphi) . \left(\frac{1}{a^2} - \frac{1}{a'''^2} \right) \right\} . a''' . \sqrt[3]{\frac{n'''^2}{n^2}} ;$$ [6794]

in this expression we may substitute $a''' . \sqrt[3]{\frac{n'''^2}{n^2}}$ for a [6789], in the divisor $\frac{1}{a^2}$. Hence it is easy to ascertain the values of a', a''. The value [6795] of $(\rho - \frac{1}{2}\varphi)$ may be determined with precision, by the motions of the orbits [6796] of the satellites. The first approximation gave its value $\rho - \frac{1}{2}\varphi = 0{,}0217794$; and this is so near to the corrected value [7152], that the difference cannot have any sensible influence on the following computations. Hence we obtain,

Mean distances of the satellites from Jupiter expressed in parts of its equatorial radius.

$$a = 5{,}698491 ; \qquad \log . a = 0{,}7557599 ;$$ [6797]

$$a' = 9{,}066548 ; \qquad \log . a' = 0{,}9574420 ;$$ [6798]

$$a'' = 14{,}461893 ; \qquad \log . a'' = 1{,}1602252 ;$$ [6799]

$$a''' = 25{,}43590 . \qquad \log . a''' = 1{,}4054471 .$$ [6800]

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two accents, we get the values of a', a''; and by reducing them to numbers, we find that they agree with [6798, 6799]. We have verified these calculations, and find for the part of [6794], which is independent of $\rho - \frac{1}{2}\varphi$, the terms in [6790h]; and that depending on $\rho - \frac{1}{2}\varphi$, as in [6790i]; the sums of these give the values of a, a', a'', a''' [6790k], agreeing with [6797—6800]. [6790g]

First term	5.697281	9.065849	14.461551	25.43590	[6790h]
Second term	1210	699	340	0	[6790i]
Sum gives	a=5.698491	a'=9.066548	a''=14.461891	a'''=25.43590	[6790k]

Hence we see that these values of a, a', a'', a''', computed in this manner, contain the term depending on $\rho - \frac{1}{2}\varphi$ [6792], and it is therefore very properly omitted in the constant terms of δr, $\delta r'$, $\delta r''$, $\delta r'''$ [6843, 6845, 6847, 6849]; so that it is evident that there is no ground for the objection to this omission, which is made by Professor Airy, in his paper in the Memoirs of the Royal Astronomical Society of London, Vol. VI. p. 88, 98. [6790l]

Combining these values together, two by two, we have deduced, by means of the formulas [963^{iv}—1008'], the following results;

First and second satellites.

I. AND II. SATELLITES, m, m'.

[6801] $$\alpha = \frac{a}{a'} = 0{,}62851829;* \qquad \log. \frac{a}{a'} = 9{,}7983179;$$

hence we deduce,

[6802] $$b_{-\frac{1}{2}}^{(0)} = 2{,}02968796; \qquad b_{-\frac{1}{2}}^{(1)} = -0{,}595719117.$$

Then,

[6803]
$$b_{\frac{1}{2}}^{(0)} = 2{,}2588400; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}7543117; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}3632143; \qquad 1$$
$$b_{\frac{1}{2}}^{(3)} = 0{,}1923542; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}1065115; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}0605324; \qquad 2$$
$$b_{\frac{1}{2}}^{(6)} = 0{,}0349955; \qquad b_{\frac{1}{2}}^{(7)} = 0{,}0204800. \qquad 3$$

[6804]
$$\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} = 1{,}099916; \qquad \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}750014; \qquad \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} = 1{,}452760; \qquad 1$$
$$\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} = 1{,}084600; \qquad \frac{db_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}773248; \qquad \frac{db_{\frac{1}{2}}^{(5)}}{d\alpha} = 0{,}537010; \qquad 2$$
$$\frac{db_{\frac{1}{2}}^{(6)}}{d\alpha} = 0{,}366639. \qquad 3$$

[6805] $$b_{\frac{3}{2}}^{(3)} = 2{,}571615.$$

First and third satellites.

I. AND III. SATELLITES, m, m''.

[6806] $$\alpha = \frac{a}{a''} = 0{,}394034909; \qquad \log. \frac{a}{a''} = 9{,}5955347;$$

hence we deduce,

[6807] $$b_{-\frac{1}{2}}^{(0)} = 2{,}078416242; \qquad b_{-\frac{1}{2}}^{(1)} = -0{,}386231350.$$

Then,

[6808]
$$b_{\frac{1}{2}}^{(0)} = 2{,}0852433; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}4194902; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}1248495; \qquad 1$$
$$b_{\frac{1}{2}}^{(3)} = 0{,}0411410; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}0142110; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}0050800. \qquad 2$$

[6809]
$$\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} = 0{,}476087; \qquad \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}208235; \qquad \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} = 0{,}681369; \qquad 1$$
$$\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} = 0{,}329802; \qquad \frac{db_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}149798. \qquad 2$$

[6801a] * (3479) The value of α [6801], is deduced from $\alpha = \frac{a}{a'}$ [6304]; by substituting the values of a, a' [6797, 6798]. In like manner, those in [6806, 6810, &c.] are the
[6801b] values of $\frac{a}{a''}$, $\frac{a}{a'''}$, &c. From these we deduce the values of $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$; $b_{\frac{1}{2}}^{(0)}$, $b_{\frac{1}{2}}^{(1)}$, &c.;
[6801c] $\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$, $\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}$, &c.; in the same manner as in the planetary calculations [4086a—e];

I. and IV. Satellites, m, m'''.

First and fourth satellites.

$$\alpha = \frac{a}{a'''} = 0{,}224033410 ; \qquad \log. \frac{a}{a'''} = 9{,}3503128 ; \qquad [6810]$$

hence we deduce,

$$b_{-\frac{1}{2}}^{(0)} = 2{,}025175212 ; \qquad b_{-\frac{1}{2}}^{(1)} = -0{,}222618894. \qquad [6811]$$

Then,

$$b_{\frac{1}{2}}^{(0)} = 2{,}025831 ; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}228387 ; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}0384562 ; \qquad 1$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}0071873 ; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}0014098 ; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}0002846 . \qquad 2 \qquad [6812]$$

$$\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} = 0{,}237381 ; \qquad \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}059579 ; \qquad \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} = 0{,}350827 ; \qquad 1$$

$$\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} = 0{,}097721 ; \qquad \frac{db_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}025197 . \qquad 2 \qquad [6813]$$

II. and III. Satellites, m', m''.

Second and third satellites.

$$\alpha = \frac{a'}{a''} = 0{,}626926714 ; \qquad \log. \frac{a'}{a''} = 9{,}7972168 ; \qquad [6814]$$

hence we deduce,

$$b_{-\frac{1}{2}}^{(0)} = 2{,}201911334 ; \qquad b_{-\frac{1}{2}}^{(1)} = -0{,}594386339. \qquad [6815]$$

Then,

$$b_{\frac{1}{2}}^{(0)} = 2{,}2570986 ; \qquad b_{\frac{1}{2}}^{(1)} = 0{,}7515340 ; \qquad b_{\frac{1}{2}}^{(2)} = 0{,}3609108 ; \qquad 1$$

$$b_{\frac{1}{2}}^{(3)} = 0{,}1906386 ; \qquad b_{\frac{1}{2}}^{(4)} = 0{,}105293 ; \qquad b_{\frac{1}{2}}^{(5)} = 0{,}059691 ; \qquad 2 \qquad [6816]$$

$$b_{\frac{1}{2}}^{(6)} = 0{,}034428 ; \qquad b_{\frac{1}{2}}^{(7)} = 0{,}020107 . \qquad 3$$

$$\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} = 1{,}093150 ; \qquad \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}743650 ; \qquad \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} = 1{,}444842 ; \qquad 1$$

$$\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} = 1{,}076290 ; \qquad \frac{db_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}765517 ; \qquad \frac{db_{\frac{1}{2}}^{(5)}}{d\alpha} = 0{,}530315 ; \qquad 2 \qquad [6817]$$

$$\frac{db_{\frac{1}{2}}^{(6)}}{d\alpha} = 0{,}361124. \qquad 3$$

$$b_{\frac{3}{2}}^{(3)} = 2{,}537577. \qquad [6818]$$

II. and IV. Satellites, m', m'''.

Second and fourth satellites.

$$\alpha = \frac{a'}{a'''} = 0{,}356447000 ; \qquad \log. \frac{a'}{a'''} = 9{,}5519949 ; \qquad [6819]$$

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retaining only such terms as are useful in the subsequent calculations, on account of their producing sensible inequalities. I have not verified these numbers, but by a very rough calculation find that they agree nearly with the results for similar values of α, in the planetary calculations [4085—4227]. [6801d]

hence we deduce,

[6820] $$b_{-\frac{1}{2}}^{(0)} = 2{,}064048552\,; \qquad b_{-\frac{1}{2}}^{(1)} = -0{,}350692291.$$

Then,

[6821]
$$b_{\frac{1}{2}}^{(0)} = 2{,}0685085\,; \quad b_{\frac{1}{2}}^{(1)} = 0{,}374917\,; \quad b_{\frac{1}{2}}^{(2)} = 0{,}1008003\,; \qquad 1$$
$$b_{\frac{1}{2}}^{(3)} = 0{,}0300272\,; \quad b_{\frac{1}{2}}^{(4)} = 0{,}0093817\,; \quad b_{\frac{1}{2}}^{(5)} = 0{,}0030277\,. \qquad 2$$

[6822]
$$\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} = 0{,}415141\,; \quad \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}164667\,; \quad \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} = 0{,}599392\,; \qquad 1$$
$$\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} = 0{,}263317\,; \quad \frac{db_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}108542. \qquad 2$$

III. and IV. Satellites, m'', m'''.

Third and fourth satellites.

[6823] $$\alpha = \frac{a''}{a'''} = 0{,}568562391\,; \qquad \log.\,\frac{a''}{a'''} = 9{,}7547781\,;$$

hence we deduce,

[6824] $$b_{-\frac{1}{2}}^{(0)} = 2{,}165200864\,; \qquad b_{-\frac{1}{2}}^{(1)} = -0{,}544549802.$$

Then,

[6825]
$$b_{\frac{1}{2}}^{(0)} = 2{,}1996536\,; \quad b_{\frac{1}{2}}^{(1)} = 0{,}6558357\,; \quad b_{\frac{1}{2}}^{(2)} = 0{,}284370\,; \qquad 1$$
$$b_{\frac{1}{2}}^{(3)} = 0{,}135969\,; \quad b_{\frac{1}{2}}^{(4)} = 0{,}068124\,; \quad b_{\frac{1}{2}}^{(5)} = 0{,}035180\,; \qquad 2$$
$$b_{\frac{1}{2}}^{(6)} = 0{,}018696\,; \quad b_{\frac{1}{2}}^{(7)} = 0{,}010398. \qquad 3$$

[6826]
$$\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} = 0{,}878931\,; \quad \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} = 1{,}545882\,; \quad \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} = 1{,}190293\,; \qquad 1$$
$$\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} = 0{,}812421\,; \quad \frac{db_{\frac{1}{2}}^{(4)}}{d\alpha} = 0{,}526520\,; \quad \frac{db_{\frac{1}{2}}^{(5)}}{d\alpha} = 0{,}330751\,; \qquad 2$$
$$\frac{db_{\frac{1}{2}}^{(6)}}{d\alpha} = 0{,}201993. \qquad 3$$

21. Using these values and the formulas in § 3 [6078—6125], we have deduced the following results; supposing, as in [6796], that

[6827] $$\rho - \tfrac{1}{2}\varphi = 0{,}0217794\,; \qquad \log.(\rho - \tfrac{1}{2}\varphi) = 8{,}3380459.$$

The values of N, N', N'', N''' [6833*a*, &c.], depend considerably upon $\rho - \frac{1}{2}\varphi$; and its effect is most sensibly seen in the expression of N [6833*a*]. These values depend also upon the masses of the satellites m, m', m'', m'''. By the first approximation we have obtained,

Approximate values of the masses of the satellites.

[6828] $$m = 0{,}0000184113\,;$$
[6829] $$m' = 0{,}0000258325\,;$$
[6830] $$m'' = 0{,}0000865185\,;$$
[6831] $$m''' = 0{,}00005590808\,;$$
[6832] Jupiter's mass $= 1$.

The smallness of the influence of the masses of the satellites upon the values of N, N', &c., renders the errors arising from any inaccuracy, in the estimated values of those masses, insensible. The mass m' is multiplied, in the expression of N^2 [6124], by the function, [6832']

$$a^2.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}.a^3.\left(\frac{ddA^{(0)}}{da^2}\right);\ * \qquad [6833]$$

We have found, in [1075], that this function can be reduced to the form,

$$a^2.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}.a^3.\left(\frac{ddA^{(0)}}{da^2}\right)=\frac{3\alpha^2.b_{-\frac{1}{2}}^{(1)}}{2.(1-\alpha^2)^2}; \qquad [6834]$$

we may therefore easily compute it, as well as the analogous functions, by means of the numerical results in [6801—6826]. This being premised, we find,

$$N\ \ =n'''.9,4269167; \qquad \log.9,4269167=0,9743697; \qquad [6835]$$

$$N'\ =n'''.4,6979499; \qquad \log.4,6979499=0,6719084; \qquad [6836]$$

$$N''\ =n'''.2,3323090; \qquad \log.2,3323090=0,3677861; \qquad [6837]$$

$$N'''=n'''.0,9999070. \qquad \log.0,9999070=9,9999596. \qquad [6838]$$

Then by supposing the time of the sideral revolution of Jupiter to be $4332^{\text{days}},602208$, we shall have,† [6839]

$$M=n'''.0,00385196; \qquad \log.0,00385196=7,5856818; \qquad [6840]$$

from these we have deduced the following formulas, in which *the quantities* ‡

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* (3480) Substituting in [6124] the value of the function [6834], we obtain,

$$N^2=n^2.\left\{1-2.\frac{(\rho-\frac{1}{2}\varphi)}{a^2}-\tfrac{3}{2}.\frac{M^2}{n^2}+\Sigma.m'.\frac{3\,\alpha^2.b_{-\frac{1}{2}}^{(1)}}{2.(1-\alpha^2)^2}\right\}. \qquad [6833a]$$

Substituting in this expression the values [6782, 6797—6800, 6827—6831], we get N [6835]; including in the terms under the sign Σ those which correspond to the satellites m', m'', m'''; the value of α, corresponding thereto, being respectively as in [6801, 6806, 6810]. Again, by changing, in [6833a], the elements of m into those of m', and the contrary, we can obtain, by a similar calculation, the value of N' [6836]; the terms under the sign Σ being, in this case, m, m'', m''', and the values of α are those in [6801, 6814, 6819]. In like manner we obtain N'', N''' [6837, 6838]. I verified the calculation of N [6835], but did not re-compute [6836—6838]. [6833b] [6833c] [6833d] [6833e] [6833f]

† (3481) From [6781, 6839] we have,

$$M:n''' :: 16^{\text{days}},689019396 : 4332^{\text{days}},602208 :: 0,00385196 : 1, \qquad [6840a]$$

as in [6840].

‡ (3482) To avoid confusion, we have here introduced the Roman letters m, m′, m″, m‴, instead of the Italic letters used by the author, for these augmented values; so [6841a]

[6841] m, m′, m″, m‴, *denote respectively the masses* m, m', m'', m''' *of the satellites, multiplied by* 10000.*

Inequalities of the longitudes of the first satellite.

[6842]
$$\delta v = \mathrm{m}'.\left\{\begin{array}{ll} + \;\; 187'',4465.\sin.\;(n't-nt+\varepsilon'-\varepsilon) & 1 \\ -21736'',4863.\sin.2.(n't-nt+\varepsilon'-\varepsilon) & 2 \\ - \;\; 70'',8315.\sin.3.(n't-nt+\varepsilon'-\varepsilon) & 3 \\ - \;\; 16'',1926.\sin.4.(n't-nt+\varepsilon'-\varepsilon) & 4 \\ - \;\; 5''.4039.\sin.5.(n't-nt+\varepsilon'-\varepsilon) & 5 \\ - \;\; 2''.1433.\sin.6.(n't-nt+\varepsilon'-\varepsilon) & 6 \end{array}\right\}$$

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that by using the approximate values [6828—6831], we shall have,

[6841b] $\mathrm{m} = 10000.m = 0,184113$;

[6841c] $\mathrm{m}' = 10000.m' = 0,258325$;

[6841d] $\mathrm{m}'' = 10000.m'' = 0,865185$;

[6841e] $\mathrm{m}''' = 10000.m''' = 0,5590808$.

[6842a] * (3483) The value of δv [6842], is computed from the formula [6119], neglecting the term [6119 line 1] depending on t, as in [6121]; and using the values of $A^{(1)}$, $A^{(2)}$, &c.; $\left(\frac{dA^{(1)}}{da}\right)$, $\left(\frac{dA^{(2)}}{da}\right)$, &c. [996—1001]. Then substituting the values of the
[6842b] elements of the satellites, and those of the functions depending on b and its differentials [6782—6840], we obtain [6842]. In like manner, the values of $\delta v'$, $\delta v''$, $\delta v'''$ [6844, 6846, 6848], are computed by changing successively, in [6119], the elements of
[6842c] m into those of m', m'', m''', and the contrary; and then using the numerical values contained in [6782—6840]. I did not examine these numerical calculations, except in a few of the most important terms. The values in [6842 lines 2, 14], were found to be correct.

Again, if we put $r = a + \delta r$ [6057a] in the first member of [6116], and neglect δr^2,
[6842d] it becomes $\frac{\delta r}{a}$. Substituting the value of k [6122], in the constant part of this expression [6116 line 1], and reducing as in [6117a, &c.], it becomes as in [6123]. If
[6842e] we now substitute, in the *periodical* terms of [6116], the numerical values [6782—6840], and then multiply by a, we shall obtain the corresponding terms of δr [6843] nearly.
[6842f] The constant parts corresponding to the function [6123], are particularly examined in [6843a—v], on account of the error of the author in the signs. In like manner, by
[6842g] changing successively the elements of m into those of m', m'', m''', in the functions
[6842h] [6116, 6123], we obtain the values of $\frac{\delta r'}{a'}$, $\frac{\delta r''}{a''}$, $\frac{\delta r'''}{a'''}$. Multiplying these by a', a'' or
[6842i] a''', we get $\delta r'$, $\delta r''$, $\delta r'''$, respectively. The periodical terms agree with those in [6845, 6847, 6849]. The constant terms are noticed in [6845i, 6847c, 6849c]. The whole of this calculation is similar to that in [4277a—p], corresponding to the planetary
[6842k] orbits, to which we may refer for the purpose of illustration. I examined the calculation of the chief term of δr [6843 line 3], and made it the same as that of the author.

$$
\begin{array}{ll}
+\mathrm{m}''.\left\{\begin{array}{l}
+21'',9334.\sin.\ (n''t-nt+\varepsilon''-\varepsilon) \\
-18'',5197.\sin.2.(n''t-nt+\varepsilon''-\varepsilon) \\
-\ \ 1'',9017.\sin.3.(n''t-nt+\varepsilon''-\varepsilon) \\
-\ \ 0'',3569.\sin.4.(n''t-nt+\varepsilon''-\varepsilon)
\end{array}\right\} & \begin{array}{l} 7 \\ 8 \\ 9 \\ 10 \end{array} \\
+\mathrm{m}'''.\left\{\begin{array}{l}
+3'',6109.\sin.\ (n'''t-nt+\varepsilon'''-\varepsilon) \\
-1'',5588.\sin.2.(n'''t-nt+\varepsilon'''-\varepsilon) \\
-0'',1079.\sin.3.(n'''t-nt+\varepsilon'''-\varepsilon)
\end{array}\right\} & \begin{array}{l} 11 \\ 12 \\ 13 \end{array} \\
+0'',1460.\sin.(2nt-2Mt+2\varepsilon-2E). & 14
\end{array}
$$
[6842]

Inequalities of the radius vector of the first satellite.

$$
\delta r = \mathrm{m}'.\left(\begin{array}{l}
-0,000041267_{c}\ * \\
+0,00046652.\cos.\ (n't-nt+\varepsilon'-\varepsilon) \\
-0,09764199.\cos.2.(n't-nt+\varepsilon'-\varepsilon) \\
-0,00040917.\cos.3.(n't-nt+\varepsilon'-\varepsilon) \\
-0,00010761.\cos.4.(n't-nt+\varepsilon'-\varepsilon) \\
-0,00003824.\cos.5.(n't-nt+\varepsilon'-\varepsilon) \\
-0,00001642.\cos.6.(n't-nt+\varepsilon'-\varepsilon)
\end{array}\right)
\qquad \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
$$
[6843]

---

* (3484) The *constant terms* in the values of $\delta r, \delta r', \delta r'', \delta r'''$ [6843 lines 1, 8, 13, 17; 6845 lines 1, 7, 13, 17; 6847 lines 1, 5, 12, 19; 6849 lines 1, 4, 8, 12], are given in conformity with the value of $\frac{\delta a}{a}$ [6123]; from which we have deduced the expression [6843*b*], by multiplying by a, and changing δa into δr, in conformity with the notation here used; [6843*a*]

$$\delta r = -\tfrac{1}{6}.\frac{M^2}{n^2}.a+\tfrac{1}{6}a.\Sigma.m'.a^2.\left(\frac{dA^{(0)}}{da}\right). \qquad [6843b]$$

We have neglected the part of δr, *which depends on* $\rho-\frac{1}{2}\varphi$, *because it has already been introduced into the values of* a, a', a'', a''' [6790*k*, &c.]. If we change successively, in [6843*b*], the elements of m into those of m', m'' or m''', and the contrary, we shall get $\delta r', \delta r'', \delta r'''$. It will be seen in [7078, 7079, &c.], that the only parts of the values of $\delta r, \delta r'$, $\delta r'', \delta r'''$, which are required to be noticed on account of their magnitudes in eclipses, are those depending on the greatest inequalities; as, for example, that in [6843 line 3] relative to the satellite m'; for, by substituting in it the value of m' [6841*c*], it becomes nearly equal to $-0,02.\cos.2.(n't-nt+\varepsilon'-\varepsilon)$; and this may become greater than any other of the variable terms of [6843]. It is also much greater than any of the constant terms, contained in the function [6843*b*, *d*]; for one of the greatest of these terms is that in [6849 line 8], which is less than 0,001. Notwithstanding these constant parts of the radii vectores are so small that they are neglected by the author in [6789', &c.], we shall compute them, by means of [6843*b*, &c.], in order to show that the numerical values given by the author have wrong signs in the original publication. We have corrected these [6843*c*] [6843*d*] [6843*e*] [6843*f*] [6843*g*] [6843*h*]

[6843]
$$+\mathrm{m}''.\left\{\begin{array}{l}-0{,}00000702_c \\ +0{,}00007780.\cos.\;(n''t-nt+\varepsilon''-\varepsilon) \\ -0{,}00010631.\cos.2.(n''t-nt+\varepsilon''-\varepsilon) \\ -0{,}00001310.\cos.3.(n''t-nt+\varepsilon''-\varepsilon) \\ -0{,}00000269.\cos.4.(n''t-nt+\varepsilon''-\varepsilon)\end{array}\right\}\begin{array}{r}8\\9\\10\\11\\12\end{array}$$

$$+\mathrm{m}'''.\left\{\begin{array}{l}-0{,}00000113_c \\ +0{,}00001478.\cos.\;(n'''t-nt+\varepsilon'''-\varepsilon) \\ -0{,}00000968.\cos.2.(n'''t-nt+\varepsilon'''-\varepsilon) \\ -0{,}00000078.\cos.3.(n'''t-nt+\varepsilon'''-\varepsilon)\end{array}\right\}\begin{array}{r}13\\14\\15\\16\end{array}$$

$$-0{,}000000158_c \qquad 17$$

$$-0{,}00000095.\cos.(2Mt-2nt+2E-2\varepsilon). \qquad 18$$

---

signs in this translation. If we substitute in the formula [6843b] the expression

[6843i] $\left(\frac{dA^{(0)}}{da}\right)=-\frac{1}{a'^2}\cdot\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$ [999], we shall get,

[6843k]
$$\delta r=-\tfrac{1}{6}\cdot\frac{M^2}{n^2}\cdot a-\tfrac{1}{6}a\cdot\left\{m'.\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}+m''.\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}+m'''.\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right\};$$

[6843l] using for α and $\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$ the values [6801, 6804 line 1], in the term depending on m'; also the values [6806, 6809 line 1], in the term depending on m''; and the values [6810, 6813 line 1], in the term depending on m'''. Now if we substitute, in the term

[6843m] $-\frac{1}{6}\cdot\frac{M^2}{n^2}\cdot a$ [6843k], the values of n, a, M [6782, 6797, 6840], it becomes $-0{,}000000158$, as in [6843 line 17]. In the original work it is $+0{,}00000095$, being six

[6843n] times its real value, and with a different sign. *It was probably computed from the formula* [6117], *without noticing the part arising from* $-2k$, *which is introduced into the*

[6843o] *expression* [6123]. In like manner we find, by substituting the values of $\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$ corresponding to the different satellites, as in [6843l, &c.], that the terms of [6843k] depending on m', m'', m''', become,

[6843p]
$$-0{,}000041267.\,\mathrm{m}'-0{,}00000702.\,\mathrm{m}''-0{,}00000113.\,\mathrm{m}'''.$$

These terms in the original work, are represented by

[6843q]
$$+0{,}000084865.\,\mathrm{m}'+0{,}00000703.\,\mathrm{m}''+0{,}00000113.\,\mathrm{m}''';$$

all the signs being wrong; and the first term is about twice its actual value. We have

[6843r] inserted in the original work [6843 lines 1, 8, 13] the corrected values [6843p], *annexing the letter* $_c$ *to the numbers to denote that this correction has been made, as in* [6021i]. If we substitute the values of m', m'', m''' [6841c, d, e] in [6843p], it becomes nearly

[6843s] equal to $-0{,}000017$, or about $\frac{1}{330000}$ of the radius a [6797], being less than two centesimal seconds, supposing the circumference of the orbit of the fourth satellite to be

Inequalities of the longitude of the second satellite.

$$
\delta v' = \text{m}.\left\{\begin{array}{l}
-6951'',4660.\sin.\ (nt-n't+\varepsilon-\varepsilon') \\
-\ \ 52'',6315.\sin.2.(nt-n't+\varepsilon-\varepsilon') \\
-\ \ 10'',5253.\sin.3.(nt-n't+\varepsilon-\varepsilon') \\
-\ \ \ 3'',3448.\sin.4.(nt-n't+\varepsilon-\varepsilon') \\
-\ \ \ 1'',2969.\sin.5.(nt-n't+\varepsilon-\varepsilon')
\end{array}\right\}
\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

$$
+\,\text{m}''.\left\{\begin{array}{l}
+\ \ 184'',5172.\sin.\ (n''t-n't+\varepsilon''-\varepsilon') \\
-12108'',9920.\sin.2.(n''t-n't+\varepsilon''-\varepsilon') \\
-\ \ \ 68'',8828.\sin.3.(n''t-n't+\varepsilon''-\varepsilon') \\
-\ \ \ 15'',7643.\sin.4.(n''t-n't+\varepsilon''-\varepsilon') \\
-\ \ \ \ 5'',2597.\sin.5.(n''t-n't+\varepsilon''-\varepsilon') \\
-\ \ \ \ 2'',0814.\sin.6.(n''t-n't+\varepsilon''-\varepsilon')
\end{array}\right\}
\begin{array}{r} 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array}
\qquad [6844]
$$

$$
+\,\text{m}'''.\left\{\begin{array}{l}
+\ 12'',3755.\sin.\ (n'''t-n't+\varepsilon'''-\varepsilon') \\
-10'',8356.\sin.2.(n'''t-n't+\varepsilon'''-\varepsilon') \\
-\ \ 1'',0646.\sin.3.(n'''t-n't+\varepsilon'''-\varepsilon')
\end{array}\right\}
\begin{array}{r} 12 \\ 13 \\ 14 \end{array}
$$

$$
+\ 0'',5881.\sin.(2n't-2Mt+2\varepsilon'-2E). \qquad 15
$$

Inequalities of the radius vector of the second satellite.

$$
\delta r' = \text{m}.\left\{\begin{array}{l}
+0,00044579^{*} \\
+0,05069318.\cos.\ (nt-n't+\varepsilon-\varepsilon') \\
+0,00059197.\cos.2.(nt-n't+\varepsilon-\varepsilon') \\
+0,00014002.\cos.3.(nt-n't+\varepsilon-\varepsilon') \\
+0,00004784.\cos.4.(nt-n't+\varepsilon-\varepsilon') \\
+0,00001928.\cos.5.(nt-n't+\varepsilon-\varepsilon')
\end{array}\right\}
\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\qquad [6845]
$$

---

expressed by $4000000''$. Now the arc q [6022n], described by the satellite during its passage from the centre to the circumference of the shadow is $111780''$ [7554], and this is described in $4945''$ of time [7558]; hence an arc of $2''$ corresponds to $\frac{1}{11}$ of a second of time in the duration of the eclipse, and as this is wholly insensible, it is neglected by the author. We may also remark that some terms, of exactly the same order, are neglected in the calculation of the values of a, a', a'', a''' [6789', &c.]; and it would not have been of the least importance if they had been wholly omitted in the expressions of δr, $\delta r'$, $\delta r''$, $\delta r'''$ [6843—6849]; they are probably inserted by the author like many other very small and insensible inequalities, merely because they had been computed by the formulas; and they serve to show, by inspection, that they have no sensible value. [6843t] [6843u] [6843v]

* (3485) The value of the constant part of $\delta r'$, is easily deduced from [6843b], by changing the elements of m into those of m', and the contrary; by this means it becomes, [6845a]

$$
\delta r' = -\tfrac{1}{6}.\frac{M^2}{n'^2}.a' + \tfrac{1}{6}.a'.\Sigma.m.a'^2.\left(\frac{d.A^{(0)}}{da'}\right). \qquad [6845b]
$$

[6845]

$$
\begin{array}{llr}
 & \left\{\begin{array}{l} -0{,}00006492_c \end{array}\right. & 7 \\
+\,\mathrm{m}''. & \left\{\begin{array}{l}
+0{,}00073255.\cos.\ (n''t-n't+\varepsilon''-\varepsilon') \\
-0{,}08670960.\cos.2.(n''t-n't+\varepsilon''-\varepsilon') \\
-0{,}00063398.\cos.3.(n''t-n't+\varepsilon''-\varepsilon') \\
-0{,}00016685.\cos.4.(n''t-n't+\varepsilon''-\varepsilon') \\
-0{,}00006067.\cos.5.(n''t-n't+\varepsilon''-\varepsilon')
\end{array}\right\} & \begin{array}{r} 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \\
+\,\mathrm{m}'''. & \left\{\begin{array}{l}
-0{,}00000797_c \\
+0{,}00007146.\cos.\ (n'''t-n't+\varepsilon'''-\varepsilon') \\
-0{,}00010133.\cos.2.(n'''t-n't+\varepsilon'''-\varepsilon') \\
-0{,}00001189.\cos.3.(n'''t-n't+\varepsilon'''-\varepsilon')
\end{array}\right\} & \begin{array}{r} 13 \\ 14 \\ 15 \\ 16 \end{array} \\
 & -0{,}000001015_c & 17 \\
 & -0{,}00000609.\cos.(2Mt-2n't+2E-2\varepsilon'). & 18
\end{array}
$$

[6845c] For the inferior satellite m, we must use the value of $a'.\left(\frac{dA^{(0)}}{da'}\right)=-A^{(0)}-a.\left(\frac{dA^{(0)}}{da}\right)$ [1002]; and by means of the formulas [996, 999], it becomes,

[6845d] $$a'.\left(\frac{dA^{(0)}}{da'}\right)=\frac{1}{a'}.b_{\frac{1}{2}}^{(0)}+\frac{a}{a'^2}.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}; \qquad \text{or} \qquad a'^2.\left(\frac{dA^{(0)}}{da'}\right)=b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha};$$

but for the superior satellite m'', we must use the formula,

[6845e] $$a'^2.\left(\frac{dA^{(0)}}{da'}\right)=-\frac{a'^2}{a''^2}.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}=-\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} \ [999],$$

and a similar one for m'''. Hence the expression [6845b] becomes,

[6845f] $$\delta r'=-\tfrac{1}{6}.\frac{M^2}{n'^2}.a'+\tfrac{1}{6}.a'.\left\{m.\left(b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right)-m''.\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}-m'''.\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right\}.$$

[6845g] In the coefficient of m, we must use the values of α, $b_{\frac{1}{2}}^{(0)}$, $\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$ [6801, 6803, 6804]; in that of m'', the values [6814, 6817]; in that of m''', the values [6819, 6822].

[6845h] Moreover in the term depending on M^2, &c., we must use the values of n', a', M [6783, 6798, 6840]. Hence the expression [6845f] becomes,

[6845i] $$\delta r'=+0{,}00044579.\mathrm{m}-0{,}00006492.\mathrm{m}''-0{,}00000797.\mathrm{m}'''-0{,}000001015.$$

In the original work the corresponding numbers are,

[6845k] $$\delta r'=-0{,}00044608.\mathrm{m}+0{,}00006497.\mathrm{m}''+0{,}00000798.\mathrm{m}'''+0{,}000006090;$$

all of them having wrong signs; and the last term, depending on M^2, being six times too

[6845l] great. I examined the three largest terms, contained in [6844 lines 1, 7; 6845 lines 2, 9], and found the two last to agree very nearly with the values given by the author; the

[6845m] coefficient in [6844 line 1] was found, by a rough calculation, to be $-6942'',5$, which differs a few seconds from the value given by La Place.

Inequalities of the longitude of the third satellite.

$$\delta v'' = \text{m}.\left\{\begin{array}{l} +24''.2648.\sin.\ (nt-n''t+\varepsilon-\varepsilon') \\ -\ 0''.7044.\sin.2.(nt-n''t+\varepsilon-\varepsilon') \\ -\ 0''.1277.\sin.3.(nt-n''t+\varepsilon-\varepsilon') \end{array}\right\} \begin{array}{r} 1 \\ 2 \\ 3 \end{array}$$

$$+\,\text{m}'.\left\{\begin{array}{l} -3478'',2675.\sin.\ (n't-n''t+\varepsilon'-\varepsilon'') \\ -\quad 50'',9399.\sin.2.(n't-n''t+\varepsilon'-\varepsilon'') \\ -\quad 10'',2071.\sin.3.(n't-n''t+\varepsilon'-\varepsilon'') \\ -\quad 3'',2305.\sin.4.(n't-n''t+\varepsilon'-\varepsilon'') \\ -\quad 1'',2551.\sin.5.(n't-n''t+\varepsilon'-\varepsilon'') \\ -\quad 0'',5453.\sin.6.(n't-n''t+\varepsilon'-\varepsilon'') \end{array}\right\} \begin{array}{r} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad [6846]$$

$$+\,\text{m}'''.\left\{\begin{array}{l} +106'',1614.\sin.\ (n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -362'',1030.\sin.2.(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -\ 25'',4655.\sin.3.(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -\quad 5'',9227.\sin.4.(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -\quad 1'',8800.\sin.5.(n'''t-n''t+\varepsilon'''-\varepsilon'') \\ -\quad 0'',6997.\sin.6.(n'''t-n''t+\varepsilon'''-\varepsilon'') \end{array}\right\} \begin{array}{r} 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{array}$$

$$+2''3870.\sin.(2n''t-2Mt+2\varepsilon''-2E). \qquad 16$$

$$\delta r'' = \text{m}.\left\{\begin{array}{l} +0,00054783_c{}^{*} \\ +0,00059147.\cos.\ (nt-n''t+\varepsilon-\varepsilon'') \\ +0,00001906.\cos.2.(nt-n''t+\varepsilon-\varepsilon'') \\ +0,00000348.\cos.3.(nt-n''t+\varepsilon-\varepsilon'') \end{array}\right\} \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad [6847]$$

---

* (3186) The expression of the constant term of $\delta r''$, similar to [6845f], is evidently represented by

$$\delta r'' = -\tfrac{1}{6}.\frac{M^2}{n''^2}.a''+\tfrac{1}{6}a''.\left\{m.\left(b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right)+m'.\left(b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right)-m'''.\alpha^2.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right\}. \qquad [6847a]$$

In the coefficient of m we must use the values of α, $b_{\frac{1}{2}}^{(0)}$, $\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$ [6806, 6808, 6809]; in those of m', the values [6814, 6816, 6817]; and in those of m''', the values [6823, 6826]. Moreover, in the term depending on M^2, we must use the values of n'', a'', M [6784, 6799, 6840]. Hence the expression [6847a] becomes, [6847b]

$$\delta r'' = 0,00054783.\text{m}+0,00070922.\text{m}'-0,00006848.\text{m}'''-0,00000657. \qquad [6847c]$$

In the original work, the corresponding numbers are,

$$\delta r'' = -0,00054798.\text{m}-0,00070942.\text{m}'+0,00006850.\text{m}'''+0,00003944; \qquad [6847d]$$

all of them having wrong signs, and the term depending on M^2 being six times too great. By a rough calculation I found the coefficient in [6846 line 4] to be $-3476'',6$; and in [6847 line 6], $+0,041358..$, agreeing nearly with the author. [6847e]

Inequalities of the radius vector of third satellite.

[6847]

$$
+\mathrm{m}'.\left\{\begin{array}{ll}
+0{,}00070922_c & 5 \\
+0{,}04137743.\cos.\ (n't-n''t+\varepsilon'-\varepsilon'') & 6 \\
+0{,}00091726.\cos.2.(n't-n''t+\varepsilon'-\varepsilon'') & 7 \\
+0{,}00021712.\cos.3.(n't-n''t+\varepsilon'-\varepsilon'') & 8 \\
+0{,}00007409.\cos.4.(n't-n''t+\varepsilon'-\varepsilon'') & 9 \\
+0{,}00002980.\cos.5.(n't-n''t+\varepsilon'-\varepsilon'') & 10 \\
+0{,}00001318.\cos.6.(n't-n''t+\varepsilon'-\varepsilon'') & 11
\end{array}\right.
$$

$$
+\mathrm{m}'''.\left\{\begin{array}{ll}
-0{,}00006848_c & 12 \\
+0{,}00075191.\cos.\ (n'''t-n''t+\varepsilon'''-\varepsilon'') & 13 \\
-0{,}0044961.\ \cos.2.(n'''t-n''t+\varepsilon'''-\varepsilon'') & 14 \\
-0{,}00039801.\cos.3.(n'''t-n''t+\varepsilon'''-\varepsilon'') & 15 \\
-0{,}00010474.\cos.4.(n'''t-n''t+\varepsilon'''-\varepsilon'') & 16 \\
-0{,}00003569.\cos.5.(n'''t-n''t+\varepsilon'''-\varepsilon'') & 17 \\
-0{,}00001379.\cos.6.(n'''t-n''t+\varepsilon'''-\varepsilon'') & 18
\end{array}\right.
$$

$$
\begin{array}{ll}
-0{,}00000657_c & 19 \\
-0{,}00003944.\cos.(2Mt-2n'''+2E-2\varepsilon'''). & 20
\end{array}
$$

Inequalities of the longitude of the fourth satellite.

[6848]

$$
\delta v'''=\mathrm{m}.\left\{\begin{array}{ll}
+14'',2458.\sin.\ (nt-n'''t+\varepsilon-\varepsilon''') & 1 \\
-\ 0'',0206.\sin.2.(nt-n'''t+\varepsilon-\varepsilon''') & 2
\end{array}\right.
$$

$$
+\mathrm{m}'.\left\{\begin{array}{ll}
+22'',4521.\sin.\ (n't-n'''t+\varepsilon'-\varepsilon''') & 3 \\
-\ 0'',3085.\sin.2.(n't-n'''t+\varepsilon'-\varepsilon''') & 4 \\
-\ 0'',0540.\sin.3.(n't-n'''t+\varepsilon'-\varepsilon''') & 5
\end{array}\right.
$$

$$
+\mathrm{m}''.\left\{\begin{array}{ll}
-32'',0439_c.\sin.\ (n''t-n'''t+\varepsilon''-\varepsilon''') & 6 \\
-15'',9570.\sin.2.(n''t-n'''t+\varepsilon''-\varepsilon''') & 7 \\
-\ 3'',3293.\sin.3.(n''t-n'''t+\varepsilon''-\varepsilon''') & 8 \\
-\ 1'',0197.\sin.4.(n''t-n'''t+\varepsilon''-\varepsilon''') & 9 \\
-\ 0'',3735.\sin.5.(n''t-n'''t+\varepsilon''-\varepsilon''') & 10
\end{array}\right.
$$

$$
\begin{array}{ll}
+12'',9881.\sin.(2n'''t-2Mt+2\varepsilon'''-2E). & 11
\end{array}
$$

[6849]

$$
\delta r'''=\mathrm{m}.\left\{\begin{array}{ll}
+0{,}00088138_c & 1 \\
+0{,}00057018.\cos.\ (nt-n'''t+\varepsilon-\varepsilon''') & 2 \\
+0{,}00000113.\cos.2.(nt-n'''t+\varepsilon-\varepsilon''')\ * & 3
\end{array}\right.
$$

[6849a] * (3487) Here all the satellites m, m', m'', are inferior to m'''; and the expression of $\delta r'''$, similar to [6847a], becomes,

$$+\mathrm{m}'.\left\{\begin{array}{ll}+0{,}00093964_c & \quad 4\\ +0{,}00091758.\cos.\ (n't-n'''t+\varepsilon'-\varepsilon''') & \quad 5\\ +0{,}00001095.\cos.2.(n't-n'''t+\varepsilon'-\varepsilon''') & \quad 6\\ +0{,}00000166.\cos.3.(n't-n'''t+\varepsilon'-\varepsilon''') & \quad 7\end{array}\right.$$

[6849]

Inequalities of the radius vector of the fourth satellite.

$$+\mathrm{m}''.\left\{\begin{array}{ll}+0{,}00114435_c & \quad 8\\ +0{,}00326071.\cos.\ (n''t-n'''t+\varepsilon''-\varepsilon''') & \quad 9\\ +0{,}00057836.\cos.2.(n''t-n'''t+\varepsilon''-\varepsilon''') & \quad 10\\ 0{,}00013614.\cos.3.(n''t-n'''t+\varepsilon''-\varepsilon''') & \quad 11\end{array}\right.$$

$$-0{,}00006290_c \qquad 12$$

$$-0{,}00037741.\cos.(2Mt-2n'''t+2E-2\varepsilon'''). \qquad 13$$

$$\delta r'''=-\tfrac{1}{6}.\frac{M^2}{n'''^2}.a'''+\tfrac{1}{6}.a'''.\left\{m.\left(b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right)+m'.\left(b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right)+m''.\left(b_{\frac{1}{2}}^{(0)}+\alpha.\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right)\right\}. \qquad [6849a']$$

In the coefficient of m, we must use the values of α, $b_{\frac{1}{2}}^{(0)}$, $\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}$ [6810, 6812, 6813]; in those of m' the values [6819, 6821, 6822]; in those of m'' the values [6823, 6825, 6826]; also those of a''', M [6800, 6840]. Hence we get, [6849b]

$$\delta r'''=+0{,}00088138.\mathrm{m}+0{,}00093964.\mathrm{m}'+0{,}00114435.\mathrm{m}''-0{,}00006290. \qquad [6849c]$$

In the original work, the corresponding numbers are,

$$\delta r'''=-0{,}00088152.\mathrm{m}-0{,}00093981.\mathrm{m}'-0{,}00114443.\mathrm{m}''+0{,}00037741; \qquad [6849d]$$

all of them having wrong signs, and the term depending on M^2 being six times too great. Professor Airy computed some of the coefficients of [6848, 6849], in the sixth volume of the Memoirs of the Royal Astronomical Society. He makes the coefficient in [6848 line 6] equal to $-32'',016$; and those in [6848 lines 7, 11; 6849 lines 10, 13], the same as the author. [6849e] I have re-computed the coefficient in [6848 line 6], using the data [6823—6826, 6784], as given by the author, and find it to be $-32'',0439$; which is inserted in this translation, instead of $-35'',4372$ in the original work; the letter $_c$ being annexed to denote that it has been so changed [6021i]. [6849f] The coefficient of the term [6849 line 9] is found by Mr. Airy to be 0,00317865. I have not examined, with much care, any of these numbers; but appearances indicate that they have not been computed with the strictest accuracy, to the full number of places of decimals inserted in these tables, though they are quite sufficient for all practical purposes; taking into consideration that the whole radius vector r''' in seconds $636620''$, appears, when viewed from the earth, under an angle of only $1530''$ [6785]; [6849g] therefore an arc of $1200''$ in $\delta v'''$ [6848], would appear to us under an angle of less than three centesimal or one sexagesimal second; and an arc of $1200''$ is described by the first satellite in its orbit in less than a sexagesimal minute of time; [6849h] so that an inequality of a few seconds in $\delta v'''$, is of no consequence in a practical point of view. [6849i]

22. *We shall now consider the perturbations depending upon the excentricities of the orbits, and shall represent the inequalities of* δv, $\delta v'$, $\delta v''$, *depending on the angle* $nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma$, *in the following manner, as in* [6725, 6726, 6739];

[6850] $$\delta v = Q.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma);$$

[6851] $$\delta v' = Q'.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma);$$

[6852] $$\delta v'' = Q''.\sin.(nt-2n't+\varepsilon-2\varepsilon'+gt+\Gamma).$$

We shall have, as in [6261, 6268, 6269],*

[6853] $$Q=-\frac{3m'.n^2}{2.(n-2n'+g)^2}.\left\{Fh+\frac{a}{a'}.Gh'\right\};$$

Values of Q, Q', Q''.

[6854] $$Q'=\frac{3n'^2}{(n-2n'+g)^2}.\left\{\begin{array}{l}+m.\left\{\frac{a'}{a}.Fh+Gh'\right\}\\ +\frac{1}{2}m''.\left\{F'h'+\frac{a'}{a''}.G'h''\right\}\end{array}\right\};$$

[6855] $$Q''=-\frac{3m'.n''^2}{(n-2n'+g)^2}.\left\{\frac{a''}{a'}.F'h'+G'h''\right\}.$$

We have given, in § 4, the analytical values of F, G, F', G'; hence we obtain the following numerical values; †

[6853a] * (3488) Putting the values of δv [6261, 6850] equal to each other, we get the expression of Q [6853]. In like manner, by comparing [6268, 6851], we get [6854];
[6853b] and from [6269, 6852], we obtain [6855], by using $n'-2n''=n-2n'$ [6154].

† (3489) From [996, 999] we get,

[6856a] $$a\,A^{(2)}=-\alpha.b_{\frac{1}{2}}^{(2)};\qquad a^2.\left(\frac{dA^{(2)}}{da}\right)=-\alpha^2.\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha};$$

substituting these in [6130] it becomes,

[6856b] $$F=\frac{2n}{n-n'}.\alpha.b_{\frac{1}{2}}^{(2)}+\alpha^2.\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha};$$

[6856c] and by using the numerical values [6783, 6782, 6801, 6803 line 1, 6804 line 1], we get
[6856d] [6856]. Substituting $-a'^2.\left(\frac{dA^{(1)}}{da'}\right)=a'A^{(1)}+aa'.\left(\frac{dA^{(1)}}{da}\right)$ [6212a] in the expression of G [6142], we get [6856f], which can be reduced to the form [6856g] by the substitution of,

[6856e] $$A^{(1)}=\frac{a}{a'^2}-\frac{1}{a'}.b_{\frac{1}{2}}^{(1)}\text{ [997]};\qquad \left(\frac{dA^{(1)}}{da}\right)=\frac{1}{a'^2}.\left\{1-\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}\right\}\text{ [1000]};$$

and α [6304]. The coefficient of α, in this last expression, vanishes; consequently it becomes as in [6856h];

[6856f] $$G=\frac{(3n'-n)}{n-n'}.\frac{a'^2}{a^2}-\frac{2n}{n-n'}.\frac{a}{a'}+\frac{(n+n')}{n-n'}.a'A^{(1)}+aa'.\left(\frac{dA^{(1)}}{da}\right)$$

[6856g] $$=\frac{(3n'-n)}{n-n'}.\frac{1}{\alpha^2}-\frac{2n}{n-n'}.\alpha+\frac{(n+n')}{n-n'}.\{\alpha-b_{\frac{1}{2}}^{(1)}\}+\alpha.\left\{1-\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}\right\}$$

[6856h] $$=\frac{(3n'-n)}{n-n'}.\frac{1}{\alpha^2}-\frac{(n+n')}{n-n'}.b_{\frac{1}{2}}^{(1)}-\alpha.\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}.$$

$F = \quad 1,483732$; $\qquad$ log. $= 0,1713555$; [6856]

$G = -0,857159$; [or $-0,856159_c$]; $\qquad$ log. $= 9,9330614$, or $9,9325544_c$; [6857]

$F' = \quad 1,466380$; $\qquad$ log. $= 0,1662466$; [6858]

$G' = -0,855370$. $\qquad$ log. $= 9,9321540$. [6859]

Now if we substitute the numerical values of n, n', &c. [6856c], we find that this last expression of G becomes $G = -0,856159$, instead of $-0,857159$, given by the author in [6857]. This causes a variation in the value of G of about $\frac{1}{856}$ part, which must produce a small change in the numerical values of Q, Q' [6860, 6861], and in those of $(2Q'-Q)$, $(2Q''-Q')$ [6744—6746]; consequently also in the fundamental equations [7170—7172]. To estimate roughly the effect of this error, we shall select the equation [6745]; and if we substitute in it the values of Q, Q' [6860, 6861], it will produce the terms in [7171] having the divisor $\left(1+\frac{g}{3001300}\right)^2$; and as this divisor is nearly equal to unity [6025o], we may neglect it in this rough estimate, and then the corresponding terms of [7171] will become $-17495'',31h - 49185'',95h' + 20804'',40h''$. Now though the error in the value of G [6856k] does not affect all the terms which are used in computing these coefficients, yet we can estimate the order or comparative magnitude of this correction, by supposing all the terms of [6856m] to be varied $\frac{1}{856}$ part. This produces $-20''.h - 57''.h' + 24.h''$; and by substituting the values of h, h'' [7184, 7185], it becomes nearly equal to $-57''.h'$; corresponding to a change in the annual motion of the perijove of the second satellite g_1 [7183], by a quantity of the same order as the coefficient of h', namely, $57''$; and this may be considered as no importance, because the excentricity of the orbit of the second satellite is insensible [6057g]. Finally, mistakes of this kind cannot be fully rectified without going over the whole of the subsequent calculations of the theory of the satellites. This seems hardly worth the labor, considering that the result will not be very materially improved by repeating the calculations; because the imperfect data we now have, for ascertaining the motions of the satellites, will probably produce much greater variations in the values of the elements of the orbits, than those which arise from this small error in the calculation. [6856i] [6856k] [6856l] [6856m] [6856n] [6856o] [6856p] [6856q] [6856r]

If we compare the values of F, F' [6130, 6147], we find that F' can be derived from F, *by changing the elements of* m *into those of* m', *and the elements of* m' *into those of* m''; and by making the same changes in the expression of F [6856b], we get, [6856s]

$$F' = \frac{2n'}{n'-n''}\cdot \alpha . b_{\frac{1}{2}}^{(2)} + \alpha^2 . \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha}. \qquad [6856t]$$

Substituting, in this expression, the numerical values of n', n'', α, $b_{\frac{1}{2}}^{(2)}$, $\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha}$, [6783, 6784, 6814, 6816, 6817], we get $F' = 1,466380$, as given by the author in [6858]. [6856u]

Again, by comparing the expressions of G, G' [6137, 6163], we find that G' can be derived from G, by changing the elements as in [6856s]; and by making the same changes in [6856h], we get, [6856v]

By means of these values, we find,*

[6860] $$Q = -\mathrm{m}' . \frac{\{16{,}850204.h - 6{,}118274.h'\}}{\left(1 + \frac{g}{3001300''}\right)^2};$$

[6861] $$Q' = +\mathrm{m} . \frac{\{13{,}307450.h - 4{,}831907.h'\}}{\left(1 + \frac{g}{3001300''}\right)^2};$$

$$+\mathrm{m}'' . \frac{\{4{,}133080.h' - 1{,}511467.h''\}}{\left(1 + \frac{g}{3001300''}\right)^2};$$

[6862] $$Q'' = -\mathrm{m}' . \frac{\{3{,}248934.h' - 1{,}188133.h''\}}{\left(1 + \frac{g}{3001300''}\right)^2}.$$

We shall determine the quantities h, h', h'', h''', and g, by means of the

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[6856w] $$G' = \frac{(3n'' - n')}{n' - n''} . \frac{1}{\alpha^2} - \frac{(n' + n'')}{n' - n''} . b_{\frac{1}{2}}^{(1)} - \alpha . \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha};$$

[6856x] Substituting the values of n', n'', &c., which are mentioned in [6856u], we get
[6856y] $G' = -0{,}854990$, differing a little from that of the author [6859], namely, $G = -0{,}855370$.

[6859a] * (3490) We have, in [6694a], $n - 2n' = 3001156''$; therefore the expression $(n - 2n' + g)^2$, which occurs in [6853—6855], may be put under the form,

[6859b] $$(n - 2n')^2 . \left\{ 1 + \frac{g}{3001156''} \right\}^2;$$

substituting this and $m' = \frac{\mathrm{m}'}{10000}$ [6841c] in [6853], it becomes,

[6859c] $$Q = -\frac{\mathrm{m}'}{\left\{ 1 + \frac{g}{3001156''} \right\}^2} . \left\{ \frac{3F.n^2}{20000.(n - 2n')^2} . h + \frac{3G.n^2}{20000.(n - 2n')^2} . \frac{a}{a'} . h' \right\}.$$

[6859d] The coefficients of h, h', may be reduced to numbers, by means of the values of n, n', $\frac{a}{a'}$, F, G, [6025k, 6801, 6856, 6857]; and then the expression of Q becomes of the same form as in [6860]. In like manner, we may reduce the values of Q', Q''
[6859e] [6854, 6855], to the forms [6861, 6862], by using the values [6859d, &c.], together with those in [6814, 6858, 6859]. There are several small errors in the numbers given by the
[6859f] author, in the expressions [6860—6862]. For example, the divisor of g in the denominator is $3001300''$, instead of $3001156''$ [6859c]. The coefficient of h, in [6860], is $16{,}853$; that of h' is $-6{,}112$; differing in the third decimal from the values of the
[6859g] author; and similar discrepancies are found in the coefficients of h, h', h'' [6861, 6862]; but, like the terms mentioned in the last note [6856k, &c.], they do not seem to be of
[6859h] sufficient importance to affect the results, in any sensible manner.

equations [6744—6747]. To reduce these equations to numbers, we shall observe that the value $\rho-\frac{1}{2}\varphi=0{,}0217794$ [6827], given by the first approximation, being liable to some degree of uncertainty, we shall put,

$$\rho-\tfrac{1}{2}\varphi=\mu.0{,}0217794;$$ [6863]

μ *being an indeterminate coefficient.* We shall then obtain, from [6217c, d],* [6863′]

Symbol μ.

$(0)=553878'',76.\mu$;	$\boxed{0}=103'',27$;	1
$(1)=109003'',20.\mu$;	$\boxed{1}=207'',29$;	2
$(2)=\ \ 21264'',89.\mu$;	$\boxed{2}=417'',63$;	3
$(3)=\ \ \ 2946'',95.\mu$;	$\boxed{3}=974'',19$.	4

[6864] Numerical values of the symbols.

$(0,1)=\mathrm{m}'.39826'',00$;	$\boxed{0,1}=\mathrm{m}'.\ 29516'',02$; †	1
$(0,2)=\mathrm{m}''.\ 5205'',05$;	$\boxed{0,2}=\mathrm{m}''.\ \ 2511'',39$;	2
$(0,3)=\mathrm{m}'''.\ \ 767'',12$;	$\boxed{0,3}=\mathrm{m}'''.\ \ \ 213'',46$;	3

[6865]

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* (3491) Substituting in [6217c, d], the values of $\rho-\frac{1}{2}\varphi$ [6863], n, n', n'', n''', M [6025k, m], and a, a', a'', a''' [6797—6800], we obtain the expressions [6864]. Upon examination, these numbers were found to be correctly computed. [6864a]

† (3492) Substituting $m'=\frac{\mathrm{m}'}{10000}$ [6841c] in (0,1) [6213], and using the values of n, α, $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$ [6025k, 6801, 6802], we get (0,1) [6865]. In like manner, from [6214] we get $\boxed{0,1}$ [6865]. By the same process, using the values [6806, 6807], we obtain (0,2), $\boxed{0,2}$ [6865]; and with the values [6810, 6811], we compute (0,3), $\boxed{0,3}$ [6865]. If we use the values [6814, 6815], we shall get (1,2), $\boxed{1,2}$; and with [6819, 6820], we get (1,3), $\boxed{1,3}$. Lastly, by using the values [6823, 6824], we get (2,3), $\boxed{2,3}$ [6867]. [6865a] [6865b]

If we divide the equations [1093, 1094] by $m'\sqrt{a'}$, and substitute the value of $\frac{a}{a'}=\alpha$ [6304], we get the expressions of (1,0), $\boxed{1,0}$ [6865d]. By changing the [6865c]

$(1,0) = \mathrm{m}.\ 31573'',71\,; \qquad \boxed{1,0} = \mathrm{m}.\ 23400'',04\,;$ 1

[6866] $(1,2) = \mathrm{m}''.19566'',65\,; \qquad \boxed{1,2} = \mathrm{m}''.14469'',66\,;$ 2

$(1,3) = \mathrm{m}'''.\ 1804'',18\,; \qquad \boxed{1,3} = \mathrm{m}'''.\ \ 790'',56\,;$ 3

$(2,0) = \mathrm{m}.\ \ 3267'',32\,; \qquad \boxed{2,0} = \mathrm{m}.\ \ 1576''.46\,;$ 1

[6867] $(2,1) = \mathrm{m}'.15492'',62\,; \qquad \boxed{2,1} = \mathrm{m}'.11456'',90\,;$ 2

$(2,3) = \mathrm{m}'''.\ 5886'',85\,; \qquad \boxed{2,3} = \mathrm{m}'''.\ 3995'',03.$ 3

$(3,0) = \mathrm{m}.\ \ \ 363'',10\,; \qquad \boxed{3,0} = \mathrm{m}.\ \ \ 101'',03\,;$ 1

[6868] $(3,1) = \mathrm{m}'.\ 1077'',15\,; \qquad \boxed{3,1} = \mathrm{m}'.\ \ 471'',99\,;$ 2

$(3,2) = \mathrm{m}''.\ 4438'',87\,; \qquad \boxed{3,2} = \mathrm{m}''.3012'',37.$ 3

Hence the equations [6744—6747] become,*

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elements of m' into those of m'', we get [6865e], and so on for other forms;

[6865d] $(1,0) = (0,1).\frac{\mathrm{m}\sqrt{a}}{\mathrm{m}'\sqrt{a'}} = (0,1).\sqrt{\bar{\mathrm{a}}}.\frac{\mathrm{m}}{\mathrm{m}'}\,; \qquad \boxed{1,0} = \boxed{0,1}.\frac{\mathrm{m}\sqrt{a}}{\mathrm{m}'\sqrt{a'}} = \boxed{0,1}.\sqrt{\bar{\mathrm{a}}}.\frac{\mathrm{m}}{\mathrm{m}'}\,;$

[6865e] $(2,0) = (0,2).\frac{\mathrm{m}\sqrt{a}}{\mathrm{m}''\sqrt{a''}}\,; \qquad \boxed{2,0} = \boxed{0,2}.\frac{\mathrm{m}\sqrt{a}}{\mathrm{m}''\sqrt{a''}}\,;$

[6865f] $(3,0) = (0,3).\frac{\mathrm{m}\sqrt{a}}{\mathrm{m}'''\sqrt{a'''}}\,; \qquad \boxed{3,0} = \boxed{0,3}.\frac{\mathrm{m}\sqrt{a}}{\mathrm{m}'''\sqrt{a'''}}\,;$

[6865g] $(2,1) = (1,2).\frac{\mathrm{m}'\sqrt{a'}}{\mathrm{m}''\sqrt{a''}}\,; \qquad \boxed{2,1} = \boxed{1,2}.\frac{\mathrm{m}'\sqrt{a'}}{\mathrm{m}''\sqrt{a''}}\,;$ &c.

[6865h] Hence we easily obtain the values of the expressions [6865—6868]. Upon re-computing the first numbers in [6865 line 1, 6866 line 1], they were found to agree nearly with the results of the author; the other expressions [6865—6868] were not examined.

[6868a] * (3493) Substituting in the four equations [6744—6747] the numerical values of the functions [6864—6868], we obtain, without any reduction, the corresponding parts, in [6868b] lines 1, 2, of the equations [6869—6872]. The terms of the equations [6869—6871], which contain the denominator, $\left(1+\frac{g}{3001300''}\right)^2$, are produced by means of the values of [6868c] Q, Q', Q'' [6860—6862]. Thus in the equation [6744] we have,

$$0=\{g-553878'',78.\mu-103'',27-39826'',00.m'-5205'',05.m''-767'',12.m'''\}.h \quad 1$$
$$+29516'',02.m'h'+2511'',39.m''h''+213'',46.m'''h''' \quad 2 \qquad [6869]$$
$$\frac{-\{688803''.m+436089''.m'\}.m'h+\{250103''.m+158343''.m'-213931''.m''\}.m'h'+78234''.m'm''.h''}{\left(1+\frac{g}{3001300''}\right)^2}; \quad 3$$

$$0=\{g-109003'',20.\mu-207'',29-31573'',71.m-19566'',65.m''-1804'',18.m'''\}.h' \quad 1$$
$$+23400'',04.mh+14469'',66.m''h''+790'',56.m'''h''' \quad 2$$
$$+\frac{\{226503''.m+143201''.m'-196037''.m''\}.mh}{\left(1+\frac{g}{3001300''}\right)^2} \quad 3 \qquad [6870]$$
$$-\frac{\{82242''.m^2+52068''.mm'-140696''.mm''+94603''.m'm''+60174''.m''^2\}.h'}{\left(1+\frac{g}{3001300''}\right)^2} \quad 4$$
$$+\frac{\{-25726''.m+34596.m''+22518''.m''\}}{\left(1+\frac{g}{3001300''}\right)^2}.m''h''; \quad 5$$

$$0=\{g-21264'',89.\mu-417'',63-3267'',32.m-15492'',62.m'-5886'',65.m'''\}.h'' \quad 1$$
$$+1576'',46.mh+11456'',90.m'h'+3995'',03.m'''h''' \quad 2 \qquad [6871]$$
$$+\frac{57703''.mm'h-\{20952''.m-28176''.m'-17921''.m''\}.m'h'-\{10279''.m'+6554''.m''\}.m'h''}{\left(1+\frac{g}{3001300''}\right)^2}; \quad 3$$

$$0=\{g-2946'',95.\mu-974'',19-363'',10.m-1077'',15.m'-4438'',87.m''\}.h''' \quad 1$$
$$+101'',03.mh+471'',99.m'h'+3012'',37.m''h''. \quad 2 \qquad [6872]$$

$$\tfrac{1}{4}m'.n.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.F.\{-2Q'+Q\}; \qquad [6868c']$$

and if we substitute, in this expression, the values of Q, Q' [6860, 6861], we shall obtain, for the numerator of the terms which have the divisor $\left\{1+\frac{g}{3001300''}\right\}^2$, the following function; [6868d]

$$\tfrac{1}{4}m'.n.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.F.\left\{\begin{array}{l}-\{26,61490.m+16,8502044.m'\}.h\\ +\{\ 9,663814.m+6,118274.m'-8,266160.m''\}.h'\\ +\ \ 3,022934.m''h''\end{array}\right\}. \qquad [6868e]$$

Now from [6025k, l] we have $2n-2n'-N=3570453''$, also $(0)=553879''$ nearly, [6868f]
[6864]; hence $1-\frac{(0)}{2n-2n'-N}=\frac{3016574}{3570453}$; multiplying this by $n=825826010''$ [6868g]
[6025k], and by $\frac{1}{4}m'.F=0,370933.m'=0,0000370933.m'$ [6856, 6841c], we get [6868h]
$\frac{1}{4}m'.n.\left\{1-\frac{(0)}{2n-2n'-N}\right\}.F=25880'',625.m'$. Substituting this in the function [6868e], [6868i]
it becomes,

$$-\{688810''.m+436094''.m'\}.m'.h+\{250106''.m+158345''.m'-213933''.m''\}.m'.h'+78235''.m'.m''.h''. \qquad [6868k]$$

[6873] When the masses m, m', m'', m''', shall be known, and also the indeterminate quantity μ [6863'], we shall have, by solving the equations [6869—6872], four values of g, and the corresponding ratios of the indeterminate quantities h, h', h'', h''', to one of them, which will remain indeterminate. *These four systems of* g, h, h', h'', h''', *will give the same*
[6873'] *number of values of* Q, Q', Q'' [6860—6862].

Moreover, *the sun's action adds to the motion of the satellite* m, *the two inequalities,*

[6874] $$\delta v = -\frac{15.M^2.h}{2n.(2M+N-n-g)}.\sin.(nt-2Mt+\varepsilon-2E+gt+\Gamma)$$

[6875] $$-\frac{3M}{n}.H.\sin.(Mt+E-I).$$

[6876] We may substitute* $N=n+g$, in the term [6874], which is only sensible

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[6868l] These numerical coefficients differ a few seconds from those given by the author in [6869 line 3]; but as they are multiplied by the small factors $\mathrm{m}\mathrm{m}'=0{,}04$, $\mathrm{m}'^2=0{,}06$, $\mathrm{m}'\mathrm{m}''=0{,}2$, nearly [6841$b, c, d$]; and also by h, h', or h'', which is of the same order as the excentricities [6057e, 6235i—m], the differences become nearly insensible. In
[6868m] the original work, the term $-213931''.\mathrm{m}''$ [6869 line 3], is printed with a wrong sign $+$; we have given it correctly in [6869]; and we may observe that it is merely a typographical mistake; since the author has used the sign $-$ in reducing it to numbers in [7170 line 1], where it makes a part of the term $-25371'',60$; the negative sign being derived from
[6868n] the part now under consideration, since the other two terms, of the coefficient of h' [6868e line 2], are positive. In the Mémoires de l'Académie Royale de Paris, 1788,
[6868o] page 359, where the author first published the theory of Jupiter's satellites, he gives the sign of this term correctly. The numbers in [6868k] are somewhat affected by the error
[6868p] in the value of G [6856i], and by the small differences in the values mentioned in
[6868q] [6859f, &c.]. The same is to be observed relative to the equations [6870, 6871], deduced
[6868r] from [6745, 6746], by the same methods of substitution. Greater differences than those which are mentioned in [6868l], are found in some of the terms of [6870 lines 3, 4, 5; 6871 line 3]; and though these quantities are much decreased by
[6868s] being connected with very small factors of the forms m^2h, $mm'h$, m''^2h', &c., yet they may not perhaps be considered as absolutely insensible. We have not, however, attempted to correct these small mistakes, for the reasons mentioned in [6856r]; considering them as
[6868t] far within the limits of the errors of the elements which are used as the basis of the calculation.

[6874a] * (3494) This assumed value of N gives $N-n-g=0$; consequently the divisor $2M+N-n-g$ [6874] is reduced to its first term $2M$; upon the supposition that $N-n-g$ is so small, in comparison with $2M$, that it may be neglected; and then the

in the third and fourth satellites;* by this means the coefficient of this [6876′]

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coefficient of the term [6874], corresponding to the first satellite, becomes simply $-\frac{15M}{4n}\cdot h = -0{,}0015312.h$ [6877]. The similar coefficients relative to the other [6874b]
satellites are easily deduced from this by accenting the letters, as in [6878—6880]. We shall see, in the next note, that these assumed values $N-n''-g_2=0$, and $N-n'''-g_3=0$, [6874c]
can be used for the third and fourth satellites; but we could not use $N-n-g=0$, $N-n'-g_1=0$, for the first and second satellites, if h, h' were of any sensible magnitude; [6874d]
because $N-n-g$, instead of vanishing, is nearly equal to $-4M$; as is evident from the values of N, n, g [6025k, l, o]. Now the object of the author being merely to show that the terms [6874], relative to the first and second satellites, are so unimportant that they [6874e]
may be neglected, taking into consideration the smallness of the values of h, h', it will be sufficiently accurate, for this purpose, to suppose that $2M+N-n-g$ is of the same [6874f]
order as $2M$; and this method is adopted in [6876, &c.], and in the following note.

* (3495) Each of the four values of g [6025o], has corresponding values of h, h', h'', h''', as in [7176—7179, 7183—7186, 7190—7193, 7195—7198]; making in [6875a]
all *sixteen* different values of h, h', &c., and producing *sixteen* inequalities of the form [6874]. Most of these inequalities are very small, as we may easily perceive by noticing [6875b]
only the largest term, corresponding to each angle. Thus with the angle $g=606989'',9$ [7176], we need only notice h [7179′]; and from the values of M, N, n, g [6025i—o], [6875c]
it is evident that, by neglecting the signs, we may consider $2M+N-n-g$ as of the [6875d]
order $2M$, as in [6874f]; so that the coefficient of the term [6874] becomes of the same order as $-\frac{15M}{4n}\cdot h = -0{,}0015.h$ nearly [6874b]; and as h is of the same order as e [6875e]
[6236], or of the order $20''$ [6057e], this becomes of the order $0{,}0015\times 20'' = 0'',03$; which is insensible. Again, with the angle $g_1 = 178141'',7$ [7183], we need only notice [6875f]
h' [7186′], and then the coefficient of the term [6874] becomes $-\frac{15M^2.h'}{2n'.(2M+N'-n'-g_1)}$, [6875g]
which may be supposed, as in [6875e], to be of the order $-\frac{15M}{4n'}\cdot h' = -0{,}003.h'$ nearly, [6878]; and as h' is of the order e' or $183''$ [6057e], this becomes of the order $0'',5$, [6875h]
which is insensible. The same process with the angle g_2 [7190] and h'' [7193′], gives a coefficient of a sensible magnitude $-\frac{15M^2.h''}{2n''.(2M+N''-n''-g_2)} = -\frac{15M.h''}{4n''}$ nearly;
observing that $N''-n''-g_2 = -58272''$ [6025k, l, o] is quite small in comparison with [6875i]
$2M = 674422''$ [6025m], so that we may change $2M+N''-n''-g_2$ into $2M$. Finally, the angle $g_3 = 7959'',105$ [7195] with the term h''' [7198′], gives in [6874] the
coefficient $-\frac{15M^2.h'''}{2n'''.(2M+N'''-n'''-g_3)} = -\frac{15M.h'''}{4n'''}$ nearly; because $N'''-n'''-g_3 = -16100''$ [6875k]
[6025k, l, o] is so small in comparison with $2M$ [6875i], that it may be neglected. [6875l]

[6876''] inequality becomes equal to $-\frac{15M}{4n}.h$. Then we have, by using the values of n, n', n'', M [6782, 6783, 6784, 6840],

[6877] $$\text{I. Satellite;}\quad \frac{15M}{4n}=0,0015312;$$

[6878] $$\text{II. Satellite;}\quad \frac{15M}{4n'}=0,0030737;$$

[6879] $$\text{III. Satellite;}\quad \frac{15M}{4n''}=0,0061926;$$

[6880] $$\text{IV. Satellite;}\quad \frac{15M}{4n'''}=0,0144449.$$

We have observed, in [6696—6698], relative to the inequality [6875], that it is modified by the reciprocal action of the satellites. We can then take,
[6881] for $H.\sin.(Mt+E-I)$, the half of the first term of the equation of the centre of Jupiter; and this term is equal to the following expression; *

[6882] $$61208'',23.\sin.(Mt+E-I).$$

This being supposed, we shall have, by noticing only the preceding inequality,

[6883] $$\delta v=-37'',49.\left\{1+\frac{3\mathrm{k}n^2}{(M^2-\mathrm{k}n^2).\left\{1+\frac{9a'.\mathrm{m}}{4a.\mathrm{m}'}+\frac{a''.\mathrm{m}}{4a.\mathrm{m}''}\right\}}\right\}.\sin.(Mt+E-I);$$

[6884] $$\delta v'=-74'',98.\left\{1-\frac{9a'.\mathrm{m}.\mathrm{k}n^2}{8a.\mathrm{m}'.(M^2-\mathrm{k}n^2).\left\{1+\frac{9a'.\mathrm{m}}{4a.\mathrm{m}'}+\frac{a''.\mathrm{m}}{4a.\mathrm{m}''}\right\}}\right\}.\sin.(Mt+E-I);$$

[6885] $$\delta v''=-149'',96.\left\{1+\frac{3a''.\mathrm{m}.\mathrm{k}n^2}{32a.\mathrm{m}''.(M^2-\mathrm{k}n^2).\left\{1+\frac{9a'.\mathrm{m}}{4a.\mathrm{m}'}+\frac{a''.\mathrm{m}}{4a.\mathrm{m}''}\right\}}\right\}.\sin.(Mt+E-I);$$

[6886] $$\delta v'''=-353'',69_c.\sin.(Mt+E-I).$$

[6882a] * (3496) The equation of the centre is $2e.\sin.(nt+\varepsilon-\varpi)$ [669]; and by changing the notation so as to correspond to [6274—6277], it becomes $2H.\sin.(Mt+E-I)$. The
[6882b] symbol H is the same as $e^{\text{iv}}=0,04807670$ [4080]; hence $2H$, in seconds, is $2\times0,04807670\times636620''=61213'',1=2H$, as in [6882] nearly. Now taking H
[6882c] equal to half this value, we get $H=30606'',5$; and by substituting it in [6696—6698], with the values of n, n', n'', M [6025k, m], they become as in [6883—6885], respectively. Lastly, the term [6875], corresponding to the fourth satellite, is
[6882d] $-\frac{3M.H}{n'''}.\sin.(Mt+E-I)$; and by using the preceding values of M, H, it becomes $-353'',69.\sin.(Mt+E-I)$, which differs a little from that of the author, whose coefficient
[6882e] is $-349'',79$. We have given the corrected value in [6886], annexing the letter $_c$ as in [6021i] to denote that it has been changed.

23. *We shall now determine the inequalities of the motions of the satellites in latitude.* The equations between λ, λ', λ'', λ''' [6347—6350], become,*

$$0=\{553878'',76.\mu+103'',27+39826'',00.m'+5205'',05.m''+767'',12.m'''\}.\lambda \quad 1$$
$$-39826'',00.m'.\lambda'-5205'',05.m''.\lambda''-767'',12.m'''.\lambda'''-103'',27\,; \quad 2 \qquad [6887]$$

$$0=\{109003'',20.\mu+207'',29+31573'',71.m+19566'',65.m''+1804'',18.m'''\}.\lambda' \quad 1$$
$$-31573'',71.m.\lambda-19566'',65.m''.\lambda''-1804'',18.m'''.\lambda'''-207'',29\,; \quad 2 \qquad [6888]$$

$$0=\{21264'',89.\mu+417'',63+3267'',32.m+15492'',62.m'+5886'',85.m'''\}.\lambda'' \quad 1$$
$$-3267'',32.m.\lambda-15492'',62.m'.\lambda'-5886'',85.m'''.\lambda'''-417'',63\,; \quad 2 \qquad [6889]$$

$$0=\{2946'',95.\mu+974'',19+363'',10.m+1077'',15.m'.+4438'',87.m''\}.\lambda''' \quad 1$$
$$-363'',10.m.\lambda-1077'',15.m'.\lambda'-4438'',87.m''.\lambda''-974'',19. \quad 2 \qquad [6890]$$

The equations [6418—6421], between l, l', l'', l''', p, become,

$$0=\{p-553878'',76.\mu-103'',27-39826'',00.m'-5205'',05.m''-767'',12.m'''\}.l \quad 1$$
$$+39826'',00.m'.l'+5205'',05.m''.l''+767'',12.m'''.l'''\,; \quad 2 \qquad [6891]$$

$$0=\{p-109003'',20.\mu-207'',29-31573'',71.m-19566'',65.m''-1804'',18.m'''\}.l' \quad 1$$
$$+31573'',71.m.l+19566'',65.m''.l''+1804'',18.m'''.l'''\,; \quad 2 \qquad [6892]$$

$$0=\{p-21264'',89.\mu-417'',63-3267'',32.m-15494'',62.m'-5886'',85.m'''\}.l'' \quad 1$$
$$+3267'',32.m.l+15494'',62.m'.l'+5886'',85.m'''.l'''\,; \quad 2 \qquad [6893]$$

$$0=\{p-2946'',95.\mu-974'',19-363'',10.m-1077'',15.m'-4438'',87.m''\}.l''' \quad 1$$
$$+363'',10.m.l+1077'',15.m'.l'+4438'',87.m''.l''. \quad 2 \qquad [6894]$$

When μ [6863'] and the masses m, m', m'', m''', are known, we can obtain, by means of the four equations [6887—6890], the values of λ, λ', λ'', λ'''. The four equations [6891—6894] give, by elimination, an equation of the fourth degree in p. With these values of p, we may obtain, by means of the equations [6427—6430], the latitudes of the satellites above the orbit of Jupiter. [6895]

We have seen, in [6362, or 6427], that *the part of the latitude* s, *which depends upon the inclination* θ' *of the equator to the orbit of Jupiter, is*

$$(\lambda-1)\,.\,\theta'.\sin.(v+\Psi'). \qquad [6896]$$

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* (3497) Substituting in [6347–6350] the values [6864–6868], we get [6887–6890]; and the like substitutions being made in [6418—6421], they become as in [6891—6894]. [6887a] Remarking, however, that the coefficient $15494'',62.m'$ [6893 lines 1, 2] should be [6887b] $15492'',62$, to conform to the value of (2,1) given in [6867], and used in [6871 line 1, 6889 lines 1, 2]; the difference of these two expressions is, however, [6887c] insensible, taking into consideration the smallness of m' [7143], as will appear in [7133f, &c.].

[6897] If we take, as in [6398], the orbit of Jupiter in 1750 for the fixed plane,
[6898] and the vernal equinox of this planet, at the same epoch, for the orgin of v and ϖ', we shall have, as in [6405],

[6899] $$\theta' = {}^{1}L + bt\,;$$

[6900] $$\varpi' = {}^{1}pt - \frac{at}{{}^{1}L}\,.$$

We shall determine a and b by means of the following equations, which depend on the differential formulas [1132, &c.]; *

[6901] $$a = \quad (4,5)\,.\,I.\cos.\Pi + (4,6)\,.\,I'.\cos.\Pi'\,;$$

[6902] $$b = -(4,5)\,.\,I.\sin.\Pi - (4,6)\,.\,I'.\sin.\Pi'\,;$$

the values of (4,5) and (4,6) being given in [4235]. We must however
[6903] decrease, in the first of these values, the mass of Saturn, in the ratio of

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[6901a] * (3498) If we take the values of $\frac{dp^{\text{iv}}}{dt}$, $\frac{dq^{\text{iv}}}{dt}$ [1132], we shall find that they contain the quantities (4,0), (4,1), (4,2), (4,3), (4,5), (4,6); and that all of them, except (4,5), (4,6), may be neglected on account of their smallness; hence we get,

[6901b] $$\frac{dp^{\text{iv}}}{dt} = \quad (4,5).\{q^{\text{v}} - q^{\text{iv}}\} + (4,6).\{q^{\text{vi}} - q^{\text{iv}}\}\,;$$

[6901c] $$\frac{dq^{\text{iv}}}{dt} = -(4,5).\{p^{\text{v}} - p^{\text{iv}}\} - (4,6).\{p^{\text{vi}} - p^{\text{iv}}\}.$$

[6901d] Now we have, as in [1032], $p^{\text{iv}} = \text{tang}.\varphi^{\text{iv}}.\sin.\theta^{\text{iv}}$, $q^{\text{iv}} = \text{tang}.\varphi^{\text{iv}}.\cos.\theta^{\text{iv}}$; φ^{iv} [1030′, 1030″]
[6901e] being the inclination of the orbit of Jupiter to the fixed plane, which is represented by γ in [6313′]; and θ^{iv} [1032″] is the longitude of its ascending node, corresponding to $\daleth$ [6314], in the present notation; hence we have very nearly,

[6901f] $$p^{\text{iv}} = \gamma.\sin.\daleth\,; \qquad q^{\text{iv}} = \gamma.\cos.\daleth.$$

Substituting these expressions in [6400, 6401], we get $p^{\text{iv}} = at$, $q^{\text{iv}} = bt$; whose differentials give

[6901g] $$\frac{dp^{\text{iv}}}{dt} = a\,; \qquad \frac{dq^{\text{iv}}}{dt} = b\,;$$

hence the equations [6901*b*, *c*] become,

[6901h] $$a = \quad (4,5).\{q^{\text{v}} - q^{\text{iv}}\} + (4,6).\{q^{\text{vi}} - q^{\text{iv}}\}\,;$$

[6901i] $$b = -(4,5).\{p^{\text{v}} - p^{\text{iv}}\} - (4,6).\{p^{\text{vi}} - p^{\text{iv}}\}.$$

[6901k] Again, if we increase the accents on p, p' [1033], by four, the quantities γ, Π [1026′] will change into I, Π [6904] respectively; and we shall get, from [1033],

[6901l] $$p^{\text{v}} - p^{\text{iv}} = I.\sin.\Pi\,; \qquad q^{\text{v}} - q^{\text{iv}} = I.\cos.\Pi.$$

[6901m] In like manner, by increasing the accents on p^{v}, q^{v} [6901*l*] by unity, the symbols I, Π [6904] will change into I', Π' [6905] respectively; and the equations [6901*l*] will become,

[6901n] $$p^{\text{vi}} - p^{\text{iv}} = I'.\sin.\Pi'\,; \qquad q^{\text{vi}} - q^{\text{iv}} = I'.\cos.\Pi'.$$

Substituting the expressions [6901*l*, *n*] in [6901*h*, *i*], we get [6901, 6902] respectively.

3359,4 to 3515,6 ; as we shall see in [9121].* I is the inclination of the orbit of Saturn to that of Jupiter in 1750. Π is, at the same epoch, the longitude of the ascending node of Saturn's orbit upon that of Jupiter, and counted from Jupiter's vernal equinox. I' and Π' are the same quantities relative to Uranus. We have by observation, very nearly, [6904] [6905]

$$ {}^1L = 3^{\circ},4444 \quad [6406] ; \qquad [6906]$$

hence we deduce, as in [6903g],

$$\frac{a}{{}^1L} = 9'',0529 ; \qquad [6907]$$

$$b = 0'',070350. \qquad [6908]$$

Then we have, as in [6366],

$$ {}^1p = \frac{3}{4i} \cdot \frac{(2C-A-B)}{C} \cdot \{ M^2 + m.n^2.\lambda + m'.n'^2.\lambda' + m''.n''^2.\lambda'' + m'''.n'''^2.\lambda''' \}. \qquad [6909]$$

If we suppose Jupiter to be an ellipsoid, we shall have, as in [6335],†

$$\frac{2C-A-B}{C} = \frac{2.(\rho - \frac{1}{2}\varphi).\int_0^1 \Pi_{,}.R^2dR}{\int_0^1 \Pi_{,}.R^4dR} ; \qquad [6910]$$

$\Pi_{,}$ *being the density of a stratum of the ellipsoid whose radius is* R; R *being equal to unity at the surface of Jupiter.* If we suppose the densities of the strata of Jupiter and the Earth, at distances proportional to the [6911]

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* (3499) The value of the mass of Saturn, which is used in [4061], is $\frac{1+\mu^{v}}{3359,4}$; that in [9121] is $\frac{1}{3534,08}$; differing a little from $\frac{1}{3515,6}$, which is used in [6904]. Putting the two first expressions equal to each other, we get $1+\mu^{v} = \frac{3359,4}{3515,6}$; substituting this in [4235 line 5], we get $(4,5) = 7^{s},3597 = 22'',715$; and by putting $\mu^{vi} = 0$, in [4235 line 6], we have $(4,6) = 0^{s},096647 = 0'',298294$. Again, by substituting the values [1032, &c.] in [6901 l, n], we get, [6903a] [6903b] [6903c]

$$I.\sin.\Pi = \text{tang}.\varphi^{v}.\sin\theta^{v} - \text{tang}.\varphi^{iv}.\sin.\theta^{iv} ; \quad I.\cos.\Pi = \text{tang}.\varphi^{v}.\cos.\theta^{v} - \text{tang}.\varphi^{iv}.\cos.\theta^{iv} ; \qquad [6903d]$$

$$I'.\sin.\Pi' = \text{tang}.\varphi^{vi}.\sin.\theta^{vi} - \text{tang}.\varphi^{iv}.\sin.\theta^{iv} ; \quad I'.\cos.\Pi' = \text{tang}.\varphi^{vi}.\cos.\theta^{vi} - \text{tang}.\varphi^{iv}.\cos.\theta^{iv}. \qquad [6903e]$$

Substituting, in these last expressions, the values of φ^{iv}, φ^{v}, φ^{vi}; θ^{iv}, θ^{v}, θ^{vi}, [4082, 4083], we get the numerical values of $I.\sin.\Pi$, $I.\cos.\Pi$, $I'.\sin.\Pi'$, $I'.\cos.\Pi'$; thence we obtain, from [6901, 6902], the values of a, b. The former, being divided by ${}^1L$ [6906], gives [6907] nearly; the latter is as in [6908]; the reduction to seconds being made by multiplying by the radius in seconds, when necessary. [6903f] [6903g]

† (3500) The equation [6910] is the same as [6335]; observing that we have changed the symbol δ, which represents the density in the original work, into $\Pi_{,}$, to conform to the notation in [6335]. [6910a]

[6912] diameters of these planets, to be in a constant ratio to each other; or, in other words, if $\Pi_{\prime}$ be represented by the same function of R, for both planets, then the value of the following function,

[6913] $$\frac{\int_0^1 \Pi_{\prime} . R^2 dR}{\int_0^1 \Pi_{\prime} . R^4 dR},$$

expressed in parts of the radius of the surface of the planet taken as unity,
[6913′] will be the same for these planets. In this hypothesis, if both planets be fluid, their ellipticities will be, as in [2068k, &c.], proportional to the respective values of φ, corresponding to each of them; or to their ellipticities
[6914] if they be homogeneous.* If we suppose the same ratio to obtain in their actual state, and we have seen in [2069] that this is nearly conformable to observation, *then the values of* $\frac{2C-A-B}{C}$ *will be, for each of these planets, respectively proportional to the ellipticities corresponding to the case of*
[6915] *homogeneity.* These ellipticities, by the same article [2068‴], are as 0,10967000 to 0,00433441; hence we have,

[6916] $$\frac{2C-A-B}{C} = \left(\frac{2C-A-B}{C}\right) . \frac{10967000}{43341};$$

[6916′] the value of $\left(\frac{2C-A-B}{C}\right)$, in the second member of this equation, corresponds to the earth. This last value is represented, in [3407], by the following expression;

[6914a] * (3501) The formula [6910] relative to Jupiter may be made to correspond to that of the earth, by changing the ellipticity ρ into ρ', the centrifugal force φ into φ', and $\frac{2C-A-B}{C}$ into $\left(\frac{2C-A-B}{C}\right)$, as in [6916′]. Then we shall have for the earth,

[6914b] $$\left(\frac{2C-A-B}{C}\right) = \frac{2.(\rho'-\frac{1}{2}\varphi').\int_0^1 \Pi_{\prime} . R^2 dR}{\int_0^1 \Pi_{\prime} . R^4 dR};$$

because the value of the function [6913] is the same for both planets [6913′]. Dividing the equation [6910] by that in [6914b], and making a slight reduction, we get,

[6914c] $$\frac{2C-A-B}{C} = \left(\frac{2C-A-B}{C}\right) . \frac{\rho-\frac{1}{2}\varphi}{\rho'-\frac{1}{2}\varphi'}.$$

The theory adopted in [6912] is the same as in [2068″]; and we have in [2068k, &c.],
[6914d] $\rho : \rho' :: \varphi : \varphi' :: \rho-\frac{1}{2}\varphi : \rho'-\frac{1}{2}\varphi'$; so that we may put $\frac{\rho-\frac{1}{2}\varphi}{\rho'-\frac{1}{2}\varphi'} = \frac{\rho}{\rho'}$ in [6914c], and it will
[6914e] become $\frac{2C-A-B}{C} = \left(\frac{2C-A-B}{C}\right) . \frac{\rho}{\rho'}$. If we now substitute in this expression, for $\frac{\rho}{\rho'}$, their values corresponding to the case of homogeneity, according to the hypothesis in [6915], we shall obtain the equation [6916].

$$\left(\frac{2C-A-B}{C}\right)=\frac{0,00519323}{1+\epsilon.0,748493}. \qquad [6917]$$

Moreover we have, in [3406, 4631c],*

$$3.(1+\epsilon)=2,566; \quad \text{or} \quad \epsilon=-0,14666\ldots. \qquad [6918]$$

By means of these data we find, for Jupiter,

$$\frac{2C-A-B}{C}=0,14735. \qquad [6919]$$

The time of Jupiter's rotation is found, by observation, to be $0^{\text{days}},41377$, and that of its sideral revolution is equal to $4332^{\text{days}},6$; hence we deduce,† [6920]

$$\frac{M}{i}=\frac{0,41377}{4332,6}. \qquad [6921]$$

Therefore we shall have,

$$\begin{aligned}{}^{1}p = {} & 3'',5591 \\ & +\text{m}.\,2134'',60.\lambda \\ & +\text{m}'.\;529'',78.\lambda' \\ & +\text{m}''.\;130'',52.\lambda'' \\ & +\text{m}'''.\;23'',99.\lambda'''.\end{aligned} \qquad [6922]$$

We have found, by a first approximation,

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* (3502) We have, in [3406], $\lambda=3.(1+\epsilon)$; the letter β being changed into ϵ, so as not to interfere with the notation which is used in [7073]. We have also $\lambda=2,566$ [4631c]. Putting these values of λ equal to each other, we get [6918]; and from this we easily deduce $\epsilon=-0,14466\ldots$. If we substitute this value of ϵ in [6917], we obtain $\left(\frac{2C-A-B}{C}\right)=0,00582385$; and by using this value in the second member of [6916], we obtain the expression [6919]. [6918a] [6918b]

† (3503) Jupiter's sideral motion is represented in [6101′] by Mt; and the rotatory motion by it [6316′]. Now as these motions are inversely as their times of revolution $4332^{\text{days}},6$, $0^{\text{day}},41377$ [6920], we get $\frac{1}{i}=\frac{1}{M}\cdot\frac{0,41377}{4332,6}$, as in [6921]. Substituting this value of $\frac{1}{i}$ in ${}^{1}p$ [6909]; using also the values of n, n', n'', n''', M [6025k, m], and those of m, m', m'', m''', deduced from [6811b—e], with the value of $\frac{2C-A-B}{C}$ [6919], we get [6922]. Finally, by substituting the values [6923—6926], also those of m, m′, m″, m‴ [6811b—e], in [6922], we obtain [6927]. Upon examination, it was found that the numerical calculations in [6919—6927] were very nearly correct. [6921a] [6921b] [6921c]

[6923] $\lambda = 0,00063534$;

[6924] $\lambda' = 0,0064232$; [Corrected in 7206—7209.]

[6925] $\lambda'' = 0,0299802$;

[6926] $\lambda''' = 0,134612$.

If we adopt the preceding values of the masses of the satellites, we shall have,

[6927] $$^1p = 9'',8788\,;$$

hence we deduce,*

[6928] $$\theta' = {}^1L + 0'',070350.t\,;$$

[6929] $$\Psi' = 0'',8259.t\,.$$

[6929′] *This last coefficient* $0'',8259$, *is therefore very nearly the annual precession of the equinoxes of Jupiter upon its orbit.*

To reduce to numbers the inequalities of the periodical motions of the satellites in latitude, computed in [6453—6482], we shall observe, that we may suppose, without any sensible error,†

[6930] $$N_{,} + \frac{p}{n} = 1\,; \qquad N'_{,} + \frac{p}{n'} = 1\,; \qquad N''_{,} + \frac{p}{n''} = 1.$$

[6929a] * (3504) Substituting in [6899, 6900] the values [6906—6908, 6927], we get [6928, 6929]; the value of Ψ' being nothing at the epoch of 1750 [6897]. If we [6929b] suppose, as in [7219a], that the value of Ψ', at the epoch of 1750, is $\Psi' = -148^{\circ},62129$, its general value, for the time $1750 + t$, will be obtained by adding the preceding value of [6929c] Ψ' to that in [6929]; hence we shall have $\Psi' = -148^{\circ},62129 + t.0'',8259$. Now if we [6929d] suppose, as in [7211], $\mathrm{I} = 200^{\circ} - \Psi'$, we shall have $\mathrm{I} = 348^{\circ},62129 - t.0'',8259$; which will be of use hereafter.

† (3505) Substituting in [6485, 6486] the values of a, a', a'', a''' [6797—6800], and that of $\rho - \frac{1}{2}\varphi$ [6827], we get nearly,

[6930a] $$N_{,} = 1,00067\,; \qquad N'_{,} = 1,00027\,; \qquad N''_{,} = 1,00010\,; \qquad N'''_{,} = 1,00003.$$

Moreover from the values [6025p, k, i], we obtain nearly,

[6930b] $$\frac{p}{n} = 0,00069\,; \qquad \frac{p_1}{n'} = 0,00032\,; \qquad \frac{p_2}{n''} = 0,00014\,; \qquad \frac{p_3}{n'''} = 0,00009\,;$$

[6930c] $$\frac{2M}{n} = 0,00081\,; \qquad \frac{2M}{n'} = 0,00164\,; \qquad \frac{2M}{n''} = 0,00330\,; \qquad \frac{2M}{n'''} = 0,00770.$$

[6930d] From the first terms in [6930b, a], we get $\frac{p}{n} + N_{,} - 1 = 0,00136$; and although the second member of this expression is small, in comparison with unity, it is larger than the [6930e] other term $\frac{2M}{n} = 0,00081$, with which it is connected, in the divisor $\frac{2M}{n} + \frac{p}{n} + N_{,} - 1$, of the term of s [6460 line 2]; so that it will not be sufficiently accurate to suppose, with

This being premised we obtain, [6930′]

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the author, as in [6930], that $\frac{p}{n}+N_{\prime}-1=0$, if the term depending on this divisor were of any sensible magnitude. Thus if we substitute the first of the expressions [6930] in the coefficient of the term of s in [6460 line 2], it becomes, [6930f] [6930g]

$$-\frac{\frac{3M^2}{4n^2}\cdot(L'-l)}{\frac{2M}{n}}=\frac{3M}{8n}\cdot(l-L')=0{,}00015.(l-L'),\quad [6930c], \qquad [6930h]$$

agreeing with the result of the author in [6931 line 2]; but this numerical value will be considerably decreased, if we use the actual value of $\frac{p}{n}+N_{\prime}-1$ [6930d]; therefore the expression [6930h] can be considered only as a rough estimate, serving to show that the term in question is, in general, so small that it may be neglected. One inequality of this kind is computed in [7521], with the coefficient $5'',26$; this is retained by the author, because it does not require a new argument, but can, in eclipses, be combined with a much larger term [7522 line 1]. This coefficient is reduced to nearly $3''$, for small values of p, by using the correct value of the divisor [6930e]. [6930i] [6930k] [6930l] [6930l′]

The similar term of s' [6480 line 2] has for its coefficient the quantity,

$$-\frac{3M^2.(L'-l')}{4n'^2.\left(\frac{2M}{n'}+N'_{\prime}+\frac{p}{n'}-1\right)}; \qquad [6930m]$$

and we have, from [6930c], $\frac{2M}{n'}=0{,}00164$, which is much larger than $N'_{\prime}-1=0{,}00027$ [6930a], so that we may, with a much greater degree of accuracy than in the preceding calculation [6930f, &c.], neglect $N'_{\prime}-1$, and also $\frac{p}{n'}$, for any small value of p, like that in [7479], or like $\frac{p_1}{n'}=0{,}00032$ [6930b], corresponding to the very small term of s' [7480]. Now if we neglect the small terms [6930n′], we find that the coefficient [6930m] is reduced to the form $\frac{3M}{8n'}\cdot(l'-L')=0{,}000307.(l'-L')$ [6930c], which is similar to that of s [6930h]. This expression is the same as that in [6932 line 2], and it would be decreased about one seventh part, for small values of p, by using the expression of $N'_{\prime}-1$, [6930n]. In like manner we may reduce the term of s'' [6481 line 2], with a considerable degree of accuracy, to the form $\frac{3M}{8n''}\cdot(l''-L')=0{,}000619.(l''-L')$ [6930c], as in [6933 line 2]. Finally, the coefficient of the term of s''' [6482] may, by a similar process, be reduced to the form $\frac{3M}{8n'''}\cdot(l'''-L')=0{,}00144.(l'''-L')$ [6930c], as in [6934]; which is sufficiently accurate for all practical purposes. [6930n] [6930n′] [6930o] [6930p] [6930q] [6930r]

The same reduction may be made in the terms in the first lines of [6460, 6480, 6481];

[6931]
$$s = \mathrm{m}'.0{,}00349437.(l'-l).\sin.(3v-4v'-pt-\textsc{a}) \qquad 1$$
$$+0{,}00015312.(l-L').\sin.(v-2U-pt-\textsc{a})\,; \qquad 2$$

[6932]
$$s' = \{\mathrm{m}.0{,}00276975.(l-l')+\mathrm{m}''.0{,}00170910.(l''-l')\}.\sin.(2v-3v'-pt-\textsc{a}) \qquad 1$$
$$+0{,}00030736.(l'-L').\sin.(v'-2U-pt-\textsc{a})\,; \qquad 2$$

[6933]
$$s'' = \mathrm{m}'.0{,}00135305.(l'-l'').\sin.(2v'-2v''-pt-\textsc{a}) \qquad 1$$
$$+0{,}00061925.(l''-L').\sin.(v''-2U-pt-\textsc{a})\,; \qquad 2$$

[6934]
$$s''' = +0{,}001447815.(l'''-L').\sin.(v'''-2U-pt-\textsc{a}).$$

24. *We shall now compute the inequalities depending upon the square of the excentricities and inclinations of the orbits*, whose analytical values we have given in [6530, 6552]. Those which finally become the most sensible are the secular equations of the satellites, depending on the secular variations
[6935] of the orbit and equator of Jupiter. But it is easy to prove that they are
[6935'] at present insensible, and that they will remain so for a long time to come. For the greatest is that of the fourth satellite, and its expression is, as in [6530, &c.],*

[6936]
$$\delta v''' = -2.\boxed{3}.H_1.ct^2+2.(1-\lambda''')^2.\boxed{3}.{}^1L.bt^2-3.(3).\lambda'''^2.{}^1L.bt^2.$$

[6930s] for we have, from [6025b, 6930a, b], $3-\frac{4n'}{n}=1{,}00727$, $\frac{p}{n}+\mathcal{N}_{\prime}=1{,}00136$; so that
[6930t] the expression of the divisor $3-\frac{4n'}{n}-\frac{p}{n}-\mathcal{N}_{\prime}$ [6460], will not be very much altered from its true value, by putting $\mathcal{N}_{\prime}+\frac{p}{n}=1$, as in [6930]; and by making this substitution
[6930u] we find that the expression [6460 line 1] becomes $\frac{m'.\alpha^2.b_{\frac{3}{2}}^{(3)}.(l'-l)}{4\times 0{,}00727}$. If we substitute in it
[6930v] the values of α, $b_{\frac{3}{2}}^{(3)}$ [6801, 6805], also $m'=\frac{\mathrm{m}'}{10000}$ [6841c], it becomes nearly as in [6931 line 1]. In like manner, the term [6480 line 1] becomes, by using the values
[6930w] [6797—6800, 6801, 6805, 6814, 6818], and $m''=\frac{\mathrm{m}''}{10000}$ [6841d], very nearly, as in [6932 line 1]. A similar calculation being made with the terms in [6481 line 1]; using the
[6930x] values [6798, 6799, 6818, &c.], we get the term in [6933 line 1].

[6935a] * (3506) The expression [6530] corresponds to the first satellite m, and by changing reciprocally the elements relative to m, into those corresponding to m''', we get $\delta v'''$ [6936]. Now it is evident that the terms of $\delta v'''$ [6936] are much larger than those of
[6935b] δv [6530]. For if we compare the first term of [6936] with that of [6530], we find that

Now we have, in [6903, 6906],

$$b = 0'',070350; \qquad {}^1L = 3^{\circ},4444; \qquad [6937]$$

and we also have,*

$$2c = 1'',9446; \quad \text{or} \quad c = 0'',9723. \qquad [6938]$$

Therefore, by making use of the values of λ''', $\boxed{3}$ and (3), given in [6926, 6864], and supposing $\mu = 1$ in (3), we find that the secular equation of the fourth satellite is represented by [6939]

$$\delta v''' = -0,000135.t^2; \qquad [6940]$$

so that it will be for a long time insensible. [6940′]

We have also, in [6552], the following inequality in the mean motion of the fourth satellite, referred to the orbit of Jupiter; †

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their ratio is as $\boxed{3}$ to $\boxed{0}$; and from [6864] we have nearly $\boxed{3} = 9.\boxed{0}$. [6935c]
The same ratio obtains very nearly between the second terms of [6936] and [6530], λ, λ', &c. being small [6923—6926]. Again, the third term of [6936] is to the third term of [6530] as $\lambda'''^2(3)$ to $\lambda^2(0)$; and by substituting the values [6926, 6923, 6864 lines 1, 4], [6935d] we find that the first term of this ratio is much greater than the second. Hence it appears that this term of $\delta v'''$ [6936] is much greater than the corresponding terms of δv, $\delta v'$, $\delta v''$, [6935e] as is observed in [6935′].

* (3507) If we compare the expression of the excentricity of Jupiter's orbit [6527] with that in [4407], we get $c = 0^s,329487$, which represents the value of the coefficient [6938a] of t in e^{iv} [4407]. This term is produced chiefly by the action of Saturn, as is evident from [4104d, &c. 4246 line 4, 4403, 4405], using the value $m^{\text{v}} = \frac{1+\mu^{\text{v}}}{3359,40}$ [4061], and [6938b] putting $\mu^{\text{v}} = 0$. If we use the value $1+\mu^{\text{v}} = \frac{3359,4}{3515,6}$ [6903b], the preceding expression [6938c] of c will be decreased in nearly the same ratio, making $c = 0^s,3149 = 0'',972$, or [6938d] $2c = 1'',941$, as in [6938]. Now substituting in [6936] the values of (3), $\boxed{3}$, λ''', b, [6938e] ${}^1L$, $2c$, [6864, 6926, 6937, 6938], also that of H or $e^{\text{iv}} = 0,0480767$ [4080], we get $\delta v'''$ [6940] nearly. This is so very small that it does not amount to two centesimal seconds in a century; and it must therefore be for a long time insensible, as in [6940′]. [6938f]

† (3508) The formula [6941] is easily deduced from the terms of [6552], depending on the angle p, by changing λ, (0), $\boxed{0}$, l, into λ''', (3), $\boxed{3}$, l''', respectively; [6941a] so as to make them conform to the fourth satellite.

[6941] $$\delta v''' = -\frac{\left\{6.(3).\lambda'''+4.(1-\lambda''').\boxed{3}-\tfrac{1}{2}.(1-\lambda''').p\right\}}{p}.\theta'.l'''.\sin.(pt+\text{A}-\Pi').$$

[6941′] The first approximation gives,*

[6942] $$pt+\text{A}-\Pi' = t.7541''+31°,9199,$$

[6943] $$l''' = 2772'';$$

hence the preceding inequality becomes,†

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* (3509) The chief term of s''' [7329], depending on the peculiar inclination of the orbit of the fourth satellite to the fixed plane, may be put under the form,

[6942a] $$s''' = 2771'',6.\sin.(v'''+283°,29861+t.7528'',01).$$

In this expression the coefficient of t is relative to the moveable vernal equinox of the

[6942b] earth [7327, &c.]; and if we refer it to the fixed equinox of 1750, we must change the coefficient of t from 7528″,01, to 7682″,64, as in [7327]. Making this change, and then putting the expression equal to that of s''' in [6300 line 4], we get,

[6942c] $$l'''.\sin.(v'''+pt+\text{A}) = 2771'',6.\sin.(v'''+283°,29861+t.7682'',64).$$

[6942d] Comparing the coefficient of the first member with that of the second, we get $l'''=2771''6$, as in [6943] nearly. Moreover, by comparing the angles under the sign *sin.*, we get,

[6942e] $$pt+\text{A} = 283°,29861+t.7682'',64.$$

[6942f] Subtracting, from this last expression, the value of $\Pi' = -148°,62129+t.0'',8259$ [6929c], and neglecting the whole circumference 400°, we obtain,

[6942g] $$pt+\text{A}-\Pi' = 31°,9199+t.7681'',8141.$$

The constant angle 31°,9199 agrees with that which we have given in [6942]; observing

[6942h] that the author has inserted, in the original work, —52875″, instead of +31°,9199, in both the formulas [6942, 6944]. This appears to be merely a typographical mistake;

[6942i] since the true value +31°,9199 is used in the subsequent parts of the work, as in [7315, 7317, 7318, &c.]. The coefficient of t, which is used by the author in

[6942k] [6942, 6944], is 7541″; being the value assumed as a first approximation, in [6941′, &c.]; and it differs a little from that in [6942g], namely, 7681″,8141. Substituting in [6941]

[6942l] the values of λ''', (3), $\boxed{3}$, $pt+\text{A}-\Pi'$, l''', &c. [6926, 6864, 6942g, 6943], also $\theta' = 3°,43519$ [7217], and dividing by the radius in seconds, it becomes very nearly equal to the following expression,

[6942m] $$\delta v''' = -49'',51.\sin.(t.7681'',8141+31°,9199);$$

being the same as in [7318 line 13]; the coefficient of t being changed as in [6942k].

[6942n] Remarking, however, that, by a rough calculation, the coefficient of this inequality appeared to be a fraction of a second less than the value given by the author.

† (3510) We have seen in the last note how the inequality [6944], corresponding to

[6944a] the angle, or value of p [6942], is computed. The terms depending on the other larger values of p, may be neglected, because they are decreased by the increased value of the

$$\delta v''' = -49'',51.\sin.(t.7541''+31^\circ,9199_c).$$ [6944]

The inequalities of this kind are insensible relative to the other satellites, [6944g—m]. [6944′]

25. *It now remains to consider the inequalities depending upon the square of the disturbing force.* We have seen, in [6765], that the second satellite is subjected to the inequality,

$$\delta v' = \tfrac{5}{16}.(\text{II})^2.\sin.(2nt - 2n't + 2\varepsilon - 2\varepsilon').$$ [6945]

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divisor p, and also by the decreased value of the coefficient l'''. This is evident by the [6944b]
inspection of the value of s''' [7352]; in which the coefficients of t, corresponding to the [6944c]
moveable equinox [7328], are $7528'',01$, $28220^s,85$, $133715'',77$; and the corresponding [6944d]
coefficients or values of l''', neglecting the signs, are $2771'',6$, $448'',93$, $4'',80$; so that the values of $\frac{l'''}{p}$, relative to these quantities, will be nearly as $\frac{2771,6}{7528,01}$, $\frac{448,93}{28220,85}$, $\frac{4,80}{133715,77}$, or as 1, 0,04, 0,0001; consequently the two last terms must be so much less than the first [6944], that they may be neglected. The coefficient $49'',51$ [6944] is the maximum of that inequality, and this arc is described by the fourth satellite in about 18 sexagesimal seconds of time [6781]; and generally the *effect in eclipses will be much less than this maximum value*, on account of the multiplication by the sine of the angle with which the coefficient is connected in [6944]. If we change the symbols in the formula [6941], so as to make it conform to the first satellite, and compute the value of δv corresponding to the expression of l, in [7522 line 3], we find, that the coefficient of this term of δv is $-2'',7$; which can be described by this satellite in a tenth of a sexagesimal second of time, and is therefore wholly insensible. In like manner, by changing the symbols in [6941], so as to correspond to the second satellite, and to the value of l' in [7482 line 4], we find, that the coefficient of this correction is $-128'',4$, which is considerably greater than that in [6944], neglecting the consideration of the signs; but as an arc of $128'',4$ is described by the second satellite in less than 10 sexagesimal seconds of time, it may, on that account, be considered as of less importance than that of $\delta v'''$ [6944f]; but it seems to be of sufficient magnitude to be noticed. Lastly, by changing the symbols in [6941], so as to correspond to the third satellite, and to the value of l'' in [7427 line 3], we find, that the coefficient of this correction is $-37'',7$; which can be described by this satellite in about 6 sexagesimal seconds of time, being considerably less than that for the fourth satellite [6944f]. The corrections for other values of l, l', l'', are decreased like those for the fourth satellite in [6944e], and may be considered insensible, as in [6944′]. We have not, in this note, taken into consideration the terms depending on the square of the disturbing force, mentioned in [6553, &c.].

[6944e] [6944f] [6944g] [6944h] [6944i] [6944k] [6944l] [6944m] [6944n]

We have by observation, very nearly,*

[6946] $$(\text{II}) = 11923'';$$

hence the preceding inequality becomes,

[6947] $$\delta v' = 69'',78.\sin.(2nt - 2n't + 2\varepsilon - 2\varepsilon').$$

The inequalities of the same kind are insensible relative to the other satellites [6947*d*, &c.].

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* (3511) We obtain from [6172*b*, 6240*q*], by successive reductions,

[6947*a*] $$\sin.(nt - n't + \varepsilon - \varepsilon') = \sin.(2n''t - 2n't + 2\varepsilon'' - 2\varepsilon') = \sin.2.(\Theta'' - \Theta') = -\sin.2.(\Theta' - \Theta'').$$

[6947*b*] Substituting this last expression in [6173], we get $\delta v' = (\text{II}).\sin.2.(\Theta' - \Theta'')$. Comparing [6947*c*] this with the corresponding term of [7450 line 2], we get $(\text{II}) = 11920'',67$, agreeing nearly with [6946]. Multiplying the square of this by $\frac{5}{16}$, and dividing the quotient by [6947*d*] the radius in seconds, it becomes as in [6947] nearly. In like manner, by comparing [7513 line 5] with [6172], and [7390 line 2] with [6174], we get $(\text{I}) = 5050'',59$, $(\text{III}) = 808'',20$. The former being substituted in [6764], gives $\delta v = 12'',5.\sin.4.(\Theta - \Theta')$; [6947*e*] and the latter being substituted in [6766] gives $\delta v'' = 0'',3.\sin.2.(\Theta' - \Theta'')$. This term of $\delta v''$ is insensible, but that of δv might be noticed, with the other similar inequalities in [6947*f*] [7513]; though it is hardly sensible in observations in eclipses, on account of the rapidity of the motions of the first satellite; since the angle $12^{s},5$ is described in about half a sexagesimal second of time [6778].

CHAPTER VIII.

ON THE DURATION OF AN ECLIPSE OF ANY SATELLITE.

26. We do not directly observe the motions of the satellites of Jupiter about that planet. The elongation of a satellite from Jupiter, as seen from the earth, is so small, that a very slight error in an observation can produce a variation of several degrees, in the place of the satellite, when referred to Jupiter's centre. The eclipses of the satellites furnish an incomparably [6948] better method of determining their motions; and it is to these observations we are indebted for the knowledge of their perturbations. The shadow of Jupiter is projected in an opposite direction to that of the sun; and the satellites are immersed in this shadow when they are nearly in opposition to the sun. The inclinations of the orbits of the three inner satellites, to [6948′] the orbit of Jupiter, and their distances from that planet are so adapted to each other, that the satellites are eclipsed at each revolution. But the fourth satellite passes often beyond the limits of the shadow, and is not eclipsed; and for this reason as well as on account of the longer duration of its revolution, the eclipses of this satellite happen less frequently than those of the other satellites.

A satellite disappears from our view before it is wholly immersed in Jupiter's shadow. Its light is diminished by the penumbra, and its disc, as [6948″] it gradually enters into the shadow of the planet, becomes invisible to us, before it is totally eclipsed. The limb of the satellite, at the moment when [6949] we cease to perceive it, is therefore at a small distance from Jupiter's shadow; Exterior or fictitious shadow. and if we suppose that there is at that distance an *exterior surface similar to that of the shadow*, the immersion of the satellite, within this *exterior surface*, will be the beginning of the eclipse, as it appears to us; and its emersion [6949′] from this *exterior surface* will be the end of the eclipse.

This exterior or fictitious shadow is not the same for all the satellites. It
[6950] depends on their apparent distance from Jupiter, whose brilliancy weakens their light, and on the greater or less power of their surfaces in reflecting the light; it also depends upon the penumbra; and probably also upon the refraction, and upon the extinction of the solar rays in Jupiter's atmosphere. The greatest duration of the eclipses of one of the satellites, cannot therefore
[6950′] give, with precision, that of the other satellites; but the comparison of these durations will throw light on the influence of the causes we have just mentioned. The variation of the distance of Jupiter from the sun or
[6951] from the earth, changes the intensity of the light we receive from the satellites, and this has an influence upon the duration of their eclipses. The elevation of Jupiter above the horizon, the clearness of the earth's
[6951′] atmosphere, and the power of the instruments which are used in the observation, have also an influence upon this duration. All these causes produce a degree of uncertainty in the observations of the eclipses of the satellites, particularly in those of the third and fourth. Fortunately, we can
[6952] observe quite often, with these two satellites, the immersion and emersion in the same eclipse; which gives the time of their conjunction with a considerable degree of accuracy, and independent of most of the causes of error above enumerated.

In the first place we shall determine the figure of Jupiter's shadow. If
[6953] this planet and the sun were spherical, the shadow would be a cone, touching the surfaces of the two bodies. But Jupiter is sensibly elliptical; therefore its shadow must differ from that of a cone.

We shall consider generally the shadow of an opake body illuminated by a luminous one, whatever be the figures of these bodies. If we draw through any point of the surface of the shadow a plane which is a tangent to that
[6954] surface, it will also be a tangent to the surfaces of both these bodies. It is evident that the three points of contact will be in a right line, which will also coincide with the surface of the shadow; therefore this surface is
[6955] formed by the intersections of a series of planes, which touch the surface of the opake and luminous bodies. We shall suppose, as in [19*d*], that

[6956] $$x = \mathrm{a}y + \mathrm{b}z + \mathrm{c},$$

Plane touching the shadow. is the general equation of these planes, a, b, c, being quantities which are variable from one plane to another. We may here apply the considerations used in [1167″, &c.], relative to the orbits of the planets considered as variable ellipses. If we vary, by infinitely small quantities, the rectangular

co-ordinates x, y, z, they may be considered as appertaining to the same plane. Hence we may take the differential of the equation, [6957]

$$x = \mathrm{a}y + \mathrm{b}z + \mathrm{c}\,; \qquad [6958]$$

considering a, b, c, as constant quantities; which gives,

$$dx = \mathrm{a}dy + \mathrm{b}dz. \qquad [6959]$$

Taking then its differential, supposing all the quantities to be variable, and subtracting the first differential from the second, we obtain,

$$0 = y.d\mathrm{a} + z.d\mathrm{b} + d\mathrm{c}\,; \qquad [6960]$$

so that if we consider b and c as functions of a, we shall have, [6961]

$$0 = y + z.\,\frac{d\mathrm{b}}{d\mathrm{a}} + \frac{d\mathrm{c}}{d\mathrm{a}}\,. \qquad [6962]$$

We shall now put,

$$\mu = 0, \qquad [6963]$$

for the equation of the surface of the luminous body; and X, Y, Z, *for the three co-ordinates of this surface,* at the point where it is touched by the plane.* In order to make this plane a tangent to the surface, it is necessary not only that the co-ordinates should satisfy the equation of this surface, but that they should also appertain to its differential, (Equation of the surface of the luminous body.)

$$0 = \left(\frac{d\mu}{dX}\right).\,dX + \left(\frac{d\mu}{dY}\right).\,dY + \left(\frac{d\mu}{dZ}\right).\,dZ. \qquad [6964]$$

Substituting for dX its value $dX = adY + bdZ$ [6959], we obtain, [6965]

$$0 = dY.\left\{\left(\frac{d\mu}{dY}\right) + \mathrm{a}.\left(\frac{d\mu}{dX}\right)\right\} + dZ.\left\{\left(\frac{d\mu}{dZ}\right) + \mathrm{b}.\left(\frac{d\mu}{dX}\right)\right\}. \qquad [6966]$$

It is evident that this last equation ought to be satisfied, whatever be the values of dY and dZ; hence we get,

$$0 = \left(\frac{d\mu}{dY}\right) + \mathrm{a}.\left(\frac{d\mu}{dX}\right); \qquad [6967]$$

$$0 = \left(\frac{d\mu}{dZ}\right) + \mathrm{b}.\left(\frac{d\mu}{dX}\right). \qquad [6968]$$

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* (3512) The equations of surfaces are treated of in [$19a$—f]. The differential of the equation of the surface $\mu = 0$ [6963], is evidently as in [6964]; and if this differential surface coincide with the plane [6958, &c.], it will satisfy the equation [6959], by changing dx, dy, dz, into dX, dY, dZ, respectively, as in [6965]. Substituting this in [6964], we get [6966]; which cannot be satisfied for all values of dX, dY, unless both their coefficients be put equal to nothing, as in [6967, 6968]. [6962a] [6962b] [6962c]

Combining these equations with the following,*

[6969] $$\mu = 0\,; \qquad X = aY + \text{b}Z + \text{c}\,;$$

[6970] μ being a function of X, Y, Z; and then eliminating X, Y, Z, *we obtain the first fundamental equation in* a, b, c.

We shall also put,

[6971] $$\mu' = 0,$$

Equation of the surface of the opake body. *for the equation of the opake body; and* X', Y', Z', *for the co-ordinates, corresponding to the points where it is touched by the plane;* μ' being

[6972] considered as a function of these co-ordinates. Then this equation will give, in like manner, the four following equations;†

[6973] $$0 = \left(\frac{d\mu'}{dY'}\right) + \text{a}.\left(\frac{d\mu'}{dX'}\right);$$

[6974] $$0 = \left(\frac{d\mu'}{dZ'}\right) + \text{b}.\left(\frac{d\mu'}{dX'}\right);$$

[6975] $$\mu' = 0\,; \qquad X' = \text{a}Y' + \text{b}Z' + \text{c}.$$

Hence we get a *second fundamental equation* between a, b, c. By means

[6976] of this, and the first fundamental equation [6970], we obtain b and c in functions of a. Substituting these functions in the two equations [6956, 6962], namely,

[6977] $$x = \text{a}y + \text{b}z + \text{c}\,;$$

[6978] $$0 = y + z.\frac{d\text{b}}{d\text{a}} + \frac{d\text{c}}{d\text{a}};$$

we shall have two equations between x, y, z, &c.; and by eliminating a,

[6979] we shall finally obtain an equation between x, y, z, which will be that of

Shadow of an opake body. the surface of the shadow. *This is the general solution of the problem for the determination of the shadow of an opake body; and the same solution*

[6980] *gives also the equation of the surface of the penumbra;* for it is evident that this surface is formed like that of the shadow, by the successive intersections

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[6969a] * (3513) The equations [6969] are the same as those in [6963, 6958], changing x, y, z, into X, Y, Z, respectively, so as to correspond to the point of contact.

[6969b] Eliminating X, Y, Z, from the four equations [6967—6969], we obtain the first fundamental equation in a, b, c; corresponding to the plane which touches the luminous body.

[6973a] † (3514) The equations [6973—6975] are similar to those in [6967—6969], changing μ, X, Y, Z, into μ', X', Y', Z', respectively; so as to correspond to the surface of the opake body.

of the planes which touch the surfaces of the luminous and opake bodies; with this difference, that *in the case of the shadow, we must consider the intersection of the planes which touch these surfaces, on the same side; whereas* [6981]
in the penumbra, we must consider the intersections of the planes which touch Penumbra.
these surfaces upon the opposite sides. We shall now apply this solution to the shadow of Jupiter.

In the first place, we shall suppose Jupiter and the sun to be spherical; Spherical form of Jupiter.
putting also,

R = the sun's semi-diameter; [6982]

R' = the semi-diameter of Jupiter; [6982']

D = the distance of the centres of the sun and Jupiter. [6982'']

Then the origin of the co-ordinates being placed at the sun's centre, we have, [6982''']
for the equation of the sun's surface, the expression,* $X^2+Y^2+Z^2-R^2=0$; [6983]
so that we shall get, from [6963],

$$\mu = X^2+Y^2+Z^2-R^2. \qquad [6984]$$

Hence we shall have the four following equations;

$$X^2+Y^2+Z^2-R^2=0; \qquad [6985]$$

$$Y+\mathrm{a}X=0; \qquad [6986]$$

$$Z+\mathrm{b}X=0; \qquad [6987]$$

$$X=\mathrm{a}Y+\mathrm{b}Z+\mathrm{c}. \qquad [6988]$$

The three first equations give,†

$$X^2.(1+\mathrm{a}^2+\mathrm{b}^2)=R^2; \qquad [6989]$$

and from the three last equations, we obtain,

$$X.(1+\mathrm{a}^2+\mathrm{b}^2)=\mathrm{c}. \qquad [6990]$$

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* (3515) The equation [6983] is similar to that in [19e], changing r', x, y, z, into R, X, Y, Z, respectively. Putting this equal to the assumed value $\mu=0$ [6963], we get $\mu=X^2+Y^2+Z^2-R^2$; whence, [6983a]

$$\left(\frac{d\mu}{dX}\right)=2X; \qquad \left(\frac{d\mu}{dY}\right)=2Y; \qquad \left(\frac{d\mu}{dZ}\right)=2Z. \qquad [6983b]$$

Substituting these in [6967, 6968], we get [6986, 6987]. The equations [6985, 6988] are the same as [6983, 6969] respectively. [6983c]

† (3516) From [6986, 6987] we get $Y=-\mathrm{a}X$, $Z=-\mathrm{b}X$; substituting these in [6985], we get [6989]; and by substituting the same values in [6988] we obtain the value of c [6990]. [6989a]

Hence we deduce,*

[6991] $$c^2 = R^2.(1 + a^2 + b^2).$$

[6992] The equation of Jupiter's surface† is $(X'-D)^2+Y'^2+Z'^2-R'^2=0$; so that from [6971] we have,

Equation of Jupiter's surface.

[6993] $$\mu' = (X'-D)^2+Y'^2+Z'^2-R'^2.$$

From what has been said, we obtain the four following equations,‡

[6994] $$(X'-D)^2+Y'^2+Z'^2-R'^2=0;$$

[6995] $$Y'+a.(X'-D)=0;$$

[6996] $$Z'+b.(X'-D)=0;$$

[6997] $$X'-D = aY'+bZ'+c-D.$$

Hence we deduce,§

[6998] $$(c-D)^2 = R'^2.(1+a^2+b^2).$$

[6991a] * (3517) Dividing the square of [6990] by [6989], and then multiplying the quotient by R^2, we get [6991].

[6992a] † (3518) If $X_{\prime}$, Y', Z', be the rectangular co-ordinates of Jupiter's surface, referred to the centre of the planet as the origin of the co-ordinates, and R' its radius, we shall [6992b] have, for the equation of its surface, as in [6985], $X_{\prime}^2+Y'^2+Z'^2-R'^2=0$; the axes of $X_{\prime}$, Y', Z', being parallel to those of X, Y, Z, respectively. If we refer these co-ordinates to the centre of the sun, and put X' for the new co-ordinate, or the line drawn from the sun's centre to the centre of the planet, and continued beyond it by the [6992c] quantity $X_{\prime}$, for positive values of the co-ordinate $X_{\prime}$, we shall evidently have $X_{\prime}=X'-D$; the co-ordinates Y' and Z' remaining unaltered. Substituting this value of $X_{\prime}$ in the preceding equation [6992b], we get [6992], corresponding to $\mu'=0$ [6971], as in [6993].

‡ (3519) The equation [6994] is the same as [6992]; putting this equal to μ', we get [6993]; whose partial differentials, relative to X', Y', Z', are,

[6994a] $$\left(\frac{d\mu'}{dX'}\right)=2.(X'-D), \qquad \left(\frac{d\mu'}{dY'}\right)=2Y', \qquad \left(\frac{d\mu'}{dZ'}\right)=2Z'.$$

Substituting these in [6973, 6974], we get [6995, 6996] respectively. Subtracting D from the last of the equations [6975] we get [6997], being the equation of the part of the surface which is touched by the tangent plane.

[6998a] § (3520) The equation [6998] can be deduced from the four equations [6994—6997], in the same manner as [6991] is from the four similar equations [6985—6988]. The same may be obtained more simply by derivation from [6991]; observing that if we change [6998b] X, Y, Z, R, c, into $X'-D$, Y', Z', R', c—D, respectively; the equations [6985-6988] will change into [6994—6997] respectively; and by making the same changes in [6991], which was deduced from [6985—6988], we get [6998].

Therefore by putting,

$$\frac{R'}{R} = \lambda_1, \quad [7000i]$$ [6999]

we shall have,*

$$\frac{c - D}{c} = \lambda_1;$$ [7000]

consequently,

$$c = \frac{D}{1 - \lambda_1}.$$ [7001]

Now if we put, [7002]

$$f^2 = \frac{D^2}{R^2.(1-\lambda_1)^2} - 1;$$ [7003]

the equation [6991] will give,†

$$b^2 = f^2 - a^2;$$ [7004]

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* (3521) Dividing the equation [6998] by [6991], we get $\left(\frac{c-D}{c}\right)^2 = \left(\frac{R'}{R}\right)^2 = \lambda_1^2$ [7000a]
[6999]; whose square root is $\frac{c-D}{c} = \pm\lambda_1$; whence $c = \frac{D}{1 \mp \lambda_1}$. Using the *upper* [7000b]
signs, we obtain the equations [7000, 7001]; but we may also use the *lower* signs; and it
is easy to prove that the former correspond to the equation of the surface of the *real* [7000c]
shadow; and the latter to that of the penumbra [6981]. For it is evident that the vertex
of the cone of the *penumbra* falls between Jupiter and the sun; so that its distance from
the sun must be *less* than D; but the vertex of the cone of the *shadow* falls beyond [7000d]
Jupiter, making its distance from the sun's centre *greater* than D. Now if we suppose, [7000e]
as in [6992c], that the axis of X is the line drawn from the centre of the sun to that of
Jupiter, it is plain that the vertex of the cone of the shadow, or that of the penumbra, will
be on this line; and we shall have, at the vertex $Y=0$, $Z=0$; whence $X=c$
[6988]. Substituting this in the value of c [7000b], we get, for the values of X, [7000f]
corresponding to these vertices, $X = \frac{D}{1 \mp \lambda_1}$. If we use the upper sign of $\mp\lambda_1$, we get [7000g]
$X > D$, corresponding to the shadow [7000e]; *and the lower sign gives $X < D$,* [7000h]
corresponding to the penumbra [7000d]. We may observe that the symbol λ_1 [6999, &c.]
is given without any accent in the original work; but we have placed the figure 1 below [7000i]
the letter, to distinguish it from the symbols λ, λ', &c. [6343—6346]. The same change
is made in [7025, &c.].

† (3522) Substituting the value of c [7001] in [6991], and then dividing by R^2,
we get $\frac{D^2}{R^2.(1-\lambda_1)^2} = 1 + a^2 + b^2$. Transposing 1, and substituting, in its first member, [7004a]
the value of f^2 [7003], we get $f^2 = a^2 + b^2$, as in [7004]. Substituting the values of
b, c, [7004, 7001], in $x - c = ay + bz$ [6958], we get [7006] for the equation of the [7004b]
tangent plane.

[7005] and the equation of the tangent plane [6958] will become,

[7006] $$x-\frac{D}{1-\lambda_1}=\mathrm{a}y+z.\sqrt{f^2-\mathrm{a}^2}.$$

Equation of the tangent plane.

Taking the partial differential of this equation relative to a, we get,*

[7007] $$0=y-\frac{\mathrm{a}z}{\sqrt{f^2-\mathrm{a}^2}};$$

hence we deduce,

[7008] $$\mathrm{a}=\frac{fy}{\sqrt{y^2+z^2}};$$

[7009] $$\mathrm{b}=\sqrt{f^2-\mathrm{a}^2}=\frac{fz}{\sqrt{y^2+z^2}};$$

consequently,

[7010] $$x-\frac{D}{1-\lambda_1}=f.\sqrt{y^2+z^2};$$

or,

[7011] $$\left(\frac{D}{1-\lambda_1}-x\right)^2=f^2.(y^2+z^2);$$

Equation of the conical surface of the shadow.

[7012] *which is the equation of the surface of a cone*,† y and z being nothing at its vertex. We shall have at this point,

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[7007a] * (3523) The equation [7006] is equivalent to that in [6958], being the equation of the tangent plane; and in the same manner as [6960] is deduced from [6958], by taking [7007b] the differential, considering a, b, c as variable, we may find [7007] from [7006]; observing that as b, c, have been eliminated, there will remain only the variable quantity a. Then the differential of [7006], considering a only as variable, gives [7007c] $0=d\mathrm{a}.y-d\mathrm{a}.\frac{\mathrm{a}z}{\sqrt{f^2-a^2}}$; dividing this by $d\mathrm{a}$, we get [7007], or $y=\frac{\mathrm{a}z}{\sqrt{f^2-\mathrm{a}^2}}$. [7007d] Squaring this and then deducing the value of a^2, we get a [7008]. Substituting this in [7007e] [7004], we get b [7009]. Substituting the value of a [7008] in the equation of the plane [7006], it becomes as in [7010]; whose square gives [7011].

[7011a] † (3524) We shall suppose, in the annexed figure, that CAV is the axis, and V the vertex of the right cone $VDGEF$. Through any point D of its surface draw the plane $DGEF$, perpendicular to the axis CV, and [7011b] intersecting the surface in the circle $DGEF$. We shall suppose CAV to be the axis of x, $CA=x$, $CV=D'$, and $VA=D'-x$; also $AB=y$, [7011c] $BD=z$; these two lines being drawn parallel to the axes of y, z, respectively. Then in the rectangular plane triangle ABD we have,

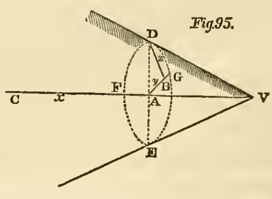

Fig.95.

[7011d] $$AD^2=AB^2+BD^2=y^2+z^2.$$

$$x = \frac{D}{1-\lambda_{1}} = \text{distance from the sun to the vertex of the cone;} \qquad [7013]$$

and this expresses the distance from the vertex of the cone to the sun's centre. Subtracting D from it, we obtain the distance from this vertex to the centre of Jupiter, which is equal to

$$\frac{D\lambda_{1}}{1-\lambda_{1}} = \text{distance from Jupiter to the vertex of the cone.} \qquad [7014]$$

Now in noticing the ellipticity of Jupiter, we shall suppose that the equator of this planet coincides with the plane of its orbit. [7015] The error arising from this supposition would vanish, if Jupiter were spherical; it must therefore be of the order of the product of the ellipticity of Jupiter, by the inclination [7015′] of its equator;* therefore it must be insensible. This being premised, we [7016] shall have, as in [6991],†

$$\mathrm{c} = R.\sqrt{1+\mathrm{a}^2+\mathrm{b}^2}\,; \qquad [7017]$$

Now by putting $\text{tang.}AVD = \frac{1}{f}$, we have, in the rectangular plane triangle VAD, [7011e] $AD = VA.\text{tang.}AVD = \frac{1}{f}.(D'-x)$. Substituting this in the preceding expression of AD^2, and then multiplying by f^2, we get $(D'-x)^2 = f^2.(y^2+z^2)$, which *represents* [7011f] *the equation of a conical surface,* whose vertex is V, and agrees with that in [7011], by putting $D' = \frac{D}{1-\lambda_{1}}$. This value of D' being the same as the value of x, or CV, [7011g] corresponding to the vertex V, where $y=0$, $z=0$; as is evident by substituting these values of y, z in [7011]; whence we get the value of x as in [7013].

* (3525) If Jupiter were spherical, the shadow would be of the same conical form, whatever be the position of the axis of revolution of the planet. [7015a] The greatest possible difference, when the form of the planet is elliptical, is of the order of the ellipticity ρ. [7015b] Now it is very evident that, if we obtain the form of the shadow, corresponding to the elliptical form of the planet, and to the supposition that its axis of revolution is perpendicular to the plane of its orbit, and then vary the inclination of the axis by a small angle; [7015c] this elliptical part of the shadow ρ, will vary by a part which may be considered as of the same order as the angle, or as the sine or tangent of this angle; so that the whole effect will be of the [7015d] order of the product of the ellipticity ρ by this small angle, as in [7015′].

† (3526) The equations [6985—6988] contain only the co-ordinates of the sun's surface, and are wholly independent of those of Jupiter, which are found in the equation [7017a] $\mu'=0$ [7020, 7018]. Therefore the equation [6991], which is deduced from the equations [6985—6988], must also hold good when Jupiter is elliptical; and by taking its square root, we get [7017].

Ellipsoidal figure of Jupiter's surface.

*and the equation of Jupiter's surface will be,**

[7018] $$(X'-D)^2+Y'^2+(1+\rho)^2.(Z'^2-R'^2)=0;$$

[7019] *R' being the semi-minor axis of Jupiter.* If we put the first member of this
[7020] equation equal to μ', we shall have, by what has been said in [7018f–h],

[7021] $$Y'+\mathrm{a}.(X'-D)=0;$$

[7022] $$(1+\rho)^2.Z'+\mathrm{b}.(X'-D)=0;$$

[7023] $$X'-D=\mathrm{a}Y'+\mathrm{b}Z'+\mathrm{c}-D.$$

Hence we deduce,†

[7024] $$\mathrm{c}-D=(1+\rho).R'.\sqrt{1+\mathrm{a}^2+\frac{\mathrm{b}^2}{(1+\rho)^2}}=R.\sqrt{1+\mathrm{a}^2+\mathrm{b}^2}-D.$$

[7018a] * (3527) The general equation of an ellipsoid, referred to its centre as the origin of the rectangular co-ordinates x, y, z, is $x^2+my^2+nz^2=k^2$ [1363]. Its semi-axes,
[7018b] parallel to x, y, z, being k, $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$, respectively [1363″]. If we put $\sqrt{m}=1$,
[7018c] $\sqrt{n}=(1+\rho)$, $k=(1+\rho).R'$, these semi-axes become $(1+\rho).R'$, $(1+\rho).R'$ and R', respectively; *R' being the semi-axis of revolution* [7019], and *$(1+\rho).R'$ the*
[7018d] *equatorial radius.* Substituting the values of m, n, k [7018c], in the equation of the
[7018e] ellipsoid [7018a], it becomes, by making a slight reduction, $x^2+y^2+(1+\rho)^2.(z^2-R'^2)=0$; now changing x, y, z, into $X'-D$, Y', Z', to conform to the present notation [6992a—e], it becomes as in [7018]. Putting this equal to μ', we get,

[7018f] $$\mu'=(X'-D)^2+Y'^2+(1+\rho)^2.(Z'^2-R'^2);$$

[7018g] $$\left(\frac{d\mu'}{dX'}\right)=2.(X'-D);\quad \left(\frac{d\mu'}{dY'}\right)=2Y';\quad \left(\frac{d\mu'}{dZ'}\right)=2Z'.(1+\rho)^2.$$

Substituting these in [6973, 6974], we get [7021, 7022] respectively; the other equation
[7018h] [7023] is the same as [6997], being the equation of the part of the surface which is touched by the plane.

[7024a] † (3528) We have $Y'=-\mathrm{a}.(X'-D)$ [7021]; $Z'=-\mathrm{b}.\frac{(X'-D)}{(1+\rho)^2}$ [7022]; substituting these in [7018, 7023], we obtain the two following equations respectively;

[7024b] $$(X'-D)^2.\left\{1+\mathrm{a}^2+\frac{\mathrm{b}^2}{(1+\rho)^2}\right\}=(1+\rho)^2.R'^2;$$

[7024c] $$(X'-D).\left\{1+\mathrm{a}^2+\frac{\mathrm{b}^2}{(1+\rho)^2}\right\}=\mathrm{c}-D.$$

Dividing the equation [7024c] by the square root of the equation [7024b],
[7024d] $(X'-D).\sqrt{\left\{1+\mathrm{a}^2+\frac{\mathrm{b}^2}{(1+\rho)^2}\right\}}=(1+\rho).R'$, we obtain,

[7024e] $$\sqrt{\left\{1+\mathrm{a}^2+\frac{\mathrm{b}^2}{(1+\rho)^2}\right\}}=\frac{\mathrm{c}-D}{(1+\rho).R'}.$$

Now putting,

$$\frac{(1+\rho).R'}{R} = \lambda_1 \ [7000i] \ ; \qquad [7025]$$

$$f^2 = \frac{D^2}{R^2.(1-\lambda_1)^2} - 1 \ ; \qquad [7026]$$

we shall have, by neglecting the square of ρ,* [7027]

Multiplying this by $(1+\rho).R'$, we get $c - D = (1+\rho).R'.\sqrt{\left\{1+a^2+\frac{b^2}{(1+\rho)^2}\right\}}$, as [7024f]
in the first expression of $c - D$ [7024]. Substituting in its first member the value of c [7024g]
[7017], we get the last member of [7024]. This equation corresponds to the shadow where $X' > D$ [7000h]; that of the penumbra is easily derived from it *by changing the* [7024h]
sign of the radical in [7024d]; or, in other words, by changing $X' - D$ into $D - X'$; [7024i]
which requires that we should change $c - D$ [7024e] into $-c + D$.

* (3529) The notation used in [7025, 7026], is similar to that in [6999, 7003]. Now if we put for a moment, for brevity, $s^2 = 1 + a^2 + b^2$, and neglect the square and higher powers of ρ, we shall have successively, [7028a]

$$1 + a^2 + \frac{b^2}{(1+\rho)^2} = 1 + a^2 + b^2 - 2\rho b^2 = s^2 - 2\rho b^2 \ ; \qquad [7028b]$$

substituting these in the two last forms of [7024], we get,

$$(1+\rho).R'.\sqrt{s^2 - 2\rho b^2} = Rs - D; \quad \text{or} \quad (1+\rho).R'.\left\{s - \frac{\rho b^2}{s}\right\} = Rs - D. \qquad [7028c]$$

Dividing this last equation by R, and substituting the value of λ_1 [7025], we get,

$$\lambda_1.\left\{s - \frac{\rho b^2}{s}\right\} = s - \frac{D}{R}; \quad \text{or,} \qquad [7028d]$$

$$(1-\lambda_1).s + \lambda_1.\frac{\rho b^2}{s} = \frac{D}{R}. \qquad [7028d']$$

Squaring this last equation, and re-substituting in its first term the value of s^2 [7028a], then dividing by $(1-\lambda_1)^2$, we get,

$$1 + a^2 + b^2 + \frac{2\lambda_1}{1-\lambda_1}.\rho b^2 = \frac{D^2}{R^2.(1-\lambda_1)^2}; \quad \text{or,} \qquad [7028e]$$

$$b^2.\left\{1 + \frac{2\lambda_1\rho}{1-\lambda_1}\right\} = \left\{\frac{D^2}{R^2.(1-\lambda_1)^2} - 1\right\} - a^2 = f^2 - a^2 \ [7026]. \qquad [7028f]$$

The square root of this last equation gives $b.\left\{1 + \frac{\lambda_1\rho}{1-\lambda_1}\right\} = \sqrt{f^2 - a^2}$; dividing it by the coefficient of b, we obtain [7028]. Substituting this value of b, in the expression of c, [7017], and then making successive reductions, using $1 + f^2 = \frac{D^2}{R^2.(1-\lambda_1)^2}$ [7026], we get, [7028g]

[7028] $$\text{b} = \left(1 - \frac{\lambda_1 \rho}{1-\lambda_1}\right) \cdot \sqrt{f^2 - \text{a}^2}\,;$$

[7029] $$\text{c} = \frac{D}{1-\lambda_1} - \lambda_1 \rho \cdot \frac{R^2}{D} \cdot (f^2 - \text{a}^2)\,;$$

which gives, *for the equation of the plane* [7028l],

[7030] Equation of the tangent plane.
$$x = \text{a}y + \left(1 - \frac{\lambda_1 \rho}{1-\lambda_1}\right) \cdot z \cdot \sqrt{f^2 - \text{a}^2} + \frac{D}{1-\lambda_1} - \frac{\lambda_1 \rho \cdot R^2}{D} \cdot (f^2 - \text{a}^2).$$

Taking its differential relatively to a only, we get,

[7031] $$0 = y - \text{a}z \cdot \frac{\left(1 - \frac{\lambda_1 \rho}{1-\lambda_1}\right)}{\sqrt{f^2 - \text{a}^2}} + \frac{2\lambda_1 \rho \cdot R^2 \text{a}}{D}.$$

Eliminating a, by means of these equations, we shall obtain the equation of the surface of the shadow. But we may simplify the computation, if we suppose,

[7032] $$\text{a} = \frac{fy}{\sqrt{y^2+z^2}} + \text{q}\rho\,.$$

q

$\frac{fy}{\sqrt{y^2+z^2}}$ being the value of a in the spherical hypothesis [7008], we shall have,*

[7028h] $$\text{c} = R \cdot \sqrt{\left\{1 + \text{a}^2 + \left(1 - \frac{2\lambda_1 \rho}{1-\lambda_1}\right) \cdot (f^2 - \text{a}^2)\right\}} = R \cdot \sqrt{\left\{1 + f^2 - \frac{2\lambda_1 \rho}{1-\lambda_1} \cdot (f^2 - \text{a}^2)\right\}}$$

[7028i] $$= R \cdot \sqrt{\left\{\frac{D^2}{R^2 \cdot (1-\lambda_1)^2} - \frac{2\lambda_1 \rho}{1-\lambda_1} \cdot (f^2 - \text{a}^2)\right\}} = \frac{D}{1-\lambda_1} \cdot \sqrt{\left\{1 - 2\lambda_1 \cdot (1-\lambda_1) \cdot \rho \cdot (f^2 - \text{a}^2) \cdot \frac{R^2}{D^2}\right\}}$$

[7028k] $$= \frac{D}{1-\lambda_1} - \lambda_1 \rho \cdot (f^2 - \text{a}^2) \cdot \frac{R^2}{D}.$$

[7028l] This last expression of c agrees with that in [7029]. Now substituting the values of b, c, [7028, 7029], in the equation of the tangent plane [6958], it becomes as in [7030]. This equation is similar to that in [7006], corresponding to the spherical form of Jupiter; and by [7028m] taking its differential relative to a, then dividing by da, we get [7031]; in the same manner as we have deduced [7007] from [7006].

* (3530) Substituting the value of a [7032], in $\sqrt{(f^2 - \text{a}^2)}$, and making successive developments and reductions, rejecting terms of the order ρ^2, we get [7033b].

[7033a] $$\sqrt{f^2 - \text{a}^2} = \sqrt{\left\{f^2 - \frac{f^2 y^2}{y^2+z^2} - \frac{2f\text{q}\rho y}{\sqrt{y^2+z^2}}\right\}} = \sqrt{\left\{\frac{f^2 z^2}{y^2+z^2} - \frac{2f\text{q}\rho y}{\sqrt{y^2+z^2}}\right\}}$$

[7033b] $$= \frac{fz}{\sqrt{y^2+z^2}} - \frac{\text{q}\rho \cdot y}{z}.$$

[7033c] Multiplying this by $1 - \frac{\lambda_1 \rho}{1-\lambda_1}$, we obtain [7033]; and by again multiplying by z, we

$$\left(1-\frac{\lambda_1\rho}{1-\lambda_1}\right)\cdot\sqrt{f^2-a^2}=\frac{fz}{\sqrt{y^2+z^2}}-\frac{q\rho.y}{z}-\frac{\lambda_1 f\rho z}{(1-\lambda_1).\sqrt{y^2+z^2}}. \qquad [7033]$$

Hence the *equation of the tangent plane becomes,*

$$x=f.\sqrt{y^2+z^2}-\frac{\lambda_1 f\rho.z^2}{(1-\lambda_1).\sqrt{y^2+z^2}}+\frac{D}{1-\lambda_1}-\frac{\lambda_1\rho.R^2}{D}\cdot\frac{f^2z^2}{y^2+z^2}. \qquad [7034]$$

Hence we deduce,

Equation of the shadow of Jupiter.

$$\left(x-\frac{D}{1-\lambda_1}\right)^2=f^2.(y^2+z^2)-\frac{2f^2.\lambda_1\rho.z^2}{1-\lambda_1}-\frac{2f^3.\lambda_1\rho.R^2z^2}{D\sqrt{y^2+z^2}}. \qquad [7035]$$

From [7026] we have $f=-\sqrt{\frac{D^2}{R^2.(1-\lambda_1)^2}-1}$; observing that the [7036] radical ought to have the sign $-$; because x is less than $\frac{D}{1-\lambda_1}$.* Thus we shall have, very nearly,

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get [7033f]. Now multiplying [7032] by y, we get [7033e]. Neglecting the part of a [7032] containing ρ, we get a value of a, whence we deduce $f^2-a^2=\frac{f^2z^2}{y^2+z^2}$; multiplying this by $-\frac{\lambda_1\rho.R^2}{D}$, we get [7033$g$]. Now adding $\frac{D}{1-\lambda_1}$ to the sum of the [7033d] first members of [7033e,f,g], we get the second member of [7030], or the value of x; [7033d'] and the term [7033d] being added to the similar sum of the second members of [7033e,f,g], becomes as in [7034], by making some slight reductions.

$$ay=\frac{fy^2}{\sqrt{y^2+z^2}}+q\rho y; \qquad [7033e]$$

$$\left(1-\frac{\lambda_1\rho}{1-\lambda_1}\right).z.\sqrt{f^2-a^2}=\frac{fz^2}{\sqrt{y^2+z^2}}-q\rho y-\frac{\lambda_1 f\rho z^2}{(1-\lambda_1).\sqrt{y^2+z^2}}; \qquad [7033f]$$

$$-\frac{\lambda_1\rho.R^2}{D}.(f^2-a^2)=\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad-\frac{\lambda_1\rho.R^2}{D}\cdot\frac{f^2z^2}{y^2+z^2}. \qquad [7033g]$$

Transposing $\frac{D}{1-\lambda_1}$ in [7034], and squaring the result, we get [7035]; which represents [7033h] the equation of the shadow of Jupiter, *free from radicals, except in the last small term of its second member.*

* (3531) The extreme point of the real shadow being at the vertex of the cone, the greatest value of x must correspond to that vertex; and if Jupiter be spherical, this [7036a] distance is $\frac{D}{1-\lambda_1}$ [7013]; therefore, in the case of $\rho=0$, the general value of x, within the limits of the shadow, must be less than $\frac{D}{1-\lambda_1}$; so that $x-\frac{D}{1-\lambda_1}$ must be negative. Now if we put $\rho=0$ in [7034], we shall get $x-\frac{D}{1-\lambda_1}=f.\sqrt{y^2+z^2}$; therefore f must [7036b]

[7037] $$f = -\frac{D}{R.(1-\lambda_1)};$$

[7038] D being much greater than R [6982, 6982″]; therefore we shall have,*

[7039] Shadow of Jupiter. $$\frac{R^2.(1-\lambda_1)^2}{D^2}\cdot\left(\frac{D}{1-\lambda_1}-x\right)^2 = y^2+z^2+\frac{2\lambda_1}{1-\lambda_1}\cdot\rho z^2.\left\{\frac{R}{\sqrt{y^2+z^2}}-1\right\};$$

which is the equation of the figure of the shadow of Jupiter. We find by the same analysis, that *the equation of the penumbra is, as in* [7039g],

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be negative, supposing the radical $\sqrt{y^2+z^2}$ to be positive. Moreover, it is evident from
[7036c] the definitions in [6982, &c., 7025], that $\frac{R'}{R}$, $\frac{R}{D}$ and λ_1, are very small quantities; and if we develop the expression of f [7036], according to the powers of $\frac{R}{D}$, we shall
[7036d] have $f = -\frac{D}{R.(1-\lambda_1)} + \frac{R.(1-\lambda_1)}{2D} + \&c.$; which, by neglecting the small terms of the order $\frac{R}{D}$, becomes as in [7037]; observing that *the symbol λ_1 denotes the ratio of the*
[7036e] *equatorial diameter of Jupiter to that of the sun* [7025, 7018d, 6982, &c.], which is very
[7036f] small; so that the expression of f [7037] is of the same order as $\frac{D}{R}$, which is very great in comparison with the neglected terms of the expression [7036d].

[7039a] * (3532) Dividing the equation [7035] by f^2, then substituting the value of f [7037], we get [7039], *which represents the equation of the surface of the real shadow of Jupiter.*
[7039b] The equation of the penumbra is obtained in a similar manner; for the equations [6983—6991] corresponding to the sun's surface, and those in [7017—7023] relative to the surface of Jupiter, require no alteration. But the quantity $c-D$ [7024] changes its
[7039c] signs in the first and third members of [7024], as in [7024h, i]. This produces a change in the signs of the second members of the equations [7028c], which were deduced from [7024], as well as in the resulting equation [7028d]; and this change may evidently be produced, by supposing the sign of λ_1 to be changed in [7028d, &c.]. Therefore the equations [7028—7035] may be made to correspond to the penumbra, by changing the sign of λ_1. Moreover, in [7036] we must change the sign of f, and put
[7039d] $f = \sqrt{\frac{D}{R^2.(1+\lambda_1)^2}-1}$, because $x > \frac{D}{1+\lambda_1}$ [7000g, &c., 7036b, &c.]; so that for
[7039e] the penumbra we must have $f = \frac{D}{R.(1+\lambda_1)}$, instead of [7037]. This change in the sign of f, produces a change in the sign of the last term of [7035], which contains the divisor
[7039f] $\sqrt{y^2+z^2}$; so that in this term the radical may be supposed to change its sign, instead of f^3. Finally, by making these changes in the signs of λ_1, $\sqrt{y^2+z^2}$, in the equation of the
[7039g] shadow [7039], we get that of the penumbra [7040].

$$\frac{R^2.(1+\lambda_1)^2}{D^2}\cdot\left(x-\frac{D}{1+\lambda_1}\right)^2=y^2+z^2+\frac{2\lambda_1}{1+\lambda_1}\cdot\rho z^2.\left\{\frac{R}{\sqrt{y^2+z^2}}+1\right\}. \qquad [7040]$$

Penumbra of Jupiter.

We shall now consider a section of Jupiter's shadow, by a plane perpendicular to the axis, at the distance r from the centre of the planet. We shall have in this case, $x=D+r$ [6982″]; therefore,* [7041]

$$\frac{R^2}{D^2}\cdot\{D\lambda_1-r.(1-\lambda_1)\}^2=y^2+z^2+\frac{2\lambda_1}{1-\lambda_1}\cdot\rho z^2.\left\{\frac{R}{\sqrt{y^2+z^2}}-1\right\}. \qquad [7042]$$

We have at first, by taking the square root of [7042], and neglecting terms of the order ρ,

$$\sqrt{y^2+z^2}=\frac{R}{D}\cdot\{D\lambda_1-r.(1-\lambda_1)\}=R.\left\{\lambda_1-\frac{r.(1-\lambda_1)}{D}\right\}. \qquad [7043]$$

Substituting this approximate value in the small term of [7042], which is multiplied by ρ, we obtain, for the equation of the shadow, the following expression;

$$(1+\rho)^2.R'^2.\left\{1-\frac{r.(1-\lambda_1)}{D\lambda_1}\right\}^2=y^2+z^2+\frac{2\rho.z^2.\left\{1+\frac{r}{D}\right\}}{1-\frac{r.(1-\lambda_1)}{D\lambda_1}}. \qquad [7044]$$

This equation appertains to an ellipsis, whose ellipticity is† $\dfrac{\rho.\left(1+\frac{r}{D}\right)}{1-\frac{r.(1-\lambda_1)}{D\lambda_1}}$. [7045]

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* (3533) The first member of [7039] may be put under the form $\frac{R^2}{D^2}\cdot\{D-x.(1-\lambda_1)\}^2$, and then substituting $x=D+r$ [7041], it becomes as in [7042]. Neglecting ρ in [7042], and extracting the square root, we get the approximate value of $\sqrt{y^2+z^2}$ [7043]. Substituting this in the small term containing ρ in [7042], and putting also $R=\frac{(1+\rho).R'}{\lambda_1}$ [7025], we get, [7042a]

$$\frac{(1+\rho)^2.R'^2}{D^2\lambda_1^2}\cdot\{D\lambda_1-r.(1-\lambda_1)\}^2=y^2+z^2+\frac{2\lambda_1}{1-\lambda_1}\cdot\rho z^2.\left\{\frac{1}{\lambda_1-\frac{r.(1-\lambda_1)}{D}}-1\right\}; \qquad [7042b]$$

which is easily put under the form [7044]; observing, that by reducing the last factor of [7042b] to a common denominator, we have,

$$\frac{1}{\lambda_1-\frac{r.(1-\lambda_1)}{D}}-1=\frac{1-\lambda_1+\frac{r.(1-\lambda_1)}{D}}{\lambda_1-\frac{r.(1-\lambda_1)}{D}}=(1-\lambda_1).\left\{\frac{1+\frac{r}{D}}{\lambda_1-\frac{r.(1-\lambda_1)}{D}}\right\}=\frac{1-\lambda_1}{\lambda_1}.\left\{\frac{1+\frac{r}{D}}{1-\frac{r.(1-\lambda_1)}{D\lambda_1}}\right\}. \qquad [7042c]$$

† (3534) We have in [2088], $k^2=x^2+\lambda_{11}{}^2y^2$, for the equation of an ellipsis, whose [7045a]

[7046] The quantity $\frac{r}{D\lambda_1}$ is very small, even in its greatest value corresponding to the fourth satellite; hence we see that *this ellipsis, or the figure of the section of the shadow, is very similar to the generating ellipsis of Jupiter*, [7045k]. We shall use the following symbols;

[7047] $\alpha =$ the semi-major axis of the section of the shadow;

[7047′] $\rho' =$ the ellipticity of the section of the shadow;

[7047″] $R' =$ the polar semi-axis of Jupiter's mass [6982′];

[7048] $R'.(1+\rho) =$ the equatorial semi-diameter of Jupiter, [7018c]; its ellipticity being ρ;

then we shall have,

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rectangular co-ordinates are x, y, greatest semi-axis k, and least semi-axis $\frac{k}{\lambda_{11}}$ [2088a]. If we change x into y, and y into z, in order to conform to the present notation, this

[7045b] equation will become $k^2 = y^2 + \lambda_{11}{}^2.z^2 = y^2 + z^2 + (\lambda_{11}{}^2 - 1).z^2$. Comparing this with the equation [7044], we find that they become identical, by putting k^2 equal to the first

[7045c] member of [7044], and $\lambda_{11}{}^2 - 1$ equal to the coefficient of z^2, in the last term of the second member of [7044]. Hence we get,

[7045d] $$k = (1+\rho).R'.\left\{1 - \frac{r.(1-\lambda_1)}{D\lambda_1}\right\} = \alpha; \ [7048']$$

[7045e] $$\lambda_{11} = \sqrt{\left\{1 + \frac{2\rho.\left\{1+\frac{r}{D}\right\}}{1 - \frac{r.(1-\lambda_1)}{D\lambda_1}}\right\}} = 1 + \frac{\rho.\left\{1+\frac{r}{D}\right\}}{1 - \frac{r.(1-\lambda_1)}{D\lambda_1}}.$$

[7045f] The greatest semi-axis k is the same as α [7048′]. The ellipticity, which is represented by the difference of the two semi-axes k, $\frac{k}{\lambda_{11}}$, divided by the first of these quantities, is equal to $1 - \frac{1}{\lambda_{11}} = \frac{\lambda_{11}-1}{\lambda_{11}} = \lambda_{11} - 1$, nearly; and by using λ_{11} [7045e], it becomes as

[7045g] in [7045 or 7049]. Now we have $\lambda_1 = \frac{1}{10}$ nearly [7547]; substituting this in the

[7045h] ellipticity of the shadow ρ' [7049], it becomes nearly $\rho' = \rho.\left(\frac{1+\frac{r}{D}}{1-9.\frac{r}{D}}\right) = \rho.\left(1+10.\frac{r}{D}\right)$;

and if we take for r its greatest value, corresponding to the distance a''' of the fourth satellite, we shall have very nearly, from [6785], $r = a''' = D.\sin.1530'',864 = \frac{1}{416}.D$;

[7045i] hence $10.\frac{r}{D} = \frac{1}{42}$ nearly; consequently the ellipticity ρ' [7045h], becomes $\rho' = \frac{43}{42}\rho$;

[7045k] which differs only $\frac{1}{42}$ part from the ellipticity ρ of Jupiter's surface [7048], as in [7046].

$$\alpha=(1+\rho).R'.\left\{1-\frac{r.(1-\lambda_1)}{D\lambda_1}\right\};$$ [7048′]

[7049] Ellipticity of the shadow.

$$\rho'=\frac{\rho.\left(1+\frac{r}{D}\right)}{1-\frac{r.(1-\lambda_1)}{D\lambda_1}}=\text{the ellipticity of the shadow};$$

we shall also have, for *the equation of the section of Jupiter's shadow*,* Equation of the section of the shadow.

$$\alpha^2-y^2=(1+\rho')^2.z^2;\quad\text{or}\quad y^2=\alpha^2-(1+\rho')^2.z^2;$$ [7050]

and 2α will be the greatest width of this section. [7051]

If we suppose λ_1 to be negative, in the values of α and ρ', the equation [7050] will become that of the section of the penumbra.† Now the greatest [7052] width of the penumbra, at the distance r from Jupiter's centre, being equal to the difference of the two values of α, relative to the shadow and the penumbra, it will be given by the following expression [7053]; R being [7052′] the semi-diameter of the sun [6982];

$$\text{greatest penumbra}=(1+\rho).R'.\frac{2r}{D\lambda_1}=\frac{2r.R}{D}\ [7052e].$$ [7053]

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* (3535) Substituting in the first member of [7011] its abridged value α^2 [7045d], and in the last term of its second member, the symbol ρ' instead of its value [7049], we get $\alpha^2=y^2+z^2+2\rho'.z^2$, or $\alpha^2-y^2=z^2(1+2\rho')=z^2.(1+\rho')^2$, nearly, as in [7050]. [7050a] Putting $z=0$, we get $y=\alpha=$ the semi-major axis, as in [7051].

† (3536) Changing the signs of λ_1, R, in the equation of the shadow [7039], we get that of the penumbra [7040]. The equation [7043] becomes,

$$\sqrt{y^2+z^2}=R.\lambda_1.\left\{1+\frac{r.(1+\lambda_1)}{D\lambda_1}\right\}.$$ [7052a]

This change in the signs of λ_1, R, requires a similar change in the values in [7048′, 7049]; and we shall put α_1, ρ_1', for the values corresponding to α, ρ', respectively; then we [7052b] shall have, from [7048′, 7049], observing that the value of R' [7025] is not altered,

$$\alpha_1=(1+\rho).R'.\left(1+\frac{r.(1+\lambda_1)}{D\lambda_1}\right);\qquad\rho_1'=\frac{\rho.\left(1+\frac{r}{D}\right)}{1+\frac{r.(1+\lambda_1)}{D\lambda_1}}.$$ [7052c]

We shall also have, for the equation of the surface of the penumbra, the following expression, which is similar to [7050];

$$\alpha_1^2-y^2=(1+\rho_1')^2.z^2.$$ [7052d]

Subtracting the value of α [7048′] from that of α_1 [7052c], we get, by using $R=\frac{(1+\rho).R'}{\lambda_1}$ [7025],

$$\alpha_1-\alpha=(1+\rho).R'.\frac{2r}{D\lambda_1}=\frac{2r.R}{D};\ \text{as in [7053].}$$ [7052e]

Symbols. We shall now put,

[7053'] $Z =$ the height of a satellite, above the orbit of Jupiter, at the moment of its conjunction ;

[7054] $r =$ the distance of the satellite from the centre of Jupiter, at the time of the eclipse ;

[7055] $v_1 =$ the angle described by the satellite, by its synodical motion, in the orbit of the planet, from the time of its conjunction ;

[7056] x. The axis of x is the projection of the radius vector of the satellite, on the orbit of Jupiter, at the instant of the conjunction ; or, in other words, it is the continuation of the radius vector of Jupiter's orbit, at the same instant ;

then we shall have,*

[7057]
$$y^2 = (r^2 - z^2).\sin.^2.v_1.$$

Hence the equation of the surface of the shadow becomes,

[7058]
$$(r^2 - z^2).\sin.^2 v_1 = \alpha^2 - (1+\rho')^2.z^2.$$

Equations of the shadow. *We shall neglect the quantities of the order* z^4, and $z^2.\sin.^2.v_1$, which reduces

[7059] this equation to the following form ;

[7060]
$$r^2.\sin.^2 v_1 = \alpha^2 - (1+\rho')^2.z^2.$$

Now we have,†

[7061]
$$z = Z + \sin.v_1.\frac{dZ}{dv_1} + \tfrac{1}{2}.\sin.^2.v_1.\frac{ddZ}{dv_1^2} + \&c. ;$$

[7057a] * (3537) The satellite being at the distance r from the centre of the planet, and at the height z above the plane of x, y, the projection of r upon this plane will be

[7057b] represented by $r_1 = \sqrt{r^2 - z^2}$; and as this projected line r_1 forms the angle v_1, with the axis of x [7055, 7056] ; and the angle $100^\circ - v_1$, with the axis of y ; we shall evidently

[7057c] have $y = r_1.\sin.v_1 = \sqrt{r^2 - z^2}.\sin.v_1$; whose square gives y^2 [7057]. Substituting this in the second of the equations [7050], we get [7058] ; and if we neglect the term $z^2.\sin.^2 v_1$, on account of its smallness, it becomes as in [7060].

[7061a] † (3538) Considering z as a function of v_1, and developing it according to the powers of v_1, by Maclaurin's theorem [607a], we get $z = Z + v_1.\frac{dZ}{dv_1} + \frac{1}{2}v_1^2.\frac{ddZ}{dv_1^2} + \&c.$; substituting $v_1 = \sin.v_1 + \frac{1}{6}.\sin.^3 v_1 + \&c.$ [46 Int.], and neglecting terms of the order

[7061b] mentioned in [7059], we get the expression of z [7061] ; observing that $\frac{dZ}{dv_1}$, $\frac{ddZ}{dv_1^2}$ are of the same order as z, as is evident from the consideration that Z is of the order as [7065] ; and if we notice only its chief term in [7522 line 1], we shall have nearly

[7061c] $Z = as = a.3^\circ,43267.\sin.(v + 51^\circ,3787)$ nearly. Now dv is to dv_1, as n to $n - M$, as

therefore we shall have, very nearly,*

$$r^2.\sin.^2.v_1 = \alpha^2-(1+\rho')^2.Z^2-2.(1+\rho')^2.\sin.v_1.Z.\frac{dZ}{dv_1}. \qquad [7062]$$

Hence we deduce,

$$\sin.v_1 = -\frac{(1+\rho')^2.Z.\frac{dZ}{dv_1}}{r^2} \pm \sqrt{\left\{\frac{\alpha}{r}+(1+\rho').\frac{Z}{r}\right\}.\left\{\frac{\alpha}{r}-(1+\rho').\frac{Z}{r}\right\}}. \qquad [7063]$$

If s be supposed to express the tangent of the latitude of the satellite, above [7064] *Jupiter's orbit, at the time of the conjunction* [6033], we shall have very nearly $Z=rs$;† and as r is nearly constant, the preceding equation becomes, [7065]

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is manifest from the definitions of v, v_1, n, M [6023c, d, 6022f, 6021z]; and as M is very small relative to n [6025i], dv_1 may be considered as of the same order as dv; [7061d] consequently $\frac{dZ}{dv_1}$ is of the same order as $\frac{dZ}{dv}$. Substituting in this last expression the value of Z [7061c], we find that $\frac{dZ}{dv}$ or $\frac{dZ}{dv_1}$ is of the same order as Z; and the same [7061e] holds good relative to $\frac{ddZ}{dv_1^2}$, &c.

* (3539) Taking the square of z [7061], and neglecting terms of the fourth order in Z, v_1, it becomes $z^2=Z^2+2.\sin.v_1.Z.\frac{dZ}{dv_1}+$ &c. Substituting this in [7060], we get [7062a] [7062], which is a quadratic equation in $\sin.v_1$, and its root gives $\sin.v_1$ [7063]; observing that if we transpose the last term of the second member of [7062] into the first member, the equation becomes $r^2.\sin.^2v_1+2.(1+\rho')^2.\sin.v_1.Z.\frac{dZ}{dv_1}=\alpha^2-(1+\rho')^2.Z^2$; which may [7062b] evidently be put under the form,

$$r^2.\left\{\sin.v_1+\frac{(1+\rho')^2.Z.\frac{dZ}{dv_1}}{r^2}\right\}^2=\{\alpha+(1+\rho').Z\}.\{\alpha-(1+\rho').Z\}; \qquad [7062c]$$

neglecting terms of the order Z^2dZ^2. Dividing this by r^2, and taking the square root, we obtain [7063].

† (3540) The expression of Z [7065] can be deduced from [6036], by neglecting terms of the order s^3. The orbit of the satellite being nearly circular, r is nearly constant; [7065a] and then the differential of Z, relative to v_1, gives $\frac{dZ}{dv_1}=r.\frac{ds}{dv_1}$ nearly. Substituting these values in [7063], we get [7066]. The remarks in [7067, 7068], relative to the [7065b] signs, are evidently correct; taking into consideration that the *positive* values of v_1 follow after the conjunction.

[7066] $$\sin.v_1 = -(1+\rho')^2.\frac{sds}{dv_1} \pm \sqrt{\left\{\frac{\alpha}{r}+(1+\rho').s\right\}.\left\{\frac{\alpha}{r}-(1+\rho').s\right\}}.$$

[7067] *This formula, with the sign $+$ prefixed to the radical, denotes the sine of the arc described by the satellite, by means of its synodical motion, from the*
[7068] *conjunction to the emersion. With the sign $-$, it denotes the negative value of the sine of the similar arc, from the immersion to the conjunction.*

We shall put,*

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* (3541) If we notice only the chief inequalities in $\frac{r}{a}$ and v, relative to the satellite m [6131, 6132], and also the chief term of the elliptical values [669], we shall have,

[7071a] $$\frac{r}{a} = 1 - e.\cos.(nt+\varepsilon-\varpi) - \frac{m'.n.F}{2.(2n-2n'-\mathcal{N})}.\cos.(2nt-2n't+2\varepsilon-2\varepsilon');$$

[7071b] $$v = nt+\varepsilon+2e.\sin.(nt+\varepsilon-\varpi) + \frac{m'.n.F}{(2n-2n'-\mathcal{N})}.\sin.(2nt-2n't+2\varepsilon-2\varepsilon');$$

and if, for brevity, we put,

[7071c] $$X' = 2e.\cos.(nt+\varepsilon-\varpi) + \frac{m'.n.F}{(2n-2n'-\mathcal{N})}.\cos.(2nt-2n't+2\varepsilon-2\varepsilon');$$

we shall find that the expression [7071a] becomes,

[7071d] $$\frac{r}{a} = 1 - \tfrac{1}{2}X'.$$

Moreover if we take the differential of [7071b], and divide it by ndt, substituting X' [7071c], we shall get [7071e], observing that we have very nearly $ndt = (2n-2n').dt$ [6151];

[7071e] $$\frac{dv}{ndt} = 1 + X'; \quad \text{or} \quad dv = ndt + ndt.X'.$$

[7071f] Now if we suppose, as in [6101'], that Mdt represents the sideral motion of Jupiter, during the time dt, we shall have, from [6023c, d], $dv = dv_1 + Mdt$, nearly. Substituting the value of dv [7071e], we obtain $ndt.X' = dv_1 - (n-M).dt$. Dividing by $(n-M).dt$,
[7071g] we get $\frac{n}{n-M}.X' = \frac{dv_1}{(n-M).dt} - 1$; so that if we put $\frac{n}{n-M}.X' = X$, it becomes as in
[7071h] [7071]. Lastly, as $n-M$ is nearly equal to n [6025i], we shall have $X' = X$ nearly; and it is evident, from [7071c], that X', or X, must be a small quantity of the order e, &c.

[7071i] From what has been said it appears, that if $\frac{r}{a}$ contain a periodical term of the form $-\frac{1}{2}X$ [7071d], $\frac{dv}{ndt}$ will contain a term of the form $+X$ [7071e]. A similar result
[7071k] holds good also with the satellites m', m'', m'''. This is evident as it respects the elliptical
[7071l] terms depending on e', e'', e''', which are of the same form as those in [7071a, b] depending on e. Moreover by comparing the coefficient of $\frac{r'\delta r'}{a'^2}$ [6160], with that of $\frac{d.\delta v'}{n'dt}$, deduced from [6161], and observing that $n-n'=n'$ nearly [6151], they become of the same
[7071m] form. Also $\frac{r''\delta r''}{a''^2}$ [6164], and $\frac{d.\delta v''}{n''dt}$ deduced from [6165], correspond in like manner.

$T =$ the time employed by the satellite in describing half the greatest width α of the shadow, by means of the synodical motion ; Symbols T, t, X [7069]

t = the time of describing the angle v_1, by the synodical motion ; [7070]

$$X = \frac{dv_1}{(n-M).dt} - 1 \text{ ; or } dv_1 = (n-M).dt.(1+X) \text{ ;} \qquad [7071]$$

X being a very small quantity. a is the mean distance of the satellite from [7072] Jupiter [6079] ; $\frac{\alpha}{a}$ is the sine of the angle under which the radius α [7073] appears at the distance a. Putting this angle equal to β, or,

$$\sin.\beta = \frac{\alpha}{a}, \qquad [7074]$$

we shall have, very nearly,*

$$t = \frac{Tv_1.(1-X)}{\beta}. \qquad [7075]$$

If we substitute in this expression for v_1 its sine, which differs but very little from the arc [7061 b] ; and for $\sin.v_1$, the preceding value [7066] ; also $\beta = \frac{\alpha}{a}$ nearly [7074] ; we shall have, [7076]

$$t = T.(1-X).\left\{-(1+\rho')^2.\frac{s}{\beta}.\frac{ds}{dv_1} \pm \sqrt{\left\{\frac{a}{r}+(1+\rho').\frac{s}{\beta}\right\}.\left\{\frac{a}{r}-(1+\rho').\frac{s}{\beta}\right\}}\right\}. \qquad [7077]$$

If we notice only the equation of the centre of the satellite, we shall have, as in [7071 c, d, e, h], nearly,

$$r = a.\{1-\tfrac{1}{2}X\}; \quad \text{or} \quad \frac{a}{r} = 1+\tfrac{1}{2}X; \qquad [7078]$$

and it also follows, from the same articles [7071 c, d, e, h], that this equation

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* (3542) During the eclipse we may suppose the variation of v_1 to be proportional to t, so that $\frac{dv_1}{dt} = \frac{v_1}{t}$; hence [7071] becomes $X = \frac{v_1}{(n-M).t} - 1$; consequently, [7075 a]

$$t = \frac{v_1}{n-M}.\frac{1}{1+X} = \frac{v_1}{n-M}.(1-X), \text{ nearly ;} \qquad [7075b]$$

or by multiplying the numerator and denominator by T; $t = \frac{Tv_1.(1-X)}{(n-M).T}$; and since $(n-M).T$ represents the synodical arc described by the satellite in the time T, which, [7075 c] by [7069, 7074], is equal to β, it becomes $t = \frac{Tv_1.(1-X)}{\beta}$, as in [7075] ; or, as it may be expressed very nearly, $t = T.(1-X).\frac{\sin.v_1}{\beta}$. Substituting the value of $\sin.v_1$ [7066], [7075 d] it becomes as in [7077], observing that $\frac{\alpha}{\beta} = a$ nearly [7076].

[7079] holds good even when we include the chief inequalities of the satellite; therefore we shall have, very nearly,*

[7080] $$t=T.(1-X).\left\{-(1+\rho')^2.\frac{s}{\beta}.\frac{ds}{dv_1}\pm\sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s}{\beta}\right\}}\right\}.$$

[7081] If we put *the whole duration of the eclipse equal to* t', we shall have,†

Symbol t'

[7082] $$t'=2T.(1-X).\sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s}{\beta}\right\}}.$$

Hence we deduce,‡

[7083] $$s=\frac{\beta.\sqrt{4T^2.(1-X)-t'^2}}{2T.(1+\rho').(1-X)}.$$

[7084] *This equation will serve to determine the arbitrary constant quantities which enter into the expression of* s, *by selecting the observations of the eclipses in which these quantities have the most influence.*

The duration of these eclipses being one of the most important parts of their theory, we shall examine particularly the preceding formulas for

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[7080a] * (3543) Substituting in the radical expression of [7077], the value $\frac{a}{r}=1+\frac{1}{2}X$ nearly [7078], we get [7080].

[7081a] † (3544) We shall suppose the time from the conjunction to the emersion to be t_1; the time from the immersion to the conjunction, considered as negative, is of the form $-t_{11}$,

[7081b] t_{11} being positive. These values may be deduced from [7080], attending to the remarks in [7067, 7068]; hence we have,

[7081c] $$t_1=T.(1-X).\left\{-(1+\rho')^2.\frac{s}{\beta}.\frac{ds}{dv_1}+\sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s}{\beta}\right\}}\right\};$$

[7081d] $$-t_{11}=T.(1-X).\left\{-(1+\rho')^2.\frac{s}{\beta}.\frac{ds}{dv_1}-\sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s}{\beta}\right\}}\right\}.$$

[7081e] Subtracting the second of these expressions from the first, and putting $t_1+t_{11}=t'$ [7081], we obtain [7082].

‡ [3545] Connecting the two factors in the radical [7082], and neglecting X^2, we get

[7083a] $$t'=2T.(1-X).\sqrt{1+X-(1+\rho')^2.\frac{s^2}{\beta^2}}.$$

Squaring this we obtain,

[7083b] $$t'^2=4T^2.(1-X)-4T^2.(1-X)^2.(1+\rho')^2.\frac{s^2}{\beta^2};$$

[7083c] hence $\frac{s^2}{\beta^2}=\frac{4T^2.(1-X)-t'^2}{4T^2.(1+\rho')^2.(1-X)^2}$. Extracting the square root, and multiplying by β, we obtain [7083].

computing them. The radius of the shadow α varies with the distances of the satellites from Jupiter, and of Jupiter from the sun. Putting $D = D' - \delta D$ for Jupiter's distance from the sun; D' being the mean [7085]
distance; and supposing, as in [7078], $r = a.(1-\frac{1}{2}X)$, we shall find that [7085']
the variations of α will be represented by the following expression;*

$$(1+\rho).R'.\left\{\tfrac{1}{2}X - \frac{\delta D}{D'}\right\}.\frac{(1-\lambda_1).a}{D'\lambda_1} = \text{variable part of } \alpha. \qquad [7086]$$

$\frac{1}{2}X$ is always very small in comparison with $\frac{\delta D}{D'}$,† and this last quantity is [7087]

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* (3546) The values of D, r [7085, 7085'], give,

$$\frac{r}{D} = \frac{a.(1-\frac{1}{2}X)}{D'-\delta D} = \frac{a}{D'}.\left\{1-\tfrac{1}{2}X+\frac{\delta D}{D'}\right\} \text{ nearly.} \qquad [7085a]$$

Substituting this in [7018'], we get,

$$\alpha = (1+\rho).R'.\left\{1-\frac{a.(1-\lambda_1)}{D'\lambda_1}\right\} + (1+\rho).R'.\left\{\tfrac{1}{2}X-\frac{\delta D}{D'}\right\}.\frac{(1-\lambda_1).a}{D'\lambda_1}. \qquad [7085b]$$

Of the two terms contained in this value of α, the first is constant, and the second variable; being the same as in [7086].

† (3547) If we put the values of D [6275, 7085] equal to each other, we shall get,

$$D'-\delta D = D'-D'H.\cos.(Mt+E-I)\,; \quad \text{whence} \quad \frac{\delta D}{D'} = H.\cos.(Mt+E-I)\,; \qquad [7086a]$$

and as $H = 0{,}048\ldots$[6882b], $\frac{\delta D}{D'}$ is of the order 0,048. On the other hand, the values [7086b]
of $\frac{1}{2}X$, corresponding to the satellites m, m', m'', m''', are found in [7527, 7488, 7432, 7377]
to be respectively of the order 0,0039; 0,0093; 0,0013; 0,0072; which are much less [7086c]
than the preceding value of $\frac{\delta D}{D'}$, as is observed in [7087]; and the author has therefore neglected the term of [7086], depending on $\frac{1}{2}X$. To estimate roughly the value of this [7086d]
neglected term, we may observe, that the chief term of α [7048'], namely $(1+\rho).R'$, is
to the term of [7086] depending on X, as 1 to $\frac{1}{2}X.\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}$, and by substituting the [7086e]
values $\frac{1-\lambda_1}{\lambda_1} = 9$ [7045g], $\frac{a}{D'} = \frac{1}{416}$ [7045i, &c.], this ratio becomes nearly as 1 to
0,01.X. Now if we take even the greatest of the preceding values of $\frac{1}{2}X = 0{,}0093$ [7086f]
[7086c], we shall have $0{,}01.X = 0{,}000186$; and its actual value is generally much less.
Multiplying this by the semi-durations of the eclipses [7562—7565], we get the greatest [7086g]
effect of this neglected term in the times of these eclipses; and this correction very rarely amounts to a second of time of the centesimal division. Hence we see that we may safely
neglect the term depending on X, in [7086]; and if we substitute for $(1+\rho).R'$ its value [7086h]
α nearly [7048'], the remaining term of [7086] becomes $-\alpha.\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}.\frac{\delta D}{D'}$; and

[7087′] equal to $H.\cos.(Mt+E-I)$ [7086a]; therefore the variation of α is very nearly represented by the following expression,

[7088] $$-\alpha.\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}.H.\cos.(Mt+E-I)=\text{variable part of }\alpha.$$

Hence it follows, that in the preceding formulas we must substitute, for $\frac{\alpha}{\beta}$, the following function,

[7089] $$\frac{\alpha}{\beta}.\left\{1-\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}.H.\cos.(Mt+E-I)\right\}.\quad\left[\text{Corrected value of }\frac{\alpha}{\beta}\right]$$

[7090] β, *in this function, corresponds to the mean motions and mean distances of the satellite from Jupiter, and of Jupiter from the sun.*

[7090′] T denotes the time that the satellite employs in describing half the width of the shadow α [7069]; this time increases in consequence of the variations of α, by the quantity,

[7091] $$-T.\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}.H.\cos.(Mt+E-I);\quad[\text{First increment of }T]$$

and it increases, because the synodical motion, in the time dt, is very nearly equal to,*

[7092] $$(n-M).dt.\left\{1+X-\frac{2M}{n-M}.H.\cos.(Mt+E-I)\right\};$$

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by substituting the value of $\frac{\delta D}{D'}$ [7086a], it becomes as in [7088]. So that if we use the

[7086i] mean values of α, β, we ought to change α into $\alpha-\alpha.\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}.H.\cos.(Mt+E-I)$; therefore $\frac{\alpha}{\beta}$ must be changed into the factor [7089]. Now as T [7069] denotes the

[7086k] time of describing the arch α, this time will be increased in consequence of the variable part [7088], by a quantity which is proportional to that variable part, as in [7091].

[7091a] * (3548) The longitude of the sun, as seen from Jupiter, is $Mt+E+2H.\sin.(Mt+E-I)$ [6101, 6102, 6882a]. Its differential, divided by dt, is $M+2HM.\cos.(Mt+E-I)$;

[7091b] which must be substituted for M, in the expression of the synodical motion $dv_1=(n-M).dt.(1+X)$ [7071, 7055], corresponding to the time dt, and by this means it becomes,

[7091c] $\{n-M-2HM.\cos.(Mt+E-I)\}.dt.(1+X)=(n-M).dt.\left\{1+X-\frac{2M}{n-M}.H.\cos.(Mt+E-I)\right\}$

nearly, as in [7092]; which is to be used instead of $(n-M).dt$, in the preceding

[7091d] calculations. Now as the time T [7069] is inversely as the synodical motion, it will be

[7091e] represented by the mean value of T, divided by $\left\{1+X-\frac{2M}{n-M}.H.\cos.(Mt+E-I)\right\}$,

which gives for the increment of T, depending upon this cause,

$$T.\left\{\frac{2M}{n-M}.H.\cos.(Mt+E-I)-X\right\};\ [7091e]$$ [Second increment of T] [7093]

Therefore by neglecting X, as we have done above, we shall find, that by the combined effect of both these causes, T changes into, [7093′]

$$T.\left\{1+\left(\frac{2M}{n-M}-\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{D'}\right).H.\cos.(Mt+E-I)\right\}.$$ [Corrected value of T] [7094]

but these two causes have no sensible effect, except upon the eclipses of the fourth satellite [7091g].

In the beginning and end of these eclipses, the quantities of the order s^3, which we have neglected in the radical part of the expression of t, [7063, 7075, &c.], may become sensible [7383i, &c.] But the only one which [7094′]
has any influence is the square of* $(1+\rho')^2.\frac{sds}{\beta.dv_1}$, which must be added to [7095]

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which augments the value of T, by the two terms given in [7093] nearly. Increasing T by the terms in [7091, 7093], and neglecting X, as in the last note, it becomes as in [7094]. The effect of this correction is, in general, insensible; and even in the eclipses of [7091f]
the fourth satellite it is hardly deserving of notice, as is shown in [7379], where the [7091g]
correction of T is found to be of the order $0{,}0006101.T$, at its maximum; and if we suppose T to be equal to $9890''$, as in [7565], it will never amount to 6 seconds of the centesimal division, and in general will be much less. [7091h]

* (3549) In finding $\sin.v_1$ [7063], we have neglected the term $\left(\frac{(1+\rho')^2.Z.\frac{dZ}{dv_1}}{r^2}\right)^2$ [7095a]
under the radical, and by substituting $\frac{Z}{r}=s$ [7065], $\frac{dZ}{rdv_1}=\frac{ds}{dv_1}$ nearly, it becomes $\left\{(1+\rho')^2.\frac{sds}{dv_1}\right\}^2$. This is to be multiplied by $\frac{T.(1-X)}{\beta}$ to produce the corresponding part of t [7075l, 7077, or 7080]; or, in other words, the function under the radical in [7095a′]
[7077, 7080], must be multiplied by $\frac{1}{\beta^2}$; by this means the neglected term in [7077, 7080] [7095b]
becomes $\frac{1}{\beta^2}.\left\{(1+\rho')^2.\frac{sds}{dv_1}\right\}^2$, as in [7095] nearly; and for brevity we may represent it by δX. If we neglect the part depending on ρ', on account of its smallness, we shall have [7095c]
$\delta X=\frac{s^2ds^2}{\beta^2.dv_1^2}$. Now the product of the two factors of the radical in [7080, or 7082] is [7095d]
nearly $1+X-(1+\rho')^2.\frac{s^2}{\beta^2}$; and if we increase this by the neglected term δX, it will be the same as to change X into $X+\delta X$, under the radical, in [7080, 7082], as is remarked in [7095, 7096]. [7095e]

the quantity contained under the radical. We may notice this by
[7096] augmenting X, by the quantity $\frac{s^2ds^2}{\beta^2.dv_1^{\,2}}$, under the radical in the expression
[7096′] of t' [7082], and *decreasing* X by the same quantity, under the radical of the expression of s* [7083].

We have confounded the arc v_1 with its sine [7076]; now we have very nearly, as in [7061*b*],†
[7097] $$v_1=\sin.v_1+\tfrac{1}{6}.\sin.^3v_1;$$

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* (3550) The author, in the original work, says, that in the expression of s [7083]
[7096*a*] we must *increase* X under the radical, by the quantity δX [7095*d*]; but this is not accurate, and we have corrected the mistake in [7096′]. For if we multiply together the two factors
[7096*b*] under the radical in [7082], neglecting X^2, and then increase X by δX, under the radical, as in [7095*e*], we get the expression of t' [7096*c*]. If we now introduce the external factor $1-X$, under the radical, it becomes as in [7096*d*]; always neglecting terms of the order X^2, or $X\delta X$. Squaring [7096*d*], and reducing, we obtain the value of s [7096*e*];

[7096*c*] $$t'=2T.(1-X).\sqrt{\left\{1+X+\delta X-(1+\rho')^2.\frac{s^2}{\beta^2}\right\}}$$

[7096*d*] $$=2T.\sqrt{\left\{1-X+\delta X-(1+\rho')^2.\frac{s^2}{\beta^2}.(1-X)^2\right\}};$$

[7096*e*] $$s=\frac{\beta\sqrt{\{4T^2.(1-X+\delta X)-t'^2\}}}{2T.(1+\rho').(1-X)}.$$

If we now compare the values of s [7083, 7096*e*], we see that $-X$ must be changed
[7096*f*] into $-X+\delta X$, under the radical of the expression of s [7083], to obtain the corrected value [7096*e*]; or, in other words, X must be *decreased* by $-\delta X$, in that part of the
[7096*g*] expression. This agrees with the corrected translation given in [7096′], but differs from the last paragraph of the original work.

† (3551) The expression of t [7075] is proportional to $\frac{v_1}{\beta}$; and if we substitute in it
[7097*a*] the value of v_1 [7097], and the similar expression of $\beta=\sin.\beta+\frac{1}{6}.\sin.^3\beta$, we get very nearly, by development,

[7097*b*] $$\frac{v_1}{\beta}=\frac{\sin.v_1}{\sin.\beta}.\frac{1+\frac{1}{6}.\sin.^2v_1}{1+\frac{1}{6}.\sin.^2\beta}=\frac{\sin.v_1}{\sin.\beta}.\left\{1+\tfrac{1}{6}.\sin.^2v_1-\tfrac{1}{6}.\sin.^2\beta\right\}$$

[7097*c*] $$=\frac{\sin.v_1}{\sin.\beta}.\left\{1+\tfrac{1}{12}.(1-\cos.2v_1)-\tfrac{1}{12}.(1-\cos.2\beta)\right\}=\frac{\sin.v_1}{\sin.\beta}.\left\{1-\tfrac{1}{12}.\cos.2v_1+\tfrac{1}{12}.\cos.2\beta\right\}.$$

Now in [7076] we have used $\frac{\sin.v_1}{\sin.\beta}$ for $\frac{v_1}{\beta}$, in the chief term of t, *which does not contain*
[7097*d*] s; and to correct this, we must multiply it by the factor $1-\frac{1}{12}.\cos.2v_1+\frac{1}{12}.\cos.2\beta$ nearly
[7097*e*] [7097*c*], as is observed in [7101]; we must also use the same factor in the expression of

therefore the preceding value of t' must be multiplied by $1+\frac{1}{6}.\sin.^2 v_1$. [7098]
Relative to the first satellite, v_1 is about ten degrees [7104, 7554]; and this renders the product of t', by $\frac{1}{6}.\sin.^2 v_1$, sensible. But this error is corrected, in a great measure, by the supposition we have made in [7076], that $\frac{\alpha}{a}=\beta$. For we have $\frac{\alpha}{a}=\sin.\beta$ [7074]; we ought therefore to have [7099]
supposed $\frac{\alpha}{a}=\beta-\frac{1}{6}.\beta^3$ [43] Int.; which amounts to nearly the same thing as to multiply the value of t' by $1-\frac{1}{6}.\sin.^2\beta$, because the term $-\frac{(1+\rho')^2.s^2}{\beta^2}$, [7100]
contained under the radical in the expression of t' [7096d], being a small fraction in the theory of the first satellite, we may neglect its product by $\frac{1}{3}.\sin.^2\beta$ [7097g]. The value t', determined by the preceding formula, must [7100′]
therefore be multiplied by $1+\frac{1}{6}.\sin.^2 v_1-\frac{1}{6}.\sin.^2\beta$, or by $1-\frac{1}{12}.\cos.2v_1+\frac{1}{12}.\cos.2\beta$ [7101]
[7097c]. The arc v_1 differs but little from β relative to the first satellite, so that the product of t' by $\frac{1}{12}.(\cos.2v_1-\cos.2\beta)$ is insensible.* [7102]

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t' [7082]. We may finally observe that the correction relative to β [7097b], produces a factor $1-\frac{1}{6}\sin.^2\beta$ without the radical in [7077]; or very nearly $1-\frac{1}{3}.\sin.^2\beta$, under the radical; but this ought not to be applied to the term depending on s, mentioned in [7097d], because the value of β [7076] was not introduced into this part of [7077]; but as this term is very small, we may neglect the effect of the correction arising from its multiplication by $\frac{1}{3}.\sin.^2\beta$, as in [7100′]. [7097f] [7097g]

* (3552) If the satellite pass through the centre of the shadow, it will describe the arch 2β [7076], during its passage; but when the latitude is large, the described arc, which we may call $2v_1$, must be less. To estimate roughly the effect of X, ζ [7527, 7529], corresponding to the first satellite, upon the time t', we shall observe, that X [7527] being much smaller than ζ [7529], we may neglect it; and then the expression of t' [7531] becomes $t'=9426''.\sqrt{1-\zeta^2}$; but when $s=0$, the value ζ [7528] vanishes, and t' becomes $9426''$. The arch described in the first case corresponds to $2v_1$ nearly; and in the second case, to 2β [7102a]. Hence $2\beta:2v_1::1:\sqrt{1-\zeta^2}$ nearly; and if we put β equal to $q=11^\circ,1780$ [7104, 7554], we shall get $2\beta=22^\circ,3560$, and $2v_1=22^\circ,3560.\sqrt{1-\zeta^2}$ nearly. The least value of $2v_1$ corresponds to the greatest value of ζ, and this is nearly $\zeta=\frac{1}{3}$ [7529]; therefore the least value of $2v_1$ is $2v_1=22^\circ,3560\sqrt{\frac{8}{9}}=21^\circ,0774$. Hence we get $\cos.2v_1-\cos.2\beta=0,0067$ and $\frac{1}{12}.(\cos.2v_1-\cos.2\beta)=0,0005$ at its maximum. Multiplying this by $9426''$ [7102c], we get the greatest error in the value of t', corresponding to the first satellite, which will not therefore exceed $5''$, and is generally much less. For the second satellite we have, in [7558], $2\beta=2q=13^\circ,9692$, and [7102a] [7102b] [7102c] [7102d] [7102e] [7102f]

[7102'] The value of T, determined by a very great number of eclipses, will give the mean distance of the satellite from the centre of Jupiter, in parts of the diameter of the equator of this planet, supposing the satellite to disappear

[7103] at the moment when its centre enters the shadow of Jupiter. For $\frac{\alpha}{a}$ [7073] *is the sine of the angle under which the half width of the shadow is seen, from Jupiter's centre, when this planet is at its mean distance from the sun, and*

[7104] *the satellite at its mean distance from Jupiter. Now if we put this angle*

q *equal to* q, *we shall have, from* [7047, 7103, &c.],*

[7105]
$$\frac{(1+\rho).R'}{a}\cdot\left\{1-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a}{D'}\right\}=\sin.q.$$

[7106] *The observed value of* T *will give that of the angle* q, *which is nothing more than the corresponding arc described by the satellite, by means of its synodical motion; therefore we shall have the four following equations;*

[7107]
$$\frac{(1+\rho).R'}{a'''}\cdot\left\{\frac{a'''}{a}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q\,;$$

[7108]
$$\frac{(1+\rho).R'}{a'''}\cdot\left\{\frac{a'''}{a'}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q'\,;$$

[7109]
$$\frac{(1+\rho).R'}{a'''}\cdot\left\{\frac{a'''}{a''}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q''\,;$$

[7110]
$$\frac{(1+\rho).R'}{a'''}\cdot\left\{1-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q'''.$$

[7102g] $2v_1=13^{\circ},9692.\sqrt{1-\xi^2}$; and as the greatest value of ξ [7491], is $\frac{1}{2}$, it becomes $2v_1=13^{\circ},9691.\sqrt{\tfrac{3}{4}}=12^{\circ},0977$. Hence $\frac{1}{12}.(\cos.2v_1-\cos.2\beta)=0\,0005$. Multiplying

[7102h] this by $11951''$ [7493], we get the greatest possible error in the value of t', corresponding to the second satellite, which does not exceed $6''$, and in general is much less. For the

[7102i] third and fourth satellites, v_1 is quite small; so that if we put $\cos 2v_1=1$, and use $\beta=q''$, or q''' [7556, 7557], we shall find that the greatest error in t' will be, by

[7102k] [7437, 7385], $14838''.\frac{1}{12}.(1-\cos.2q'')=11''$, and $19780''.\frac{1}{12}.(1-\cos.2q''')=5''$. Both these expressions will be decreased by using the actual values of $\cos.2v_1$; and in general these errors will be incomparably less than the limits here computed.

* (3553) The value of α [7048'] being divided by a, putting also D' for D [7085],

[7107a] and a for r, gives $\frac{\alpha}{a}=\sin.q$ [7104], corresponding to the first satellite, as in [7105].

[7107b] Changing the symbols a, q, relative to the first satellite, successively into a', q'; a'', q''; a''', q''', corresponding respectively to the second, third and fourth satellites, we obtain

[7107c] the equations [7107—7110], the forms being altered a little by the introduction of the

Each of these four equations gives a value of $\frac{a'''}{(1+\rho).R'}$, that is, the value [7111]
of a''' in parts of the radius $(1+\rho).R'$ of Jupiter's equator [7048]; for there is but little uncertainty in the ratio $\frac{a'''}{D'}$, given by Pound's observations, [7111']
which are quoted by Newton [7107*d*]; and the ratios $\frac{a'''}{a}$, $\frac{a'''}{a'}$, $\frac{a'''}{a''}$, are well determined in [6797—6800]. The differences of these values of a''' will make known the errors of the supposition that the satellites are eclipsed [7112]
at the moment of the entrance of their centres into the shadow. The penumbra, the magnitude as well as the brightness of the disc, and the refraction which the sun's rays may suffer in Jupiter's atmosphere, are [7112']
sources of error whose effect it will be difficult to ascertain with correctness.

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quantities a, a', &c. between the braces. Before closing this note we may remark, that we have already mentioned, in [6787*b*, &c.], that observations similar to those of Pound [7107*d*]
[7111'], have been made with great accuracy by Professor Airy.

CHAPTER IX.

DETERMINATION OF THE MASSES OF THE SATELLITES AND THE OBLATENESS OF JUPITER.

27. The formulas [6782—6947] contain a great number of indeterminate constant quantities, whose values are indispensable in the determination of the theory of each satellite. The chief of these are the masses of the four satellites and the oblateness of Jupiter: we shall first attend to these points. To determine their values, we must have, by observation, *five data*. We

Data used in finding the masses of the satellites and the oblateness of Jupiter.

[7113] shall take for the *first datum*, the principal inequality of the first satellite, whose greatest term, according to the results of Delambre's researches, is

First datum.

[7114] equal to $223'',471$, in time; that is to say, it retards or accelerates the eclipses of that satellite, by that quantity, when at its *maximum*. To reduce this to an arc of the circle, we must multiply it by the whole circumference, or 400°, and divide it by the time of the synodical revolution of the first

[7115] satellite, which is $1^{\text{days}},769861$.* Hence we have, for this term,

[7116] $$5050'',59.$$

The coefficient of the greatest term of this inequality is as in [6842 line 2],

[7117] $$m'.21736'',4863.$$

Putting the expressions [7116, 7117] equal to each other, we get,

[7118] $$m' = 0,232355.$$

[7119] We shall take for the *second datum*, the chief inequality of the second satellite, whose greatest term, according to the researches of Delambre, is

Second datum.

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* (3551) Multiplying the time of the sideral revolution [6778], by the value of

[7115a] $\frac{n}{n-M}$ [6025i], we get the time of the synodical revolution [7115], as in [6781i] nearly. The rest of the calculation in [7116—7118] is easily made. In the same manner we must

[7115b] proceed with the second satellite, using the value [6779], and multiplying it by $\frac{n'}{n'-M}$ [6025i], to obtain [7120].

equal to $1059'',18$ in time. To reduce it to an arc of the circle, we must [7119′]
multiply it by 400°, and divide it by the time of the synodical revolution
of the second satellite, which is nearly equal to $3^{\rm days},554095$ [6779, 6781k]; [7120]
hence we have, for this term, the following expression;

$$11920'',68. \qquad [7121]$$

The greatest coefficients of the terms of this inequality are in [6844 lines 1, 7]; and by taking these terms at their greatest positive values, and adding them together, we get,*

$$\mathrm{m}.6951'',466 + \mathrm{m}''.12108'',992. \qquad [7122]$$

Putting these two quantities [7121, 7122] equal to each other, we obtain,

$$\mathrm{m} = 1,714843 - \mathrm{m}''.1,741934. \quad (1) \qquad [7123]$$

Third datum.

The *third datum* which we shall use, is the annual sideral motion of the perijove of the fourth satellite; which, according to the computations of Delambre, is equal to $7959'',105$ [7195]. Therefore we shall suppose, in [7124]
the equation [6872], that

$$g = 7959'',105\,; \qquad [7124']$$

then it becomes, by dividing by h''',

$$\begin{aligned} 0 = 6984'',915 - 2946'',95.\mu - 363'',10.\mathrm{m} - 1077'',15.\mathrm{m}' - 4438'',87.\mathrm{m}'' \\ + 101'',03.\mathrm{m}.\frac{h}{h'''} + 471'',99.\mathrm{m}'.\frac{h'}{h'''} + 3012'',37.\mathrm{m}''.\frac{h''}{h'''}. \end{aligned} \qquad [7125]$$

To reduce this equation, so that it may contain only the indeterminate quantities μ, m, m″, we must eliminate the fractions $\frac{h}{h'''}, \frac{h'}{h'''}, \frac{h''}{h'''}$. The comparison of a great number of eclipses of the third satellite with the theory, makes it evident, that the expression of its motion contains two distinct equations of the centre,† and that one of them corresponds to the [7126]

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* (3555) The argument of the term in [6844 line 7], may be reduced to the same form as that in [6844 line 1], as is evident from [6172b]; and the maximum value corresponds [7123a]
to $\sin.(nt - n't + \varepsilon - \varepsilon') = \sin.2.(n''t - n't + \varepsilon'' - \varepsilon') = -1$; which makes the two terms become as in [7122]. Putting this expression equal to the value by observation [7121], [7123b]
and then dividing by $6951'',466$, we get [7123].

† (3556) This has already been observed in [6241t, &c.]. The terms of $\delta v''$ [6241i], and $\delta v'''$ [6241k], depending on the perijove of the fourth satellite ϖ''', are respectively [7127a]
represented by $\delta v'' = -0,0816578.2h'''.\sin.(\circleddash'' - \varpi''')$, $\delta v''' = -1,0000000.2h'''.\sin.(\circleddash''' - \varpi''')$ [6241i, k]; and by putting $2h''' = -9265'',56$, these coefficients become $756'',605$, [7127b]
$9265'',56$, as in [7127]; being derived from observation.

perijove of the fourth satellite. Delambre finds that this equation is
[7127] 756",605 [7389], and the equation of the centre of the fourth satellite 9265",56 [7318 line 1]. Therefore we have,

[7128] $$\frac{h''}{h'''} = \frac{756'',605}{9265'',56} = 0{,}0816578\,;$$

Fourth datum.

and this is the *fourth datum*, which we have derived from observation, in order to determine the masses. Hence the equation [7125] becomes, by the substitution of [7128],

[7129] $$\left.\begin{array}{l} 0 = 6734'',634 - 2946'',95.\mu - 363'',10.\mathrm{m} + 101'',03.\mathrm{m}.\frac{h}{h'''} \\ \quad -4192'',89.\mathrm{m}'' + 109'',67.\frac{h'}{h'''} \end{array}\right\}; \quad (2)$$

The equations [6869—6871] become, by substituting for g, m', $\frac{h''}{h'''}$, the preceding values [7124', 7118, 7128], and then dividing by h''',*

[7130] $$\left.\begin{array}{l} 0 = -\{24817'',58 + 553878'',78.\mu + 159201'',5.\mathrm{m} + 5205'',05.\mathrm{m}'' + 767'',12.\mathrm{m}'''\}.\frac{h}{h'''} \\ \quad +\{15361'',81 + 57805'',9.\mathrm{m} - 49445'',3.\mathrm{m}''\}.\frac{h'}{h'''} \\ \quad +1681'',67.\mathrm{m}'' + 213'',46.\mathrm{m}''' \end{array}\right\}; (3)$$

[7131] $$\left.\begin{array}{l} 0 = \{56497'',7.\mathrm{m} + 225306'',4.\mathrm{m}^2 - 195001'',4.\mathrm{mm}''\}.\frac{h}{h'''} \\ \quad + \left\{\begin{array}{l} 7751'',815 - 109003'',2.\mu - 43608'',1.\mathrm{m} - 41432'',0.\mathrm{m}'' - 1804'',18.\mathrm{m}''' \\ -81807'',5.\mathrm{m}^2 + 139953''.\mathrm{mm}'' - 59856'',1.\mathrm{m}''^2 \end{array}\right\}.\frac{h'}{h'''} \\ \quad +1834'',50.\mathrm{m}'' + 790'',56.\mathrm{m}''' - 2089'',63.\mathrm{mm}'' + 1829'',06.\mathrm{m}''^2 \end{array}\right\}; (4)$$

[7132] $$\left.\begin{array}{l} 0 = 14913'',3.\mathrm{m}.\frac{h}{h'''} + \{4175'',23 - 4842'',59.\mathrm{m} + 4142'',04.\mathrm{m}''\}.\frac{h'}{h'''} \\ \quad +276'',79 - 1736'',44.\mu - 266'',80.\mathrm{m} - 123'',70.\mathrm{m}'' + 3514'',34.\mathrm{m}''' \end{array}\right\}; \quad (5)$$

Lastly, the *fifth datum* which we shall use, is the annual and sideral motion

Fifth datum.

of the node of the orbit of the second satellite upon the fixed plane. This
[7133] motion is retrograde, and equal to 133870",4, according to the last computations of Delambre [7233], and it is the value of p.† Substituting

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[7130a] * (3557) Dividing the equations [6869, 6870, 6871] by h''', then substituting the values of g, m', $\frac{h''}{h'''}$ [7124', 7118, 7128], we get the equations [7130–7132] respectively.

[7133a] † (3558) The expression of the latitude of the satellite $s = l.\sin.(v + pt + \Lambda)$ [6300 line 1], is similar to that of the moon $s = \gamma.\sin.\{v + (g-1)v - \theta\}$ [6298c];
[7133b] changing γ or $18542^s,79$ into l, $-\theta$ into Λ, and $(g-1)v$ into pt; so that $(g-1)v$

it in the equation [6892], and then dividing by l', we get,

$$\left.\begin{array}{l} 0=133663'',11-109003'',20.\mu-31573'',71.\text{m}.\left(1-\frac{l}{l'}\right)-19566'',65.\text{m}''.\left(1-\frac{l''}{l'}\right) \\ \quad -1804'',18.\text{m}'''.\left(1-\frac{l'''}{l'}\right) \end{array}\right\}. \quad (6)$$ [7134]

The equations [6891, 6893, 6894] become, by dividing them by l', and substituting for p and m' their values [7133, 7118],

$$\left.\begin{array}{l} 0=9253'',79+\{124513'',3-553878'',76.\mu-5205'',05.\text{m}''-767'',12.\text{m}'''\}.\frac{l}{l'} \\ \quad +5205'',05.\text{m}''.\frac{l''}{l'}+767'',12.\text{m}'''.\frac{l'''}{l'} \end{array}\right\}; \quad (7)$$ [7135]

$$\left.\begin{array}{l} 0=3600'',26+3267'',32.\text{m}.\frac{l}{l'}+\{129852'',51-21264'',89.\mu-3267'',32.\text{m}-5886'',85.\text{m}'''\}.\frac{l''}{l'} \\ \quad +5886'',85.\text{m}'''.\frac{l'''}{l'} \end{array}\right\}; \quad (8)$$ [7136]

$$\left.\begin{array}{l} 0=250'',28+363'',10.\text{m}.\frac{l}{l'}+4438'',87.\text{m}''.\frac{l''}{l'} \\ \quad +\{132645'',93-2946'',95.\mu-363'',10.\text{m}-4438'',87.\text{m}''\}.\frac{l'''}{l'} \end{array}\right\}. \quad (9)$$ [7137]

To deduce from these *nine* equations the values of the *nine* unknown quantities μ, m, m″, m‴, $\frac{h}{h'''}$, $\frac{h'}{h'''}$, $\frac{l}{l'}$, $\frac{l''}{l'}$, $\frac{l'''}{l'}$, we shall observe that the last five of them being small,* we may, at first, suppose them to be [7138] nothing in the equations [7129, 7132, 7134]. Then eliminating m from these three equations, by means of its value in m″, given by the equation [7123], we obtain three equations between μ, m″, m‴; which may be used [7138′]

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is equivalent to pt. Now $(g-1)v$ [4817] represents the retrograde motion of the moon's nodes; therefore *pt must express the retrograde motion of the node of this satellite.* [7133c] Moreover as t is expressed in Julian years [7284], p will represent the annual retrograde motion of the node, as in [7133]. Substituting this value of $p=133870'',4$ in [6892], and then dividing by l', we get [7134]. In like manner, if we divide the equations [7133d] [6891, 6893, 6894] by l', and then substitute the preceding values of p, m′ [7133, 7118], [7133e] we obtain the equations [7135, 7136, 7137] respectively. If we change the coefficient 15494″,62 [6893] into 15492″,62, as in [6887b, c], it would produce, in the second member of [7136], a correction that may be represented by $-0'',46+0'',46.\frac{l''}{l'}$, which [7133f] is insensible, and may be neglected, as we have already observed in [6887c].

* (3559) The smallness of these terms is evident from the inspection of the values [7138a] [7145′—7150].

in finding these three unknown quantities; finally, m can be determined by the equation [7123].

Substituting these first approximate values of μ, m, m$''$, m$'''$, in the
[7139] equations [7130, 7131], we obtain two equations for finding the values of $\frac{h}{h'''}$, $\frac{h'}{h'''}$. Similar substitutions in [7135, 7136, 7137], give three equations for the determination of the values of $\frac{l}{l'}$, $\frac{l''}{l'}$, $\frac{l'''}{l'}$. Substituting these
[7140] values of $\frac{h}{h'''}$, $\frac{h'}{h'''}$, $\frac{l}{l'}$, $\frac{l''}{l'}$, $\frac{l'''}{l'}$, in [7129, 7132, 7134], we obtain, in like manner as in [7138'], three equations containing the four unknown quantities μ, m, m$''$, m$'''$. We may eliminate m by means of the equation [7123]; and by resolving these equations we get the values of μ, m$''$, m$'''$, consequently also that of m; these values being more accurate than those which were obtained by the first process in [7138'].

We must repeat, with these approximate values, the same calculations as
[7140'] before, and continue this process, until the two consecutive approximate values of each unknown quantity differ but very little from each other, which will happen after a few operations.* In this manner we have found,

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[7140a] * (3560) The method of computing the unknown quantities, as it is explained in [7138—7140], is quite simple, and has no other difficulty than the tediousness of these repeated numerical operations. We did not go over the calculations in this manner, but
[7140b] verified the results in [7141—7150], by the following method. Substituting in the
[7140c] equation [7131] the values [7141—7150], excepting that of μ, we obtain an equation for the determination of μ; and as the coefficient of μ is quite large, we can determine very nearly the value of μ; this result agrees very nearly with [7141]. In like manner, by the substitution of m$''$ [7144] in [7123], we get m, as in [7142]; m$'$ is the same as in [7118]; m$''$ is found from [7129], by substituting the values of all the other quantities
[7140d] as above. In the same way the quantities m$'''$, $\frac{h}{h'''}$, $\frac{h'}{h'''}$, $\frac{h''}{h'''}$, $\frac{l}{l'}$, $\frac{l''}{l'}$, $\frac{l'''}{l'}$, were computed from the equations [7132, 7130, 7131, 7128, 7135, 7136, 7137] respectively, by
[7140e] substituting successively all except the required value; and the results were found to agree very nearly with those in [7141—7150]. The values of m$'''$, h [7145, 7145'], were decreased a little, and that of μ [7141] somewhat increased, so as to make $\rho-\frac{1}{2}\varphi$ [7152]
[7140f] become 0,021903. From this verification it is evident that there is no mistake of any
[7140g] importance in this part of the calculation. Finally we must observe that we have inserted the logarithms in [7141—7150], which are not in the original work.

$\mu = 1,0055974$; $\log.\mu = 0,0024242$; [7141]

$\mathrm{m} = 0,173281 = m.10^4$; $\log.\mathrm{m} = 9,2387509$; [7142]

$\mathrm{m}' = 0,232355 = m'.10^4$; $\log.\mathrm{m}' = 9,3661520$; [7143]

$\mathrm{m}'' = 0,884972 = m''.10^4$; $\log.\mathrm{m}'' = 9,9469295$; [7144]

$\mathrm{m}''' = 0,426591 = m''',10^4$; $\log.\mathrm{m}''' = 9,6300117$; [7145]

$h = h'''.0,00206221$; $\log.\frac{h}{h'''} = 7,3143329$; [7145']

$h' = h'''.0,0173350$; $\log.\frac{h'}{h'''} = 8,2389238$; [7146]

$h'' = h'''.0,0816578$; $\log.\frac{h''}{h'''} = 8,9119976$; [7147]

$l = l'.0,0207938$; $\log.\frac{l}{l'} = 8,3179338$; [7148]

$l'' = -l'.0,0342530$; $\log.\frac{l''}{l'} = 8,5346986_n$; [7149]

$l''' = -l'.0,000931164$; $\log.\frac{l'''}{l'} = 6,9690262_n$. [7150]

The oblateness of Jupiter can be determined by means of the quantity μ. For this purpose we shall observe that we have, in [6863],

$\rho - \frac{1}{2}\varphi = \mu.0,0217794$; log.coeff. $= 8,3380459$; [7151]

substituting the value of μ [7141], we obtain,

$\rho - \frac{1}{2}\varphi = 0,0219013$; log. $= 8,3404699$. [7152]

To determine φ, we shall put the time of Jupiter's rotation equal to t; and the time of the sideral revolution of the fourth satellite equal to T; then we shall have very nearly,* [7153]

$$\varphi = \frac{\mathrm{T}^2}{a'''^3 . t^2}. \quad [7154]$$

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* (3561) We shall put T' equal to the time required by a satellite to revolve about Jupiter, in a circular orbit, at a distance equal to *the radius of its equator, or unity* [6786]. [7154a] [7154b]
Then we shall have, as in [387], $\mathrm{T}^2 : T'^2 :: a'''^3 : 1$; or $T'^2 = \frac{\mathrm{T}^2}{a'''^3}$. Moreover the [7154c]
centrifugal force of this satellite is to the centrifugal force of a particle, situated on the surface of Jupiter's equator, as $\frac{1}{T'^2}$ to $\frac{1}{t^2}$; or as 1 to $\frac{T'^2}{t^2}$ [54']; and by using the preceding value of T'^2 [7154c], i becomes as 1 to $\frac{\mathrm{T}^2}{a'''^3 t^2}$. Now the centrifugal force of the satellite [7154d]
being equal to its gravity, the preceding ratio will express that of the gravity of a particle of Jupiter's equator to its centrifugal force; which is represented in [6044'] by 1 to φ; [7154e]

Now we have, in [6787, 6781],

[7155] $a''' = 25,4359$; $\log. a''' = 1,4054471$;

[7156] $T = 16^{days},689019$; $\log. T = 1,2224308$;

and according to the observations of Cassini,

[7157] $t = 0^{day},413889$; $\log. t = 9,6168839$.

Hence we have,

[7158] $\varphi = 0,0987990$; $\log. \varphi = 8,9947525$;

and from this we get, as in [7154*f*—*h*],

[7159] $\rho = 0,0713008$. $\log. \rho = 8,8530944$.

The semi-diameter of Jupiter's equator being taken for unity [7154*b*], the semi-polar axis will be $1-\rho$ [6044]; therefore it will be represented by
[7159'] $1-\rho = 0,9286992$. The ratio of the polar to the equatorial axis has been several times measured. The mean between the various measures is 0,929,
[7160] which differs only by an insensible quantity from the preceding calculation. But if we consider the great influence of the value of ρ, upon the motions of the nodes and the apsides of the orbits of the satellites, we must be
[7160'] convinced that the ratio of Jupiter's axes is given, by observations of the eclipses of the satellites, with greater certainty than by the most accurate direct measurement. *The agreement of these measures with the result of the*
[7161] *theory, proves, in a very striking manner, that the gravity of Jupiter is composed of the combined attractions of all its particles; since the variation of the attractive force of Jupiter, arising from the observed oblateness of this*
[7161'] *planet, represents accurately the motions of the nodes and apsides of the orbits of the satellites.*

We shall now collect together the preceding results. If we divide the values of m, m', m'', m''' [7142—7145] by 10000, we shall have, as in [6841], the masses of the satellites; *that of Jupiter being taken for unity;*

[7154*f*] hence we get $\varphi = \frac{T^2}{a'''^3 t^2}$, as in [7154]. Substituting the values of a''', T, t [7155-7157], we get φ [7158]; and then from [7152] we obtain ρ [7159]. This agrees very well with
[7154*g*] the measures lately made by Struve, with the assistance of Fraunhofer's great telescope. These observations are given in the fifth volume of Schumacher's *Astronomische Nachrichten*, page 14; and vary from $\rho = 0,0714$ to $\rho = 0,0742$. He assumed, as the
[7154*h*] most probable result, $\rho = 0,0728 = \frac{1}{13,74}$.

Mass of the first Satellite.... m	$=0,0000173281$;	$\text{m} =0,173281$;	$\log.\text{m} =9,2387509$;	[7162]
" second Satellite.... m'	$=0,0000232355$;	$\text{m}' =0,232355$;	$\log.\text{m}' =9,3661520$;	[7163]
" third Satellite.... m''	$=0,0000884972$;	$\text{m}'' =0,884972$;	$\log.\text{m}'' =9,9469295$;	[7164]
" fourth Satellite.... m'''	$=0,0000426591$;	$\text{m}''' =0,426591$;	$\log.\text{m}''' =9,6300117$;	[7165]
Jupiter's mass $=1$.				[7165']
Polar semi-axis of Jupiter	$=0,9286992$; [7159']		$\log. =9,9678750$;	[7166]
Equatorial semi-axis of Jup.	$=1,0000000$.			[7166']

If we use the values of the masses of Jupiter and the earth, given in [4061], we find that the mass of the third satellite is* 0,027337, that of the earth being taken for unity. [7167] We have found, in [4631], that the moon's mass is equal to $\frac{1}{68,5} = 0,014599$; *hence the mass of Jupiter's third satellite is nearly double that of the moon; and the mass of the fourth satellite is nearly equal to that of the moon.* [7168] [7169]

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* (3562) The values m'', m^{iv} [4061], show that Jupiter's mass is $\frac{329630}{1067,09}$, expressed in parts of the earth's mass. [7167a] Multiplying this by 0,0000884972 [7164], we obtain the mass of the third satellite, expressed in the same parts $=0,027337$, as in [7167]. These results would vary a little, by using the corrected values in [4061*d*, &c.]. [7167b]

CHAPTER X.

ON THE EXCENTRICITIES AND INCLINATIONS OF THE ORBITS OF THE SATELLITES.

28. AFTER *having determined the oblateness of Jupiter, and the masses of the satellites, we shall now determine, in numbers, the secular inequalities of*
[7169′] *the elements of their orbits.* The excentricities and motions of the apsides depend on the resolution of the equations in g [6869—6872]. If we substitute the preceding values of μ, m, m′, m″, m‴ [7141—7145], they will become,*

[7170a] * (3563) Substituting the values of the elements [7141—7150], in the equations [6869—6872], we obtain the four equations [7170—7173]. Upon verifying the
[7170b] calculations of the author, we have found that the term $276'',18.h$ [7172 line 1], should be
[7170c] $273'',17.h$; and the numerator of the last term in [7171 line 1], instead of being $-49185'',95$, should be $-49570'',03$. The other coefficients are very nearly accurate.
[7170d] The first of these mistakes is wholly unimportant, taking into consideration the smallness of h [6241l, &c.]; it may therefore be left uncorrected. The second mistake augments the
[7170e] second member of the equation [7171], by the term $-\frac{384'',08.h'}{\left(1+\frac{g}{3001300''}\right)^2}$, and it is evident that its general effect is much decreased by the smallness of h' [6241l, &c.]; so that
[7170f] almost the whole change it produces is in the second value of g, or g_1 [7183]; and if we substitute this value of g_1 in the expression of this error [7170e], it becomes $-342'',2.h'$.
[7170g] Now it is plain that if we use the method of approximating to the value of g_1, which is mentioned in [7181], we shall find that this value of g_1 will be augmented, by the quantity $342'',2$ nearly, making it $g_1 = 178141'',7 + 342'',2 = 178483'',2$, instead of $178141'',7$,
[7170h] [7183]. This correction has however no sensible effect in the place of the second satellite,
[7170i] because the excentricity of its orbit is very small [6057e]; we have not therefore thought
[7170k] it to be necessary to repeat these calculations, in order to obtain the value of g_1 to the nearest second; particularly as terms of the same kind have been neglected in [6856q, &c.].

$$0=\left\{g-571269'',6-\frac{51277'',10}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h+\left\{6858'',2-\frac{25371'',60}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h'+\left\{2222'',5+\frac{16087'',10}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h''+91'',060.h''' \,;\quad (1)$$ [7170]

$$0=\left\{4054'',77-\frac{17495'',31}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h+\left\{g-133377'',2-\frac{49185'',95}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h'+\left\{12805'',3+\frac{20804'',40}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h''+337'',25.h''' \,;\quad (2)$$ [7171]

$$0=\left\{276'',18+\frac{2322'',70}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h+\left\{2662'',1+\frac{4362'',65}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h'+\left\{g-28478'',7-\frac{1902'',60}{\left(1+\frac{g}{3001300''}\right)^2}\right\}.h''+1704'',20.h''' \,;\quad (3)$$ [7172]

$$0=17'',506.h+109'',67.h'+2665'',86.h''+(g-8179'',12).h'''. \quad (4)$$ [7173]

These equations give a final equation in g, of a very high degree. To each of these values of g, corresponds a system of constant quantities h, h', h'', h''', in which three of these constant quantities are given, by means of the fourth, which remains arbitrary; and as the nature of the problem requires but four arbitrary quantities [6225], the equation in g has but four useful roots. [7174] *The great influence of the oblateness of Jupiter upon the motions of the apsides of the satellites, renders the values of g but little different from what they would be by means of that oblateness alone.* [7174'] Hence we can obtain the first approximation to these values, by putting, in each equation, the term connected with g separately equal to nothing. [7174''] This consideration facilitates extremely the determination of the values of g, which we can obtain by a rapid approximation in the following manner.

We shall in the first place observe, that the greatest value of g differs but little from $620000''$; therefore we shall put $g=620000''$ in the equations [7171, 7172, 7173], and after having divided them by h, we can deduce the values of $\frac{h'}{h}$, $\frac{h''}{h}$, $\frac{h'''}{h}$. [7175] We must then substitute these values in the equation [7170], putting also $g=620000''$ in the divisor

[7175'] $\left(1+\frac{g}{3001300''}\right)^2$; and we shall obtain a more accurate value of g than that which was first assumed. We must repeat this operation, with this last value, and so on, until the consecutive values of g agree nearly with each other. A few essays will suffice for this purpose; so that we shall be sure
[7175''] that the equations [7170—7173] will be satisfied; and this calculation may be verified by substituting for g, $\frac{h'}{h}$, $\frac{h''}{h}$, $\frac{h'''}{h}$, their values. Thus we find,*

g, h, peculiar to the first satellite.

[7176] $$g = 606989'',9; \qquad g+154'',63 = 607144'',53;$$
[7177] $$h' = 0,0185238.h; \qquad \text{log.coeff.} = 8,26773;$$
[7178] $$h'' = -0,0034337.h; \qquad \text{log.coeff.} = 7,53576_n;$$
[7179] $$h''' = -0,00001735.h. \qquad \text{log.coeff.} = 5,23930_n.$$

[7179'] *The values of h', h'', h''', corresponding to this value of g, being less than*
[7180] *h, we may consider h as the peculiar excentricity of the orbit of the first satellite, whose apsides have an annual sideral motion of 606989'',9.*

The second value of g, or g_1, is given by approximation, by putting the
[7181] term connected with g, in the equation [7171], equal to nothing. This value of g is nearly equal to $180000''$; therefore we shall put $g = 180000''$ in the equations [7170, 7172, 7173], and then dividing by h', we can deduce the values of the fractions $\frac{h}{h'}$, $\frac{h''}{h'}$, $\frac{h'''}{h'}$. We must then substitute these values in the equation [7171], after dividing it by h', and substituting $g = 180000''$
[7182] in the divisor $\left(1+\frac{g}{3001300''}\right)^2$. We shall thus obtain a more accurate value of g, which is to be used, like the preceding value [7181], in repeating the operation; and by continuing the calculation in this manner, we shall finally get,

g_1, h', peculiar to the second satellite.

[7176a] * (3564) Instead of going over the calculation, according to the directions here given, we have verified the computation, by substituting successively the values [7176—7179, 7183—7186, 7190—7193, 7195—7198], in the equations [7170—7173];
[7176b] and we have found that these equations are very nearly satisfied by these values. The values of g correspond to the *sideral* motion of the perigee, and by adding the annual
[7176c] precession 154'',63 [3380], we get the annual motion from the moveable equinox of the earth. As this motion is frequently used, we have inserted the values of $g+154'',63$ in [7176, 7183, &c.].

$g_1 = 178141'',7$;* $\quad g_1+154'',63 = 178296'',33$; [7183]

$h = -0,0375392.h'$; $\quad$ log.coeff. $= 8,57449_n$; [7184]

$h'' = -0,0436686.h'$; $\quad$ log.coeff. $= 8,64017_n$; [7185]

$h''' = 0,00004357.h'$. $\quad$ log.coeff. $= 5,63919$. [7186]

[7186′] *The values of h, h'', h''', being less than h', we may consider h' as the peculiar excentricity of the orbit of the second satellite, whose apsides have an annual sideral motion of* $178141'',7$. [7187]

The third value of g, or g_2, is given by approximation, by putting the term connected with g, in the equation [7172], equal to nothing. This value is very nearly equal to $30000''$. Therefore we shall suppose $g = 30000''$, in [7188] the equations [7170, 7171, 7173]; and, after dividing them by h'', we can deduce the values of $\frac{h}{h''}$, $\frac{h'}{h''}$, $\frac{h'''}{h''}$. These must be substituted in the equation [7172], after dividing it by h'', and substituting $g = 30000''$ in the divisor $\left(1+\frac{g}{3001300''}\right)^2$. Thus we shall obtain a more correct value of [7189] g, which must be used, like the preceding value [7188], in repeating the operation; and by continuing the calculation, in this way, we shall finally obtain,

g_2, h'', peculiar to the third satellite.

$g_2 = 29009'',8$; $\quad g_2+154'',63 = 29164'',43$; [7190]

$h = 0,0238111.h''$; $\quad$ log.coeff. $= 8,37678$; [7191]

$h' = 0,2152920.h''$; $\quad$ log.coeff. $= 9,33303$; [7192]

$h''' = -0,1291564.h''$; $\quad$ log.coeff. $= 9,11212_n$. [7193]

These values of h, h', h''', being less than h'', we may consider h'' as being the peculiar excentricity of the third satellite, whose apsides have an annual sideral motion of $29009'',8$. [7194]

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* (3565) We have seen, in [7170h], that this value of g ought to be increased to $178483'',9$ nearly. Moreover the coefficient [7196] is printed 0,0020622 in the original; [7183a] we have corrected it for an error in the fifth decimal place. We may remark that the comparison of the values of g, g_1, g_2, g_3 [7176, 7183, 7190, 7195], corresponding to the [7183b] peculiar motions of the perijoves of the first, second, third and fourth satellites respectively, shows that they decrease rapidly with the mean distance of the satellite from Jupiter. This peculiarity does not take place in the planetary orbits, as is observed in [7199′], and [7183c] as we may see by inspecting the first lines of the formulas [4242—4248]. The reason of this difference is, that the motions of the perijoves of the satellites are produced chiefly by [7183d] the ellipticity of Jupiter's mass [7174′].

g_3, h''', peculiar to the fourth satellite. Lastly, the fourth value of g, or g_3, is that which is given by observation in [7124], for the annual sideral motion of the apsides of the fourth satellite; and we have found, in [7124, 7145′—7147], that in this case we get,

[7195] $$g_3 = 7959'',105; \qquad g_3 + 154'',63 = 8113'',735;$$

[7196] $$h = 0{,}0020522.h''' \text{ [7183a]}; \qquad \text{log.coeff.} = 7{,}31222;$$

[7197] $$h' = 0{,}0173350.h'''; \qquad \text{log.coeff.} = 8{,}23892;$$

[7198] $$h'' = 0{,}0816578.h'''. \qquad \text{log.coeff.} = 8{,}91200.$$

[7198′] *These values of h, h', h'', being less than h''', we may consider h''' as the peculiar excentricity of the fourth satellite, whose apsides have an annual*
[7199] *sideral motion of 7959″,105.*

Hence we see that each satellite has an excentricity which is particularly
[7199′] *adapted to it. This peculiarity, which does not take place in the theory of the planets* [7183c], *depends on the oblateness of Jupiter; the effects of this*
[7200] *oblateness on the perijoves of the satellites being very great. It now remains to find the excentricities which are peculiar to each satellite, and the positions*
[7200′] *of their apsides, at a given epoch. We shall give, in explaining the theory of each satellite, what has been discovered by observation relative to this subject.*

We shall now consider the inclinations and motions of the nodes of the orbits of the satellites. These elements depend upon the equations in λ and l,
[7201] [6887—6894], which we shall here resume. The equations in λ [6887—6890] become, by substituting the values of μ, m, m′, m″, m‴ [7141—7145],

[7202] $$0 = -103'',27 + 571269'',64.\lambda - 9253'',80.\lambda' - 4606'',32.\lambda'' - 327'',25.\lambda''';$$

[7203] $$0 = -207'',29 - 5471'',12.\lambda + 133377'',33.\lambda' - 17315'',94.\lambda'' - 769'',65.\lambda''';$$

[7204] $$0 = -417'',63 - 566'',16.\lambda - 3599'',79.\lambda' + 28478'',73.\lambda'' - 2511'',25.\lambda''';$$

[7205] $$0 = -974'',19 - 62'',92.\lambda - 250'',28.\lambda' - 3928'',28.\lambda'' + 8179'',11.\lambda'''.$$

λ, λ', λ'', λ'''. Resolving these equations, we find,*

[7206] $$\lambda = 0{,}00057879; \quad \log.\lambda = 6{,}7625210; \quad \log.(1-\lambda) = 9{,}9997485;$$

[7207] $$\lambda' = 0{,}00585888; \quad \log.\lambda' = 7{,}7678146; \quad \log.(1-\lambda') = 9{,}9974480;$$

[7208] $$\lambda'' = 0{,}02708801; \quad \log.\lambda'' = 8{,}4327771; \quad \log.(1-\lambda'') = 9{,}9880736;$$

[7209] $$\lambda''' = 0{,}13235804; \quad \log.\lambda''' = 9{,}1217504; \quad \log.(1-\lambda''') = 9{,}9383406.$$

[7206a] * (3566) For the purpose of verifying these calculations, the values of λ', λ'', λ''', [7207—7209], were substituted in [7202], and the resulting value of λ was found to be 0,00057; which differs from [7206] by an insensible quantity. In a similar way we have
[7206b] deduced $\lambda'=0{,}005858$, from [7203]; $\lambda''=0{,}027088$, from [7204]; and $\lambda'''=0{,}13230$, from [7209]; which agree nearly with the results of the author in [7206—7209].

These values of λ, λ', λ'', λ''', determine the parts of the latitudes of the satellites, which depend upon the inclination of the equator of Jupiter to its orbit. [7209'] For we have, as in [6361], $200^\circ - \Upsilon'$ *equal to the longitude of the ascending node of the equator of Jupiter upon the orbit of this planet, and for* [7210] *brevity we shall represent it by* $\mathrm{I} = 200^\circ - \Upsilon'$; *moreover,* θ' [6360] *represents the inclinations of these planes to each other.* [7211] Then it follows, from [6362], that the parts of the latitudes of the satellites above Jupiter's orbit, mentioned in [7209'], will be represented by the following expressions;* [7212]

$$s = (1-\lambda).\ \theta'.\sin.(v-\mathrm{I});\qquad [7213]$$

$$s' = (1-\lambda').\ \theta'.\sin.(v'-\mathrm{I});\qquad [7214]$$

$$s'' = (1-\lambda'').\ \theta'.\sin.(v''-\mathrm{I});\qquad [7215]$$

$$s''' = (1-\lambda''').\theta'.\sin.(v'''-\mathrm{I}).\qquad [7216]$$

The inclination θ' of the equator of Jupiter to the orbit of this planet, and the longitude I of the ascending node upon the same orbit, are determined by observation. [7216'] Delambre obtained, for the epoch of 1750,

$$\theta' = 3^\circ{,}43519;\qquad \log.\theta' = 0{,}5359507;\qquad [7217]$$

$$\mathrm{I} = 348^\circ{,}62129.\qquad [7218]$$

These values of θ' and I are not rigorously constant. We have seen, in [6928, 6929], that the value of θ' increases annually by $0'',07035$, and that [7219] the value of I decreases annually by $0'',8259$, relative to the fixed equinox.† *These quantities are so small that we need not take notice of them, during the interval in which the eclipses of the satellites have been observed;* [7220] but it is easy to introduce them in the calculation, if it be thought proper.

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* (3567) Substituting in the expression of the part of the latitude of the satellite m [6362], the value of $\Upsilon' = 200^\circ - \mathrm{I}$ [7211], it becomes,

$$(\lambda-1).\theta'.\sin.(v+200^\circ-\mathrm{I}) = (1-\lambda).\theta'.\sin.(v-\mathrm{I}),\qquad [7212a]$$

as in [7213]; and by changing successively the quantities v, λ, corresponding to the satellite m, into v', λ'; v'', λ''; v''', λ''', corresponding respectively to the other satellites, [7212b] we obtain the parts of the values of s', s'', s''' [7214, 7215, 7216].

† (3568) The value of I [7218] being substituted in $\Upsilon' = 200^\circ - \mathrm{I}$ [7211], gives at the epoch of 1750, $\Upsilon' = -148^\circ{,}62129$; which is used in [6929b]. [7219a] Moreover we have shown, in [6929d], that the general expression of I is $\mathrm{I} = 348^\circ{,}62129 - 0'',8259.t$; so [7219b] that its annual decrement is $0'',8259$, as in [7219].

The equations in l [6891—6894],* become, by substituting the values of μ, m, m′, m″, m‴ [7141, &c.],

[7221] $$0=(p-571269'',64).l+9253'',80.l'+4606'',32.l''+327'',25.l''';\qquad(5)$$
[7222] $$0=5471'',12.l+(p-133377'',33).l'+17315'',94.l''+769'',65.l''';\qquad(6)$$
[7223] $$0=566'',16.l+3599'',79.l'+(p-28478'',73).l''+2511'',25.l''';\qquad(7)$$
[7224] $$0=62'',92.l+250'',28.l'+3928'',28.l''+(p-8179'',11).l'''.\qquad(8)$$

These four equations give an equation in p, of the fourth degree.† To obtain the roots we can use the approximate method, which is employed in finding the values of g [7175, &c.]. In this manner we shall have the first value of p, relative to the orbit of the first satellite, by putting the coefficient
[7225] of l, in the equation [7221], equal to zero; which gives $p=571269'',64$. Substituting this in the equations [7222, 7223, 7224], we may thence deduce the values of $\frac{l'}{l}$, $\frac{l''}{l}$, $\frac{l'''}{l}$. Then substituting these values in the equation [7221], after dividing it by l, we shall obtain a more approximate value of
[7225′] p. This value must then be used instead of the former, and the operation must be repeated, till the two consecutive values of p shall differ but very little from each other. By this means we shall obtain, after a few operations,‡

[7221a] * (3569) The coefficients of the equations [6891—6894], are the same as those in [6887—6890], which are computed in [7202—7204], and agree with the numerical values in [7221—7224]. We may moreover observe, that the equations [6891—6894] can be
[7221b] derived from [6887—6890], by changing the last terms of these equations, namely; $-103'',27$; $-207'',29$; $-417'',63$; $-974'',19$, into pl, pl', pl'', pl''', respectively; also λ, λ', λ'', λ''', into $-l$, $-l'$, $-l''$, $-l'''$, respectively. Now making the same
[7221c] changes in [7202—7205], which are derived from [6887—6890], we get, without any reduction, the equations [7221—7224].

[7224a] † (3570) Finding l from [7221], and substituting it in [7222—7224], we get three equations in l', l'', l'''. Then finding l' from the first of these equations, and substituting it in the others, we get two equations in l'', l'''. From the first of these, we get l''; then
[7224b] substituting it in the second, and dividing it by l''', we get an equation of the fourth degree in p.

‡ (3571) To verify the values [7226—7229], we have reduced the equations
[7226a] [7221—7224] by dividing them by l; then using these new forms, we have substituted in [7221] the expressions of $\frac{l'}{l}$, $\frac{l''}{l}$, $\frac{l'''}{l}$ [7227—7229], and have obtained the value of

$$p = 571389'',32; \qquad p - 154'',63 = 571234'',69;$$ [7226]

$$l' = -0,0124527.l; \qquad \text{log.coeff.} = 8,09526_n;$$ [7227]

$$l'' = -0,0009597.l; \qquad \text{log.coeff.} = 6,98214_n;$$ [7228]

$$l''' = -0,0000995.l. \qquad \text{log.coeff.} = 5,99782_n.$$ [7229]

*The values of l', l'', l''', being in this case less than l, we may consider the quantity l as expressing the peculiar inclinations of the orbit of the first satellite, upon a plane which passes always through the nodes of Jupiter's equator, between that equator and the orbit of the planet, and inclined to the equator by the angle $\lambda\theta'$.** If we substitute the preceding values of λ, θ' [7206, 7217], we shall find this inclination to be $\lambda\theta' = 19'',88$. The preceding value of p [7226] then expresses the annual *retrograde motion of the nodes of the orbit upon this plane*;† consequently this motion is 571389″,32 [7226].

p, l, peculiar to the first satellite.

[7230]

[7231]

[7232]

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p [7226]. From the equation [7222], we get $\frac{l'}{l}$ [7227], by the substitution of the values of p, $\frac{l''}{l}$, $\frac{l'''}{l}$ [7226, 7228, 7229]. In like manner, from [7223] we find $\frac{l''}{l}$ [7228], by the substitution of the values of p, $\frac{l'}{l}$, $\frac{l'''}{l}$ [7226, 7227, 7229]. Finally we obtain $\frac{l'''}{l}$ [7229], from [7224], by the substitution of the values of p, $\frac{l'}{l}$, $\frac{l''}{l}$ [7226—7228]. A similar process was used in the verification of the other expressions [7233—7248]; observing that in the verification of the values [7233—7236], we have divided the equations [7221—7224] by l'; in those of [7238—7241] we have divided the equations [7221—7224] by l''; and finally, in the verification of the values [7245—7248], we have divided the equations [7221—7224] by l'''. The results of these verifications confirm, to a sufficient degree of accuracy, the correctness of the numerical results, deduced by the author from the proposed system of equations [7221—7224].

[7226b]

[7226c]

[7226d]

[7226e]

* (3572) This agrees with what is said in [6357—6362]; where it is shown that the proposed fixed plane is inclined to the orbit of Jupiter by the angle $(1-\lambda).\theta'$; subtracting this from θ', the angle of inclination of Jupiter's orbit and equator [6360], we obtain $\lambda\theta'$ for the inclination of this plane to the plane of Jupiter's equator, as in [7230]; and by substituting the values of λ, θ' [7206, 7217], it becomes 19″,88, as in [7231]. The inclinations of the similar planes corresponding to the second, third and fourth satellites, are $\lambda'\theta'$, $\lambda''\theta'$, $\lambda'''\theta'$, respectively, as in [7242, 7249, &c.]; and by using the values [7208, 7209], we find that the two last of these quantities become as in [7243, 7250].

[7230a]

[7230b]

[7230c]

† (3573) We have seen, in [7133c], that pt expresses the *retrograde* motion of the node of the satellite upon the fixed plane, and from the *fixed* equinox [7328, &c.]; and its

[7231a]

p_1, l', peculiar to the second satellite. *The second value of p [7233] corresponds to the orbit of the second satellite.* It is given by observation [7133]; and we have seen in the preceding article [7133, 7148—7150], that we have in this case,

[7233] $p_1 = 133870'',4$; $p_1 - 154'',63 = 133715'',77$;

[7234] $l = 0,0207938.l'$; log.coeff. $= 8,31793$;

[7235] $l'' = -0,0342530.l'$; log.coeff. $= 8,53470_n$;

[7236] $l''' = -0,0009312.l'$. log.coeff. $= 6,96904_n$.

The third value of p [7238] corresponds to the orbit of the third satellite; we shall have the first approximation to its value, by putting the coefficient of l'' in the equation [7223] equal to nothing, which gives $p = 28478'',73$.

[7237] Substituting this in [7221, 7222, 7224], we may thence deduce the values of $\frac{l}{l''}$, $\frac{l'}{l''}$, $\frac{l'''}{l''}$. These values being substituted in [7223], after dividing it by l'', give a second value of p; which is to be used, in repeating the process, in a second operation; and by proceeding in this manner, we finally obtain,

p_2, l'', peculiar to the third satellite.

[7238] $p_2 = 28375'',48$; $p_2 - 154'',63 = 28220'',85$;

[7239] $l = 0,0111626.l''$; log.coeff. $= 8,04777$;

[7240] $l' = 0,1640530.l''$; log.coeff. $= 9,21498$;

[7241] $l''' = -0,1965650.l''$; log.coeff. $= 9,29351_n$.

These values of l, l', l''', are less than l'', *so that this quantity may be considered as expressing the peculiar inclination of the orbit of the third satellite upon a plane, which passes always through the nodes of Jupiter's*

[7242] *equator, between the equator and the orbit of the planet, and is inclined to the equator by the quantity $\lambda''\theta'$* [7230c]. *Substituting for λ'' and θ' their values*

[7243] [7208, 7217], *we find this inclination to be* $930'',52$. *The annual retrograde*

[7243'] *motion of the nodes of the orbit of the third satellite upon this plane, is* $28375'',48$ [7238].

Lastly, *the fourth value of p corresponds to the orbit of the fourth satellite.* We obtain a first approximation to its value, by putting the coefficient of l'''

[7244] in the equation [7224], equal to nothing, which gives $p = 8179'',11$. Substituting this in [7221—7223], we get three equations for the determination of $\frac{l}{l'''}$, $\frac{l'}{l'''}$ $\frac{l''}{l'''}$. These values being substituted in the

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mean value is the same as that on the plane mentioned in [7232]; by *subtracting* the

[7231b] precession $154'',63.t$, we get its value from the *variable* vernal equinox of the earth, so that $p - 154'',63$, *represents this annual retrograde motion.*

equation [7224], after dividing it by l''', give a second value of p, which must be used like the first value in repeating this process. In this manner we finally obtain,

$$p_3 = 7682'',64\,; \qquad p_3 - 154'',63 = 7528'',01\,; \qquad [7245]$$

$$l \;= 0,0019856.l'''\,; \qquad \text{log.coeff.} = 7,29789\,; \qquad [7246]$$

$$l' = 0,0234108.l'''\,; \qquad \text{log.coeff.} = 8,36942\,; \qquad [7247]$$

$$l'' = 0,1248622.l'''\,; \qquad \text{log.coeff.} = 9,09643. \qquad [7248]$$

These values of l, l', l'', are less than l'''; therefore l''' may be supposed p_3, l''', peculiar to the fourth satellite.
to express the peculiar inclination of the fourth satellite, upon a plane which passes always through the nodes of Jupiter's equator, between the equator and the orbit of the planet, and inclined to the equator by the angle $\lambda'''\theta'$ [7230*c*].
Substituting for λ''' and θ' their values [7209, 7217], *we find this inclination* [7249]
to be 4546″,74. *The annual retrograde motion of the nodes of the orbit of the fourth satellite upon this plane, is* 7682″,64 [7245]. [7250]

Hence we see that the orbit of each satellite has an inclination which is peculiarly adapted to it; a circumstance depending upon the oblateness of Jupiter, whose influence upon the motions of the nodes of the orbits of the satellites, is very great [7183*c*, *d*]. It now remains to find the inclinations corresponding to each orbit, and the positions of the nodes. We shall soon see what has been discovered by observation, relative to this subject. [7251]

CHAPTER XI.

ON THE LIBRATION OF THE THREE INNER SATELLITES OF JUPITER.

29. We have seen, in [6629], that the mean motions of the three inner satellites of Jupiter are subjected to the following law, which holds good relative to any variable axis, moving according to any law whatever [6632].

La Place's first law.

[7252] *The mean motion of the first satellite, plus twice that of the third, is exactly equal to three times the mean motion of the second satellite.*

To show how accurately this law agrees with observation, we shall give the mean secular motions of these three bodies, as Delambre has determined them, by the discussion of an immense number of eclipses. He has found that, in one hundred Julian years, these motions, relative to the variable equinox, are,*

Mean motions.

[7253]	First Satellite,	8258261°,63313;	
[7254]	Second Satellite,	4114125°,81277;	
[7255]	Third Satellite,	2042057°,90398;	
[7255′]	[Fourth Satellite,	875427°,45956].	[7281]

[7253a] * (3574) The motions of the satellites from the variable equinox in 100 Julian years, is given in [7253—7255]. Dividing these by 100, we get the motions in *one Julian year, which is taken for the unit of time in* [7283′]. Subtracting the annual precession 154″,63 [4357], we get the motions from the fixed equinox 825826009″, 411412427″, 204205635″, 87542591″, which agree nearly with the values of n, n', n'', n''' [6025k].

[7253b] With the preceding value of n''', we obtain, from [6840], the expression of $M = 337211''$ [6025m]; agreeing nearly with Bouvard's tables 337212″,094. These values of n, n', n'', n''', agree very nearly with those used by Delambre, in his new tables

[7253c] [6781h, &c.], as is evident by subtracting the precession for 100 years 1°,5463 from the numbers in [6781o—r]; then dividing by 100, and reducing to seconds.

The mean motion of the first, *minus* three times that of the second, *plus* twice that of the third, is therefore equal to 27″,8. This difference is so [7256] small that it excites surprise, at the very near agreement of the theory with the observations; and as the tables must be strictly subjected to the preceding law, the results in [7253—7255] have been slightly altered, by [7257] Delambre, to attain this object.

We have seen, in [6630], that the epochs of the mean motions of the three satellites are subjected to the following law: La Place's second law.

The epoch of the first satellite, minus three times that of the second, plus twice that of the third, is exactly equal to the semi-circumference, or 200°. [7258]

Delambre has determined these epochs, by the discussion of a very great number of eclipses, and has obtained the following results, corresponding to the commencement of the first of January, 1750, at midnight; Epochs.

First	Satellite,	16°,69584;	[7259]
Second	Satellite,	346°,0521;	[7260]
Third	Satellite,	11°,41354;	[7261]
[Fourth	Satellite,	$366°,89437_c$]. [7282]	[7261′]

From these results of observation it appears, that the epoch of the first satellite, minus three times that of the second, plus twice that of the third, is equal to 200°,01962,* which exceeds the semi-circumference by 196″,2. [7262]

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* (3575) The numbers [7259—7261] give,

$$16°,69584+3(400°-346°,0521)+2\times 11°,41354 = 201°,36662, \qquad [7259a]$$

instead of 200°,01962, which is given by the author in [7262]. This difference probably arises from a mistake in the angle [7260], which is too small by about half a degree. These angles are afterwards changed by the author, into 16°,68093, 346°,48931, [7259b] 11°,39319, in [7495, 7439, 7386], respectively; which satisfy the second law, or theorem, on the epochs [7258], within 0°,00002. We may observe that *these angles are not* [7259c] *explicitly given in Delambre's tables*; for, instead of them, he uses the times of the mean conjunction, or middle of the eclipse. From these times we find the epochs or angles [7259d] Θ, Θ', Θ'', Θ''', corresponding to the commencement of the year 1750, to be nearly 16°,67860, 346°,51677, 11°,39696, 366°,89437, respectively. The first three of [7259e] these values agree nearly with those used by the author [7259b]; but the last, or the value $\Theta''' = 366°,89437$, corresponding to the commencement of the year 1750, exceeds the value 80°,61249, given by the author in [7282, 7281a], by the quantity 286°,28188. [7259f] This mistake of the author has been noticed by Professor Airy, in Vol. 6, page 98, of the Memoirs of the Royal Astronomical Society of London. It has however no effect on the [7259g] subsequent calculations of the coefficients of the inequalities, but may be considered as

Therefore the observations do not satisfy, quite so well, this second law
[7262'] relative to the epochs, as they do the first law, relative to the mean motions; and it would not have been strange if there had been found a still greater difference, for the following reasons.

We are situated at so great a distance from the satellites of Jupiter, that they disappear from our sight before they are wholly immersed in the
[7263] shadow of the planet; and they do not again become visible until they have partly emerged from it. To determine the time of the conjunction of a satellite, we suppose, that, at the moment of the immersion, its centre is at the same distance from the conical shadow as at the moment of the emersion. Now it may happen, that the part of the disk of the satellite, which first enters into the shadow, and of course re-appears the first, may be more or less adapted to reflect the sun's light, than the part which is eclipsed the last. In this case it is evident, that, at the moment of the immersion, the distance of the centre of the satellite from the conical shadow, will be greater or less than at the time of the emersion; and the time of conjunction, deduced from these observations, will therefore be more
[7264] or less advanced than the true time. The epochs of the mean longitudes of the three inner satellites, deduced from the observations of their eclipses, may differ, on this account, from their real values, and therefore may not satisfy accurately the second law above mentioned [7258]. It is true that we have here supposed that the part of the disk, which is first eclipsed, is, in all cases, sensibly the same. Now this is really the case; for it is well known that the satellites present always the same face towards Jupiter, as the moon does to the earth. The circumstance we have just mentioned does not prevent the observations from satisfying the law of the mean motions [7252]. For these motions are determined by means of the difference of the epochs at very distant intervals of time, and are therefore independent of
[7265] the inequalities, which might exist in the light of the different parts of the disks of the satellites, particularly when we notice as many immersions as emersions.

The difference between the result of the observations and the law of the
[7266] epochs [7258—7262], being very small, Delambre has thought it best to

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[7259h] nothing more than an inaccurate deduction from the numbers in the tables; and that the mistake can be wholly corrected, by merely changing the angle into its corrected value, wherever it occurs in the formulas.

subject the epochs of his tables strictly to this law; the corrections necessary [7266′]
to be made in the observations being within the limits of the errors to which they are liable.

The two preceding theorems give rise, as we have seen in [6657, &c.], *to a particular inequality, which we have denoted by the name of the libration of the satellites.* We have given its analytical expression in [6646, 6652—6654, &c.], and to reduce it to numbers, we have, as in [6858, 6857], [7267]

$F' = 1,466380$; $\log. = 0,1662466$; 1

$G = -0,857159$, [or $-0,856159_c$]; $\log. = 9,9330614$, or $9,9325544_c$. 2 [7268]

Hence the expression k [6609] becomes,*

$$k = 123,855. \left\{ \frac{a}{a'} . m'm'' + \tfrac{9}{4}.mm'' + \frac{a''}{4a'}.mm' \right\}; \qquad [7269]$$

therefore the value of k *is positive, as we have observed in* [6619]; where we have shown that the sign of k determines whether the mean longitude of the first satellite, *minus* three times that of the second, *plus* twice that of the third, is equal to nothing, or to the semi-circumference. *The negative sign corresponds to the first case* [6616], *and the positive sign to the second case* [6615]. [7270] [7271]

Value of k

If we substitute, in [7269], the preceding values of m, m', m'', [7162—7164], we shall obtain,

$$k = 0,000000607302. \qquad [7272]$$

We have observed, in [6623], that if the law of the mean motions of the three first satellites [7252] were not rigorously correct, the observations would vary from it, by 100°, *in an interval of time which is less than*†

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* (3576) Substituting in k [6609] the values of F', G, given by the author in [7268], and those of n', N', in terms of n [6025b,f]; or, for greater accuracy, those in [6025k,l]; we obtain, very nearly, the expression of k [7269]; and if we use the values of a, a', a''; m, m', m'' [6797—6799, 7162—7164], we obtain the expressions of k [7269, 7272]. This will be decreased about $\frac{1}{856}$, by correcting the mistake in the estimated value of G [6857, or 7268 line 2]; and by this means the coefficient 123,855 [7269] is reduced to 123,7 nearly; but this correction is of no importance in the remaining part of the calculation, where no great accuracy is required. [7269a] [7269b] [7269c]

† (3577) Putting $\pi = 200°$, in the expression of this interval of time [6623], it becomes, by successive reductions, as in [7273, 7274], [7273a]

[7273] $\frac{100^\circ}{n.\sqrt{2k}}$. If we put T for the time of the sideral revolution of the first satellite, we shall have $nT = 400^\circ$; hence the preceding interval can be

[7273′]

[7274] reduced to the form $\frac{T}{4.\sqrt{2k}}$; and by substituting the value of T [6778],

[7274′] it becomes $401^{\text{days}},314$; *therefore it is less than two years, as we have observed in* [6623].

Libration of the satellites.

The expressions of v, v', v'', depending upon the libration, which we have found in [6652—6654], become, by the substitution of the values of m, m', m'',*

[7275] $$v = P.\sin.(nt.\sqrt{k} + A)$$

[7276] $$v' = -P.0,889912.\sin.(nt.\sqrt{k} + A);$$

[7277] $$v'' = P.0,062115.\sin.(nt.\sqrt{k} + A);$$

[7278] P and A being two arbitrary quantities, which can be determined only by

Periodical time of the libration.

observation. The duration of the period of this inequality† is $\frac{400^\circ}{n.\sqrt{k}}$, or

[7279] $\frac{T}{\sqrt{k}}$; which is equal to $2270^{\text{days}},18$, *or rather more than six years.*

After having thus taken into consideration the whole system of the satellites, we shall now develop the particular theories of each one of them, beginning with the fourth.

[7273b] $$\frac{100^\circ}{n.\sqrt{2k}} = \frac{100^\circ.T}{nT.\sqrt{2k}} = \frac{100^\circ.T}{400^\circ.\sqrt{2k}} = \frac{T}{4.\sqrt{2k}}.$$

[7273c] Substituting in this last expression the value of $T = 1^{\text{day}},7691\ldots$ [6778], and that of k [7272], we obtain very nearly the same expression as in [7274′].

[7275a] * (3578) If we put $P = \frac{\beta_{,}}{1 + \frac{9a'.m}{4a.m'} + \frac{a''.m}{4a.m''}}$, the coefficients of the values of v, v', v'',

[7275b] [6652—6654], will be represented by P; $-\frac{3a'.m}{4a.m'}.P$; $\frac{a''.m}{8a.m''}.P$, respectively; and by substituting the values of a, a', a''; m, m', m'' [6797—6799, 7162—7164], they become as in [7275—7277].

† (3579) Putting T' for the period of this inequality, we shall have

[7278a] $T'.n.\sqrt{k} = 400^\circ = nT$ [7273′]; or by dividing by $n.\sqrt{k}$, $T' = \frac{T}{\sqrt{k}}$, as in [7279]. Substituting in this expression the values of T, k [7273c, 7272], it becomes as in [7279].

CHAPTER XII.

THEORY OF THE FOURTH SATELLITE.

30. Delambre has found, by the discussion of all the observed eclipses of the fourth satellite, that its mean motion, referred to the earth's vernal equinox in a hundred Julian years, is,* [7280]

$$875427^\circ,45956. \qquad [7281]$$

He has also found that the mean longitude of this satellite, referred to the same equinox at the moment of the midnight, commencing the first of January, 1750, *which we shall hereafter call the epoch of* 1750, is equal to,† [7281′]

Epoch of 1750.

$$366^\circ,89437_c. \qquad [7282]$$

Therefore we shall put,

Mean motion.

$$\circledcirc''' = 366^\circ,89437_c + t.8754^\circ,2745956\,; \qquad [7283]$$

t = the number of Julian years elapsed since 1750; and, [7284]

$\circledcirc'''$ = the mean longitude of the fourth satellite, viewed from the centre of [7285]

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* (3580) This motion of the fourth satellite is increased to 875427°,46850, in the latest edition of Delambre's tables, as we have remarked in [6781r]; so that in these tables the annual motion is greater by about 0″,89, than that given by the author in [7283]. [7280a]

† (3581) We have already observed, in [7259f], that the epoch 80°,61249, given by the author in the original work in [7282], is too small by 286°,28188; and a similar mistake occurs in the angle 303°,76542 [7336, 7340, &c.], which is too great by 18°,56542, as Delambre has observed in the last edition of his tables [7336a]. These mistakes are corrected in this translation, by inserting the true values, with the letter $_c$ annexed, as in [6021i], namely, by putting 366°,89437$_c$ for 80°,61249, in [7282, &c.]; and 285°,20$_c$ for 303°,76542, in [7336, &c.]. This change in the value of the angle [7282], affects the angle [7290, &c.], which is therefore corrected. [7281a] [7281b] [7281c] [7281d] [7281e]

Jupiter, and counted from the earth's moveable vernal equinox.*

Delambre has also found that the perijove of the fourth satellite has an
[7286] annual sideral motion of $7959'',105$; or a motion of $8113'',735$ relative to the vernal equinox;† and that the mean longitude of this perijove, in 1750, was,

[7287] $$200^\circ,38054.$$

Longitude of the perijove.

Therefore if we put,‡

[7288] $$\varpi''' = 200^\circ,38054 + t.8113'',735.$$

[7289] $\circleddash''' - \varpi'''$ will be the mean anomaly of the fourth satellite, counted from the perijove; and we shall have, as in [7283, 7288],

[7290] $$\circleddash''' - \varpi''' = 166^\circ,51383_c + t.8753^\circ,4632221.$$

Mean anomaly.

We have seen, in [7127], that the coefficient of the greatest term of the
[7291] equation of the centre, is equal to $9265'',56$. It is easy to deduce, from this, the elliptical part of the longitude of the fourth satellite, as in the following expression;§

[7285a] * (3582) In like manner, $\circleddash$, $\circleddash'$, $\circleddash''$ [7496, 7440, 7301, &c.], represent the mean longitudes of the first, second and third satellites respectively, viewed from the centre of Jupiter, and referred to the earth's vernal equinox.

[7286a] † (3583) The annual sideral motion of the apsides of the fourth satellite, is $7959'',105$ [7195], from the fixed equinox; or $7959'',105 + 151'',63 = 8113'',735$ from the earth's moveable equinox [7176e]. This agrees with [7286]. A similar alteration is made in
[7286b] [7295], for reducing the sideral motion $29009'',8$ to the tropical motion $29164'',43$; observing that the longitudes of *the perijoves of the first, second, third and fourth satellites*
[7286c] *are respectively represented by* ϖ, ϖ', ϖ'', ϖ'''; *and that, in the rest of this book, these longitudes are counted from the earth's moveable vernal equinox.*

‡ (3584) The last tables of Delambre augment the angle [7287] about $1^\circ,7$, making
[7288a] the longitude of the perijove, at the commencement of the year 1750, nearly equal to $202^\circ,10$. Using the epoch and annual motion, as given by the author in [7287, 7286], we
[7288b] get the expression of ϖ''' [7288]; subtracting this from $\circleddash'''$ [7283], we obtain [7290]; the angle $280^\circ,23194$, given by the author, being changed into $166^\circ,51383_c$, in consequence of the change in [7283].

§ (3585) We have, in [668, 669], the following expression of the equation of the centre, neglecting e^4,

[7292a] $$(2e - \tfrac{1}{4}.e^3).\sin.(\circleddash''' - \varpi''') + \tfrac{5}{4}.e^2.\sin.2(\circleddash''' - \varpi''') + \tfrac{13}{12}.e^3.\sin.3(\circleddash''' - \varpi''');$$

and by putting its first term equal to $9265'',56$, as in [7291], we get $e = 4633''$ nearly.
[7292b] Substituting this value of e, in the second and third terms of [7292a], it becomes as in [7292].

$$\begin{aligned}\Theta''' &+ 9265'',56.\sin.\ (\Theta'''-\varpi''') \\ &+ \quad 42'',14.\sin.2.(\Theta'''-\varpi''') \\ &+ \quad 0'',27.\sin.3.(\Theta'''-\varpi''').\end{aligned}$$ [7292]

The fourth satellite participates a little in the equation of the centre of the third satellite. Delambre has found the coefficient of this equation of the third satellite to be 1709'',05 [7388], and the longitude of the corresponding perijove, in 1750,* [7293]

$$343^{\circ},82067.$$ [7294]

The annual sideral motion of this perijove is 29009'',8 [7190]; hence its annual tropical motion is 29164'',43 [7286b]; so that we shall have, [7295]

Perijove of the third satellite.

$$\varpi'' = 343^{\circ},82067 + t.29164'',43.$$ [7296]

We shall put Θ'' *for the mean tropical longitude of the third satellite.* Then [7297]
$\Theta''-\varpi''$ will be its mean longitude, counted from the perijove. To determine [7298]
Θ'', we shall observe, that Delambre has found the annual motion of the satellite, in a hundred Julian years, to be,†

$$2042057^{\circ},9040;$$ [7299]

and its mean longitude, at the epoch of 1750,

$$11^{\circ},39349.$$ [7300]

Hence we have,

$$\Theta'' = 11^{\circ},39349 + t.20420^{\circ},579040;$$ [7301]

consequently,

Mean anomaly of the second satellite.

$$\Theta''-\varpi'' = 67^{\circ},57282 + t.20417^{\circ},662597. \quad [7301, 7296]$$ [7302]

Thus the equation of the centre of the third satellite [7293, 7303], will be, as in [7304a],

$$\delta v'' = 1709'',05.\sin.(\Theta''-\varpi'').$$ [7303]

We have, as in [7193], relatively to this equation of the centre,‡

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* (3586) The longitude of the perijove of the third satellite [7294], is increased, in the last tables of Delambre, by nearly the same quantity as that of the fourth satellite, [7288a]. [7294a]

† (3587) The annual motion [7299] is increased about 89'', in the new tables of Delambre [6781q]; and the longitude of the epoch [7300] is increased about 34''. From the values given by the author in [7299, 7300], we easily deduce the expression of Θ'' [7301], and by subtracting the value of ϖ'' [7296], we obtain [7302]. [7300a]

‡ (3588) The chief term of $\delta v''$ [6241i] is $-2h''.\sin.(\Theta''-\varpi'')$. Putting this equal to the expression [7303], we find $-2h''=1709'',05$. Substituting this in the term of $\delta v'''$ [7304a]

[7304] $$h''' = -0{,}1291564.h''.$$

Hence the equation of the centre of the fourth satellite, which depends on the perijove of the third satellite, is represented by $\delta v'''$ [7305]; and by reducing the coefficient, it becomes of the form [7306];

[7305] $$\delta v''' = -1709'',05 \times 0{,}1291564.\sin.(\circleddash'''-\varpi'')$$

[7306] $$= -220'',73.\sin.(\circleddash'''-\varpi'').$$

Symbol Π [7307] We shall put Π *for the mean longitude of Jupiter, referred to the moveable vernal equinox of the earth.* Then substituting, in [6848], the value of m″ [7144], and neglecting the terms depending upon m, m′, which may be

[7307′] done without any sensible error,* we shall find that this expression of $\delta v'''$ becomes,

[7308]
$$\begin{array}{rlr}
\delta v''' = & -28'',36_c.\sin.\ (\circleddash''-\circleddash''') & 1 \\
& -14'',12.\sin.2.(\circleddash''-\circleddash''') & 2 \\
& -\ 2'',95.\sin.3.(\circleddash''-\circleddash''') & 3 \\
& -\ 0'',90.\sin.4.(\circleddash''-\circleddash''') & 4 \\
& -\ 0'',33.\sin.5.(\circleddash''-\circleddash''') & 5 \\
& +12'',99.\sin.(2\circleddash'''-2\Pi). & 6
\end{array}$$

[7309] If we then observe that $-2h'''$ [6241k line 2] is the coefficient of the greatest term of the equation of the centre of the fourth satellite,† we shall

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[7304b] [6241k], depending on $2h''$, we get the corresponding part of the equation of the centre of v''' [7305]. Multiplying together the two factors of the coefficient of [7305], we obtain

[7304c] [7306]. The terms of $\delta v'''$, depending on h, h' [6241k], are insensible [6241l]; that depending on h''' is given in [7310, or 7318 line 1].

[7307a] * (3589) Substituting the value of m [7142] in [6848 line 1], and m′ [7143] in [6848 line 3], we obtain $2'',4$; and $5'',2$ for the greatest terms of $\delta v'''$, depending on

[7307b] m, m′; and these coefficients are so small that they are hardly worth the trouble of forming two new tables for the sake of noticing them. The remaining terms in [6848 lines 6—11], produce the expression [7308], by substituting the values of m″ [7144],

[7307c] $n''t-n'''t+\varepsilon''-\varepsilon''' = \circleddash''-\circleddash'''$ [6240q], and $2n'''t-2Mt+2\varepsilon'''-2E = 2\circleddash'''-2\Pi$ [6240s']. The coefficient in [7308 line 1], according to the author, is $31'',36$; we have decreased it

[7307d] $3'',00$, to correct for the mistake mentioned in [6849e].

[7309a] † (3590) We have seen, in [6240d, e], that the greatest term of the equation of the centre of the first satellite, $2e.\sin.(nt+\varepsilon-\varpi)$, is changed into $-2h.\sin.(nt+\varepsilon-gt-\Lambda)$. Adding three accents to these symbols, so as to make them correspond to the fourth

[7309b] satellite, we find that the greatest term of the equation of the centre of the fourth satellite is $2e'''.\sin.(n'''t+\varepsilon'''-\varpi''')$; and that it can, in like manner, be changed into

have, by considering the greatest term of the equation of the centre [7292 line 1],

$$-2h''' = 9265'',56\,; \quad \text{or} \quad h''' = -4632'',78\,; \qquad [7310]$$

we shall have, as in [6874 line 1, 6875*k*, &c.], the inequality,

$$\delta v''' = 0{,}0144449.\tfrac{1}{2}.9265'',56.\sin.(\Theta'''+\varpi'''-2\Pi), \qquad [7311]$$

which may be reduced to the following form ;

$$\delta v''' = 66'',94.\sin.(\Theta'''+\varpi'''-2\Pi). \qquad [7312]$$

Putting, as in [6275*e*, &c.], *the mean anomaly of Jupiter, counted from the* [7313]
perihelion, equal to $Mt+E-I=V$, we shall have, for the inequality [6886], the following expression ;

$$\delta v''' = -353'',69_c.\sin.V. \qquad [7314]$$

Lastly we have, as in [6944], the inequality,*

$$\delta v''' = -49'',51.\sin.(t.7541''+31^\circ,9199)\,; \qquad [7315]$$

7541″ being the assumed annual sideral motion of the node of the fourth satellite [6942] ; but we have found, in [7245], that this motion is greater, being equal to 7682″,64. We must subtract from it the annual variation of the longitude of the node of Jupiter's equator, Ψ', which is equal to 0″,8259 [6929] ; hence the preceding inequality becomes, [7316]

$$\delta v''' = -49'',51.\sin.(t.7681'',81+31^\circ,9199). \qquad [7317]$$

Connecting together all these inequalities, we shall have, *for the longitude*

$-2h'''.\sin.(n'''t+\varepsilon'''-g_3t-\Gamma_3)$, using the values of g, Γ [7286, &c.], peculiar to the greatest term of the equation of the centre for this satellite. Putting this last expression [7309*c*] equal to the chief term of the equation of the centre of the fourth satellite [7292 line 1], we get $-2h'''=9265'',56$, as in [7310]. Now the inequality in $\delta v'''$, corresponding to that in δv [6874 line 1], is represented, as in [6875*k*], by

$$\delta v''' = -\frac{15Mh'''}{4n'''}.\sin.(n'''t-2Mt+\varepsilon'''-2E+g_3t+\Gamma_3)\,; \qquad [7309d]$$

hence if we change the angle into $\Theta'''+\varpi'''-2\Pi$, as in [6240*s*′], and substitute the numerical value [6880], it becomes $\delta v'''=-0{,}0144449.h'''.\sin.(\Theta'''+\varpi'''-2\Pi)$. Substituting [7309*e*] the value of h''' [7310], it becomes as in [7311] ; or by reduction, as in [7312] nearly.

* (3591) The argument of the equation [6944] is the same as that in [7315] ; and by increasing the annual sideral motion of the node from 7541″ to 7682″,64 [7316], then subtracting the precession 0″,8259 [6929], it changes into 7681″,81 ; hence the expression of $\delta v'''$ [6944], becomes as in [6942*m*, or 7317] ; being the same as in [7318 line 13]. [7316*a*] [7316*b*]

[7317'] v''' *of the fourth satellite, counted upon its orbit from the vernal equinox of the earth,**

Longitude of the fourth satellite from the moveable vernal equinox of the earth.

[7318]

$$\begin{array}{rll} v''' = \varpi''' & +9265'',56.\sin.\ (\text{ɵ}'''-\varpi''') & 1 \\ & +\ 42'',14.\sin.2.(\text{ɵ}'''-\varpi''') & 2 \\ & +\ 0'',27.\sin.3.(\text{ɵ}'''-\varpi''') & 3 \\ & -\ 28'',36_c.\sin.\ (\text{ɵ}''-\text{ɵ}''') & 4 \\ & -\ 14'',12.\sin.2.(\text{ɵ}''-\text{ɵ}''') & 5 \\ & -\ 2'',95.\sin.3.(\text{ɵ}''-\text{ɵ}''') & 6 \\ & -\ 0'',90.\sin.4.(\text{ɵ}''-\text{ɵ}''') & 7 \\ & -\ 0'',33.\sin.5.(\text{ɵ}''-\text{ɵ}''') & 8 \\ & -\ 220'',73.\sin.\ (\text{ɵ}'''-\varpi'') & 9 \\ & +\ 12'',99.\sin.(2\text{ɵ}'''-2\Pi) & 10 \\ & +\ 66'',94.\sin.(\text{ɵ}'''+\varpi'''-2\Pi). & 11 \\ & -\ 353'',69_c.\sin.V & 12 \\ & -\ 49'',51.\sin.(t.7681'',81+31^{\circ},9199). & 13 \end{array}$$

We shall now consider the motion in latitude. This motion depends upon the inclination of the equator of Jupiter to its orbit, and upon the longitude of the ascending node of the equator, at a given epoch. Delambre found, by the discussion of a very great number of eclipses, chiefly of the third and fourth satellites, that *the inclination of Jupiter's equator to the orbit of this*
[7319] *planet, was* $\theta' = 3^{\circ},4352$ in 1750 [7217]; and at the same epoch, the longitude of its ascending node was $348^{\circ},6213$ [7218]. Moreover the
[7320] annual precession of the equinoxes being $154'',63$ [7176c]; and the annual precession of Jupiter's equinoxes $0'',8259$ [6929']; the annual motion of
[7321] the last equinox, referred to the first, is $153'',8$; *so that the longitude of the ascending node of Jupiter's equator, relative to the earth's vernal equinox, is represented by*

Longitude of the ascending node of Jupiter's equator.

[7322]
$$348^{\circ},6213+t.153'',8.$$

[7322'] The term $(\lambda'''-1).\theta'.\sin.(v'''+\Psi')$ in the expression of the latitude s''' of the fourth satellite [6430 line 1], will therefore become,†

[7318a] * (3592) Adding the mean longitude $\text{ɵ}'''$, at the epoch [7285], to the sum of the inequalities, which are computed in [7292, 7306, 7308, 7312, 7314, 7317], we get the
[7318b] complete value of v''' [7318].

[7323a] † (3593) From [7216', 7322] we have $I = 348^{\circ},6213+t.153'',8$; substituting this in the value of Ψ' [7211], we get $\Psi' = 200^{\circ}-I = -148^{\circ},6213-t.153'',8$; hence,

$$s''' = (1-\lambda''').3^\circ{,}4352.\sin.(v'''-348^\circ{,}6213-t.153'',8).$$ [7323]

Substituting the value of λ''' [7209], we shall have,

$$s''' = 2^\circ{,}98051.\sin.(v'''+51^\circ{,}3787-t.153'',8).$$ [7324]

The term $l'''.\sin.(v'''+p_3t+\Lambda_3)$, which occurs in the expression of s''' [6430 line 5, 6298x], corresponds to the peculiar inclination of the orbit of the fourth satellite to the fixed plane; it is therefore necessary to ascertain the values of l''' and Λ_3. Delambre has found, [7324′]

$$l''' = -2771'',6; \quad [7352 \text{ line } 2]$$ [7325]

and in 1750,

$$\Lambda_3 = 83^\circ{,}29861. \quad [7353 \text{ line } 2]$$ [7326]

The value of p_3 relative to this term, is 7682″,64 [7245]. *To reduce it to the moveable vernal equinox of the earth, we must subtract the precession* 154″,63 [7176c]. Hence the term of s''' [7324′] becomes, [7327] [7328]

$$s''' = -2771'',6.\sin.(v'''+83^\circ{,}29861+t.7528'',01).$$ [7329]

Delambre has found, by comparing the eclipses of the third satellite, that the value of l'', relative to the orbit of the third satellite, is $l'' = -2283'',9$ [7415]; and the value of Λ_2, at the commencement of the year 1750, is $\Lambda_2 = 208^\circ{,}32562$ [7415]. Moreover, the corresponding value of p is $p_2 = 28375'',48$ [7238]; and by subtracting from it the precession 154″,63 [7328], we obtain 28220″,85, *for the annual tropical motion of the node of the orbit of the third satellite, upon its fixed plane.* Therefore the part of s'', relative to this motion, is,* [7330] [7331] [7332]

$$\sin.(v'''+\Upsilon') = -\sin.(v'''+\Upsilon'+200^\circ) = -\sin.(v'''+51^\circ{,}3787-t.153'',8);$$ [7323b]

consequently the value of s''' [7322′] becomes,

$$s''' = (\lambda'''-1).\theta'.\sin.(v'''+\Upsilon') = (1-\lambda''').\theta'.\sin.(v'''+51^\circ{,}3787-t.153'',8).$$ [7323c]

Now $51^\circ{,}3787 = 400^\circ - 348^\circ{,}6213$, and $\theta' = 3^\circ{,}4352$ [7319]; hence the last of the expressions [7323c], becomes as in [7323]; and by substituting the value of λ''' [7209], it becomes as in [7324]. We may observe that, if we place three accents on l, λ, v, in the value of s [6357f], we shall get the corresponding value of s''' [7323f]; and by substituting its value [7323c], also θ' [7323d], and λ''' [7209], we get [7323g], which will be used in [7314a, &c.]; [7323d] [7323e]

$$s''' = \Sigma'.(l'''-L').\sin.(v'''+pt+\Lambda) = (1-\lambda''').\Sigma'.(L-L').\sin.(v'''+pt+\Lambda) = (\lambda'''-1).\theta'.\sin.(v'''+\Upsilon')$$ [7323f]

$$= (1-\lambda''').3^\circ{,}4352.\sin.(v'''+51^\circ{,}3787-t.153'',8) = 2^\circ{,}98051.\sin.(v'''+51^\circ{,}3787-t.153'',8).$$ [7323g]

* (3594) The term of s'', here treated of, is that in [6129 line 4]; and by substituting $\zeta_2'' l_2 = l''$ [6422, &c.], it becomes $s'' = l''.\sin.(v''+p_2t+\Lambda_2)$. We must substitute, in this [7332a]

[7333] $s'' = -2283'',9.\sin.(v''+208^\circ,32562+t.28220'',85).$

To obtain the corresponding part of s''', we must multiply the coefficient of

[7334] this term by $\frac{l'''}{l''}$, and this fraction is $-0,1965650$ [7241]; hence we get the following term of s''';*

[7335] $s''' = 448'',93.\sin.(v'''+208^\circ,32562+t.28220'',85).$

Delambre found that the value of l', relative to the peculiar inclination of the orbit of the second satellite to its fixed plane, is $l' = -5152'',2$

[7336] [7471]; and the corresponding value of Λ, in 1750, is $\Lambda_1 = 285^\circ,20_c$.† The value of p relative to this inclination is $p_1 = 133870'',4$ [7233]. Subtracting from it the precession $154'',63$ [7176c], *we obtain* $133715'',77$,

[7337] *for the annual tropical motion of the node of the orbit of the second satellite upon its fixed plane.* Therefore the part of s' relative to this motion is,

[7338] $s' = -5152'',2.\sin.(v'+285^\circ,20_c+t.133715'',77).$

[7332b] expression, the value of $p_2t+\Lambda_2$, which is computed in [7331, 7332], namely, $p_2t+\Lambda_2 = 208^\circ,32562+t.28220'',85$; also for l'' its value $-2283'',9$ [7330], then it becomes as in [7333]. For the sake of distinctness, we have placed the proper indices

[7332c] $_3$, $_2$, below the quantities p, Λ, in [7324', &c., 7331], though they are not inserted in this part of the original work, notwithstanding they are used in [6430, &c.].

* (3595) If we suppose the term of s'' [7333] to correspond to that in [6429 line 4], we

[7335a] shall have $-2283'',9 = \zeta_2''l_2 = l''$ [6422, &c.]; and $p_2t+\Lambda_2 = 208^\circ,32562+t.28220'',85$. The corresponding term of s''' [6430 line 4], becomes, by the substitution of $\zeta_2'''l_2 = l'''$

[7335b] [6422, &c.], $s''' = l'''.\sin.(v'''+p_2t+\Lambda_2)$; using the preceding value of $p_2t+\Lambda_2$, or that which depends on p_2 [7238]. Now for the angle p_2 we have $l''' = -0,1965650.l''$

[7335c] [7241], as in [7334]; and if we substitute it in s''' [7335b], it becomes

[7335d] $s''' = -0,1965650.l''.\sin.(v'''+p_2t+\Lambda_2)$; and by using $l'' = -2283'',9$ [7335a], it is reduced to the form [7335]. We must proceed in the same manner in finding the term of s''' [7340], corresponding to the values p_1 [7233] and l''' [7236]. We may neglect the

[7335e] term depending on p, l''' [7226, 7229], considering the smallness of the coefficient $-0,0000995$ [7229]; and also that the corresponding value of l is so small that it is wholly neglected in the general expression of the latitude s [7522].

† (3596) This angle, in the original work, is $303^\circ,76542$, being too great by

[7336a] $18^\circ,56542$, as we have already observed in [7281b, &c.]. This correction is given in page xxii of the last Tables of the Satellites, by Delambre. We have inserted the corrected

[7336b] value $285^\circ,20_c$, using the letter $_c$ as in [6021i]; observing that this corrected angle occurs in several places, as in [7338, 7340, 7352, &c.].

To obtain the corresponding part of s''', we must multiply the coefficient of this term by $\frac{l'''}{l'}$; and this fraction is found in [7236] to be equal to $-0{,}0009312$; hence we get, in s''', the following term; [7339]

$$s''' = 4'',80.\sin.(v'''+285°,20_c+t.133715'',77). \qquad [7340]$$

It now remains to consider the inequality,

$$s''' = 0{,}001447815.(l'''-L').\sin.(v'''-2U-pt-\text{A}); \qquad [7341]$$

which is given in [6934]. If we suppose that the value of p corresponds to the motions of the equator and orbit of Jupiter, we shall have, as in [6346, 6355d],

$$L-l''' = \lambda'''.(L-L'); \qquad [7342]$$

$$l'''-L' = (1-\lambda''').(L-L'); \qquad [7343]$$

therefore,*

$$s''' = (1-\lambda''').(L-L').\sin.(v'''+pt+\text{A}), \qquad [7344]$$

is the latitude of the satellite above Jupiter's orbit, supposing it to be moved upon its fixed plane [6357—6358]; and we have found, in [7324, &c.], that this latitude [7344] is represented by [7344']

$$s''' = 2°,98051.\sin.(v'''+51°,3787-t.153'',8). \qquad [7345]$$

Hence the preceding inequality of s''' [7341] becomes,

$$s''' = 2°,98051\times 0{,}001447815.\sin.(v'''-2U-51°,3787+t.153'',8); \qquad [7346]$$

or, by reduction,

$$s''' = 43'',15.\sin.(v'''-2U-51°,3787+t.153'',8). \qquad [7347]$$

Among the other terms contained in the expression,

$$s''' = 0{,}001447815.(l'''-L').\sin.(v'''-2U-pt-\text{A}); \quad [6934] \qquad [7348]$$

the only one which is sensible, is that which corresponds to the peculiar inclination of the orbit of the satellite to its fixed plane. In this case $L'=0$ [6415], since the position of Jupiter's orbit is not sensibly altered by the action of the satellites. Moreover we have, by what has been said in [7324', 7329], [7349]

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* (3597) The expression of s''' [7344] is the same as that in [7323f], using the symbol Σ' as in [6324']. The terms under this symbol in [7323f], are reduced to one term of the form [7323g], which is the same as that in [7345]. Comparing the first of the expressions [7323f] with the last of [7323g], we find that we may change $\Sigma'.(l'''-L')$ into 2°,98051, and $pt+\text{A}$ into 51°,3787—t.153'',8. Making the same changes in [7341], after having prefixed the symbol Σ' to the terms in the second member, we get [7346]; which is easily reduced to the form [7347]. [7344a] [7344b] [7344c] [7344d] [7344e]

[7350] $l'''.\sin.(v'''+p_3t+\text{A}_3)=-2771'',6.\sin.(v'''+83^\circ,29861+t.7528'',01)$;

hence the preceding term [7348] becomes,*

[7351] $s'''=-4'',01.\sin.(v'''-2U-83^\circ,29861-t.7528'',01).$

Connecting together these different terms of the latitude s''' of the fourth satellite above Jupiter's orbit, we obtain,†

[7352]
$$\begin{array}{llr} s'''= & 2^\circ,98051\ .\sin.(v'''+51^\circ,3787-t.153'',8) & 1 \\ & -2771'',6\ .\sin.(v'''+83^\circ,29861+t.7528'',01) & 2 \\ & +\ 448'',93.\sin.(v'''+208^\circ,32562+t.28220'',85) & 3 \\ & +\ \ 4'',80.\sin.(v'''+285^\circ,20_c+t.133715'',77) & 4 \\ & +\ \ 43'',15.\sin.(v'''-2U-51^\circ,3787+t.153'',8) & 5 \\ & -\ \ 4'',01.\sin.(v'''-2U-83^\circ,29861-t.7528'',01). & 6 \end{array}$$

[7353] In the eclipses of the fourth satellite, and in the eclipses of Jupiter by the same satellite, these expressions of v''' and s''' become more simple; for we

[7353′] may suppose‡ $2\Pi=2\circleddash'''$, $2U=2v'''$; and then we shall have, at the times of these eclipses,§

[7351a] * (3598) We obtain from [7350] $pt+\text{A}$, or rather $p_3t+\text{A}_3=83^\circ,29861+t.7528'',01$, and $l'''=-2771'',6$, or $l'''-L'=-2771'',6$ [7349]; hence,

[7351b] $$0,001447815.(l'''-L')=-4'',01.$$

Substituting this in [7348], it becomes as in [7351]; using the value of $pt+\text{A}$, or rather

[7351c] $p_3t+\text{A}_3$ [7351a]. It is unnecessary to notice the other values of p, depending on the terms in [7352 lines 3, 4, &c.], as the coefficients are too small to be noticed; the greatest

[7351d] of them is $448'',93$ [7352 line 3], being hardly a sixth part of the preceding term; and, of course, the coefficients will not be one sixth part of that in [7351].

[7352a] † (3599) The expression of s''' [7352] is the sum of the terms which are contained in [7324, 7329, 7335, 7340, 7347, 7351] respectively.

[7353a] ‡ (3600) It is evident from the definitions in [6023x, 6024f], that, in these eclipses, $\circleddash'''$, Π are equal, or differ by 200°; so that we may put generally $2\Pi=2\circleddash'''$. Also

[7353b] from [6022y, 6023c], v''', U are equal, or differ by 200°; therefore we may put $2U=2v'''$.

[7353c] In like manner, in the eclipses of the *third* satellite, we may put $2\Pi=2\circleddash''$, $2U=2v''$;

[7353d] in those of the *second* satellite, $2\Pi=2\circleddash'$, $2U=2v'$; and in those of the *first* satellite,

[7353e] $2\Pi=2\circleddash$, $2U=2v$.

[7354a] § (3601) Substituting $2\Pi=2\circleddash'''$ [7353′] in [7318 line 10], it vanishes. The same substitution being made in the term [7318 line 11], it becomes,

[7354b] $$+66'',94.\sin.(\varpi'''-\circleddash''')=-66'',94.\sin.(\circleddash'''-\varpi''');$$

[7354c] connecting this with the term in [7318 line 1], we obtain $+9198'',62.\sin.(\circleddash'''-\varpi''')$, as in [7354 line 1]; and the whole expression of v''' becomes as in [7354]. Again, if we

Values of v''', s''', in eclipses; the longitude being counted from the moveable vernal equinox of the earth.

$$
\begin{aligned}
v''' = \circleddash''' &+ 9198'',62.\sin.\ (\circleddash'''-\varpi''') && 1\\
&+ 42'',14.\sin.2.(\circleddash'''-\varpi''') && 2\\
&+ 0'',27.\sin.3.(\circleddash'''-\varpi''') && 3\\
&- 28'',36_c.\sin.\ (\circleddash''-\circleddash''') && 4\\
&- 14'',12.\sin.2.(\circleddash''-\circleddash''') && 5\\
&- 2'',95.\sin.3.(\circleddash''-\circleddash''') && 6\\
&- 0'',90.\sin.4.(\circleddash''-\circleddash''') && 7\\
&- 0'',33.\sin.5.(\circleddash''-\circleddash''') && 8\\
&- 220'',73.\sin.\ (\circleddash'''-\varpi'') && 9\\
&- 353'',69_c.\sin.V && 10\\
&- 49'',51.\sin.(t.7681'',81+31^\circ,9199). && 11
\end{aligned}
$$ [7354]

$$
\begin{aligned}
s''' = 2^\circ,97620\ &.\sin.(v'''+51^\circ,3787-t.153'',8) && 1\\
-2767'',6\ &.\sin.(v'''+83^\circ,29861+t.7528'',01) && 2\\
+\ 448'',93&.\sin.(v'''+208^\circ,32562+t.28220'',85) && 3\\
+\ 4'',80&.\sin.(v'''+285^\circ,20_c+t.133715'',77). && 4
\end{aligned}
$$ [7355]

This expression of s''' gives the explanation of a singular phenomenon, which has been observed in the inclination of the orbit of the fourth satellite, and in the motion of its nodes. The inclination of the orbit of this satellite to the orbit of Jupiter, appeared to be nearly constant, and equal to $2^\circ,7$, from the [7356]

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substitute $2U=2v'''$ [7353′] in [7352 line 5], it becomes,

$$43'',15.\sin.(-v'''-51^\circ,3787+t.153'',8)=-43'',15.\sin.(v'''+51^\circ,3787-t.153'',8);$$ [7354d]

connecting this with the term in [7352 line 1], we obtain,

$$2^\circ,976195.\sin.(v'''+51^\circ,3787-t.153'',8),$$ [7354e]

as in [7355 line 1], nearly. Making the same substitution of $2U=2v'''$, in the term [7352 line 6], it becomes $+4'',01.\sin.(v'''+83^\circ,29861+t.7528'',01)$; and by connecting it with the term in [7352 line 2], we obtain $-2767'',6.\sin.(v'''+83^\circ,29861+t.7528'',01)$, [7354f] nearly, as in [7355 line 2]. We may observe that the arguments of the inequalities in [7355 lines 1, 2, 3, 4], are represented by Delambre, for brevity, by H, I, K, L, at the [7354g] times of the middle of the eclipse or opposition. The same arguments occur in the values of s'', s', s [7427, 7482, 7522, &c.]; and by using this notation, and putting V''' equal to [7354h] the mean longitude of Jupiter at the conjunction, corrected for the great inequality, we have,

$$H=V'''+\ 51^\circ,3787-t.153'',8;$$ [7354i]

$$I=V'''+\ 83^\circ,29861+t.7528'',01;$$ [7354k]

$$K=V'''+208^\circ,32562+t.28220'',85;$$ [7354l]

$$L=V'''+285^\circ,20_c+t.133715'',77.$$ [7354m]

year 1680 till about 1760. During this interval the nodes had a direct
[7356′] motion, upon the orbit of the planet, of eight minutes in a year. Since 1760, the inclination has very sensibly increased. We may obtain the inclination of the orbit, and the position of its nodes, at any given epoch, by
[7357] giving to t the value which agrees with that epoch. We shall put the preceding expression of s''' under the form,*

[7358] $$s''' = A.\sin.v''' - B.\cos.v'''.$$

[7359] We can determine A and B, by putting successively† $v''' = 100^\circ$, and $v''' = 200^\circ$, in the expression of s'''. $\frac{B}{A}$ will be the tangent of the longitude
[7360] of the node, and $\sqrt{A^2+B^2}$ the inclination of the orbit.‡ This being

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[7357a] * (3602) Any one of the lines of the second member of [7355], may be put under the form $A'.\sin.(v'''-B')$; A' being the constant coefficient of that line, and $-B'$ the constant angle, connected with v'''. If we suppose the sign Σ of finite differences to
[7357b] include the terms in the four lines of [7355], we shall have $s''' = \Sigma.A'.\sin.(v'''-B')$. Developing this by [22] Int., we obtain,

[7357c] $$s''' = \sin.v'''.\Sigma.A'.\cos.B' - \cos.v'''.\Sigma.A'.\sin.B'.$$

[7357d] Now putting $\Sigma.A'.\cos.B' = A$, $\Sigma.A'.\sin.B' = B$, it becomes as in [7358].

[7359a] † (3603) Putting $v''' = 100^\circ$ in [7358], we get $s''' = A$; and if we put $v''' = 200^\circ$, it gives $s''' = B$. Hence it appears that we can compute the value of A, by substituting $v''' = 100^\circ$ in [7355]; and the value of B, by substituting $v''' = 200^\circ$ in [7355].

[7362a] ‡ (3604) It appears, from [533a], that if γ be the inclination of the orbit, and $\theta_{,}$ the longitude of the ascending node, we shall have very nearly, for the expression of the
[7362b] latitude $s''' = \gamma.\sin.(v'''-\theta_{,}) = \gamma.\cos.\theta_{,}.\sin.v''' - \gamma.\sin.\theta_{,}.\cos.v'''$ [22] Int. Comparing this last expression of s''' with that in [7358], we get,

[7362c] $$A = \gamma.\cos.\theta_{,}; \qquad B = \gamma.\sin.\theta_{,}.$$

Dividing this value of B by that of A, we get the following expression of $\text{tang}.\theta_{,}$, and the sum of their squares gives the value of γ^2; hence we have, as in [7359, 7360],

[7362d] $$\text{tang}.\theta_{,} = \frac{B}{A}; \qquad \gamma = \sqrt{A^2+B^2}.$$

[7362e] With these formulas, and those in [7359], we may compute the values of A, B, γ, θ' [7362]; for $t = -70$, $t = -30$, and $t = 10$; which correspond respectively to the years 1680, 1720, 1760; the epoch being 1750 [7281′]. The values of A [7359, 7355],
[7362f] corresponding to these times, are *positive*; therefore the values of $\theta_{,}$ must be in the *first* or *fourth* quadrant, as is evident from the expression of $A = \gamma.\cos.\theta_{,}$ [7362c], γ being positive; and to know which of these must be selected, we must refer to the value of B,
[7362g] which is *negative*; so that $\gamma.\sin.\theta_{,}$ [7362c] must be negative; consequently $\theta_{,}$ must be in

premised, if we put successively $t=-70$, $t=-30$, $t=10$, which correspond to the years 1680, 1720 and 1760, we shall have, [7361]

	Inclination.	Longitude of the node.	
1680,	$2^{\circ},7515$;	$346^{\circ},0191$;	1
1720,	$2^{\circ},7210$;	$348^{\circ},1186$;	2
1760,	$2^{\circ},7123$;	$352^{\circ},3238$.	3

[7362]

If we represent the inclination by the formula $2^{\circ},7515+Nt+Pt^2$, t being the number of Julian years elapsed since 1680, we shall have, by comparing this formula with the three preceding inclinations,* [7363]

$$N=-0^{\circ},001035; \qquad P=0^{\circ},0000068125. \tag{7364}$$

The *minimum* of the formula corresponds to† $t=75^{\text{years}},963$, or to the year 1756. The mean of the three preceding inclinations is $2^{\circ},7283$; the [7365]

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the *fourth* quadrant of the circle. The calculations of the author in [7362] have been verified, and found to be very nearly correct; though some slight discrepancies were found in the fourth decimal place of the inclination, and in the third decimal place of the longitude of the node. [7362h]

* (3605) We shall suppose the general expression of the inclination of the orbit γ, to be represented, as in [7363], by [7364a]

$$\gamma=2^{\circ},7515+Nt+Pt^2; \tag{7364b}$$

t being the time elapsed since the year 1680. Putting $t=0$, it becomes $\gamma=2^{\circ},7515$, as in [7362 line 1]; then putting $t=40$, and $\gamma=2^{\circ},7210$ [7362 line 2], we get [7364d]; lastly, putting $t=80$ and $\gamma=2^{\circ},7123$ [7362 line 3], we get [7364e]; [7364c]

$$2^{\circ},7515+40N+1600P=2^{\circ},7210, \quad \text{or} \quad 0^{\circ},0305+40N+1600P=0; \tag{7364d}$$

$$2^{\circ},7515+80N+6400P=2^{\circ},7123, \quad \text{or} \quad 0^{\circ},0392+80N+6400P=0. \tag{7364e}$$

Subtracting a quarter part of the second of these equations from the first, we obtain $0^{\circ},0207+20N=0$; whence $N=-0^{\circ},001035$, and $40N=-0^{\circ},0414$. Substituting this in [7364d], we obtain $1600P=0^{\circ},0414-0^{\circ},0305=0^{\circ},0109$; consequently $P=0,0000068125$. Hence the expression of γ [7364b] becomes, as in [7364, &c.], [7364f] [7364g]

$$\gamma=2^{\circ},7515-0^{\circ},001035.t+0^{\circ},0000068125.t^2. \tag{7364h}$$

† (3606) The minimum value of γ [6364h], is found by putting its differential equal to nothing, which gives $0=-0^{\circ},001035+0^{\circ},000013625.t$; whence $t=75,963$, as in [7365]; and this value would be somewhat augmented, by correcting the calculation for the small mistakes which are spoken of in [7362h]. Substituting this value of t in [7364h], we get $\gamma=2^{\circ},71$ nearly, for the minimum value of γ. [7365a] [7365b] [7365c]

[7365′] mean annual motion of the node, from 1680 to 1760, is $7',88$.* These results agree perfectly with those which have been found by astronomers,
[7366] from the observed eclipses during this interval. Since the year 1760 the inclination has varied by a very sensible quantity. The preceding value of
[7367] s''' [7355], makes the inclination, in 1800, equal to $2^\circ,8657$,† and the longitude of the node equal to $355^\circ,8817$. Observations, by confirming these results, compel us to give up the hypothesis of a constant inclination;
[7368] and we should have found it difficult to discover the law of these variations, without the assistance of the theory.

To obtain the duration of an eclipse of the fourth satellite, we shall resume the formula [7080];

[7369] $$t=T.(1-X).\left\{-(1+\rho')^2.\frac{s'''}{\beta}.\frac{ds'''}{dv'''}\pm\sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s'''}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s'''}{\beta}\right\}}\right\};$$

[7370] in which T represents the half of the mean duration of an eclipse of the satellite in its nodes [7069]. Delambre has found, by the mean of all the observations which he has used, that this semi-duration is equal to $9942''$;
[7371] and since the discovery and use of achromatic telescopes, he finds this semi-duration to be diminished about $52''$, by the discussion of all the
[7372] eclipses observed since that epoch. Therefore we shall suppose
[7373] $T=9890''[=2^h,22^m,25^s \text{ sex.}]$, as in [7565]. The symbol β [7073] represents the mean synodical motion of the satellite, during the time T;
[7374] and we have‡ $\beta=23613''$. The value of ρ' [7049] is

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[7366a] * (3607) The longitudes of the node in 1680, 1760, are $346^\circ,0191$, and $352^\circ,3238$ [7362 lines 1, 3] respectively; so that the motion in 80 years is

[7366b] $$352^\circ,3238-346^\circ,0191=6^\circ,3047;$$

dividing this by 80, we get the annual motion $7',88$, as in [7365′].

[7367a] † (3608) Putting $t=50$ in [7355], and then successively $v'''=100^\circ$, $v'''=200^\circ$, we obtain the values of s''', which are denoted respectively by A, B, in [7359]. With
[7367b] these values of A, B, we compute, by means of the formulas [7362d], the corresponding inclination γ, and the longitude of $\theta_{,}$; which are found to be very nearly the same as those
[7367c] given by the author in [7367]. We may observe that the formula [7364h] is not used for
[7367d] this purpose, because the value of $t=1800-1680=120$, is so great that it becomes necessary to notice terms of the order t^3, which are neglected in that formula.

[7374a] ‡ (3609) Dividing the expression of $T=9890''$ [7373], by the number of seconds in a Julian year, $36525000''$, we get $T=0,0002707735$, expressed in parts of a Julian

$$\rho' = \frac{\rho.\left(1+\frac{a'''}{D'}\right)}{1-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}}\,; \qquad [7375]$$

and we have found, in [7159], $\rho = 0,0713008$; hence we deduce,* [7375′]

$$\rho' = 0,0729603. \qquad [7376]$$

The value of X [7071] is very nearly represented by† $X = \frac{dv'''}{n'''.dt} - 1$; [7376′]

therefore if we notice only the greatest term of v''', we shall have,

$$X = 0,0145543.\cos.(\Theta'''-\varpi'''). \qquad [7377]$$

We have also seen, in [7094], that the value of T must be multiplied by the factor,‡

$$1+\left\{\frac{2M}{n'''-M} - \frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}.H.\cos.V = \text{factor of } T\ [7369]\,; \qquad [7378]$$

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year; multiplying this by the annual synodical motion $n'''-M = 87205380''$ [6025n], we get the synodical motion in the time T equal to $(n'''-M).T = 23613'' = \beta$, as in [7374]. [7374$b$]

*(3610) We have $\frac{a'''}{D'} = \sin.1530'',864$ [7045i]; $\lambda_1 = 0,105469$ [7547]. Substituting these and $\rho = 0,0713008$ [7375′], in ρ' [7375], it becomes as in [7376]. [7375a]

† (3611) We have, as in [7071], $X = \frac{dv_1'''}{(n'''-M).dt} - 1$; moreover, dv''' is to dv_1''' as n''' to $n'''-M$, nearly [6023c, d, 7374b]; hence X becomes very nearly as in [7376′]. [7376a]
Substituting, in this expression of X, the chief term of v''' [7318 line 1], it produces, by using [7290], the expression, [7376b]

$$9265'',56.\frac{d.\sin.(\Theta'''-\varpi''')}{n'''dt} = 9265'',56 \times \frac{8753^{\circ},463221.\cos.(\Theta'''-\varpi''')}{n'''}. \qquad [7376c]$$

Substituting $n''' = 8754^{\circ},2591$ [6025k], and dividing by the radius in seconds, 636620″, it becomes $X = 0,014553.\cos.(\Theta'''-\varpi''')$, as in [7377] nearly.

‡ (3612) If we change n into n''', and a into a''', in the expression [7094], we shall obtain the factor of T corresponding to the fourth satellite, namely,

$$1+\left(\frac{2M}{n'''-M} - \frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right).H.\cos.(Mt+E-I)\,; \qquad [7378a]$$

and by using the symbol $V = Mt+E-I$ [6023b], it becomes as in [7378]. Finally, substituting the values of $\frac{a'''}{D'}$, λ_1 [7375a], M [6840], and $2H = 61213'',1$ [6882b], [7378b]
or $H = 30606'',5$, it becomes $1-0,0006087.\cos.V$; being nearly the same as in [7379]. [7378c]

H being the excentricity of Jupiter's orbit. Hence this factor becomes,

[7379] $$1-0{,}0006101.\cos.V = \text{factor of } T \text{ [7369]}.$$

[7380] We shall put $\zeta = \frac{(1+\rho').s'''}{\beta}$, and we shall have,*

Symbol ζ

[7381]
$$\begin{aligned} \zeta = {} & 1{,}352380\ .\sin.(v'''+\ 51°{,}3787-t.153'',8) && 1 \\ & -0{,}125759.\sin.(v'''+\ 83°{,}29861+t.7528'',01) && 2 \\ & +0{,}020399.\sin.(v'''+208°{,}32562+t.28220'',85) && 3 \\ & +0{,}000218.\sin.(v'''+285°{,}20_c+t.133715'',77). && 4 \end{aligned}$$

This being premised, if we neglect the square of X, the quantity under the

[7382] radical, in the expression of t [7369], will become $1+X-\zeta^2$; and if we also neglect the products of X and H by $\frac{\zeta d\zeta}{dv'''}$, we shall obtain,†

Time of the emersion or immersion

[7383] $$t = -366'',832.\frac{\zeta d\zeta}{dv'''} \pm 9890''.(1-X-0{,}0006101.\cos.V).\sqrt{1+X-\zeta^2}.$$

[7381a] * (3613) From the values of β, ρ' [7374, 7376], we obtain $\frac{1+\rho'}{\beta} = \frac{1{,}0729603}{23613''}$; multiplying this by s''' [7355], it becomes as in [7381], using the abridged symbol [7380].

[7383a] † (3614) The factor $T.(1-X) = 9890''.(1-X)$ [7373], which occurs in the expression of t [7369], being multiplied by the factor of T [7379], neglecting terms of the order HX, becomes,

[7383b] $$9890''.(1-X-0{,}0006101.\cos.V);$$

and this is to be substituted for $T.(1-X)$ in [7369]. Now the radical expression in

[7383c] [7369] may be put under the form $\pm\sqrt{1+X-\zeta^2}$, as is evident by multiplying together the factors composing this radical, neglecting X^2, and substituting ζ [7380]; hence we get the term depending on the radical in [7383]. In the term of [7369] without the radical, which is very small, we may suppose the factor [7383b] to be simply equal to

[7383d] $9890''$; so that the term itself becomes $-\frac{9890''}{\beta}.(1+\rho')^2.\frac{s'''ds'''}{dv'''}$. But from [7380] we have $(1+\rho')^2.s'''^2 = \beta^2\zeta^2$, whose differential, divided by 2, is $(1+\rho')^2.s'''ds''' = \beta^2.\zeta d\zeta$;

[7383e] hence the differential expression [7383d] becomes $-9890''.\beta.\frac{\zeta d\zeta}{dv'''}$; and by substituting the value of β [7374], then dividing by the radius in seconds, it becomes as in the first term of

[7383f] [7383]. The whole duration of the eclipse t' [7082], is derived from that of t [7080], by retaining only the radical part of the expression, and multiplying it by ± 2. The same

[7383g] process, being performed in the expression of t [7383], gives the whole duration of the eclipse t', as in [7385]. If we now examine into some of the largest terms, which have

[7383h] been neglected in computing the value of t, or t' [7383, 7385], we shall find that they are not in general of any importance. The most noted term is that spoken of in [7096],

[7383i] requiring that X should be increased under the radical, by the quantity $\pm\frac{s'''^2 ds'''^2}{\beta^2 dv_1'''^2}$, which

It is easy to deduce, from this expression, the times of the immersion and emersion of the satellite; observing that t expresses the time elapsed from [7384]
the conjunction of the satellite [7070, 7055]; the conjunction being estimated by means of its projected place upon the orbit of Jupiter; and the time from the conjunction can be determined by means of the tables of Jupiter, and the preceding expressions of v''', s''' [7354, 7355]. The whole duration t' of the eclipse is represented by

Duration of the eclipse.

$$t'=19780''.(1-X-0{,}0006101.\cos.V).\sqrt{1+X-\zeta^2} = \text{duration of the eclipse.}$$ [7385]

---

is nearly of the order $\beta^2.\left(\frac{\zeta d\zeta}{dv'''}\right)^2$, as is evident from [7383e]; observing that ρ' [7376] is so small that it may be neglected. Now $\left(\frac{\zeta d\zeta}{dv'''}\right)$ may be considered as of the order 1, at [7383k]
its maximum, and is generally much less. For if we notice only the chief term of ζ [7381 line 1], and represent it, for brevity, by $\zeta = A.\sin.(v'''+B)$, we shall have, very [7383l]
nearly, $\frac{d\zeta}{dv'''}=A.\cos.(v'''+B)$; whence,

$$\frac{\zeta d\zeta}{dv'''}=A^2.\sin.(v'''+B).\cos.(v'''+B)=\tfrac{1}{2}A^2.\sin.(2v'''+2B),$$ [7383l']

which we shall put equal to $b^{\frac{1}{2}}$, for brevity, so that $b=\left(\frac{\zeta d\zeta}{dv'''}\right)^2=\frac{1}{4}A^4.\sin.^2(2v'''+2B)$; [7383m]
and this varies from 0 to its maximum $\frac{1}{4}A^4=\frac{1}{4}.(1{,}35)^4$, which is less than unity. Moreover β [7374] is about $\frac{1}{27}$ of the radius; hence $\beta^2=(\frac{1}{27})^2=\frac{1}{729}$ nearly; so that if $X-\zeta^2$ be very small in comparison with unity, the radical $\sqrt{1+X-\zeta^2}$ [7383] will be [7383n]
varied by $\frac{1}{2}\beta^2 b=\frac{b}{1458}$ nearly, in consequence of this variation in the value of X. Multiplying this by the factor $T=9890''$ [7383], it becomes $\frac{1}{2}T\beta^2 b=7''.b$ nearly; [7383o]
which, on account of the smallness of b, will not be in general of any importance. This term however must be noticed when the radical $\sqrt{1+X-\zeta^2}$ becomes very small in consequence of the great latitude of the satellite, as is observed in [7094'—7096]. We [7383p]
may find, by a similar process, that the terms of the order $\sin.^2 r_1.\left(\frac{dZ}{dv}\right)^2$, &c., which are neglected in [7062, 7059, &c.], produce nothing of importance in the values of t, t'; and the same is to be observed relative to the other quantities, which are neglected in these calculations. [7383q]

CHAPTER XIII.

THEORY OF THE THIRD SATELLITE.

31. We have found, in the preceding chapter [7301, 7296], that

[7386] $$\theta'' = 11^{\circ},39349 + t.20420^{\circ},579040;$$

[7387] $$\varpi'' = 343^{\circ},82067 + t.29164'',43.$$

We have also seen, in [7303], that the equation of the centre, corresponding to this satellite, is

[7388] $$\delta v'' = 1709'',05.\sin.(\theta''-\varpi'').$$

This satellite has another equation of the centre corresponding to the perijove of the fourth satellite [7127], and represented by

[7389] $$\delta v'' = 756'',61.\sin.(\theta''-\varpi''').$$

The expression of $\delta v''$ [6846] becomes, by the substitution of the values of m, m', m''' [7142—7145],*

---

* (3615) We have, as in [6240n, p, q, s'],

[7390a] $$nt-n''t+\varepsilon-\varepsilon''=\theta-\theta''; \qquad n't-n''t+\varepsilon'-\varepsilon''=\theta'-\theta'';$$

[7390b] $$n'''t-n''t+\varepsilon'''-\varepsilon''=\theta'''-\theta''; \qquad 2n''t-2Mt+2\varepsilon''-2E=2\theta''-2\Pi;$$

[7390c] $$nt-2n't+\varepsilon-2\varepsilon'+g_2t+\Gamma_2=\theta-2\theta'+\varpi''; \qquad nt-2n't+\varepsilon-2\varepsilon'+g_3t+\Gamma_3=\theta-2\theta'+\varpi''';$$

[7390d] $$n''t-2Mt+\varepsilon''-2E+g_2t+\Gamma_2=\theta''-2\Pi+\varpi''; \qquad n''t-2Mt+\varepsilon''-2E+g_3t+\Gamma_3=\theta''-2\Pi+\varpi'''.$$

Substituting the values [7390a, b] in [6846], also the values of m, m', m''' [7142, 7143, 7145], we get, by a very easy numerical calculation, the values [7390].

$$
\begin{aligned}
\delta v'' = {} & \quad 4'',21.\sin.\ (\theta - \theta'') && 1 \\
& -808'',20.\sin.\ (\theta' - \theta'') && 2 \\
& - \ 11'',84.\sin.2.(\theta' - \theta'') && 3 \\
& - \quad 2'',37.\sin.3.(\theta' - \theta'') && 4 \\
& - \ 45'',29.\sin.\ (\theta'' - \theta''') && 5 \\
& +154'',47.\sin.2.(\theta'' - \theta''') && 6 \\
& + \ 10'',86.\sin.3.(\theta'' - \theta''') && 7 \\
& + \quad 2'',53.\sin.4.(\theta'' - \theta''') && 8 \\
& + \quad 2'',39.\sin.(2\theta'' - 2\Pi). && 9
\end{aligned}
$$ [7390]

The theorem on the epochs of the three inner satellites [6623g], becomes,*

$$\theta - \theta'' = 200^\circ + 3\theta' - 3\theta'';$$ [7391]

therefore the two terms,

$$
\begin{aligned}
& 4'',21.\sin.\ (\theta - \theta'')\,; \quad [7390 \text{ line } 1] && 1 \\
& -2'',37.\sin.3.(\theta' - \theta'')\,; \quad [7390 \text{ line } 4] && 2
\end{aligned}
$$ [7392]

may be connected together in one single term, $\delta v'' = -6'',58.\sin.(3\theta' - 3\theta'')$. [7393]

Substituting in the expression of Q'' [6862], the value of $g = 29009'',8$ [7190], relative to the apsides of the third satellite, also $\frac{h'}{h''} = 0,2152920$ [7192], corresponding to this value of g, and $-2h'' = 1709'',05$ [7388], we shall find, that the inequality $\delta v'' = Q''.\sin.(nt - 2n't + \varepsilon - 2\varepsilon' + gt + \Gamma)$ [6852], becomes,† [7393′] [7394] [7394′]

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* (3616) Adding $3\theta' - 3\theta''$ to both members of the equation [6623g], we get the expression of $\theta - \theta''$ [7391]; whose sine gives $\sin.(\theta - \theta'') = -\sin.3(\theta' - \theta'')$; substituting this in [7392 line 1], it becomes $-4'',21.\sin.3(\theta' - \theta'')$. The sum of this, and that in [7392 line 2], becomes as in [7393]. [7392a] [7392b]

† (3617) The part of the equation of the centre of the third satellite $\delta v''$ [6243], depending on the angle g or $g_2 = 29009'',8$ [7393′ or 7190], has for its coefficient the quantity $-2\beta_2'' h_2$, which is represented by $-2h''$ in [6229d line 5]; and the coefficient of this term of $\delta v''$ is $1709'',5$ [7388]; hence we have $-2h'' = 1709'',05$, or $h'' = -854'',525$. Now for this value of g we have, in [7192], [7394a] [7394b]

$$h' = 0,2152920.h'' = -0,2152920 \times 854'',525 = -183'',97.$$ [7394c]

Substituting these values of g_2, h', h'', and that of $m' = 0,232355$ [7143], in Q'' [6862], we get $Q'' = -95'',18$; hence the inequality [6852 or 7394′] becomes, [7394d]

$$\delta v'' = -95'',18.\sin.(nt - 2n't + \varepsilon - 2\varepsilon' + g_2 t + \Gamma_2)\,;$$ [7394e]

which is easily reduced to the form [7395], by using the first of the expressions [7390c]; and we finally obtain the form [7396], by substituting the value of $\theta - 2\theta'$ [7395′]. [7394f]

[7395] $$\delta v'' = -95'',18.\sin.(\circleddash-2\circleddash'+\varpi'').$$

[7395'] Now we have $\circleddash-2\circleddash' = 200^\circ+\circleddash'-2\circleddash''$ [7391]; hence the preceding inequality becomes,

[7396] $$\delta v'' = 95'',18.\sin.(\circleddash'-2\circleddash''+\varpi'').$$

[7396'] Substituting in Q'' [6862] the value of g or $g_3 = 7959'',105$ [7195], relative to the apsides of the fourth satellite, and for $\frac{h'}{h'''}$, $\frac{h''}{h'''}$, the quantities [7197, 7198], depending upon this value of g, observing also that we have as in [7127, 7127b],

[7397] $$-2h''' = 9265'',56,$$

we find that the same inequality [7394'] becomes, for this case,*

[7398] $$\delta v'' = 43'',58.\sin.(\circleddash'-2\circleddash''+\varpi''').$$

The inequality [6885],

[7399] $$\delta v'' = -149'',96.\left\{1+\frac{3a''\mathrm{m}.\mathrm{k}n^2}{32.a\,\mathrm{m}''.(M^2-\mathrm{k}n^2).\left(1+\frac{9a'.\mathrm{m}}{4a.\mathrm{m}'}+\frac{a''.\mathrm{m}}{4a.\mathrm{m}''}\right)}\right\}.\sin.(Mt+E-I),$$

becomes, by substituting the values of m, m′, m″, k,†

[7400] $$\delta v'' = -147'',42.\sin.V.$$

The inequality [6874, 6876], corresponding to the third satellite, is,‡

[7401] $$\delta v'' = -\frac{15.M.h''}{4n''}.\sin.(n''t-2Mt+\varepsilon''-2E+gt+\Gamma);$$

[7398a] * (3618) We get from [7397] $h''' = -4632'',78$; and for the value of g or g_3 [7396'], we have, as in [7197, 7198],

[7398b] $$h' = 0,0173350.h''' = -80'',31; \qquad h'' = 0,0816578.h''' = -378'',30.$$

Substituting these values of h', h'', g, also m′ [7143], in [6862], we get $Q'' = -43'',58$;

[7398c] hence the inequality [6852] becomes $\delta v'' = -43'',58.\sin.(nt-2n't+\varepsilon-2\varepsilon'+g_3t+\Gamma_3)$; and by using the second expression [7390c], it becomes $\delta v'' = -43'',58.\sin.(\circleddash-2\circleddash'+\varpi''')$.

[7398d] Now substituting the value of $\circleddash-2\circleddash'$ [7395'], it becomes as in [7398].

[7400a] † (3619) Substituting $V = Mt+E-I$ [7313] in [7399], using also the values of n, M, k, a, a', a'', m, m′, m″ [6782, 6840, 7272, 6797–6799, 7142–7144], it becomes as in [7400]; as we have found by verifying the calculation.

[7401a] ‡ (3620) Substituting $N-n-g=0$ [6876] in [6874], it becomes, for the third
[7401b] satellite, as in [7401]; and the coefficient of this inequality is reduced to the form
[7401c] $-0,0061926.h''$, by using [6879]. Substituting $h'' = -854'',525$ [7394b], corresponding
[7401d] to the root g_2, it becomes $5'',29.\sin.(n''t-2Mt+\varepsilon''-2E+g_2t+\Gamma_2)$; which is easily reduced to the form in [7402 line 1], by using the first equation [7390d]. In like manner,

and on account of the double excentricity of the third satellite, it produces the two following inequalities,

$$\begin{aligned} \delta v'' &= 5'',29.\sin.(\circleddash''-2\Pi+\varpi'') && 1 \\ &\quad +2'',34.\sin.(\circleddash''-2\Pi+\varpi'''). && 2 \end{aligned}$$ [7402]

It now remains to consider the equation of the libration of the third satellite; but it follows, from [7275—7277], that this equation [7277] is not a tenth part of that of the first or second satellite [7275, 7276]; and as neither of these is of sufficient magnitude to become sensible by observation, it is evident that this equation of the third satellite must be wholly insensible. [7403] Therefore, by collecting together all these inequalities of the third satellite, we obtain the following expression of its longitude in eclipses, where we may suppose $2\Pi=2\circleddash''$ [7353c]; * [7404]

Longitude v'' in eclipses; counted from the earth's moveable vernal equinox.

$$\begin{aligned} v'' = \circleddash'' &+1703'',76.\sin.\ (\circleddash''-\varpi'') && 1 \\ &+\ 754'',27.\sin.\ (\circleddash''-\varpi''') && 2 \\ &-\ 808'',20.\sin.\ (\circleddash'-\circleddash'') && 3 \\ &-\ \ 11'',84.\sin.2.(\circleddash'-\circleddash'') && 4 \\ &-\ \ \ 6'',58.\sin.3.(\circleddash'-\circleddash'') && 5 \\ &-\ \ 45'',29.\sin.\ (\circleddash''-\circleddash''') && 6 \\ &+\ 154'',47.\sin.2.(\circleddash''-\circleddash''') && 7 \\ &+\ \ 10'',86.\sin.3.(\circleddash''-\circleddash''') && 8 \\ &+\ \ \ 2'',53.\sin.4.(\circleddash''-\circleddash''') && 9 \\ &+\ \ 95'',18.\sin.\ (\circleddash'-2\circleddash''+\varpi'') && 10 \\ &+\ \ 43'',58.\sin.\ (\circleddash'-2\circleddash''+\varpi''') && 11 \\ &-\ 147'',42.\sin.V && 12 \end{aligned}$$ [7405]

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by using $-2h''=756'',61$, or $h''=-378'',30$ [7398b], corresponding to the root g_3, [7401e] we find that the coefficient $-0,0061926.h''$ [7401c], becomes equal to $2'',34$; hence we [7401f] get the expression in [7402 line 2], by using the second of the equations [7390d].

* (3621) Substituting $2\Pi=2\circleddash''$ [7404], in the term in [7390 line 9], we find that it vanishes. The term in [7402 line 1] becomes $5'',29\ \sin.(-\circleddash''+\varpi'')=-5'',29.\sin.(\circleddash''-\varpi'')$; [7404a] connecting this with that in [7388], it becomes $1703'',76.\sin.(\circleddash''-\varpi'')$, as in [7405 line 1]. [7404b] Lastly, the term [7402 line 2] becomes $2'',34.\sin.(-\circleddash''+\varpi''')=-2'',34.\sin.(\circleddash''-\varpi''')$; [7404c] and by connecting it with that in [7389], we get $754'',27.\sin.(\circleddash''-\varpi''')$, as in [7405 line 2]. [7404d] The rest of the terms in [7405] are the same as those in [7390, 7396, 7398, 7400]; observing that the two terms in [7390 lines 1, 4] are reduced to one in [7393], being the same as that in [7405 line 5]. [7404e]

In the motions of the third satellite there have been discovered some singular variations, arising from the double equation of the centre in the theory of this satellite. To explain these variations, Wargentin had recourse to two
[7406] particular equations, whose periods in eclipses were twelve and a half, and fourteen years, and which are, in fact, two equations of the centre, referred to apsides, which move with different velocities; but having been compelled, by observation, to abandon this theory, he introduced, instead of it, the
[7407] supposition of a variable excentricity. The first hypothesis of this learned astronomer is, as we have just seen, conformable to nature; but he was deceived with respect to the periods and magnitudes of these equations, because he did not know that one of them must be referred to the apsides of the fourth satellite. We have, as in [7387, 7288],

[7408] $$\varpi'' = 343^\circ{,}82067 + t . 29164'',43;$$

[7409] $$\varpi''' = 200^\circ{,}38054 + t . \ 8113'',735.$$

Comparing these two equations together, we find that the perijoves of the
[7410] third and fourth satellites coincided in 1682,* and then the coefficient of the equation of the centre was equal to the sum of the coefficients of the two
[7410′] partial equations [7405 lines 1, 2], that is, 2458″,03. In 1777 the perijove of the third satellite was further advanced by 200° than that of the fourth,†
[7411] and then the coefficient of the equation of the centre was equal to the difference of the coefficients of the partial equations [7405 lines 1, 2], or
[7411′] 949″,49. These results are entirely conformable to observation.

[7409a] * (3622) Putting the expressions of ϖ'', ϖ''' [7408, 7409], equal to each other, we get $t = -68$ nearly; and as the epoch is 1750 [7281′], this time corresponds to $1750 - 68 = 1682$. In this case the arguments of both the inequalities [7405 lines 1, 2], are equal to $\circleddash'' - \varpi''$; and their sum is

[7409b] $$(1703'',76 + 754'',27) . \sin . (\circleddash'' - \varpi'') = 2458'',03 . \sin . (\circleddash'' - \varpi''),$$

as in [7410].

† (3623) Putting the expression of ϖ'' [7408] equal to

[7410a] $$\varpi''' + 200^\circ = 400^\circ{,}38054 + t . 8113'',735 \quad [7409],$$

we get $t = 27$, corresponding to the year $1750 + 27 = 1777$, as in [7410′]. Substituting this value of $\varpi''' = \varpi'' - 200^\circ$, in the term [7405 line 2], it becomes,

[7410b] $$754'',27 . \sin . (\circleddash'' - \varpi'' + 200^\circ) = -754'',27 . \sin . (\circleddash'' - \varpi'').$$

Connecting this with the similar term in [7405 line 1], it becomes,

[7410c] $$(1703'',76 - 754'',27) . \sin . (\circleddash'' - \varpi'') = 949'',49 . \sin . (\circleddash'' - \varpi'');$$

as in [7411].

We shall now consider the motion of this satellite in latitude. The term of s'' [6429 line 1], namely,

$$s'' = (\lambda''-1).\theta'.\sin.(v''+\Psi'),$$ [7412]

becomes, by the substitution of the values of λ'', θ', Ψ',*

$$s'' = 3^{\circ},34213.\sin.(v''+51^{\circ},3787-t.153'',8).$$ [7413]

The term $s''=l''.\sin.(v''+pt+\Lambda)$, or $s''=l''.\sin.(v''+p_2t+\Lambda_2)$, which is [7414]
included in the same expression of s'' [6429 line 4, 6422, &c.], and corresponds to the peculiar inclination of the orbit of the third satellite, is represented in [7333] in the following manner,

$$s'' = -2283'',9.\sin.(v''+208^{\circ},32562+t.28220'',85).$$ [7415]

If we substitute, in the term of s'' [6429 line 5, &c.], the values of p_3, Λ_3 [7329], corresponding to the peculiar inclination of the orbit of the fourth satellite, it becomes,†

$$s'' = -2771'',6.\frac{l''}{l'''}.\sin.(v''+83^{\circ},29861+t.7528'',01).$$ [7416]

Now we have in this case, by [7248],

$$\frac{l''}{l'''} = 0,1248622\,;$$ [7417]

hence the preceding term [7416] becomes,

$$s'' = -346'',07.\sin.(v''+83^{\circ},29861+t.7528'',01).$$ [7418]

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* (3624) The term of s'' [6429 line 1], is the same as in [7412]. It is incorrectly [7413a]
printed in this part of the original work; its second member being given under the form $(1-\lambda'').\theta'.\sin.(v''+pt+\Lambda)$. Now by proceeding as in [7323a—g], we get the expressions [7413b]
of s'', similar to those in [7323f, g], and which may be derived from them, by changing [7413c]
λ''' into λ'', and v''' into v''. In this way, the three last of the formulas [7323f, g] give,

$$s''=(\lambda''-1).\theta'.\sin.(v''+\Psi')=(1-\lambda'').3^{\circ},4352.\sin.(v''+51^{\circ},3787-t.153'',8)=3^{\circ},34213.\sin.(v''+51^{\circ},3787-t.153'',8).$$ [7413d]

The first of these expressions is the same as [7412]; the third is deduced from the second, by using λ'' [7208], and is the same as that in [7413] representing the latitude of the [7413e]
satellite, supposing it to move in a fixed plane, similar to that spoken of in [6358], and corresponding to the third satellite.

† (3625) The coefficient of the term of s'' [6429 line 5], can be deduced from that of [7414a]
s''' [6430 line 5], corresponding to the same value of p [7245], by multiplying the expression of s''' by $\frac{l''}{l'''}=0,1248622$ [7248 or 7417], and changing v''' into v''; hence [7414b]
the expression of s''' [7329] gives that in [7416], which is easily reduced to the form [7418] by substituting [7417].

The term s'' [6429 line 3] becomes also, by substituting the values of p_1, A_1 [7338, &c.], corresponding to the orbit of the second satellite,*

[7419] $$s'' = -5152'',2.\frac{l''}{l'}.\sin.(v''+285^\circ,20_c+t.133715'',77).$$

We have in this case, as in [7235],

[7420] $$\frac{l''}{l'} = -0,034253\,;$$

then the preceding term becomes,

[7421] $$s'' = 176'',48.\sin.(v''+285^\circ,20_c+t.133715'',77).$$

The only sensible term of s'', among those we have given in [6933 line 2], is the following;

[7422] $$s'' = 0,00061925.(l''-L').\sin.(v''-2U-pt-\text{A}).$$

Substituting for $l''-L'$, p, A, their values corresponding to Jupiter's equator, it becomes,†

[7423] $$s'' = 20'',7.\sin.(v''-2U-51^\circ,3787+t.153'',8).$$

If we substitute also for l'', p, A, their values corresponding to the orbit of

[7424] the third satellite [7415], and then put $L' = 0$ [6415], we shall have,‡

[7419a] * (3626) The expression [7419], corresponding to the second satellite, is similar to that in [7416], depending upon the fourth, changing the symbols relative to the fourth [7329 or 7350] into those corresponding to the second [7338]; by this means we get the term

[7419b] s'' [7419]; and by substituting [7420], it becomes as in [7421]. It is not necessary to notice the term of s'' [7414], corresponding to the value of p [7226]; because, in this case,

[7419c] the factor $\frac{l''}{l} = -0,0009597$ [7228] is very small, and the term is also insensible on account of the smallness of l.

[7423a] † (3627) The expression of the inequality of s'' [7422], corresponding to the third satellite, is similar to that of s''' [7341], relative to the fourth satellite; changing λ''' into

[7423b] λ'', l''' into l'', v''' into v'', and 0,001447815 [6934] into 0,00061925 [6933 line 2];

[7423c] observing also, that by this means the factor $(1-\lambda''').3^\circ,4352=2^\circ,98051$ [7323g, or 7345], corresponding to the fourth satellite, becomes for the third satellite $(1-\lambda'').3^\circ,4352=3^\circ,34213$,

[7423d] as in [7413]. Now making the same changes in the expression of s''' [7346], which is derived from [7341], we get the corresponding term of s'', namely,

[7423e] $$s''=3^\circ,34213\times 0,00061925.\sin.(v''-2U-51^\circ,3787+t.153'',8)\,;$$

which is easily reduced to the form [7423].

[7425a] ‡ (3628) When we consider the terms of s'', spoken of in [6415], we may put $L'=0$ [7424]; and then the inequality [6933 line 2, or 7422] becomes,

$$s'' = -1'',42.\sin.(v''-2U-208^{\circ},32562-t.28220'',85).$$ [7425]

Connecting together all these terms of the latitude, we shall have in eclipses, where $2U=2v''$ [7353c], very nearly,* [7426]

Latitude of the third satellite in eclipses.

$$\begin{aligned} s'' = 33400'',6.\ &\sin.(v''+51^{\circ},3787-t.153'',8) && 1 \\ -\ 346'',07.&\sin.(v''+83^{\circ},29861+t.7528'',01) && 2 \\ -2282'',5.\ &\sin.(v''+208^{\circ},32562+t.28220'',85) && 3 \\ +\ 176'',48.&\sin.(v''+285^{\circ},20_c+t.133715'',77). && 4 \end{aligned}$$ [7427]

To obtain the duration of the eclipses of the third satellite, we shall resume the formula [7080],†

$$s'' = 0,00061925.\,l''.\sin.(v-2U-pt-\text{A}).$$ [7425a′]

If we use the values of l'', $pt+\text{A}$, or rather $p_2t+\text{A}_2$, corresponding to [7415], it becomes,

$$s'' = -0,00061925\times 2283'',9.\sin.(v''-2U-208^{\circ},32562-t.28220'',85)\ ;$$ [7425b]

and by reducing the coefficient we obtain [7425]. The other terms of s'' [7427 lines 2, 4], being small, will not produce any sensible terms of this form. [7425c]

* (3629) Putting $2U=2v''$ [7426] in [7423], it becomes, by a slight reduction, $-20'',7.\sin.(v''+51^{\circ},3787-t.153'',8)$. Connecting this with the similar term in [7413], [7427a] we obtain the part of s'' which is contained in [7427 line 1]. The same substitution of $2U=2v''$ being made in [7425], it becomes $1'',42.\sin.(v''+208^{\circ},32562+t.28220'',85)$; [7427b] connecting this with the term in [7415], depending on the same argument, we obtain the part of s'', which is contained in [7427 line 3]. Finally, the terms in [7418, 7421] [7427c] correspond respectively to [7427 lines 2, 4].

In the above calculation [7427b, &c.], we have neglected the terms in [7427 lines 2, 4], because they are much smaller than the term in [7425 line 3], which we have retained; [7427d] and this term produces only a very small quantity [7425], which would not be worth the trouble of computing, if it depended on a different argument from the other inequalities; [7427e] but as it depends, at the time of the conjunction, on the same argument with the inequality [7427f] [7415], it is computed and combined with it in [7427 line 1]. For similar reasons we may neglect the term of s'' in [6933 line 1], which, by using the value of m' [7143], becomes [7427g] nearly $0,0003.(l'-l'').\sin.(2v'-2v''-pt-\text{A})$. Now taking for l', l'', any of the [7427h] coefficients of s', s'' [7482, 7427], corresponding to the same angle, we see that the resulting terms are quite small, and not of sufficient importance to be introduced by a table [7427i] with a new argument, like those mentioned in [7427e].

† (3630) Changing s into s'', v_1 into v_1'' in [7080], we get the expression of t [7428a] [7428], corresponding to the third satellite; being similar to that for the fourth satellite [7428b] [7369]. The expression of T in [7428] represents the time of the mean duration of the eclipses of the *third* satellite, when in its node; in like manner as for the *fourth* satellite in [7428c]

[7428] $$t=T.(1-X).\left\{-(1+\rho')^2.\frac{s''}{\beta}.\frac{ds''}{dv_1''}\pm\sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s''}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s''}{\beta}\right\}}\right\};$$

[7429] In this formula T represents the half of the mean duration of the eclipses of the third satellite in its nodes [7428c]. This time has been found by
[7429′] Delambre to be $T=7419''$ [7564], from the observations made since the invention of the achromatic telescope; and we shall assume this as the value of T. The mean synodical motion of the third satellite, during the time T,
[7430] is represented by* β [7428d]; and we have $\beta=41410''$ [7430b]. The value of ρ' in this case is $\rho'=0,072236$;† the value of X is, by [7376a],
[7431] very nearly equal to $\frac{dv''}{n''dt}-1$; so that if we notice only the greatest terms of v'', we shall have,‡

[7370]. Moreover β, instead of being, as in [7373], for the *fourth* satellite, represents,
[7428d] in [7428], the mean synodical motion of the *third* satellite during the time T, corresponding to the third satellite.

[7430a] * (3631) In this case we have, in like manner as in [7374b], $(n''-M).T=\beta$. Now from [6025n] we have $n''-M=203868424''$; and the expression of $T=7419''$ [7429′] being divided by the number of seconds in a Julian year 36525000, gives $T=0,000203121$. Substituting this last value of T, and that of $n''-M$, in the
[7430b] expression of β [7430a], it becomes $\beta=41410''$, as in [7430].

[7431a] † (3632) The value of ρ' corresponding to the third satellite, is easily deduced from that in [7375], for the fourth, by changing a''' into a''. By this means $\frac{a'''}{D'}$ is changed into $\frac{a''}{D'}$; and by using [7375a], it may be put under the form,

[7431b] $$\frac{a''}{D'}=\frac{a''}{a'''}.\frac{a'''}{D'}=\frac{a''}{a'''}.\sin.1530'',864;$$

hence we have,

[7431c] $$\rho'=\frac{\rho.\left\{1+\frac{a''}{a'''}.\sin.1530'',864\right\}}{1-\frac{(1-\lambda_1)}{\lambda_1}.\frac{a''}{a'''}.\sin.1530'',864}.$$

Substituting the values of $\frac{a''}{a'''}$, λ_1, ρ [6823, 7375a], it becomes as in [7431].

[7432a] ‡ (3633) Changing n''' into n'', and v_1''' into v_1'', in the expression of X [7376a], corresponding to the fourth satellite, we get the similar value of X for the third satellite,

$$
\begin{aligned}
X = {} & 0{,}00268457.\cos.(\text{ө}''-\varpi'') && 1 \\
& +0{,}00118848.\cos.(\text{ө}''-\varpi''') && 2 \\
& -0{,}00128817_c.\cos.(\text{ө}'-\text{ө}'') && 3
\end{aligned}
$$ [7432]

We have seen also, in [7094], that the value of T must be multiplied by the factor,*

$$1+\left\{\frac{2M}{n''-M}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a''}{D'}\right\}.H.\cos.V;$$ [7433]

$X=\frac{dv''}{n''dt}-1$ [7376']. If we retain only the greatest terms of v'', which are given in [7388, 7389, 7390 line 2], we shall have,

$$X=\frac{1}{n''dt}.d.\{1709'',05.\sin.(\text{ө}''-\varpi'')+756'',61.\sin.(\text{ө}''-\varpi''')-808'',20.\sin.(\text{ө}'-\text{ө}'')\};$$ [7432b]

and the three terms, of which this expression is comprised, produce the three terms in [7432 lines 1, 2, 3] respectively. For the first term of [7432b] is

$$1709'',05.\frac{d.(\text{ө}''-\varpi'')}{n''dt}.\cos.(\text{ө}''-\varpi'');$$ [7432c]

and if we substitute the values of $\text{ө}''-\varpi''$, n'' [7302, 6025k], it becomes,

$$1709'',05\times\frac{204176626}{204205635}.\cos.(\text{ө}''-\varpi'');$$ [7432d]

which is easily reduced to the form in [7432 line 1], by dividing by the radius in seconds 636620″. Again, by subtracting the value of ϖ''' [7288] from that of $\text{ө}''$ [7301], we get $\text{ө}''-\varpi'''=211°,0129+t.204197677''$; and its differential being divided by the value of n'' [6025k], gives $\frac{d.(\text{ө}''-\varpi''')}{n''dt}=\frac{204197677}{204205635}$; multiplying this by the second coefficient [7432e] [7432b], or $\frac{756'',61}{636620''}$, we get the coefficient of the term in [7432 line 2]. Lastly, we have, from [7301, 7440], $\text{ө}'-\text{ө}''=335°,0958+t.207206791''$; hence $\frac{d.(\text{ө}'-\text{ө}'')}{n''dt}=\frac{207206791}{204205635}$; multiplying this by $-\frac{808'',20}{636620''}$ [7432b], we get the coefficient of [7432f] the third term of X, in [7432 line 3]; which is $-0{,}00128817$; differing a little from the author, who makes it $-0{,}00126952$; having probably used the factor 204197677 [7432e], [7432g] instead of 207206791 [7432f]. We have corrected this in [7432 line 3], annexing the small letter $_c$ to the number $-0{,}00128817_c$, to denote that it varies from the original work. We may however remark, that this correction is of very little importance, since it [7432h] produces only a variation of a small fraction of a second of time in the value of t [7436, &c.]. [7432i]

* (3634) The factor [7433] is similar to that in [7094 or 7378]. The last of these expressions corresponds to the fourth satellite; and by changing n''' into n'', also a''' into [7433a] a'', we get the similar factor for the third satellite; then substituting the value of $\frac{a''}{D'}$, [7431b], it becomes,

and by reduction this factor becomes of the following form,

[7434] $$1-0{,}00039871.\cos.V.$$

[7434′] Putting $\zeta=\frac{(1+\rho').s''}{\beta}$, we shall have,*

[7435]
$$\begin{aligned}\zeta = {} & 0{,}864850.\sin.(v''+\ 51^{\circ}{,}3787-t.153''{,}8) && 1\\ & -0{,}008961.\sin.(v''+\ 83^{\circ}{,}29861+t.7528''{,}01) && 2\\ & -0{,}059101.\sin.(v''+208^{\circ}{,}32562+t.28220''{,}85) && 3\\ & +0{,}004570.\sin.(v''+285^{\circ}{,}20_{c}+t.133715''{,}77). && 4\end{aligned}$$

This being premised, we shall have,†

[7436] $$t=-482''{,}6_{c}.\frac{\zeta d\zeta}{dv''}\pm 7419''.(1-X-0{,}00039871.\cos.V).\sqrt{1+X-\zeta^{2}};$$

hence we may easily determine the times of immersion and emersion; and

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[7433b] $$1+\left\{\frac{2M}{n''-M}-\frac{(1-\lambda_{1})}{\lambda_{1}}.\frac{a''}{a'''}.\sin.1530''{,}864\right\}.H.\cos.V.$$

[7433c] Substituting the values $\frac{a''}{a'''}$, M, n'', λ_{1}, ρ, H [6823, 6025k,m, 7375a, 7378b], it becomes

[7433d] $1-0{,}0003984.\cos.V$, being very nearly as in [7434].

[7435a] * (3635) From [7430, 7431] we have $\frac{1+\rho'}{\beta}=\frac{1{,}072236}{41410''}$; multiplying the expression of s'' [7427] by this factor, we obtain the expression of ζ [7434′] in the form given in [7435].

[7436a] † (3636) The expression [7436] for the third satellite is easily deduced from the similar one for the fourth satellite [7383], observing that the first coefficient $-366''{,}832$ [7383]
[7436b] may be put under the form $-T\beta=-9890''.\beta$, as in [7383e]. Then changing $T=9890''$, which corresponds to the fourth satellite [7373], into $T=7419''$ [7429′] for the third, we find that the expression of t for the third satellite, derived from [7383], will be,

[7436c] $$t=-7419''.\beta.\frac{\zeta d\zeta}{dv''}\pm 7419''.(1-X-0{,}000398171.\cos.V).\sqrt{1+X-\zeta^{2}};$$

observing that the coefficient of $-\cos.V$, in [7379, 7383], changes from 0,0006101 to
[7436d] 0,00039871 in [7434, 7436]. Substituting the value $\beta=41410''$ [7430], in the factor of the first term of [7436c], namely, $-7419''.\beta$, and dividing by the radius in seconds, it becomes $-482''{,}6$, which is inserted, with the usual mark $_{c}$, in [7436], instead of the
[7436e] number given by the author in the original work, which is $-517''{,}4$; being too great by nearly a fourteenth part.

the whole duration t' of the eclipse will be,*

$$t' = 14838''.(1-X-0,00039871.\cos.V).\sqrt{1+X-\zeta^2}. \qquad [7437]$$

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* (3637) The expression t' [7437] is deduced from [7436], in the same manner as [7385], is derived from [7383] in [7383f]; namely, *by retaining only the part depending on the radical, and multiplying it by* ± 2; by this means it becomes as in [7437]. We may make an estimate of the value of the neglected terms, in the same way as we have done for the fourth satellite, in [7383e—q]. The most important of these neglected terms is that which is computed in [7383i], producing in [7436] the term $\frac{1}{2}T\beta^2.b=7419''\times\frac{1}{2}\beta^2.b$ [7383o], supposing $X-\zeta^2$ to be small, as in [7383n]. Now β [7430] is about $\frac{1}{15}$ part of the radius, or, more accurately, $\beta^2=\frac{1}{236}$; hence $7419''\times\frac{1}{2}\beta^2.b=16''.b$ nearly; and if we use the value of b [7383m] corresponding to this case, namely, $b=\frac{1}{4}A^4.\sin.^2(2v''+2B)$, it becomes $4''.A^4.\sin.^2(2v''+2B)$; A being, as in [7383l], the greatest coefficient in the value of ζ [7435], or $A=0,86..$; hence $4''.A^4=2''$, and the preceding correction becomes equal to $2''.\sin.^2(2v''+2B)$; so that at its maximum it is only $2''$, and is generally less, except when the latitude of the satellite is very great, or $X-\zeta^2$ large; then it requires some notice, as in the similar case for the fourth satellite in [7383p]. In like manner we may prove that the other neglected terms [7062, 7059, &c.], are in general unimportant.

[7437a] [7437b] [7437c] [7437d] [7437e] [7437f]

CHAPTER XIV.

THEORY OF THE SECOND SATELLITE.

32. The discussion of the eclipses of the second satellite gives, for its mean motion from the earth's vernal equinox in one hundred Julian years, the following expression,

[7438] $$4114125°,812765\,;$$

and its mean longitude at the epoch of 1750, equal to*

[7439] $$346°,48931_c\,;$$

we shall therefore put,

[7440] $$\circleddash' = 346°,48931_c + t.41141°,25812765.$$

The different equations of the centre are included in the term,†

[7441] $$\delta v' = -2h'.\sin.(n't+\varepsilon'-gt-\Gamma).$$

[7441′] *The expressions of* h, h', *relative to the two first values of* g [7176, 7183], *have appeared to Delambre to be insensible, notwithstanding the attempts he*

[7439a] * (3638) In the original work the mean longitude is stated to be 146°,48931, instead of 346°,48931; we have given the corrected value, with the small letter $_c$ appended, to
[7439b] denote that it has been corrected. The value [7439] differs a little from that obtained in [7259e], from the last edition of Delambre's tables of the satellites, printed in 1817.

[7441a] † (3639) The expression of the equation of the centre of the *second* satellite [7441], is similar to that for the *third*, in [7394a, &c.]; and the term which is considered as peculiar to
[7441b] the second satellite, is easily deduced from the second term of [6242], by substituting $\beta_1' h_1 = h'$ [6229d line 4]. We may also observe that the equation of the centre, which is considered as peculiar to the *first* satellite, is given by the first term of [6241], and is
[7441c] $\delta v = -2h.\sin.(nt+\varepsilon-gt-\Gamma)$; or, as it may be written, $\delta v = -2h.\sin.(\circleddash-\varpi)$ [6240r].
[7441d] These are used hereafter in [7457, 7497, &c.]. Both these values of h, h', are found to be insensible by observation [7441′].

made to discover them. The orbits of the first and second satellites do not therefore seem to have any excentricities peculiar to them; but they participate sensibly in the excentricities of the orbits of the third and fourth satellites, [7442]
[7467 lines 1, 2, &c.]. We have, in [7192 or 6241h], relative to the third value of g, or g_2, the excentricity peculiar to the orbit of the third satellite, [7442′]

$$h' = 0{,}2152920.h''; \qquad [7443]$$

now we have, as in [7394],

$$-2h'' = 1709'',05; \quad \text{or} \quad h'' = -854'',525. \qquad [7444]$$

Hence the equation of the centre of the second satellite, relative to this value of g, or g_2, is* [7444′]

$$\delta v' = 367'',95.\sin.(\circleddash' - \varpi''). \qquad [7445]$$

We have, relative to the fourth value of g, or g_3 [7195], [7445′]

$$h' = 0{,}0173350.h'''; \quad [7197] \qquad [7446]$$

but we have, as in [7310],

$$-2h''' = 9265'',56; \quad \text{or} \quad h''' = -4632'',78; \qquad [7447]$$

therefore the equation of the centre of the second satellite, corresponding to this value of g, is

$$\delta v' = 160'',62.\sin.(\circleddash' - \varpi'''). \qquad [7448]$$

If we substitute the values of m, m″, m‴ [7142—7145], in the expression of $\delta v'$ [6844], observing also that the law of the epochs gives, as in [7391],

$$\circleddash - \circleddash' = 200^{\circ} + 2\circleddash' - 2\circleddash'', \qquad [7449]$$

we shall obtain,†

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* (3640) Substituting h'' [7444] in [7443], we get,

$$-2h' = 1709'',05 \times 0{,}2152920 = 367'',95.$$

We also have, in like manner as in [6240s], $n't + \varepsilon' - g_2 t - \Gamma_2 = \circleddash' - \varpi''$; hence the [7445$a$]
expression of $\delta v'$ [7441] becomes as in [7445]. In like manner, from [7446, 7447], we obtain $-2h' = 9265'',56 \times 0{,}0173350 = 160'',62$; substituting this and $n't + \varepsilon' - g_3 t - \Gamma_3 = \circleddash' - \varpi'''$ [7445b]
[6240s], in [7441], we get $\delta v'$ [7448]; observing that $g_2 t + \Gamma_2$ [7445a], corresponds to the *third* satellite [7442′], whose perijove is in the longitude ϖ'' [6021s, 6205h]; and [7445b] corresponds to the *fourth* satellite [7445′], whose perijove has the longitude ϖ''' [6021s].

† (3641) We have, as in [6240p, q, s', 7449],

$$nt - n't + \varepsilon - \varepsilon' = \circleddash - \circleddash' = 200^{\circ} + 2\circleddash' - 2\circleddash''; \qquad [7450a]$$

$$n''t - n't + \varepsilon'' - \varepsilon' = -(\circleddash' - \circleddash''); \qquad [7450b]$$

$$n'''t - n't + \varepsilon''' - \varepsilon' = -(\circleddash' - \circleddash'''); \qquad [7450c]$$

$$2n't - 2Mt + 2\varepsilon' - 2E = 2\circleddash' - 2\Pi. \qquad [7450d]$$

$$\begin{array}{llr} \delta v' = - & 163'',29.\sin.\ (\theta'-\theta'') & 1 \\ & +11920'',67.\sin.2.(\theta'-\theta'') & 2 \\ + & 60'',96.\sin.3.(\theta'-\theta'') & 3 \\ + & 4'',83.\sin.4.(\theta'-\theta'') & 4 \\ + & 4'',66.\sin.5.(\theta'-\theta'') & 5 \\ + & 3'',66.\sin.6.(\theta'-\theta'') & 6 \\ - & 5'',28.\sin.\ (\theta'-\theta''') & 7 \\ + & 4'',62.\sin.2.(\theta'-\theta''') & 8 \\ + & 0'',59.\sin.\ (2\theta'-2\Pi). & 9 \end{array}$$ [7450]

[7451] We must connect these terms with the inequality $\delta v' = 69'',78.\sin.2(\theta-\theta')$ [6947, 7450a], or

[7452] $$\delta v' = 69^s,78.\sin.(4\theta'-4\theta''). \quad [7449]$$

The values of Q' [6861], corresponding to the different values of g, are,*

[7453] $Q' = 1,634693.h$; [corresponds to g]

[7454] $Q' = 2,488106.h'$; " g_1

[7455] $Q' = -0,662615.h''$; " g_2

[7456] $Q' = -0,055035.h'''$. " g_3

Hence it follows, that *the excentricity, appertaining to the first satellite, is*

Substituting $m = 0,173281$ [7142] and the expression [7450a], in [6844 lines 1, 2, 3], we obtain,

[7450e] $$+1204'',56.\sin.2(\theta'-\theta'')-9'',11.\sin.4(\theta'-\theta'')+1'',82.\sin.6(\theta'-\theta''),$$

for the terms depending on m; the expressions [6844 lines 4, 5] being neglected on account of their smallness. Again, by substituting $m''=0,884972$ [7144], and the expression [7450b], in [6844 lines 6—11], we get, for the terms depending on m'',

[7450f] $$-163'',29.\sin.(\theta'-\theta'')+10716'',11.\sin.2(\theta'-\theta'')+60'',96.\sin.3(\theta'-\theta'')$$
$$+13'',94.\sin.4(\theta'-\theta'')+4,66.\sin.5(\theta'-\theta'')+1,84.\sin.6(\theta'-\theta'').$$

Adding together the terms in [7450e, f], we get the terms in [7450 lines 1—6]. Now

[7450g] substituting $m'''=0,426591$ [7145] in [6844 lines 12, 13], we get [7450 lines 7, 8] respectively. The term in [6844 line 14] is neglected, on account of its smallness. Finally, the term in [6844 line 15] is the same as that in [7450 line 9], using [6240s'].

* (3642) Putting for brevity $\frac{1}{F} = \left(1+\frac{g}{3001300''}\right)^2$, we may reduce the expression of

[7453a] Q' [6861] to the form [7453b]. This may be reduced to the forms [7453c, d, e], by multiplying successively the factor Fh, without the braces, by $\frac{h}{h'}$, $\frac{h''}{h}$ or $\frac{h'''}{h}$, and dividing the factor within the braces by the same quantity respectively; hence we get,

more sensible in the eclipses of the second, than in those of the first. For the equation of the centre of the first, is $-2h.\sin.(\odot-\varpi)$ [7441e]; and although the coefficient $2h$ is greater than the value of Q' [7453], which corresponds to it, yet the motion of the second satellite being less swift, by one half, than that of the first, the inequality depending on Q' [7453, 6850] produces, in time, a greater variation in the eclipses of the second satellite than the [7457]

Remarkable results relative to the excentricities.

$$Q'=\left\{13{,}307450.\quad \mathrm{m}-4\,831907.\tfrac{h'}{h}.\mathrm{m}+4{,}133080.\tfrac{h'}{h}.\mathrm{m}''-1{,}511467.\tfrac{h''}{h}.\mathrm{m}''\right\}.Fh; \qquad [7453b]$$

$$=\left\{13{,}307450.\tfrac{h}{h'}.\mathrm{m}-4{,}831907.\quad \mathrm{m}+4{,}133080.\quad \mathrm{m}''-1{,}511467.\tfrac{h''}{h'}.\mathrm{m}''\right\}.Fh'; \qquad [7453c]$$

$$=\left\{13{,}307450.\tfrac{h}{h''}.\mathrm{m}-4{,}831907.\tfrac{h'}{h''}.\mathrm{m}+4{,}133080.\tfrac{h'}{h''}.\mathrm{m}''-1{,}511467.\quad \mathrm{m}''\right\}.Fh''; \qquad [7453d]$$

$$=\left\{13{,}307450.\tfrac{h}{h'''}.\mathrm{m}-4{,}831907.\tfrac{h'}{h'''}.\mathrm{m}+4{,}133080.\tfrac{h'}{h'''}.\mathrm{m}''-1{,}511467.\tfrac{h''}{h'''}.\mathrm{m}''\right\}.Fh'''. \qquad [7453e]$$

We must substitute, in all these expressions, the values of m, m″ [7142, 7144]. Then in [7453b], we must substitute the values of $\frac{h'}{h}=0{,}0185238$, $\frac{h''}{h}=-0{,}0034337$, and [7453f]
$g=606989'',9$ [7176—7178], and we shall obtain the expression of Q' [7453]. Again, substituting the values [7183—7185] in [7453c], we get [7454]; also substituting the values [7190—7192] in [7453d], we get [7455]. Lastly, by substituting the values [7195—7198] in [7453e], we get $Q'=-0{,}00553.h'''$; which differs a little from the expression given by the author in [7456]; however this difference is hardly sensible in the resulting inequality in [7462 line 2]. [7453g] [7453h]

We must now substitute, in the expression of Q' [7453—7456], the values of h, h', h'', h''', which are peculiar to the *first*, *second*, *third* and *fourth* satellites respectively; [7453i]
but the values of h, h' being insensible [7441′], may be neglected. The value of h'' [7460] being substituted in [7455], gives $Q'=566'',22$; hence the inequality [6851] becomes, by using the corresponding angle $g_2t+\Gamma_2$ [7442′], [7453k]

$$\delta v'=566'',22.\sin.(nt-2n't+\varepsilon-2\varepsilon'+g_2t+\Gamma_2). \qquad [7453l]$$

Now we have, in [6240p, s], $2nt-2n't+2\varepsilon-2\varepsilon'=2\odot-2\odot'$, $nt+\varepsilon-g_2t-\Gamma_2=\odot-\varpi''$; and by subtracting the second of these equations from the first, we get,

$$nt-2n't+\varepsilon-2\varepsilon'+g_2t+\Gamma_2=\odot-2\odot'+\varpi''; \qquad [7453m]$$

hence we obtain the inequality $\delta v'$ [7462 line 1]. In like manner the value of h''' [7461], with the corresponding angle $g_3t+\Gamma_3$ [7445′, &c.], being substituted in [7456], gives [7453n]
$Q'=254'',97$; and for this value of Q', the inequality [6851] becomes,

$$\delta v'=254'',97.\sin.(nt-2n't+\varepsilon-2\varepsilon'+g_3t+\Gamma_3). \qquad [7453o]$$

This is easily reduced to the form [7462 line 2], using the above values, and observing that $nt+\varepsilon-g_3t-\Gamma_3=\odot-\varpi'''$ [6240s]. [7453p]

[7458] equation of the centre of the first does in its eclipses.* *It is also curious to observe that the equation of the centre of the second satellite is more sensible* [7459] *by the inequality depending on* Q' *than by itself;* since its coefficient is $-2h'$ [7441], whilst that which depends on Q' is $2{,}488106.h'$ [7454].

We have, by what has been said in [7444, 7447],

[7460] $$-2h'' = 1709'',05; \quad \text{or} \quad h'' = -\ 854'',525;$$

[7461] $$-2h''' = 9265'',56; \quad \text{or} \quad h''' = -4632'',78;$$

the two inequalities depending on Q', relative to h'', h''', will therefore be, as in [7453*l*, *o*, &c.],

[7462] $$\delta v' = \ 566'',22.\sin.(\circleddash - 2\circleddash' + \varpi'') \qquad 1$$
$$+254'',97.\sin.(\circleddash - 2\circleddash' + \varpi'''). \qquad 2$$

The inequality in [6884],

[7463] $$\delta v' = -74'',98.\left\{1 - \frac{9a'\text{m}.\text{k}n^2}{8a\text{m}'.(M^2 - \text{k}n^2).\left(1 + \frac{9a'.\text{m}}{4a.\text{m}'} + \frac{a''.\text{m}}{4a.\text{m}''}\right)}\right\}.\sin.(Mt + E - I),$$

becomes, by substituting the values of m, m′, m″ [7142—7144],†

[7464] $$\delta v' = -111'',34.\sin.V;$$

[7464′] the other inequalities of the same article are insensible. Lastly, the coefficient of the equation of libration, relative to the second satellite, is, as in [7276],

[7465] $$-0{,}889912.P;$$

[7466] P being the coefficient of the similar inequality [7275], relative to the first satellite. *Hence it follows, that the inequality of libration must be most*

[7458*a*] * (3643) The mean motions of the satellites m, m', in the time t, are represented by nt, $n't$, respectively [6022*f*]. Hence the time required by the *first* satellite to describe the arc $2h$, which is the maximum value of its equation of the centre [7441*c*], is $\frac{2h}{n}$;
[7458*b*] and the time required by the *second* satellite, to describe the arc $Q' = 1{,}634693.h$ [7453], is $\frac{1{,}634693.h}{n'}$; now as n is nearly equal to $2n'$ [6151], these times are to each other as 1
[7458*c*] to 1,634693, or as 3 to 5 nearly; so that the time of describing Q', is considerably greater than that of describing $2h$.

† (3644) Substituting in [7463] the expression $Mt + E - I = V$, and the other values
[7464*a*] mentioned in [7400*a*], we get [7464]. The terms of $\delta v'$ depending on [6874] are insensible, as is observed in [6875*e*—*h*]; therefore they are neglected as in [7464′].

sensible in the motion of the second satellite; but it has not however been discovered by observation.* [7466′]

Connecting together all these inequalities, we shall have, in the eclipses of the second satellite,†

Longitude of the second satellite in eclipses, from the earth's moveable vernal equinox.

$$
\begin{array}{llr}
v' = \circleddash' + & 367'',95.\sin.\ (\circleddash' - \varpi'') & 1 \\
+ & 160'',62.\sin.\ (\circleddash' - \varpi''') & 2 \\
- & 163'',29.\sin.\ (\circleddash' - \circleddash'') & 3 \\
+ & 11920'',67.\sin.2.(\circleddash' - \circleddash'') & 4 \\
+ & 60'',96.\sin.3.(\circleddash' - \circleddash'') & 5 \\
+ & 74'',61.\sin.4.(\circleddash' - \circleddash'') & 6 \\
+ & 4'',66.\sin.5.(\circleddash' - \circleddash'') & 7 \\
+ & 3'',66.\sin.6.(\circleddash' - \circleddash'') & 8 \\
- & 5'',28.\sin.\ (\circleddash' - \circleddash''') & 9 \\
+ & 4'',62.\sin.2.(\circleddash' - \circleddash''') & 10 \\
+ & 566'',22.\sin.\ (\circleddash - 2\circleddash' + \varpi'') & 11 \\
+ & 254'',97.\sin.\ (\circleddash' - 2\circleddash'' + \varpi''') & 12 \\
- & 111'',34.\sin.V. & 13
\end{array}
$$ [7467]

We shall now consider the motion of the second satellite in latitude. The part,‡

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* (3645) Proceeding in the same manner as in [7458a—c], we find that the time of describing the arc P [7466], by the first satellite, is to that of describing the arc $P.0,889912$ [7465] by the second satellite, as $\frac{P}{n}$ to $\frac{P}{n'} \times 0,889912$; and as $n = 2n'$ nearly [7458c], this ratio is represented by that of $\frac{1}{2}$ to $0,889912$; so that the time of the beginning or end of an eclipse is affected nearly twice as much, by the libration of the *second* satellite, as by that of the *first* satellite. [7466a] [7466b]

† (3646) Connecting together the terms of $\delta v'$, which are computed in this chapter, and adding to them the value of $v' = \circleddash'$ at the epoch, we get the expression [7467], corresponding to the time of an eclipse of the second satellite, where we have $2\Pi = 2\circleddash'$ [7353d]. For the term [7445] is the same as [7467 line 1]; [7448] is the same as [7467 line 2]; [7450 lines 1, 2, 3] correspond respectively to [7467 lines 3, 4, 5]; the sum of the expressions in [7450 line 4, 7452] gives [7467 line 6]; the terms in [7450 lines 5, 6, 7, 8] correspond respectively to [7467 lines 7, 8, 9, 10]; the term [7450 line 9] vanishes, by putting $2\circleddash' = 2\Pi$ [7467b]; lastly, the terms in [7462, 7464] correspond respectively to the terms in [7467 lines 11, 12, 13]. [7467a] [7467b] [7467c] [7467d]

‡ (3647) In the original work, the second member of [7468] is given under the form $(1-\lambda').\theta'.\sin.(v'+pt+\text{A})$; which is changed in like manner as in the similar case for the [7468a]

[7468] $$s'=(\lambda'-1).\theta'.\sin.(v'+\Psi'),$$

of the expression of s' [6428 line 1], becomes, by the substitution of the values of λ', θ', Ψ',

[7469] $$s'=3^\circ{,}41507.\sin.(v'+51^\circ{,}3787-t.153'',8).$$

[7470] The term $s'=l'.\sin.(v'+p_1t+\text{A}_1)$, in the expression of s' [6428 line 3, 6425, &c.], corresponding to the peculiar inclination of the orbit of the second satellite, is given in [7338] under the form,

[7471] $$s'=-5152'',2.\sin.(v'+285^\circ{,}20_c+t.133715'',77).$$

We have, in [7227], relative to the first of the values of p [6428 line 2, 7226],*

[7472] $$l'=-0{,}0124527.l\,; \quad [7227]$$

but as no inclination of the orbit of the first satellite, or value of l, has been discovered by observation [7516'], we may neglect it. Again, with the third value of p, or p_2 [6428 line 4, 7238], we have,

[7473] $$\frac{l'}{l''}=0{,}1640530\,; \quad [7240]$$

and the term $s'=l'.\sin.(v'+p_2t+\text{A}_2)$ [6428 line 4], relative to the value p_2, becomes, by means of [7415],

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[7468b] third satellite, in [7413a]. Now proceeding as in [7413d], we may change [7468] into $s'=(1-\lambda').3^\circ{,}4352.\sin.(v'+51^\circ{,}3787-t.153'',8)$; and by substituting the value of λ' [7207], it becomes as in [7469].

[7472a] * (3648) The process here used for finding the terms of s' [7472—7477], is similar to that for the fourth satellite [7338—7340, &c.], changing v''', s''', l''', into v', s', l',

[7472b] respectively, where it may be necessary, and retaining the same values of the angles $pt+\text{A}$, $p_1t+\text{A}_1$, &c. Thus the expression s''' [7335d], changes into,

[7472c] $$s'=0{,}164053.l''.\sin.(v'+p_2t+\text{A}_2)\,;$$

observing that the numerical coefficient $-0{,}1965650.l''$ [7241], corresponding to the

[7472d] fourth satellite, must be changed into $0{,}164053.l''$ [7240], corresponding to the second satellite. Substituting the value of l'' [7335d] in s' [7472c], it becomes as in [7474']. In the same way we get the term [7477], corresponding to the angle $p_3t+\text{A}_3$; as we have done for the third satellite in [7416]; which, by changing v'', s'', l'', into v', s', l', respectively, becomes,

[7472e] $$s'=-2771'',6.\frac{l'}{l'''}.\sin.(v'+83^\circ{,}29861+t.7528'',01)\,;$$

[7472f] and by substituting the value of $\frac{l'}{l'''}=0{,}0234108$ [7247 or 7476], corresponding to the present case, which gives $-2771'',6.\frac{l'}{l'''}=-64'',88$, it becomes as in [7477].

$$s' = -2283'',9.\frac{l'}{l''}.\sin.(v'+208^\circ,32562+t.28220'',85)$$ [7474]

$$= -2283'',9\times 0,164053.\sin.(v'+208^\circ,32562+t.28220'',85)\,;$$ [7474']

therefore by reduction we have, in s', the following inequality,

$$s' = -374'',68.\sin.(v'+208^\circ,32562+t.28220'',85).$$ [7475]

The value of $\frac{l'}{l'''}$, corresponding to the fourth value of p, or p_3, [7245, 6428 line 5], is

$$\frac{l'}{l'''} = 0,0234108\,; \quad [7247]$$ [7476]

Therefore the term [6428 line 5] becomes, relative to this value of p,

$$s' = -64'',88.\sin.(v'+83^\circ,29861+t.7528'',01). \quad [7472e,f]$$ [7477]

We may neglect, without any sensible error, all the terms of s' [6932], except the following,* [7477']

$$s' = 0,00030736.(l'-L').\sin.(v'-2U-pt-\text{A}).$$ [7478]

This becomes, relative to the inclination of the orbit of Jupiter to that of the satellite,†

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* (3649) If we substitute the values of m, m″ [7142, 7144], in the terms of s' [6932 line 1], they become $s' = \{0,0005.(l-l')+0,0015.(l''-l')\}.\sin.(2v-3v'-pt-\text{A})$ nearly. Now if we take any one of the values of p, as for example the second [7233], we shall have very nearly $l = -105''$ [7522 line 3], $l' = -5150''$ [7482 line 4], $l'' = 176''$ [7427 line 4]. Substituting these in [7478a], it becomes $10''.\sin.(2v-3v'-pt-\text{A})$. The maximum of this inequality is an arc of 10″, which is described by the second satellite in about one centesimal second of time, as is evident from the time of its sideral revolution [6779]. We have selected this second value of p because it corresponds to some of the greatest numerical values of $l-l'$, $l''-l'$ [7478a], as appears by comparing the terms of s, s', s'' [7522, 7482, 7427], which correspond respectively to these angles. Thus if we had used the third value of p, or p_2 [7238], the resulting inequality [7478a] would be less than 3″ of space, or 0″,3 of time. Hence we see that this term may be neglected, as in [7477']; and then the expression of s' [6932] becomes as in [7478].
[7478a] [7478b] [7478c] [7478d] [7478e] [7478f]

† (3650) The terms of s' [7478] are of the same order as those which we have neglected in the preceding note; but the inequalities deduced from it, in [7479, 7480], can, in eclipses, be reduced to the same arguments as the large terms in [7482 lines 1, 4], without requiring any new arguments or tables; they are therefore taken into consideration and computed in the following manner. Proceeding as in [7423a, &c.], we find that the expression of s' [7478], may be derived from that of s''' [7341], by changing l''' into l', λ''' into λ', v''' into v', and the coefficient 0,001447815 into 0,00030736; observing also
[7479a] [7479b] [7479c]

[7479] $$s' = 10'',49.\sin.(v'-2U-51°,3787+t.153'',8);$$

and relative to the inclination of the second satellite,

[7480] $$s' = -1'',58.\sin.(v'-2U-285°,20_c-t.133715'',77).$$

[7481] Connecting together all these terms, we shall have, in eclipses of the second satellite, in which we may suppose $2U = 2v'$ [7353d],*

Latitude of the second satellite.

[7482]
$$\begin{aligned} s' = {} & 34140'',2.\ \sin.(v'+\ 51°,3787-t.153'',8) && 1 \\ & -\ 64'',88.\sin.(v'+\ 83°,29861+t.7528'',01) && 2 \\ & -\ 374'',68.\sin.(v'+208°,32562+t.28220'',85) && 3 \\ & -5150'',6.\ \sin.(v'+285°,20_c+t.133715'',77) && 4 \end{aligned}$$

To obtain the duration of the eclipses of the second satellite, we shall resume the formula [7080],

[7483] $$t = T.(1-X).\left\{-(1+\rho')^2.\frac{s'}{\beta}.\frac{ds'}{dv_1'} \pm \sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s'}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s'}{\beta}\right\}}\right\};$$

[7484] In this formula, T *is the half of the mean duration of the eclipses of the satellite in its nodes* [7069], where s' vanishes. Delambre has found this

[7485] semi-duration to be $T = 5975'',7$ [7563], by observations made since the discovery of achromatic telescopes; and we shall use this value of T. The

[7486] mean synodical motion of the second satellite, during the time T,

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[7479d] that the factor $(1-\lambda''').3°,4352 = 2°,98051$ [7423c] becomes for the second satellite

[7479e] $(1-\lambda').3°,4352 = 3°,41507$, as in [7469]. Now making the same changes in the expression of s''' [7346], we get the corresponding inequality in s', namely,

[7479f] $$s' = 3°,41507 \times 0,00030736.\sin.(v'-2U-51°,3787+t.153'',8);$$

which is easily reduced to the form [7479]. Again, in noticing the terms depending on the

[7479g] mutual attraction of the satellites, we may put $L' = 0$ [6415], and then [7478] becomes $s' = 0,00030736.l'.\sin.(v'-2U-pt-\Lambda)$. Substituting the value of $l' = -5150''$ [7482 line 4], and the corresponding angle $p_1t+\Lambda_1 = 285°,20_c+t.133715'',77$, it becomes,

[7479h] $$s' = -0,00030736 \times 5150''.\sin.(v'-2U-285°,20_c-t.133715'',77);$$

and by reduction we get [7480]. The other values of l' [7482] produce, in [7478], no terms of a sensible magnitude.

[7482a] * (3651) Substituting $2U = 2v'$ in [7479], and making a slight reduction, it becomes $s' = -10'',49.\sin.(v'+51°,3787-t.153'',8)$; connecting this with the term in [7469], it becomes as in [7482 line 1]. Again, by making the same substitution of $2U = 2v'$, in [7480], it becomes $s' = 1'',58.\sin.(v'+285°,20_c-t.133715'',77)$; connecting this with the

[7482b] term in [7471], we get that in [7482 line 4]. The term [7475] is the same as [7482 line 3]; and the term [7477] is the same as [7482 line 2].

is* $\beta = 67254'',2$; and we have, in this case,† $\rho' = 0,0718862$. The value [7486']
of X [7376'] becomes for this case $X = \frac{dv'}{n'dt} - 1$; so that if we notice only [7487]
the greatest terms of v', in which the argument differs but little from Θ', we shall have,‡

$$\begin{array}{lll} X = & 0,00057797.\cos.\ (\Theta'-\varpi'') & 1 \\ & +0,018861_{c}\ .\cos.2.(\Theta'-\Theta'') & 2 \end{array} \qquad [7488]$$

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* (3652) For the second satellite we have, in like manner as in [7430a], $(n'-M).T = \beta$; substituting $n'-M = 411075216''$ [6025n], and $T = 5975'',7$ [7485], or $T = 0^{year},0001636058$, it becomes as in [7486']. [7486a]

† (3653) The formula similar to [7431c], corresponding to the second satellite, is

$$\rho' = \frac{\rho.\left\{1 + \frac{a'}{a'''}.\sin.1530'',864\right\}}{1 - \frac{(1-\lambda_1)}{\lambda_1}.\frac{a'}{a'''}.\sin.1530'',864}. \qquad [7487a]$$

substituting the values $\frac{a'}{a'''}$, λ_1, ρ [6819, 7375a], we get $\rho' = 0,0718845$, being nearly as in [7486']. [7487b]

‡ (3654) If we retain only the two greatest terms of v' [7467 lines 1, 4], we shall get from [7487],

$$X = \frac{1}{n'dt}.d.\{367'',95.\sin.(\Theta'-\varpi'') + 11920'',67.\sin.2.(\Theta'-\Theta'')\}; \qquad [7488a]$$

and these produce respectively the two terms of X [7488]. For the first term of [7488a] is $367'',95.\frac{d.(\Theta'-\varpi'')}{n'dt}.\cos.(\Theta'-\varpi'')$; and if we subtract the value of ϖ'' [7408] from that [7488b]
of Θ' [7440], we get $\Theta'-\varpi'' = 2°,66864 + t.411383417''$; substituting this and n' [6025k] [7488c]
in the preceding expression, and then dividing by the radius in seconds $636620''$, it becomes $\frac{367'',95}{636620''}.\frac{411383417}{411412427}.\cos.(\Theta'-\varpi'')$, or $0,00057793.\cos.(\Theta'-\varpi'')$, as in [7488 line 1] [7488d]
nearly. Again, from [7386, 7440], we obtain $\Theta'-\Theta'' = 335°,09582 + t.207206791''$; using this and the preceding value of n', we find that the second term of [7488a] gives in [7488e]
X the term $\frac{11920'',67}{636620''}.2.\frac{207206791''}{411412427''}.\cos.2.(\Theta'-\Theta'')$; which, by reduction, becomes [7488f]
$0,018861.\cos.2.(\Theta'-\Theta'')$. This differs a little from that given by the author in the original work, $0,0187249.\cos.2.(\Theta'-\Theta'')$; we have given the corrected value with the letter $_c$ annexed, to denote that this correction has been made; we have also changed *sin.* into [7488g]
cos. in the formula [7488], as it was incorrectly given in the original work.

[7488′] In this theory, and in that of the first satellite, we may neglect, without any sensible error, the factor,*

[7489] $$1+\left\{\frac{2M}{n'-M}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'}{D'}\right\}.H.\cos.V,$$

[7490] which is contained in the value of T [7094]. Now putting $(1+\rho')\cdot\frac{s'}{\beta}=\zeta$, we shall have,†

[7491]
$$\begin{aligned}\zeta = {} & 0{,}544120_c.\sin.(v'+\ 51°{,}3787-t.153''{,}8) && 1\\ & -0{,}001034_c.\sin.(v'+\ 83°{,}29861+t.7528''{,}01) && 2\\ & -0{,}005971_c.\sin.(v'+208°{,}32562+t.28220''{,}85) && 3\\ & -0{,}082089_c.\sin.(v'+285°{,}20_c+t.133715''{,}77). && 4\end{aligned}$$

This being premised, we shall have,‡

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* (3655) The factor [7489] for the second satellite is easily deduced from that for the
[7489a] third satellite [7433], by changing n'' into n', also a'' into a'; and if we make these changes in the expression [7433b], we shall find that this factor will become, for the second satellite,

[7489b] $$1+\left\{\frac{2M}{n'-M}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'}{a'''}.\sin.1530''{,}864\right\}.H.\cos.V.$$

Substituting the values [7433c, 6819], it becomes very nearly equal to $1-0{,}00027.\cos.V$;
[7489c] and the part depending on $\cos.V$ produces in t [7492], a term of the order $1''.\cos.V$; which is so small that it may be neglected. It is also plain that the term depending on
[7489d] $\cos.V$ may be neglected for the *first* satellite, because each of the two terms of the expression [7489b] will be decreased nearly one half, by changing a' into a, and n' into n, as is evident from the values given in [6801, 6025b].

† (3656) The expression of ζ [7490] is similar to that which is assumed in [7434′, &c.]
[7491a] for the other satellites. Substituting in it the values of ρ', β [7486′], we get $\zeta=\frac{1{,}0718862}{67254''{,}2}.s'$; so that the value of ζ is obtained by multiplying the expression of s'
[7491b] [7482], by the factor $\frac{1{,}0718862}{67254''{,}2}$; hence we obtain [7491]. In computing these coefficients
[7491c] the author has neglected to multiply the numbers by the factor $1+\rho'=1{,}0718862$, so that the numbers given in the original work, namely $0{,}507629$, $-0{,}0009214$,
[7491d] $-0{,}005571$, $-0{,}076569$, are too small. We have given the corrected values in [7491], and have, as usual, annexed the small letter $_c$ to denote that they have been corrected.

‡ (3657) The numerical coefficient of the *first* term of [7492], is represented by $-T\beta$
[7492a] in [7436b], and that of the *second* by T; so that the expression of t, which is similar to [7436], becomes for the second satellite, when the term depending on $\cos.V$ is neglected [7488′],

$$t = -631'',29.\frac{\xi d\xi}{dv'} \pm 5975'',7.(1-X).\sqrt{1+X-\xi^2}. \qquad [7492]$$

The total duration t' of the eclipse will be, Duration of an eclipse.

$$t' = 11951'',4.(1-X).\sqrt{1+X-\xi^2}. \qquad [7493]$$

$$t = -T\beta.\frac{\xi d\xi}{dv'} \pm T.(1-X).\sqrt{1+X-\xi^2}. \qquad [7492b]$$

Substituting $T = 5975'',7$ [7485], also the value of β, in parts of the radius, namely $\beta = \frac{67254'',2}{636620''} = 0,10565$ [7486'], it becomes as in [7492]. The expression of t' [7493] is found as in [7437b], by multiplying the second term of [7492] by ± 2. The terms which are neglected in [7492] may be estimated as in [7437b–f]; thus the greatest neglected term is $\frac{1}{2}T\beta^2.b$ [7437c]; and by substituting the values of T, β [7492c], it becomes $33''.b$. Now the maximum value of b [7383m], is $\frac{1}{4}A^4$; A being, as in [7383l], the chief term of ξ, or nearly $A = \frac{1}{2}$ [7491 line 1]; hence this value of b is $b = \frac{1}{4}A^4 = \frac{1}{64}$ nearly, and the preceding term $33''.b$ is less than a second. The other terms, similar to those in [7383q, &c.], may evidently be neglected from what has been said in the notes on that article.

[7492c] [7492d] [7492e] [7492f]

CHAPTER XV.

THEORY OF THE FIRST SATELLITE.

33. The discussion of the eclipses of this satellite has given for its mean secular motion, relative to the vernal equinox of the earth, the following expression,

[7494] $$8258261^\circ{,}63035\,;$$

and for its mean longitude at the epoch of 1750, the value,

[7495] $$16^\circ{,}68093\,;$$

we shall therefore put,

[7496] $$\circleddash = 16^\circ{,}68093 + t.82582^\circ{,}6163035.$$

The different equations of the centre of the satellite are included in the term [7441c],

[7497] $$\delta v = -2h.\sin.(nt+\varepsilon-gt-\Gamma)\,;$$

and we have seen, in [7442], that it is only necessary to notice, in this
[7497'] theory, the third and fourth values of g. We have, in [7191], relative to the third value of g, or g_2,

[7498] $$h = 0{,}0238111.h''.$$

Now we have, as in [7394],

[7499] $$-2h'' = 1709'',05\,; \quad \text{or} \quad h'' = -854'',525\,;$$

therefore the equation of the centre of the first satellite, relative to the value of g or g_2, is*

[7499a] *(3658) Substituting h'' [7499] in [7498], we get $h = -20'',347$, or $-2h = 40'',69$ nearly. Using the value of g, namely g_2, in [7497], and putting, as in [6240s],
[7499b] $nt+\varepsilon-g_2t-\Gamma_2 = \circleddash-\varpi''$, it becomes as in [7500]. In a similar manner by substituting h''' [7502], corresponding to the fourth value of g or g_3, in [7501], we get $-2h = 19'',01$. Hence the expression [7497] becomes for this value of g as in [7503], by using the same

$$\delta v = 40'',69.\sin.(\circleddash - \varpi'').$$ [7500]

We have, relative to the fourth value of g [7195, 7196], [7500′]

$$h = 0,0020522_c.h''';$$ [7501]

and in [7310],

$$-2h''' = 9265'',56\,; \quad \text{or} \quad h''' = -4632'',78\,;$$ [7502]

therefore the equation of the centre, relative to this value of g or g_3, is as in [7499b], [7502′]

$$\delta v = 19'',01_c.\sin.(\circleddash - \varpi''').$$ [7503]

If we substitute, in the expression of δv [6842], the values of m′, m″, [7143, 7144], and neglect the terms depending on m‴*, using also the law of the epochs [7391],†

$$2\circleddash - 2\circleddash'' = 200^\circ + 3\circleddash - 3\circleddash',$$ [7504]

we shall obtain,‡

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reductions as above, and observing, as in [6240s], that for this case we have $nt+\varepsilon-g_3t-\Gamma_3 = \circleddash - \varpi'''$. We may remark that the correction, made in the coefficient 0,0020522$_c$ in [7183a], changes a little the coefficient of the inequality [7503], which, in the original work, is 19″,11 instead of 19″,01$_c$. [7499c] [7499d]

* (3659) The greatest coefficient of the terms depending on m‴, is that in [6842 line 11], namely, m‴.3″.6109; which, by using the value of m‴ [7145], becomes 1″,5; and as this may be neglected on account of its smallness, we may much more safely neglect the smaller terms, contained in [6842 lines 12, 13]. [7503a]

† (3660) Changing the signs of all the terms of [7504], and then adding to both members the expression $3\circleddash - 3\circleddash'$, it becomes, by a slight reduction, $\circleddash - 3\circleddash' + 2\circleddash'' = -200^\circ$; differing in the sign of the second member from the expression [6628g], which is used for the third and second satellites in [7391, 7449]. These last expressions correspond to the case of $i=0$, in the equation [6628q]; but in the equation [7504a or 7504], we have $i=-1$. [7504a] [7504b]

‡ (3661) We have, in [6240p], $n't - nt + \varepsilon' - \varepsilon = -(\circleddash - \circleddash')$; substituting this, and the value of m′ [7143], in [6842 lines 1—5], we obtain the following terms of δv, [7505a]

$$\delta v = -43'',55.\sin.(\circleddash - \circleddash') + 5050'',59.\sin.2.(\circleddash - \circleddash') + 16'',46.\sin.3.(\circleddash - \circleddash')$$
$$+ 3'',76.\sin.4.(\circleddash - \circleddash') + 1'',3.\sin.5.(\circleddash - \circleddash')\,;$$ [7505b]

the term in [6842 line 6] being neglected on account of its smallness. In like manner, by substituting $n''t - nt + \varepsilon'' - \varepsilon = -(\circleddash - \circleddash'')$ [6240p], and m″ [7144] in [6842 lines 7, 8], we obtain the two following terms of δv; the terms in [6842 lines 9, 10] being neglected on account of their smallness; [7505c]

[7505]
$$
\begin{aligned}
\delta v = &- 43'',56.\sin.(\circleddash - \circleddash') && 1\\
&\mp 19'',41_c.\cos.\tfrac{3}{2}.(\circleddash - \circleddash') && 2\\
&+5050'',59.\sin.2.(\circleddash - \circleddash') && 3\\
&+ 0'',07.\sin.3.(\circleddash - \circleddash') && 4\\
&+ 3'',76.\sin.4.(\circleddash - \circleddash') && 5\\
&+ 1'',58.\sin.5.(\circleddash - \circleddash') && 6
\end{aligned}
$$

The values of Q, corresponding to the different values of g, are,*

[7505d] $$\delta v = -19'',41.\sin.(\circleddash - \circleddash'') + 16'',39.\sin.2.(\circleddash - \circleddash'').$$

If we change the signs of all the terms of the equation [6628q], and then add $3\circleddash - 3\circleddash'$ to both members, we shall get generally, i being an integral number, positive, negative or zero,

[7505e] $$2\circleddash - 2\circleddash'' = (-200^\circ + 400^\circ.i) + 3\circleddash - 3\circleddash'.$$

Taking the sine of this expression and developing the second member by [21] Int., we get [7505f]; and in like manner the sine of one half of the same expression gives [7505g];

[7505f] $$\sin.(2\circleddash - 2\circleddash'') = \sin.(-200^\circ + 400^\circ.i).\cos.(3\circleddash - 3\circleddash') + \cos.(-200^\circ + 400^\circ.i).\sin.(3\circleddash - 3\circleddash');$$

[7505g] $$\sin.(\circleddash - \circleddash'') = \sin.(-100^\circ + 200^\circ.i).\cos.\tfrac{3}{2}.(\circleddash - \circleddash') + \cos.(-100^\circ + 200^\circ.i).\sin.\tfrac{3}{2}.(\circleddash - \circleddash').$$

[7505h] Now i being an integer or zero [6628p], we evidently have $\sin.(-200^\circ + 400^\circ.i) = 0$, $\cos.(-200^\circ + 400^\circ.i) = -1$; $\sin.(-100^\circ + 200^\circ.i) = \pm 1$, $\cos.(-100^\circ + 200^\circ.i) = 0$; hence the expressions [7505f, g] become,

[7505i] $$\sin.(2\circleddash - 2\circleddash'') = -\sin.(3\circleddash - 3\circleddash'); \qquad \sin.(\circleddash - \circleddash'') = \pm \cos.\tfrac{3}{2}.(\circleddash - \circleddash');$$

and by substituting these values in [7505d], it changes into,

[7505k] $$\delta v = \mp 19'',41.\cos.\tfrac{3}{2}.(\circleddash - \circleddash') - 16'',39.\sin.(3\circleddash - 3\circleddash').$$

[7505l] Connecting together the terms of δv [7505b, k], we get, very nearly, the same expression as in [7505]. For the first, second, fourth and fifth terms of [7505b], correspond respectively to those in [7505 lines 1, 3, 5, 6]; the third term of [7505b], being connected with the second of [7505k], gives [7505 line 4]; and the first term of [7505k] is the same

[7505m] as that in [7505 line 2]. The coefficient of this term in [7505 line 2], is given by the author, in the original work, in the form $-19'',41$, without noticing the double sign $\mp$, of which he seems not to have been aware; we have inserted the double sign, putting the

[7505n] usual mark $_c$ to denote that the coefficient has been altered. If this coefficient were important, it would be necessary to resort to the usual methods of investigating it, as its sign could not be determined by means of the argument $\circleddash - \circleddash'$; and the method of reducing the terms of δv [7505d] to the same form as those in [7505b], would fail. Fortunately,

[7505o] however, this correction is so small that it hardly affects the time of an eclipse $\frac{3}{4}$ of a second; so that Delambre neglects it in his tables.

[7506a] * (3662) Substituting $\frac{1}{F} = \left(1 + \frac{g}{3001300''}\right)^2$ [7453a], in the expression of Q [6860], it becomes of the form [7506b]; and by proceeding as in [7453a—c], we may deduce from it the expressions [7506c, d, e].

$$Q = -2,690499.h\,;$$ [7506]
$$Q = \quad 1,397738.h'\,;$$ [7507]
$$Q = \quad 0,208780.h''\,;$$ [7508]
$$Q = \quad 0,016482.h'''.$$ [7509]

*It is also remarkable that the excentricity of the orbit of the first satellite is more sensible in the inequality depending upon Q, than by its direct action.** [7510]
Substituting $h'' = -854'',525$ [7499], and $h''' = -4632'',78$ [7502], we shall have the two inequalities, [7510']

$$Q = F.\mathrm{m}'.\left\{-16,850204 \quad +6,118274.\frac{h'}{h}\right\}.h\,;$$ [7506b]
$$= F.\mathrm{m}'.\left\{-16,850204.\frac{h}{h'}+6,118274 \quad \right\}.h'\,;$$ [7506c]
$$= F.\mathrm{m}'.\left\{-16,850204.\frac{h}{h''}+6,118274.\frac{h'}{h''}\right\}.h''\,;$$ [7506d]
$$= F.\mathrm{m}'.\left\{-16,850204.\frac{h}{h'''}+6,118274.\frac{h'}{h'''}\right\}.h'''.$$ [7506e]

We must substitute $\mathrm{m}' = 0,232355$ [7143] in all these expressions. Then in [7506b] we must put $g = 606989'',9$ and $\frac{h'}{h} = 0,0185238$ [7176, 7177], and we shall get the value of Q, as in [7506] nearly. Again substituting the values of g_1, $\frac{h}{h'}$ [7183, 7184], in [7506c], we get [7507]; also substituting the values of g_2, $\frac{h}{h''}$, $\frac{h'}{h''}$ [7190, 7191, 7192] in [7506d], we get [7508]. Lastly, substituting g_3, $\frac{h}{h'''}$, $\frac{h'}{h'''}$ [7195, 7196, 7197] in [7506e], we get [7509]. [7506f] [7506g]

* (3663) The coefficient of the equation of the centre is $-2h$ [7497]; and that of the inequality [6850] is $Q = -2,690499.h$ [7506]; and this is greater than the numerical value of the preceding expression. The values of h, h' being insensible [7441'], we may neglect the inequalities depending on the values of Q [7506, 7507], as we have done for the second satellite in [7453i]. The value h'' [7510'] being substituted in [7508], gives $Q = -178'',41$; hence the expression [6850] becomes, [7510a] [7510b] [7510c]

$$\delta v = -178'',41.\sin.(nt-2n't+\varepsilon-2\varepsilon'+g_2t+\Gamma_2)\,;$$ [7510d]

which is easily reduced to the form [7511 line 1], by substituting the value [7453m]. Again, by substituting the value of h''' [7510'] in [7509], we get $Q = -76'',36$; hence the expression [6850] becomes, [7510e]

$$\delta v = -76'',36.\sin.(nt-2n't+\varepsilon-2\varepsilon'+g_3t+\Gamma_3)\,;$$ [7510f]

and by using the values [7453m, p], it is reduced to the form [7511 line 2].

[7511]
$$\delta v = -178'',41.\sin.(\Theta - 2\Theta' + \varpi'') \qquad 1$$
$$- \; 76'',36.\sin.(\Theta - 2\Theta' + \varpi'''). \qquad 2$$

[7512] As the inequality [6883] depending upon V is reduced to only $3''$, we may neglect it.*

[7512′] Connecting together all these inequalities, we shall have in eclipses, in which we may suppose $2\Pi = 2\Theta$ [7353e],†

Longitude of the first satellite in eclipses, from the earth's vernal equinox.

[7513]
$$\begin{array}{lll}
v = \Theta + & 40'',69.\sin.\;(\Theta - \varpi'') & 1 \\
\quad + & 19'',01_c.\sin.\;(\Theta - \varpi''') & 2 \\
\quad - & 43'',56.\sin.\;(\Theta - \Theta') & 3 \\
\quad \mp & 19'',41_c.\cos.\tfrac{3}{2}.(\Theta - \Theta') \;\; [\text{this term is } -19'',41.\sin.(\Theta-\Theta'')] & 4 \\
\quad + & 5050'',59.\sin.2.(\Theta - \Theta') & 5 \\
\quad + & 16'',26_c.\sin.4.(\Theta - \Theta') & 6 \\
\quad + & 1'',58.\sin.5.(\Theta - \Theta') & 7 \\
\quad - & 178'',41.\sin.\;(\Theta - 2\Theta' + \varpi'') & 8 \\
\quad - & 76'',36.\sin.\;(\Theta - 2\Theta' + \varpi'''). & 9
\end{array}$$

We shall now consider the motion of the first satellite in latitude. The part,‡

[7514]
$$s = (\lambda - 1).\theta'.\sin.(v + \Upsilon'),$$

of the expression of s [6427 line 1], becomes, by substituting the values of λ, θ', Υ',

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[7512a] * (3664) Substituting the values of the quantities mentioned in [7400a] in [6883], it becomes $\delta v = 3''.\sin.V$ nearly; this is so small that it is not worth the trouble of a new argument and another table to notice it; the author has therefore neglected it in [7512].

[7513a] † (3665) The term [6842 line 14, 6240s′] is $0'',1460.\sin.2.(\Theta - \Pi)$, which vanishes in eclipses where $2\Pi = 2\Theta$ [7512′]; we may also observe that this term is always insensible. Now connecting together the terms in [6947e, 7500, 7503, 7505, 7511], neglecting the very small term in [7505 line 4], and adding to the sum the longitude of the epoch Θ, we
[7513b] get v [7513]. We may observe that the author omitted the term in [6947e], making the coefficient in [7513 line 6] $+3'',76$, instead of the corrected value $16'',26_c$.

[7514a] ‡ (3666) In the original work the formula [7514] is given under the form $(1-\lambda).\theta'.\sin.(v + pt + \Lambda)$; we have altered it, so as to make it conform to [6427 line 1]; in
[7514b] the same manner as we have done for the second and third satellites, in [7468a, 7413a, &c.]; and by proceeding as in [7468b, 7413d], we may change the formula [7514] into
[7514c] $s = (1-\lambda).3°,4352.\sin.(v + 51°,3787 - t.153'',8)$. Now substituting the values of λ [7206], it becomes as in [7515].

$$s = 3°,43320.\sin.(v+51°,3787-t.153'',8).$$ [7515]

The term $s = l.\sin.(v+pt+\text{A})$, of the same expression of s [6427 line 2], [7516]
which corresponds to the peculiar inclination of the orbit of the first satellite, *has hitherto been found to be insensible.* We have, relative to the *second* [7516′]
and *third* values of p [7233, 7238],

$$l = 0,0207938.l' ; \quad [7234]$$ [7517]

$$l = 0,0111626.l''. \quad [7239]$$ [7517′]

The term $s = l.\sin.(v+pt+\text{A})$ [7516], becomes, relative to these values,*

$$\begin{aligned} s = &- 25'',49 .\sin.(v+208°,32562+t.28220'',85) ; & 1 \\ &-107'',14_c.\sin.(v+285°,20_c+t.133715'',77) ; & 2 \end{aligned}$$ [7518]

as it regards the other values of p, this term is insensible in eclipses [7518*d*, *e*]. We might therefore neglect, without any sensible error, all the terms of s which are contained in [6931].† We shall however retain the term in [6931 line 2], [7519]

$$s = 0,00015312.(l-L').\sin.(v-2U-pt-\text{A}).$$ [7520]

which becomes, relative to the inclination of the equator to the orbit of Jupiter,‡

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* (3667) Substituting in [7517] the value of $l' = -5152'',2$ [7471], corresponding to the *second* value of p [7233], we get $l = -107'',14$, producing in s [7516], the term $s = -107'',14.\sin.(v+285°,20_c+t.133715'',77)$, as in [7518 line 2]; observing that in the original work the coefficient is $-105'',04$, which we have changed into $-107'',14_c$. In like manner, by substituting in [7517′] the value of $l'' = -2283'',9$ [7415], corresponding to the *third* value of p [7238], we get $l = -25'',49$, producing in s [7516] the term [7518 line 1]. We may neglect the *first* value of p, because the term l [7516′] is insensible. We may also neglect the term depending on the fourth value of p [7245], and $l = 0,0019856.l'''$ [7246]; which, by using $l''' = -2771'',6$ [7329], gives $l = -5''$; which is so small that it is not worth the trouble of a new argument and another table, in order to notice it. [7518*a*] [7518*b*] [7518*c*] [7518*d*] [7518*e*]

† (3668) Substituting the value of m' [7143] in the coefficient of the term of s [6931 line 1], it becomes $s = 0,0008(l'-l).\sin.(3v-4v'-pt-\text{A})$; and as the greatest value of $l'-l$, corresponding to either of the angles p, in [7482, 7522], is about $5000''$, this inequality will not exceed $4''$, which is too small to be introduced by a new argument and a new table; and by neglecting it, the expression is reduced to the term in [6931 line 2], which is the same as [7520]. [7519*a*] [7519*b*]

‡ (3669) The expression of s [7520] is similar to that of s''' [7341], and may be derived from it by changing λ''' into λ, v''' into v, l''' into l, and 0,001447815 into [7521*a*]

[7521] $$s = 5'',26.\sin.(v-2U-51°,3787+t.153'',8);$$

[7521′] Connecting together all these terms [7515, 7518, 7521], we shall have in eclipses, in which we may suppose $2U=2v$ [7353e],

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[7522]
$$\begin{array}{lr} s = 34326',7.\sin.(v+51°,3787-t.153'',8) \quad [7521g] & 1 \\ \quad -\ 25'',49.\sin.(v+208°,32562+t.28220'',85) & 2 \\ \quad -107'',14_c.\sin.(v+285°,20_c+t.133715'',77). & 3 \end{array}$$

To obtain the duration of the eclipses of the first satellite, we shall resume the formula [7080],

[7523] $$t = T.(1-X).\left\{-(1+\rho')^2.\frac{s}{\beta}.\frac{ds}{dv_1} \pm \sqrt{\left\{1+\tfrac{1}{2}X+(1+\rho').\frac{s}{\beta}\right\}.\left\{1+\tfrac{1}{2}X-(1+\rho').\frac{s}{\beta}\right\}}\right\}.$$

In this formula, T *is the mean semi-duration of the eclipses of the satellite*
[7523′] *in its nodes* [7069]. Delambre has found this semi-duration to be $T=4713''$ [7562], by means of the observations which have been made since the use of achromatic telescopes. Therefore we shall assume this value
[7524] of T. Moreover β is the mean synodical motion of the satellite, during the time T [7373]; and we have* $\beta = 106516''$ [7524a]; also in this
[7525] case† $\rho' = 0,0716667$. The value of X, as in [7376′], is represented very

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[7521b] 0,00015312; observing also that the factor $(1-\lambda''').3°,4352 = 2°,98051$ [7323g or 7346], becomes for the first satellite $(1-\lambda).3°,4352 = 3°,43320$, as in [7514c, 7515]. Now
[7521c] making the same changes in the expression of s''' [7346], which is derived from s''' [7341], we get the corresponding term of s, namely,

[7521d] $$s = 3°,43320\times 0,00015312.\sin.(v-2U-51°,3787+t.153'',8),$$

which is easily reduced to the form [7521]. In eclipses, where $2U=2v$ [7353e], the
[7521e] expression [7521] becomes $s = -5'',26.\sin.(v+51°,3787-t.153'',8)$; and by connecting
[7521f] it with the term in [7515], it becomes as in [7522 line 1]. We may observe that the coefficient $5'',26$ [7521], ought to be reduced to about $3''$, as we have shown in [6930l′]; and this would increase the factor in [7522 line 1] by the small quantity 0,00023 nearly,
[7521g] making it $34329'',0_c$. The terms in [7522 lines 2, 3] are the same as those in [7518 lines 1, 2] respectively. As it regards the terms depending on the mutual action of the satellites, we may put $L'=0$ [6415], and then [7520] becomes,

[7521h] $$s' = 0,00015312.l.\sin.(v-2U-pt-\Lambda);$$

and by using the values of l [7518, &c.], it becomes insensible.

* (3670) We have, in like manner as in [7374b], $\beta = (n-M).T$; substituting
[7524a] $n-M = 825488799''$ [6025n], and $T = \frac{4713''}{36525000''} = 0,0001290349$ [7523′], in parts of a Julian year, it becomes $\beta = 106516''$, as in [7525].

[7526a] † (3671) Changing in [7487a] a' into a, we get the value of ρ', corresponding to the first satellite,

nearly by $X=\frac{dv}{ndt}-1$; so that by noticing only the greatest term of v, [7526]

we shall have very nearly,*

$$X=0,0079334.\cos.2.(\Theta-\Theta'). \qquad [7527]$$

Putting $\zeta=(1+\rho').\frac{s}{\beta}$, we shall have,† [7528]

$$\begin{aligned}\zeta = {} & 0,345364.\sin.(v+51^{\circ},3787-t.153'',8) & 1\\ & -0,000256.\sin.(v+208^{\circ},32562+t.28220'',85) & 2 \qquad [7529]\\ & -0,001078_{c}.\sin.(v+285^{\circ},20_{c}+t.133715'',77). & 3\end{aligned}$$

This being premised, we shall have,‡

$$t=-788'',55.\frac{\zeta d\zeta}{dv}\pm 4713''.(1-X).\sqrt{1+X-\zeta^2}; \qquad [7530]$$

$$\rho'=\frac{\rho.\left\{1+\frac{a}{a'''}.\sin.1530'',864\right\}}{1-\frac{(1-\lambda_1)}{\lambda_1}.\frac{a}{a'''}.\sin.1530'',864}. \qquad [7526b]$$

Substituting the values $\frac{a}{a'''}$, λ_1, ρ [6810, 7375a], we get ρ', as in [7525].

* (3672) Noticing only the great inequality of v, contained in [7513 line 5], namely $v=5050'',59.\sin.2.(\Theta-\Theta')$, we get, by substituting it in the expression of X [7526], [7527a]
$X=5050'',59.\frac{2d.(\Theta-\Theta')}{ndt}.\cos.2.(\Theta-\Theta')$. Now we have, from [7449, 7488e], [7527b]

$$d.(\Theta-\Theta')=2d.(\Theta'-\Theta'')=2\times 207206791''. \qquad [7527c]$$

Substituting this and n [6025k], we get, by dividing by the radius in seconds 636620'',

$$X=\frac{5050'',59}{636620''}\times\frac{4\times 207206791''}{825826010}.\cos.2.(\Theta-\Theta')=0,0079622.\cos.2.(\Theta-\Theta'); \qquad [7527d]$$

which differs a little from [7527]; but this difference has no sensible effect on the value of t [7530].

† (3673) From [7525] we get $\frac{1+\rho'}{\beta}=\frac{1,0716667}{106516''}$. Multiplying s [7522] by this factor, we get the expression of ζ [7528], as in [7529]. The coefficient of the third line [7529a]
of the original is $-0,001057$, instead of $-0,001078_c$; the difference arises from the correction in the coefficient of s [7522 line 3], spoken of in [7518b]. [7529b]

‡ (3674) The expression of t is easily deduced from that in [7492b], corresponding to the second satellite, by changing v' into v; hence we have,

$$t=-T\beta.\frac{\zeta d\zeta}{dv}\pm T.(1-X).\sqrt{1+X-\zeta^2}. \qquad [7530a]$$

Substituting $T=4713''$ [7523], and in parts of the radius $\beta=\frac{106516}{636620}$ [7525], or [7530b]

Duration of an eclipse.

and the whole duration of the eclipse will be,

[7531] $$t' = 9426''.(1-X).\sqrt{1+X-\xi^2}.$$

$T\beta = 788'',55$, it becomes as in [7530]. Multiplying the term of [7530], connected with the radical, by ± 2, we get, as in [7437b], the expression of t' [7531].

[7530c] The terms which are neglected in [7530], may be estimated as in [7492d, 7437b, &c.]; thus the chief term $\frac{1}{2}T\beta^2.b$ [7437c], by substituting $T = 4713''$ [7523'], and in parts of
[7530d] the radius $\beta = \frac{1}{6}$ nearly, becomes $65''.b$. The maximum value of b being $\frac{1}{4}.A^4$ [7383m]; and A is nearly equal to the chief term of ξ, which is about equal to $\frac{1}{3}$ [7529 line 1]; hence this maximum value becomes $b = \frac{1}{2}.(\frac{1}{3})^4 = \frac{1}{162}$; therefore the preceding term
[7530e] $65''.b$ is less than half a second at its maximum. The other terms may be neglected as in [7383q, &c.].

CHAPTER XVI.

ON THE DURATION OF THE ECLIPSES OF THE SATELLITES.

34. We have given, in [7107—7110], the expression of the sine of the angle q [7104], described by each satellite in half the time of the duration of the eclipse; supposing the planet to be at its mean distance from the sun, and the satellite in its node, and at its mean distance from Jupiter; also that [7532] the satellite is eclipsed at the moment its centre enters into the shadow of the planet. Dividing this angle q by the circumference 400°, and multiplying the quotient by the time of the synodical revolution of the satellite, we obtain the half time of the duration of the eclipse. The difference, between this and the observed time, represents the whole effect of the error of the preceding supposition, and of the errors of the other elements, which enter into the calculation. We shall now resume the expressions [7107—7110];

$$\frac{(1+\rho).R'}{a'''}\cdot\left\{\frac{a'''}{a}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q; \qquad [7533]$$

$$\frac{(1+\rho).R'}{a'''}\cdot\left\{\frac{a'''}{a'}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q'; \qquad [7534]$$

$$\frac{(1+\rho).R'}{a'''}\cdot\left\{\frac{a'''}{a''}-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q''; \qquad [7535]$$

$$\frac{(1+\rho).R'}{a'''}\cdot\left\{1-\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{a'''}{D'}\right\}=\sin.q'''. \qquad [7536]$$

$(1+\rho).R'$ being, as in [7018c, d], the semi-diameter of Jupiter's equator; [7537] we shall have, by means of [6785, 6786],

$$\frac{(1+\rho).R'}{a'''}=\frac{\frac{1}{2}.120'',3704}{1530'',864}. \qquad [7538]$$

We have, as in [6787], $a'''=25{,}43590$. This value of a''' is deduced

[7539] from the preceding equation, by putting $(1+\rho).R'$ equal to unity [6786]; therefore we shall have,

[7540] $$\frac{(1+\rho).R'}{a'''}.\,a'''=1;$$

hence we obtain,*

[7541] $$\frac{(1+\rho).R'}{a'''}.\frac{a'''}{a}=\frac{1}{a};$$

[7542] the value of a being that which is given in [6797]. Then we have, as in [7025],

[7543] $$\lambda_1=\frac{(1+\rho).R'}{R};$$

R being the sun's semi-diameter, when viewed from Jupiter.† Its diameter, when observed at the mean distance of the earth from the sun, is
[7544] $5936''=32^m,3^s,264$; therefore, when viewed from Jupiter, it is
[7545] $2R=\frac{5936''}{D'}$; D' being the mean distance of Jupiter from the sun [7085], that of the earth from the sun being taken for unity. Now we have, by means of
[7546] [6786, 7537], $2.(1+\rho).R'=120'',3704$; hence we get,‡

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* (3675) Multiplying [7540] by $\frac{1}{a}$, we get the expression of the first term of
[7541a] [7533], equal to $\frac{1}{a}$, as in the second member of [7541]. In like manner the first terms of [7534, 7535, 7536], are represented by $\frac{1}{a'}$, $\frac{1}{a''}$, $\frac{1}{a'''}$ respectively; using the values of a, a', a'', a''' [6797—6800].

[7544a] † (3676) In [6982] R is taken for the sun's actual semi-diameter, and in [7018c] R' is the polar semi-axis, also $2.(1+\rho).R'$ the equatorial diameter of Jupiter; but as only the ratio of R' to R occurs in [7543], under the form $\frac{R'}{R}$, we may substitute the apparent
[7544b] semi-diameters of these bodies as viewed from the same distance, as for example, the mean distance of Jupiter from the sun, which is the same as the mean distance of that planet from
[7544c] the earth; so that we may put, as in [7545], $2R=\frac{5936''}{D'}$; this being the sun's diameter as seen from Jupiter; and as in [6786 or 7546] $2.(1+\rho').R'=120'',3704$ equal to the
[7544d] equatorial diameter of Jupiter, as seen from the sun or from the earth, at the mean distance of Jupiter from either of these bodies.

[7547a] ‡ (3677) The expression of λ_1 [7543], may be put under the form, $\lambda_1=\{2.(1+\rho).R'\}.\frac{1}{2R}$; now substituting the values of $2.(1+\rho).R'=120'',3704$ [7546],

$$\lambda_1 = \frac{120'',3701 \,.\, D'}{5936''} = 0,105469. \qquad [7547]$$

Moreover we have,*

$$\frac{(1+\rho)\,.\,R'}{a'''} \,.\, \frac{a'''}{D'} = \frac{(1+\rho)\,.\,R'}{D'} = \tfrac{1}{2}.\sin.120'',3704 = 0,000945387. \qquad [7548]$$

Hence we deduce from [7547, 7548],

$$\frac{(1+\rho)\,.\,R'}{a'''} \,.\, \frac{a'''}{D'} \,.\, \frac{1-\lambda_1}{\lambda_1} = 0,000801823. \qquad [7549]$$

Therefore the four equations [7533—7536] become,†

$$\frac{1}{a} - 0,000801823 = \sin.q\,; \qquad [7550]$$

$$\frac{1}{a'} - 0,000801823 = \sin.q'\,; \qquad [7551]$$

$$\frac{1}{a''} - 0,000801823 = \sin.q''\,; \qquad [7552]$$

$$\frac{1}{a'''} - 0,000801823 = \sin.q'''. \qquad [7553]$$

$\frac{1}{2R} = \frac{D'}{5936''}$ [7545], we get $\lambda_1 = \frac{120'',3701.D'}{5936''}$, as in the first expression [7547]; and by [7547a'] using $D' = 5,20116636$ [4079], it becomes $\lambda_1 = 0,105469$, as in [7547]; whence $\frac{(1-\lambda_1)}{\lambda_1} = 8,481427.$ [7547b]

* (3678) The factor in the first member of [7548] being multiplied by $\frac{1-\lambda_1}{\lambda_1}$, [7548a] produces, in [7549], the numerical value of the second term of each of the equations [7533—7536]; and we have obtained, in [7547b], the value of $\frac{1-\lambda_1}{\lambda_1}$; also that of the [7548b] factor [7548] may be reduced to its second form $\frac{(1+\rho).R'}{D'}$, by rejecting a''', which occurs in its numerator and denominator. [7548c] Now $(1+\rho).R'$ being Jupiter's equatorial semi-diameter [7544a], and D' the mean distance of Jupiter from the earth, the expression $\frac{(1+\rho).R'}{D'}$ will express nearly the sine of half the semi-diameter, or very nearly half the sine of the whole diameter 120'',3704 [7544c], as in [7548]. [7548d] Multiplying this by $\frac{1-\lambda_1}{\lambda_1}$ [7547b], we get [7549].

† (3679) The first terms of the equations [7533—7536] are given in [7541a], and the second term in [7549]; substituting these in [7533—7536], we get [7550—7553] [7550a] respectively. Substituting the values of a, a', a'', a''' [6797—6800], we get the values of q, q', q'', q''' [7554—7557].

Substituting the values of a, a', a'' a''' [6797—6800], we obtain,

[7554] $$q = 111780'';$$

[7555] $$q' = 69846'';$$

[7556] $$q'' = 43544'';$$

[7557] $$q''' = 24524'';$$

which give the following values for the half duration of the eclipses ;*

[7558] I. Satellite, 4945″,87 ;

[7559] II. Satellite, 6205″,93 ;

[7560] III. Satellite, 7801″,30 ;

[7561] IV. Satellite, 10271″,64.

The observed semi-durations are,

[7562] I. Satellite, 4713″ ; [7524]

[7563] II. Satellite, 5976″ ; [7485]

[7564] III. Satellite, 7419″ ; [7429]

[7565] IV. Satellite, 9890″. [7372]

They are all of them less than the calculated semi-durations, which ought to be the case on account of the disks of the satellites; for although these disks are small, they are of a sensible magnitude, when viewed from Jupiter's centre. A satellite does not therefore disappear at the moment of the [7566] entrance of the centre of its disk into Jupiter's shadow, and the semi-duration of the eclipse is diminished by the time which elapses between the entrance of the centre, and the disappearance of the satellite. It may also be decreased by the refraction of the sun's light, in passing through Jupiter's atmosphere; but it is increased by the penumbra. These various causes are not however sufficient to account for the difference between the observed and calculated semi-durations. This will appear by considering the eclipses

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[7558a] * (3680) The synodical motion of the first satellite is $n-M$ seconds [6025n] in one Julian year, or 36525000″ ; hence the time of describing a synodical arc of one second is $\frac{36525000''}{n-M}$. Multiplying this by the arc q [7554], we get the time [7558]. In like [7558b] manner, the time [7559] is represented by $\frac{36525000''}{n'-M}\cdot q'$; that in [7560] by $\frac{36525000''}{n''-M}\cdot q''$; lastly, that in [7561] by $\frac{36525000''}{n'''-M}\cdot q'''$. Substituting the values of $n-M$, &c. [6025n], [7558c] we get the values [7558—7561], as given by the author very nearly.

of the first satellite, in which the effects of the penumbra, and of the refracted light in Jupiter's atmosphere, are but of small moment. To obtain the width of the disk, viewed from Jupiter's centre, we shall suppose the density [7567] of the satellite to be the same as that of the planet; then taking, for unity, the semi-diameter of Jupiter, we shall find that the apparent semi-diameter of the satellite, viewed from Jupiter's centre, will be equal to* $\frac{\sqrt[3]{m}}{a}$. [7568] Substituting the values of a, m [6797, 7162], we shall obtain $2890'',93$, for this semi-diameter. This angle being multiplied by the time of the synodical revolution of the satellite, and the quotient divided by 400°, gives $127'',913$ [7569] for the diminution of the semi-duration of the eclipse, depending upon the magnitude of its disk. Subtracting this quantity from $4945'',870$ [7558], [7570] we get $4817'',957$ for the calculated semi-duration. This semi-duration is greater than the observed time [7562], and yet there is reason to believe that the satellite disappears before it is wholly immersed in the shadow. [7571]

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* (3681) Jupiter's mass being 1, and that of the satellite m, their radii will be as [7568a] $1 : \sqrt[3]{m}$, supposing their densities to be equal; so that Jupiter's radius being 1 [6786], that of the satellite will be $\sqrt[3]{m}$. Dividing this by the distance of the satellite from Jupiter a, we obtain very nearly the sine of the angle under which that radius or half-diameter [7568b] appears when viewed from Jupiter's centre $=\frac{\sqrt[3]{m}}{a}$, as in [7568]. Substituting the values of a, m [6797, 7162], we find that it corresponds to $\sin.2890'',93$, as in [7568]; observing that, in the original work, this semi-diameter is erroneously marked $2809'',93$; and the [7568c] whole diameter is given in [7574] equal to $5619'',86$, instead of $5781'',86$. The time of [7568d] describing the arc $2890'',93$, by the synodical motion, may be obtained by the method pointed out in [7569], or by multiplying it by the factor $\frac{36525000''}{n-M}$ [7558b], which gives [7568e] $127'',913$ for the diminution of the semi-duration of the eclipse, as in [7569]; and by [7568f] doubling it we get $255'',826$ for the time requisite for the whole disk to enter into the shadow, as in [7574]. Proceeding in the same manner with the other satellites, we have, as in [7568b], for their diameters, the expressions $2.\frac{\sqrt[3]{m'}}{a'}$, $2.\frac{\sqrt[3]{m''}}{a''}$, $2.\frac{\sqrt[3]{m'''}}{a'''}$; and by [7568g] using the values of a', a'', a''' [6798—6800], and those of m', m'', m''' [7163—7165], [7568h] they become as in the first column of the table in [7575—7577]. Multiplying these [7568i] expressions, as in [7568e], by $36525000''$, and then dividing the products by $n'-M$, $n''-M$, $n'''-M$ [6025n] respectively, we get very nearly the times in the second column [7568k] of the table [7575—7577].

[7572] Hence it appears that we must decrease, by at least $\frac{1}{50}$th part, the assumed diameter of Jupiter 120″,3704 [6786], so as to reduce it to 118″.

[7573] If we compute, in the same manner, the disks of the satellites viewed from Jupiter's centre, and the times they require to enter perpendicularly into the shadow, we shall obtain the following results.

		Disks of the Satellites, viewed from Jupiter's centre, at their mean distances from the planet.	Times required for the whole disk to enter into the shadow.
[7574]	I. Satellite,	5781″,86	255″,826;
[7575]	II. Satellite,	4007″,30	356″,055;
[7576]	III. Satellite,	3923″,44	702″,914;
[7577]	IV. Satellite,	1749″,04	732″,567.

[7578] Hence it is easy to determine the times when the satellites, or their shadows, enter into or quit the disk of Jupiter. Comparing these times with those which are observed, we shall obtain the densities of the satellites of Jupiter, when their masses shall be well known. The observations of these eclipses of Jupiter by his satellites, may throw much light upon their theory. We can almost always observe the beginning and end of them; and these observations can be made with the satellites and with their shadows, so that they are really equivalent to four observations; whilst, in general, we cannot [7579] observe more than the beginning or the end of an eclipse of a satellite. This kind of observations, which has been very much neglected, deserves therefore the utmost attention of astronomers.

CHAPTER XVII.

ON THE SATELLITES OF SATURN.

35. THE theory of the satellites of Saturn is very imperfect, since we have not a sufficient number of observations to determine their elements. The impossibility of observing their eclipses, and the difficulty of measuring their elongations from Saturn, have prevented the determination, with much accuracy, of any of the elements, except the times of revolution and the mean distances; and even with respect to these distances there is a degree of uncertainty, which leaves some doubt upon the value which results for [7580]
the mass of this planet. Now as we do not know the values of the excentricities of the orbits of these satellites, it is impossible to give the [7580′]
theory of their perturbations. *But there is one phenomenon which deserves the attention of mathematicians and astronomers, namely, the constant position* [7581]
of the orbits of the satellites in the plane of the ring; excepting, however, the plane of the orbit of the outer satellite, which varies sensibly from the plane of the ring. [7581′]
This phenomenon is analogous to that which we have explained, in the last chapter of the fifth book [3689, &c.], *relative to the permanency of Saturn's rings in the same plane.* We have already observed, in [3689, 3692, &c.], that *both these phenomena depend on the same cause, namely, the oblateness of Saturn; whose action keeps the rings and the satellites in the plane of its equator.* [7582]
We shall now explain the reason why the orbit of the outer satellite varies from that plane in a very sensible manner.

We shall resume the equation [6295],

$$0=\frac{dds}{dv^2}+s-\frac{r^2}{h^2}\cdot\left\{\left(\frac{dR}{dv}\right)\cdot\frac{ds}{dv}-\left(\frac{dR}{ds}\right)\right\}. \qquad [7583]$$

If we neglect the ellipticity of the orbit, we shall have $r=a$, *and* $h^2=a$, [7584]

[7584′] [6299a]. *Moreover, if we take the primitive orbit of the satellite for the fixed plane, s will be of the order of the disturbing forces; therefore by neglecting*
[7585] *the square of these forces, we may reject the products of* s *and* $\frac{ds}{dv}$ *by these forces.* Hence the equation [7583] becomes,*

[7586] $$0=\frac{dds}{dv^2}+s+a.\left(\frac{dR}{ds}\right).$$

We shall suppose that this equation is for the outer satellite of Saturn, and we shall proceed to ascertain the corresponding value of R. In the first place we shall have, by noticing only the sun's action,†

Part of R depending on the sun.

[7587] $$R=-\frac{S}{D}+\frac{Sr^2}{2D^3}-\frac{3S.\{xX+yY+zZ\}^2}{2D^5}.$$

We shall use the following symbols:

[7587′] The axis of x is the line drawn from the centre of Saturn to the *ascending* node of the primitive orbit of the satellite upon the orbit of Saturn;

[7588] λ is the inclination of the primitive orbit of the satellite to the orbit of Saturn;

[7588′] X', Y', are the rectangular co-ordinates of the sun's place, referred to the centre of Saturn, and to the relative orbit of the sun about Saturn; the axis of X being the same as that of x;

[7588″] U is the angle included between the sun's radius vector D, and the axis x or X.

[7586a] * (3682) In this case R and $\left(\frac{dR}{dv}\right)$ [6030, 6039, 6042, &c.], are each of the order of the disturbing masses; hence $\left(\frac{dR}{dv}\right).\frac{ds}{dv}$, is of the order of the square of these masses; and by rejecting it from [7583], we obtain [7586].

† (3683) The expression of R, corresponding to the action of the sun S upon the
[7587a] outer satellite m of Saturn, as it is given in [7587], is of exactly the same form as that in [6042e], relative to the sun's action upon Jupiter's satellite m; the symbols being the same
[7587b] as in [6021—6033], *merely changing what relates to Jupiter and its satellite into the corresponding quantities relative to Saturn and its satellite* respectively. Thus X, Y, Z
[7587c] [6026], in the present article, represent the rectangular co-ordinates of the sun, referred to the centre of gravity of Saturn as their origin, supposing this centre to be at rest, and the sun to describe a relative orbit about Saturn. In like manner, x, y, z [6022] represent
[7587d] the rectangular co-ordinates of the outer satellite m, referred to the centre of gravity of Saturn as their origin, &c.

Then we have,*

$$X=X';$$ [7589]

$$Y=Y'.\cos.\lambda;$$ [7590]

$$Z=-Y'.\sin.\lambda.$$ [7591]

$$X'=D.\cos.U;$$ [7592]

$$Y'=D.\sin.U;$$ [7593]

$$x=a.\cos.v; \qquad y=a.\sin.v; \qquad z=as;$$ [7594]

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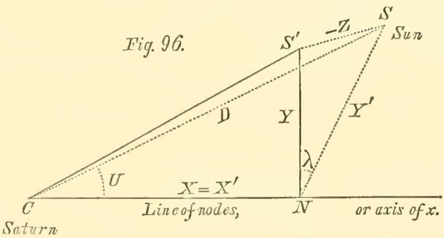
* (3684) We shall suppose in the annexed figure, that C represents the place of the centre of the planet Saturn, which is considered as the origin of the co-ordinates. The plane of the figure is the primitive orbit of the outer satellite, which is taken for the fixed plane of xy, [7584′]; the axis of x being the line CN, drawn from the centre of Saturn to the *ascending* node of the primitive orbit above the sun's relative orbit about Saturn, being the same as the *descending node of the sun's orbit* below the fixed plane of xy [7587′]. S is the place of the sun, whose rectangular co-ordinates are represented, as in [7587c], by [7589a] [7589b] [7589c] [7589d]



$$CN=X, \qquad NS'=Y, \qquad S'S=-Z;$$ [7589e]

the negative sign being prefixed to Z, because the line NS is supposed to fall *below* the plane of xy [7589d]. We have also, as in [7588′], [7589f]

$$CN=X', \qquad NS=Y', \qquad \text{angle } SNS'=\lambda.$$ [7589g]

Now in the rectangular triangle $NS'S$, we have $NS'=NS.\cos.SNS'=Y'.\cos.\lambda$; [7589$h$] $SS'=NS.\sin.SNS'=Y'.\sin.\lambda$. Substituting these in the values of Y, $-Z$ [7589e], we get [7590, 7591] respectively. The equation [7589] is obtained by putting the two expressions of CN [7589e, g] equal to each other. Again we have $CS=D$ [7589i] [6027, 7587b], angle $SCN=U$ [7588″]; hence in the rectangular triangle CNS, we have,

$$CN=CS.\cos.SCN=D.\cos.U; \qquad NS=CS.\sin.SCN=D.\sin.U;$$ [7589k]

and by substituting the values of CN, NS [7589g], we get [7592, 7593] respectively. If we substitute the values of X', Y' [7592, 7593] in [7589—7591], we shall get, [7589l]

$$X=\quad D.\cos.U;$$ [7589m]

$$Y=\quad D.\sin.U.\cos.\lambda;$$ [7589n]

$$Z=-D.\sin.U.\sin.\lambda.$$ [7589o]

Finally, the values of x, y, z [7591], are easily deduced from [6034—6036], by rejecting terms of the order s^2 [7585], and substituting $r=a$, $r'=a'$ [7584 or 6091]; neglecting the excentricities, which are unknown [7580′]. [7589p]

[7595] *therefore, by retaining in* $a.\left(\frac{dR}{ds}\right)$ *no other terms than those depending on* $\sin.v$ *or* $\cos.v$, *which are the only ones producing the secular motions of the orbit* [6298],* we shall have,

[7596]
$$a.\left(\frac{dR}{ds}\right)=\frac{3S.a^3}{2D^3}.\sin.\lambda.\cos.\lambda.\sin.v.$$

To determine the part of $a.\left(\frac{dR}{ds}\right)$, *which depends on the oblateness of Saturn*, we shall observe that this part of R is found, in [6050, 6052], to be,†

Part of R depending on the oblateness of Saturn.

[7597]
$$R=(\rho-\tfrac{1}{2}\varphi).(\mathrm{v}^2-\tfrac{1}{3}).\frac{\mathrm{M}B^2}{r^3}.$$

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* (3685) If we substitute the values of x, y, z, X, Y, Z [7594, 7589m—o] in the first member of [7596a], it becomes as in its second member; and its partial differential relative to s being divided by ds, gives [7596b].

[7596a]
$$\{xX+yY+zZ\}^2=a^2.D^2.\{\cos.U.\cos.v+\sin.U.\cos.\lambda.\sin.v-s.\sin.U.\sin.\lambda\}^2;$$

[7596b]
$$\frac{d.\{xX+yY+zZ\}^2}{ds}=-2a^2.D^2.\sin.U.\sin.\lambda.\{\cos.U.\cos.v+\sin.U.\cos.\lambda.\sin.v-s.\sin.U.\sin.\lambda\}.$$

[7596c] Now if we put $r=a$ [7584] in the second term of [7587], and then substitute this value of R in the first member of [7596], we shall find that the first and second terms of [7587] produce nothing, and the third term becomes like the first member of [7596b], multiplied by $-\frac{3S.a}{2D^5}$; therefore the second member of [7596b], being multiplied by the same factor, gives,

[7596d]
$$a.\left(\frac{dR}{ds}\right)=\frac{3S.a^3}{D^3}.\sin.U.\sin.\lambda.\{\cos.U.\cos.v+\sin.U.\cos.\lambda.\sin.v-s.\sin.U.\sin.\lambda\}.$$

[7596e] In noticing the terms depending on the secular equations, *it will only be necessary to retain, as in* [6298], *the terms depending on* $\sin.v$, $\cos.v$, s. We may neglect the term depending on s [7596d], because it is of the order of the disturbing forces [7585], and is multiplied by the factor $\frac{3S.a^3}{D^3}$, of the same order. We may also neglect the first term of

[7596f] the second member of [7596d], containing the factor $\sin.U.\cos.U.\cos.v=\frac{1}{2}.\sin.2U.\cos.v$, producing angles of the form $v\mp 2U$, which differ from $\sin.v$, $\cos.v$ [7596e]. The

[7596g] remaining term of [7596d] is the second, $\frac{3S.a^3}{D^3}.\sin.\lambda.\cos.\lambda.\sin.v.\sin.^2U$; and by substituting $\sin.^2U=\frac{1}{2}-\frac{1}{2}.\cos.2U$, and neglecting the term containing $2U$, it becomes as in [7596].

† (3686) Substituting $R=-\delta V$ [6052] in [6050], we get [7597]; M being the

[7597a] mass of Saturn [6028, 7587b]; $B=$ the radius of the equator of Saturn [6045]; v $=$ the sine of the declination of the outer satellite above Saturn's equator [6045′, 7587b].

M = the mass of Saturn, which is taken for unity, or $\mathrm{M}=1$; [7598]

B = the mean radius of Saturn's mass, which is taken for unity, or $B=1$; [7598′]

Symbols.

γ = the inclination of the primitive orbit of the outer satellite to the plane of the ring; [7599]

Ψ = the distance of the *descending* node of the primitive orbit of the outer satellite, relative to the plane of the ring, from the *ascending* node of the same primitive orbit upon the orbit of Saturn; the *first* of these nodes being supposed more advanced, according to the order of the signs, than the *second*; [7600]

$v-\Psi$ = the distance of the outer satellite from the *descending* node of its orbit upon the ring. [7600′]

Then we easily find, by neglecting the square of s,*

$$\mathrm{v}^2=\sin.^2\gamma.\sin.^2(v-\Psi)-2s.\sin.\gamma.\cos.\gamma.\sin.(v-\Psi). \qquad [7601]$$

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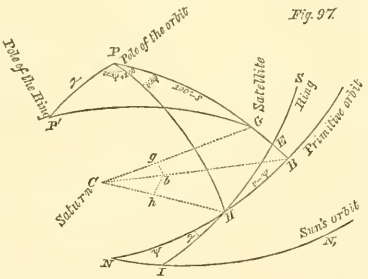
* (3687) In the annexed figure 97, C represents the centre of Saturn, about which is drawn the spherical surface $NIHBGPP'$. The plane of the sun's relative orbit about Saturn intersects this surface in the arc $NIN_{,}$; the plane of the primitive orbit of the outer satellite intersects this surface in the arc NHB; and the plane of the rings intersects it in the arc $IHES$; P is the pole of the orbit NHB; and P' the pole of the plane of the rings IHE. Then from the above notation we have, by supposing G to be the place of the outer satellite, [7601a] [7601b] [7601c] [7601d]



$NB=v$; $NH=\Psi$; $HB=v-\Psi=$ angle BPH, or GPH; [7601e]

$PP'=\gamma=$ angle EHB; $BG=s$; $PG=100^\circ-s$; [7601f]

angle $GPP'=GPH+100^\circ=v-\Psi+100^\circ$; $P'G=100^\circ$-dec.; $\cos.P'G=\mathrm{v}$ [6045′]. [7601g]

Then in the spherical triangle PGP', we have, from [63] Int., the formula [7601h]; which, by using the symbols [7601e—g], become as in [7601i],

$$\cos.P'G=\cos.PG.\cos.PP'+\sin.PG.\sin.PP'.\cos.GPP'; \qquad [7601h]$$

$$\mathrm{v}=\sin.s.\cos.\gamma-\cos.s.\sin.\gamma.\sin.(v-\Psi). \qquad [7601i]$$

Neglecting terms of the order s^2, as in [7589p], we may put $\sin.s=s$, $\cos.s=1$ [43, 44] Int.; substituting these in [7601i], then squaring the result, and neglecting s^2, we get [7601].

Hence we get,*

[7602] $$a.\left(\frac{dR}{ds}\right) = -\frac{2.(\rho-\frac{1}{2}\varphi)}{a^2}.\sin.\gamma.\cos.\gamma.\sin.(v-\Upsilon).$$

It now remains to consider the action of the rings and the six inner satellites.

[7603] If we consider an inferior satellite, whose radius is r' and mass m', supposing its orbit to be in the plane of the ring, or in that of the equator of Saturn, we shall have, as in [6030 line 1], relative to this satellite,

[7604] $$R = \frac{m'.(xx'+yy'+zz')}{r'^3} - \frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}.$$

[7605] If we now take, for the axis of x, the intersection of the plane of the primitive orbit with that of Saturn's equator or ring, we shall have,†

[7602a] * (3688) Putting $r=a$ [7589p], $M=1$, $B=1$ [7598, 7598'], in [7597], then taking its differential relative to s, which is found in v^2 [7601], we get,

[7602b] $$a.\left(\frac{dR}{ds}\right) = \frac{(\rho-\frac{1}{2}\varphi)}{a^2}.\left(\frac{d.(v^2)}{ds}\right).$$

Substituting the value of v^2 [7601], we obtain [7602].

[7606a] † (3689) The change of the axis of x from the line CN [7587'], in fig. 96, page 317, to the line CH, fig. 97, page 319 [7605], has no effect on the function R [7604]; as we have seen in [949', &c.], where it is proved that this function R is wholly independent

[7606b] of the planes of x, y, z; but the values of x, y, z, will differ from those in [7594]. To obtain the values corresponding to the present case, we shall draw the lines CH, CB, CG, and upon CG shall take $Cg=r$. From g let fall upon CB the perpendicular

[7606c] gb; and from b let fall upon CH the perpendicular bh; then we shall have $x=Ch$, $y=hb$, $z=bg$. Now in the triangle Cgb, we have $bg=Cg.\sin.GCB=r.\sin.GCB=rs$

[7606d] nearly, as in [7608]; also $Cb=Cg.\cos.GCB=r.\cos.GCB$, or $Cb=r$, neglecting terms of the order s^2 [44] Int. Then in the triangle Chb we have $Ch=Cb.\cos.BCH$,

[7606e] $hb=Cb.\sin.BCH$, or in symbols $x=r.\cos.(v-\Upsilon)$, $y=r.\sin.(v-\Upsilon)$, as in [7606, 7607].

[7606f] We may find the values of x', y', z' [7609—7611], corresponding to the satellite m', in the same manner as we have found those of X, Y, $-Z$, for the sun [7589m—o],

[7606g] merely changing the sign of Z, because the satellite m' is supposed to be at the point S, on the plane of the rings, above the plane of the primitive orbit x, y. In this case we have

[7606h] the arc $HS=v'$ [7612], the angle $SHB=\gamma$, and the radius vector of the satellite equal to r'. Hence it is evident that we may derive the values of x', y', z' [7609–7611],

[7606i] from those of X, Y, $-Z$ [7589m—o], by changing D, U, λ, into r', v', γ, respectively.

$$x = r.\cos.(v-\Psi);$$ [7606]

$$y = r.\sin.(v-\Psi);$$ [7607]

$$z = r.s;$$ [7608]

$$x' = r'.\cos.v';$$ [7609]

$$y' = r'.\cos.\gamma.\sin.v';$$ [7610]

$$z' = r'.\sin.\gamma.\sin.v';$$ [7611]

v' being the angular distance of the satellite m' from the descending node of the primitive orbit upon the plane of the ring. If we change r, r' into [7612] a, a' respectively, in $a.\left(\frac{dR}{ds}\right)$, rejecting the terms which do not depend [7613] upon $\sin.v$, $\cos.v$, or those which are multiplied by s [7596e], we shall have,* [7614]

Action of the satellites.

$$a.\left(\frac{dR}{ds}\right) = -\frac{m'.a^2a'.\sin.\gamma.\sin.v'}{\{a^2+a'^2-2aa'.\cos.(v-\Psi).\cos.v'-2aa'.\cos.\gamma.\sin.(v-\Psi).\sin.v'\}^{\frac{3}{2}}}.$$ [7615]

If we suppose a' to be small in comparison with a, as is the case with respect to the inferior satellites and the different points of the ring, we shall [7616] have, by neglecting terms of the order† $\frac{a'^4}{a^4}$,

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* (3690) If we use for brevity $W=xx'+yy'+zz'$, also $r^2=x^2+y^2+z^2$ [6023], [7615a] $r'^2=x'^2+y'^2+z'^2$ [6039d], we shall find that the expression of R [7604] will become $R=\frac{m'.W}{r'^3}-\frac{m'}{\{r^2+r'^2-2W\}^{\frac{1}{2}}}$. Its differential relative to s, which is found only in W, [7615b] gives $a.\left(\frac{dR}{ds}\right)=\frac{m'.a}{r'^3}.\left(\frac{dW}{ds}\right)-\frac{m'.a}{\{r^2+r'^2-2W\}^{\frac{3}{2}}}.\left(\frac{dW}{ds}\right)$. Now if we substitute the values [7615c] [7606—7611] in W [7615a], it becomes as in [7615d]; and its partial differential, relative to s, gives [7615e],

$$W=rr'.\{\cos.(v-\Psi).\cos.v'+\cos.\gamma.\sin.(v-\Psi).\sin.v'+s.\sin.\gamma.\sin.v'\};$$ [7615d]

$$\left(\frac{dW}{ds}\right)=rr'.\sin.\gamma.\sin.v'.$$ [7615e]

Substituting [7615e] in [7615c], and neglecting the first term which does not contain v [7596e], we get,

$$a.\left(\frac{dR}{ds}\right)=-\frac{m'.a}{\{r^2+r'^2-2W\}^{\frac{3}{2}}}.rr'.\sin.\gamma.\sin.v'.$$ [7615f]

Now putting, as in [7589p], $r=a$, $r'=a'$, and then substituting the value of W [7615d], we get [7615]; observing that we may neglect s, which occurs in the term W [7615g] of the denominator, because it produces only terms of the same order as those which we have usually neglected.

† (3691) If we put for brevity $W'=\cos.(v-\Psi).\cos.v'+\cos.\gamma.\sin.(v-\Psi).\sin.v'$, we [7616a] shall find that the expression [7615] becomes, by development,

[7617] $$a.\left(\frac{dR}{ds}\right) = -\frac{3m'.a^3a'^2}{2.(a^2+a'^2)^{\frac{5}{2}}}.\sin.\gamma.\cos.\gamma.\sin.(v-\Psi).$$

[7618] If we consider the rings as a collection of an infinite number of satellites, we shall have, in virtue of their mutual action, and of that of the satellites which are within the orbit of the outer satellite,

[7619] $$a.\left(\frac{dR}{ds}\right) = -B.\sin.\gamma.\cos.\gamma.\sin.(v-\Psi);$$

[7620] B being a constant coefficient depending upon the mass and constitution of the rings, and also upon the masses and mean distances of the satellites from Jupiter. Now if we put,

[7621] $$K = \frac{3S.a^3}{4D^3}; \qquad K' = \frac{(\rho-\frac{1}{2}\varphi)}{a^2} + \tfrac{1}{2}B;$$

[7616b] $$a.\left(\frac{dR}{ds}\right) = -\frac{m'.a^2a'.\sin.\gamma.\sin.v'}{(a^2+a'^2)^{\frac{3}{2}}}.\left\{1-\frac{2aa'}{a^2+a'^2}.W'\right\}^{-\frac{3}{2}}$$

[7616c] $$= -\frac{m'.a^2a'.\sin.\gamma.\sin.v'}{(a^2+a'^2)^{\frac{3}{2}}}.\left\{1+\frac{3aa'}{a^2+a'^2}.W'+\frac{15a^2a'^2}{2.(a^2+a'^2)^2}.W'^2+\&c.\right\}.$$

The first term of [7616c] which does not contain W', is independent of v, and

[7616d] may therefore be neglected [7614]. We may also neglect the term containing W'^2, because the square of W' [7616a], when reduced by [1, 6, 31] Int. relative to $\sin.^2v'$, $\sin.v'.\cos.v'$, $\cos.^2v'$, will become of the form $A_{,}+B_{,}.\sin.2v'+C_{,}.\cos.2v'$; $A_{,}$, $B_{,}$, $C_{,}$,

[7616e] being independent of v'. This expression of W'^2 is multiplied by $\sin.v'$, in [7616c]; so that if we reduce the product, by means of [17, 18] Int., it will produce terms of the

[7616f] forms $\begin{smallmatrix}\sin.\\ \cos.\end{smallmatrix}v'$, $\begin{smallmatrix}\sin.\\ \cos.\end{smallmatrix}3v'$, &c., and no term independent of v'; therefore it cannot be of either of the forms $\sin.v$ or $\cos.v$, which are retained in [7614]. Terms depending on

[7616g] W^3, W^4, &c. are neglected, as in [7616], because they are multiplied by $\frac{a'^4}{a^4}$ and the higher powers of $\frac{a'}{a}$. Hence it is evident, that the only term to be retained in [7616c] is the second, depending on the first power of W', which gives,

[7616h] $$a.\left(\frac{dR}{ds}\right) = -\frac{3m'.a^3a'^2}{(a^2+a'^2)^{\frac{5}{2}}}.W'.\sin.\gamma.\sin.v'.$$

Multiplying W' [7616a] by $\sin.v'$, and substituting, in the products, the expressions

[7616i] $\sin.v'.\cos.v' = \frac{1}{2}.\sin.2v'$, $\sin.^2v' = \frac{1}{2}-\frac{1}{2}.\cos.2v'$ [31, 1] Int., then retaining only the terms which are independent of v', we get $\sin.v'.W' = \frac{1}{2}.\cos.\gamma.\sin.(v-\Psi)$. Substituting this in [7616h], it becomes as in [7617]. Now putting B equal to the sum of all the terms, similar to $\frac{3m'.a^3a'^2}{2.(a^2+a'^2)^{\frac{5}{2}}}$, corresponding to the rings and to the inner satellites, we get, for the

[7616k] complete expression of $a.\left(\frac{dR}{ds}\right)$ [7617], a function of the form [7619].

the differential equation in s will become,* Differential equations in s.

$$0=\frac{dds}{dv^2}+s+2K.\sin.\lambda.\cos.\lambda.\sin.v-2K'.\sin.\gamma.\cos.\gamma.\sin.(v-\Upsilon). \quad [7622]$$

Hence we obtain by integrating and neglecting the arbitrary constant terms, as may be done in this case,†

$$s=Kv.\sin.\lambda.\cos.\lambda.\cos.v-K'.v.\sin.\gamma.\cos.\gamma.\cos.(v-\Upsilon). \quad [7623]$$

Through the centre of Saturn, we shall suppose an *intermediate* plane to be drawn, between the orbit of this planet and its equator, so as to pass through the line of intersection of these two last planes, or the line of nodes; then we shall put, [7624]

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* (3692) Connecting together the parts of $a.\left(\frac{dR}{ds}\right)$, which have been computed in [7596, 7602, 7619], and substituting the abridged symbols [7621], we obtain the following complete value of the terms of this form and order; [7621a]

$$a.\left(\frac{dR}{ds}\right)=2K.\sin.\lambda.\cos.\lambda.\sin.v-2K'.\sin.\gamma.\cos.\gamma.\sin.(v-\Upsilon). \quad [7621b]$$

Substituting this in [7586], we get the differential equation in s [7622].

† (3693) If in the differential equation [6049k] we have $m_{,}=a_{,}$, or, in other words, if we have any term of the form $K_{,}.\sin.(a_{,}t+\varepsilon_{,})$, the corresponding term of y [6049l] will be represented, as in [871″], by [7622a]

$$y=\frac{K_{,}t}{2a_{,}}.\cos.(a_{,}t+\varepsilon_{,}); \quad [7622b]$$

which can be derived from the term of the differential equation [7622a], by multiplying it by $\frac{t}{2a_{,}}$, and changing $\varepsilon_{,}$ into $\varepsilon_{,}+100^{\circ}$; observing that this last process is equivalent to that of changing *sin.* into *cos.* Now the equation [7622] is of the same form as [6049k], putting $y=s$, $t=v$, $a_{,}=1$; hence the factor $\frac{t}{2a_{,}}$ [7622c] becomes $\frac{1}{2}v$; so that if we multiply the terms of [7622], containing K, K', by $\frac{1}{2}v$, changing also $\sin.v$ into $\cos.v$, and $\sin.(v-\Upsilon)$ into $\cos.(v-\Upsilon)$, we shall get the corresponding terms of s [7623]. In this value of s, the symbols λ, γ, represent respectively the inclinations of the *primitive* orbit of the outer satellite to that of Saturn, or of that of the ring [7588, 7599]. Now as these inclinations vary only by quantities which are of the order of the disturbing forces, we may, without much error, suppose these inclinations to correspond to the actual orbit, by neglecting terms of the order of the square of the disturbing forces. [7622c] [7622d] [7622e] [7622f]

[7625] $\theta =$ the angle formed by the intermediate plane and the plane of Saturn's equator and rings;

[7626] $\varpi =$ the angle of inclination of the variable orbit of the satellite to the intermediate plane;

[7627] $\Gamma =$ the arc of the variable orbit of the satellite, counted according to the order of the signs, from the ascending node of the satellite's orbit upon the intermediate plane, to the ascending node of the same orbit upon the orbit of Saturn, or upon the sun's relative orbit about Saturn;

[7628] $\Pi =$ the arc of the intermediate plane, counted according to the order of the signs, from the ascending node of the variable orbit of the satellite upon the intermediate plane, to the ascending node of the equator of Saturn upon its orbit.*

[7628a] * (3694) For illustration we have marked these symbols in the annexed figure 98, which is similar to fig. 97, page 319, with the addition of the intermediate plane or arc KIK', which is drawn, as in [7624], through the node K and the equinoctial point of
[7628b] Saturn's orbit I, to the opposite node K'; the orbit of the satellite KN being also continued to the same node K'. Moreover the arc $N_{,}KN'$ *is drawn through* K,
[7628b'] *perpendicular to the arc* NK of the satellite's orbit, and meeting $NIN_{,}$, IHN', in $N_{,}$, N', respectively. Then we have, in like manner as in [7601e—g, 7625—7628],

[7628c] $NB = v,\quad NH = \psi,\quad HB = v - \psi,\quad K'I = \Pi,\quad IK = 200^{\circ} - \Pi,$

[7628d] $K'N = \Gamma,\quad NK = 200^{\circ} - \Gamma,\quad K'B = v + \Gamma,\quad K'H = \psi + \Gamma,\quad HK = 200^{\circ} - \psi - \Gamma,$

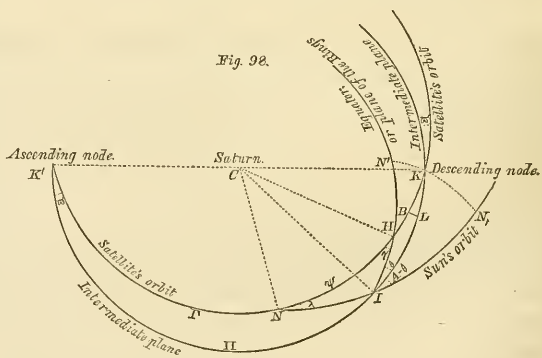
[7628e] $HNI = \lambda,\quad NHI = N'HK = \gamma,\quad IHK = 200^{\circ} - \gamma,\quad HKI = BK'L = \varpi,\quad IKN' = 100^{\circ} + \varpi,$

[7628f] $HIN_{,} = A,\quad IHK = \theta,\quad KIN_{,} = A - \theta,\quad NIK = 200^{\circ} - A + \theta.$

[7628g] Then if the satellite be at B, its latitude above the intermediate plane will be represented by the arc $BL = s$, which is let fall perpendicularly upon this plane; and in the rectangular spherical triangle $K'LB$, we shall have by the usual rule,
[7628h] in like manner as in [6328d],

$$BL = BK'L . \sin . K'B,$$

[7628i] nearly, or $s = \varpi . \sin . (v + \Gamma)$. Taking the variation of



[7628k] this expression, supposing $v + \Gamma$ to be constant, and s, ϖ variable, we get $\delta s = \delta \varpi . \sin . (v + \Gamma)$, as in [7628']. Again, on account of the smallness of the angle $BK'I$, we have very nearly

This being premised, if we vary ϖ by $\delta\varpi$, and suppose Π to be constant, we shall obtain, in s, a term equal to $\delta\varpi.\sin.(v+\Gamma)$. If ϖ be supposed [7628′]
constant, and Π vary by $\delta\Pi$, we shall obtain in s a term equal to [7629]
$\delta\Pi.\sin.\varpi.\cos.(v+\Gamma)$; therefore we shall have, by making ϖ, Π, vary at the [7630]
same time,

$$s=\delta\varpi.\sin.(v+\Gamma)+\delta\Pi.\sin.\varpi.\cos.(v+\Gamma). \qquad [7631]$$

If we put this value equal to that in [7623], we shall obtain,

$$\delta\varpi.\sin.(v+\Gamma)+\delta\Pi.\sin.\varpi.\cos.(v+\Gamma)=Kv.\sin.\lambda.\cos.\lambda.\cos.v-K'v.\sin.\gamma.\cos.\gamma.\cos.(v-\Psi). \quad (1) \qquad [7632]$$

If we continue the approximation only as far as the first power of v, we [7633]
shall have,*

$$\delta\varpi=v.\frac{d\varpi}{dv}; \qquad \delta\Pi=v.\frac{d\Pi}{dv}; \qquad [7634]$$

Substituting, in the second member of the equation [7632], $\cos.(v-\Gamma+\Gamma)$ for $\cos.v$, and $\cos.(v-\Gamma-\Psi+\Gamma)$ for $\cos.(v-\Psi)$, and then developing it [7635]
relative to the sine and cosine of $v+\Gamma$, we find that the comparison of the [7636]
coefficients with those of the first member, give the two following equations; †

$K'B=K'L$; hence [7628h] becomes $BL=HKI.\sin.K'L$, nearly. Now if we [7628l]
suppose the node K to move towards I, upon the arc KI, by the quantity $\delta\Pi$, without changing the inclination HKI, the arc $K'I=\Pi$ will increase by the increment $\delta\Pi$, and the corresponding increment or differential of BL [7628l], will be represented by [7628m]
$\delta\Pi.HKI.\cos.K'L$, nearly; and if we change HKI into $\sin.HKI$, or $\sin.\varpi$, on account of
its smallness; putting also $K'L=K'B=v+\Gamma$, as above; it becomes $\delta\Pi.\sin.\varpi.\cos.(v+\Gamma)$, [7628n]
as in [7630]. Adding together the two terms of δs [7628k, n], we get the expression [7628o]
[7631]. Putting this equal to the terms of s [7623], which produce the motions $\delta\varpi$, $\delta\Pi$, we get the equation [7632].

* (3695) As we neglect the excentricities of the orbit [7584], v will be nearly proportional to t; and in considering the secular motions we may, in the usual manner, [7629a]
suppose ϖ to be a function of t or v; and by developing it, according to the powers of v, by means of Maclaurin's theorem [607a], we get,

$$\varpi=\varpi_{,}+\frac{d\varpi}{dv}.v+\tfrac{1}{2}.\frac{d^2\varpi}{dv^2}.v^2+\&c.; \qquad [7629b]$$

$\varpi_{,}$ being the value of ϖ, when $v=0$. Neglecting v^2, v^3, &c. and putting $\varpi-\varpi_{,}=\delta\varpi$, we get $\delta\varpi=\frac{d\varpi}{dv}.v$, as in [7634]; and in like manner we get $\delta\Pi$ [7635]. [7629c]

† (3696) We have identically $v=(v+\Gamma)-\Gamma$; $v-\Psi=(v+\Gamma)-(\Psi+\Gamma)$; substituting [7636a]
these in the first members of [7636b, c], and then developing, by [24] Int., we get,

[7637] $$\frac{d\varpi}{dv} = K.\sin.\lambda.\cos.\lambda.\sin.\Gamma - K'.\sin.\gamma.\cos.\gamma.\sin.(\Psi+\Gamma);$$

[7637′] $$\frac{d\Pi}{dv}.\sin.\varpi = K.\sin.\lambda.\cos.\lambda.\cos.\Gamma - K'.\sin.\gamma.\cos.\gamma.\cos.(\Psi+\Gamma). \qquad (2)$$

[7638] If we put A for the inclination of the equator of Saturn to its orbit, we shall have the following trigonometrical formulas ;*

[7636b] $$\cos.v = \cos.(v+\Gamma).\cos.\Gamma + \sin.(v+\Gamma).\sin.\Gamma;$$

[7636c] $$\cos.(v-\Psi) = \cos.(v+\Gamma).\cos.(\Psi+\Gamma) + \sin.(v+\Gamma).\sin.(\Psi+\Gamma).$$

Substituting the expressions [7634, 7636b, c] in [7632], and then dividing by v, we obtain,

[7636d] $$\frac{d\varpi}{dv}.\sin.(v+\Gamma) + \frac{d\Pi}{dv}.\sin.\varpi.\cos.(v+\Gamma) = \{K.\sin.\lambda.\cos.\lambda.\sin.\Gamma - K'.\sin.\gamma.\cos.\gamma.\sin.(\Psi+\Gamma)\}.\sin.(v+\Gamma)$$
$$+\{K.\sin.\lambda.\cos.\lambda.\cos.\Gamma - K'.\sin.\gamma.\cos.\gamma.\cos.(\Psi+\Gamma)\}.\cos.(v+\Gamma).$$

As this equation must be satisfied, for all values of v, we must put the terms depending on
[7636e] $\sin.(v+\Gamma)$, $\cos.(v+\Gamma)$, separately equal to nothing; and by this means we shall obtain the two equations [7637, 7637′].

[7639a] * (3697) We may demonstrate the formulas [7639—7644] in the following manner, using fig. 98, page 324, with the symbols [7628c—f] ; and as [7639—7641] can be easily derived from [7642—7644], as we shall hereafter see, we shall, in the first place, demonstrate the formulas [7642—7644]. Now in the spherical triangle IHK we have
[7639b] $\sin.IHK : \sin.IK :: \sin.HIK : \sin.HK$ [1345[15]] ; and by substituting the symbols [7628c—f], it becomes $\sin.(200^\circ-\gamma) : \sin.(200^\circ-\Pi) :: \sin.\theta : \sin.(200^\circ-\Psi-\Gamma)$, or
[7639c] $\sin.\gamma : \sin.\Pi :: \sin.\theta : \sin.(\Psi+\Gamma)$, which is easily reduced to the form [7642]. Again, in referring to the formula [1345[22]], we shall suppose the angular points A, B, C, to correspond, in the triangle $N'KI$, to N', I, K, respectively, in fig. 98, page 324 ; and
[7639d] this formula will become $\cos.IN'K = \cos.IK.\sin.N'IK.\sin.IKN' - \cos.N'IK.\cos.IKN'$. In the first member of this equation we may change $IN'K$ into $HN'K$; and then in the rectangular spherical triangle HKN', we have, as in [1345[32]],

[7639e] $$\cos.HN'K = \sin.N'HK.\cos.HK.$$

Substituting this in the first member of [7639d], and then introducing the symbols [7628c—f], we get [7639f], which is easily reduced to the form [7643], by changing the signs of all the terms, and making some slight reductions,

[7639f] $$\sin.\gamma.\cos.(200^\circ-\Psi-\Gamma) = \cos.(200^\circ-\Pi).\sin.\theta.\sin.(100^\circ+\varpi) - \cos.\theta.\cos.(100^\circ+\varpi).$$

In like manner, in the triangle IHK we have, as in [7639d],

[7639g] $$\cos.IHK = \cos.IK.\sin.HIK.\sin.HKI - \cos.HIK.\cos.HKI;$$

substituting the values [7628e—f], it becomes,

[7639h] $$\cos.(200^\circ-\gamma) = \cos.(200^\circ-\Pi).\sin.\theta.\sin.\varpi - \cos.\theta.\cos.\varpi.$$

Changing the signs and making some slight reductions, we get [7644].

The three formulas [7642—7644] represent certain relations between the three angles

Formulas in spherics.

$$\sin.\lambda.\sin.\Gamma = \sin.\Pi.\sin.(A-\theta)\,;$$ [7639]

$$\sin.\lambda.\cos.\Gamma = \cos.\Pi.\sin.(A-\theta).\cos.\varpi+\cos.(A-\theta).\sin.\varpi\,;$$ [7640]

$$\cos.\lambda = -\cos.\Pi.\sin.(A-\theta).\sin.\varpi+\cos.(A-\theta).\cos.\varpi\,;$$ [7641]

$$\sin.\gamma.\sin.(\Upsilon+\Gamma) = \sin.\Pi.\sin.\theta\,;$$ [7642]

$$\sin.\gamma.\cos.(\Upsilon+\Gamma) = \cos.\Pi.\sin.\theta.\cos.\varpi-\cos.\theta.\sin.\varpi\,;$$ [7643]

$$\cos.\gamma = \cos.\Pi.\sin.\theta.\sin.\varpi+\cos.\theta.\cos.\varpi.$$ [7644]

Therefore by putting,*

$$K.\sin.(A-\theta).\cos.(A-\theta) = K'.\sin.\theta.\cos.\theta,$$ [7645]

which gives the following equation to determine θ,

$$\text{tang}.2\theta = \frac{K.\sin.2A}{K'+K.\cos.2A}, \quad [7645f]$$ [7646]

we shall obtain,†

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and two of the sides of the triangle HIK, namely, the sides $IK=200^\circ-\Pi$, [7639i]
$HK=200^\circ-\Upsilon-\Gamma$, and the angles $HIK=\theta$, $HKI=\varpi$, $IHK=200^\circ-\gamma$;
and the same formulas hold good relative to the triangle NIK, the points IK remaining [7639k]
unaltered, and H changing into N. By this means the side $HK=200^\circ-\Upsilon-\Gamma$ changes into $NK=200^\circ-\Gamma$ [7628d]; the angle $HIK=\theta$ changes into $NIK=200^\circ-A+\theta$
[7628f]; and the angle $IHK=200^\circ-\gamma$ changes into $INK=\lambda$ [7628e]; or, in other [7639l]
words, we must change $\Upsilon+\Gamma$ into Γ, θ into $200^\circ-A+\theta$, and γ into $200^\circ-\lambda$.
Now making these changes in the equations [7642, 7643, 7644], we obtain the equations [7639m]
[7639, 7640, 7641] respectively; the signs of all the terms of this last equation being [7639n]
changed.

* (3698) The reasons for assuming the equation [7645] for the determination of θ,
will be seen hereafter in [7707—7711], where we shall find that the expression of s [7645a]
[7623 or 7710] is made to vanish by means of the equation [7645 or 7711]. Now [7645b]
substituting $\sin.\theta.\cos.\theta=\frac{1}{2}.\sin.2\theta$, $\sin.(A-\theta).\cos.(A-\theta)=\frac{1}{2}.\sin.(2A-2\theta)$, [31] Int.,
in the assumed equation [7645], for the determination of θ, we get, by multiplying by 2,
$K.\sin.(2A-2\theta)=K'.\sin.2\theta$. But from [22] Int. we have, [7645c]

$$\sin.(2A-2\theta)=\sin.2A.\cos.2\theta-\cos.2A.\sin.2\theta\,;$$ [7645d]

substituting this in the equation [7645c], and then dividing by $\cos.2\theta$, we get,

$$K.\{\sin.2A-\cos.2A.\text{tang}.2\theta\} = K'.\text{tang}.2\theta\,;$$ [7645e]

whence we easily deduce $\text{tang}.2\theta$ [7646], which may be put under the form,

$$\text{tang}.2\theta = \frac{\sin.2A}{\frac{K'}{K}+\cos.2A}.$$ [7645f]

† (3699) Multiplying the product of the two equations [7639, 7641] by K, and the
product of the two equations [7642, 7644] by $-K'$, then taking the sum of these two [7647a]

[7647] $$\frac{d\varpi}{dv} = -\tfrac{1}{2}.\{K.\sin.^2(A-\theta)+K'.\sin.^2\theta\}\sin.\varpi.\sin.2\Pi\,;$$

[7648] $$\frac{d\Pi}{dv} = \{K.\cos.^2(A-\theta)+K'.\cos.^2\theta\}.\cos.\varpi - \{K.\sin.^2(A-\theta)+K'.\sin^2\theta\}.\cos.\varpi.\cos.^2\Pi.$$

Now if we put,

[7649] $$q=\tfrac{1}{4}.\{K+K'-\sqrt{K^2+2KK'.\cos.2A+K'^2}\}\,;$$

[7650] $$p=\tfrac{1}{2}.\{K+K'+\sqrt{K^2+2KK'.\cos.2A+K'^2}\}-q\,;$$

we shall have,*

[7647b] results, we find that the first member of the sum is equal to the expression of $\frac{d\varpi}{dv}$ [7637], and its second member becomes as in [7647d], and by reduction as in [7647e]. In like manner, if we multiply the product of the two equations [7640, 7641] by K, and the product of the two equations [7643, 7644] by $-K'$, then taking the sum of the products,

[7647c] we find that the first member of the sum is equal to the expression of $\frac{d\Pi}{dv}.\sin.\varpi$ [7637′], and its second member becomes as in [7647f, g, h].

[7647d] $$\frac{d\varpi}{dv}=K.\sin.\Pi.\sin.(A-\theta).\{-\cos.\Pi.\sin.(A-\theta).\sin.\varpi+\cos.(A-\theta).\cos.\varpi\} -K'.\sin.\Pi.\sin.\theta.\{\cos.\Pi.\sin.\theta.\sin.\varpi+\cos.\theta.\cos.\varpi\}$$

[7647e] $$=-\{K.\sin.^2(A-\theta)+K'.\sin.^2\theta)\}.\sin.\Pi.\cos.\Pi.\sin.\varpi+\sin.\Pi.\cos.\varpi.\{K.\sin.(A-\theta).\cos.(A-\theta)-K'.\sin.\theta.\cos.\theta\}.$$

[7647f] $$\frac{d\Pi}{dv}.\sin.\varpi=\sin.\varpi.\cos.\varpi.\{[K.\cos.^2(A-\theta)+K'.\cos.^2\theta]-\cos.^2\Pi.[K.\sin.^2(A-\theta)+K'.\sin.^2\theta]\}$$

[7647g] $$+\cos.\Pi.\sin.^2\varpi.\{-K.\sin.(A-\theta).\cos.(A-\theta)+K'.\sin.\theta.\cos.\theta\}$$

[7647h] $$+\cos.\Pi.\cos.^2\varpi.\{\;K.\sin.(A-\theta).\cos.(A-\theta)-K'.\sin.\theta.\cos.\theta\}.$$

Now the coefficient of $\sin.\Pi.\cos.\varpi$ [7647e] vanishes, by using the equation [7645]; therefore if we neglect this term, and substitute $\sin.\Pi.\cos.\Pi=\frac{1}{2}.\sin.2\Pi$ [31] Int. in the

[7647i] other term of [7647e], it will become as in [7647]. Again, the terms in [7647g, h] separately vanish, by using [7645]; therefore if we neglect these lines, and divide the remaining part of the expression [7647f] by $\sin.\varpi$, we shall get [7648].

[7650a] * (3700) Putting for brevity $k=\sqrt{K^2+2KK'.\cos.2A+K'^2}$, we get, from [7646],

[7650b] $$\sin.2\theta=\frac{K.\sin.2A}{k}\,;\qquad \cos.2\theta=\frac{K'+K.\cos.2A}{k}\,;$$

because the sum of their squares is easily reduced to the form $\frac{k^2}{k^2}=1$; and the value of

[7650c] $\sin.2\theta$, being divided by that of $\cos.2\theta$, gives the expression of $\tang.2\theta$ [7646]. Again, from [1] Int. we have the first development of [7650d, e]; and from [22] Int. we get the second form of [7650e]. Substituting in these the values of $\sin.2\theta$, $\cos.2\theta$ [7650b], we

$$\frac{d\varpi}{dv} = -q.\sin.\varpi.\sin.2\Pi\,; \qquad [7651]$$

$$\frac{d\Pi}{dv} = (p-q.\cos.2\Pi).\cos.\varpi. \qquad [7652]$$

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get the same expressions in terms of K, K' [7650d,f]. Performing the multiplications indicated in [7650f], and putting $\cos.^2 2A+\sin.^2 2A=1$, we get [7650g].

$$\sin.^2\theta = \tfrac{1}{2}-\tfrac{1}{2}.\cos.2\theta = \tfrac{1}{2}.\left\{1-\frac{K'}{k}-\frac{K}{k}.\cos.2A\right\} \qquad [7650d]$$

$$\sin.^2(A-\theta) = \tfrac{1}{2}-\tfrac{1}{2}.\cos.(2A-2\theta) = \tfrac{1}{2}-\tfrac{1}{2}.\cos.2A.\cos.2\theta-\tfrac{1}{2}.\sin.2A.\sin.2\theta \qquad [7650e]$$

$$= \tfrac{1}{2}-\tfrac{1}{2}.\cos.2A.\frac{(K'+K.\cos.2A)}{k}-\tfrac{1}{2}.\sin.2A.\frac{K.\sin.2A}{k} \qquad [7650f]$$

$$= \tfrac{1}{2}.\left\{1-\frac{K}{k}-\frac{K'}{k}.\cos.2A\right\}. \qquad [7650g]$$

Multiplying [7650g] by K, and [7650d] by K', then taking the sum of the products, we get [7650h]. Multiplying together the factors of [7650h], we get [7650i]; which is reduced to the form [7650k], by using k^2 [7650a]. Substituting $2q$ [7649], it becomes as in [7650l];

$$K.\sin.^2(A-\theta)+K'.\sin.^2\theta = \tfrac{1}{2}.K.\left\{1-\frac{K}{k}-\frac{K'}{k}.\cos.2A\right\}+\tfrac{1}{2}.K'.\left\{1-\frac{K'}{k}-\frac{K}{k}.\cos.2A\right\} \qquad [7650h]$$

$$= \tfrac{1}{2}.\left\{K+K'-\frac{(K^2+2KK'.\cos.2A+K'^2)}{k}\right\} \qquad [7650i]$$

$$= \tfrac{1}{2}.\{K+K'-k\} \qquad [7650k]$$

$$K.\sin.^2(A-\theta)+K'.\sin.^2\theta = 2q. \qquad [7650l]$$

Again, we have identically $K.\{\sin.^2(A-\theta)+\cos.^2(A-\theta)\}+K'.\{\sin.^2\theta+\cos.^2\theta\}=K+K'$; [7650$m$]
subtracting from this the equation [7650l], we obtain [7650n],

$$K.\cos.^2(A-\theta)+K'.\cos.^2\theta = K+K'-2q. \qquad [7650n]$$

Now multiplying [7649] by 2, and adding the product to [7650], we get, by a slight reduction, $p+3q=K+K'$; substituting this value of $K+K'$ in the second member of [7650n], we get, [7650o]

$$K.\cos.^2(A-\theta)+K'.\cos.^2\theta = p+q. \qquad [7650p]$$

Substituting [7650l] in [7647], we get [7651]. Again, substituting [7650p,l] in [7648], we obtain [7650r]; and by using $\cos.^2\Pi=\tfrac{1}{2}+\tfrac{1}{2}.\cos.2\Pi$, it is reduced to the form [7650q]
[7650s], being the same as in [7652];

$$\frac{d\Pi}{dv} = (p+q).\cos.\varpi-2q.\cos.\varpi.\cos.^2\Pi \qquad [7650r]$$

$$= p.\cos.\varpi-q.\cos.\varpi.\cos.2\Pi. \qquad [7650s]$$

We may observe that, if we substitute the value of q [7649] in [7650], we get, by a slight reduction,

$$p = \tfrac{1}{4}.\{K+K'+3.\sqrt{K^2+2KK'.\cos.2A+K'^2}\}, \qquad [7650t]$$

which is the form of p given by La Place, in his paper on this subject in the *Connaisance des tems* for 1829, page 247.

Hence we deduce,*

[7653] $$\frac{d\varpi.\cos.\varpi}{\sin.\varpi}=-\frac{q.d\Pi.\sin.2\Pi}{p-q.\cos.2\Pi}.$$

Integrating this, considering p, q as constant, we obtain,

[7654] $$\sin.\varpi=\frac{b}{\sqrt{p-q.\cos.2\Pi}};$$

[7655] b being an arbitrary constant quantity. Again, from [7652] we obtain,†

[7656] $$dv=\frac{d\Pi}{\sqrt{(p-q.\cos.2\Pi).(p-b^2-q.\cos.2\Pi)}};$$

and the integral of this equation depends upon the rectification of the conic sections. We may put it under a more simple form, by supposing,

[7657] $$\text{tang}.\Pi=\sqrt{\frac{p-q}{p+q}}.\text{tang}.\Pi';$$

and then it becomes,‡

[7653a] * (3701) Dividing the equation [7651] by [7652], we get $\frac{d\varpi}{d\Pi}=\frac{-q.\sin.\varpi.\sin.2\Pi}{(p-q.\cos.2\Pi).\cos.\varpi}$; multiplying this by $\frac{d\Pi.\cos.\varpi}{\sin.\varpi}$, we obtain [7653]; which may be put under the form,

[7653b] $$\frac{d.(\sin.\varpi)}{\sin.\varpi}=-\tfrac{1}{2}.\frac{d.(p-q.\cos.2\Pi)}{p-q.\cos.2\Pi}.$$

Integrating, as in [59] Int. and adding the arbitrary constant quantity $\log.b$, we obtain,

[7653c] $$\log.\sin.\varpi=\log.b-\tfrac{1}{2}.\log.(p-q.\cos.2\Pi)=\log.\frac{b}{\sqrt{p-q.\cos.2\Pi}};$$

being the same as the logarithm of the expression in [7654].

† (3702) From [7654] we obtain,

[7656a] $$\cos.\varpi=\left\{1-\sin.^2\varpi\right\}^{\frac{1}{2}}=\left\{1-\frac{b^2}{p-q.\cos.2\Pi}\right\}^{\frac{1}{2}}=\frac{\{p-b^2-q.\cos.2\Pi\}^{\frac{1}{2}}}{\{p-q.\cos.2\Pi\}^{\frac{1}{2}}}.$$

Multiplying this by $p-q.\cos.2\Pi$, we get,

[7656b] $$(p-q.\cos.2\Pi).\cos.\varpi=\left\{(p-q.\cos.2\Pi).(p-b^2-q.\cos.2\Pi)\right\}^{\frac{1}{2}}.$$

Substituting this in [7652], we easily deduce the value of dv [7656].

‡ (3703) Squaring [7657], and adding 1 to both sides of the equation, we get [7657a], which is successively reduced by using [34′, 34″, &c.] Int.;

[7657a] $$1+\text{tang}.^2\Pi=\frac{1}{\cos.^2\Pi}=1+\frac{p-q}{p+q}.\frac{\sin.^2\Pi'}{\cos.^2\Pi'}=\frac{p.(\cos.^2\Pi'+\sin.^2\Pi')+q.(\cos.^2\Pi'-\sin.^2\Pi')}{(p+q).\cos.^2\Pi'}=\frac{p+q.\cos.2\Pi'}{(p+q).\cos.^2\Pi'}.$$

[7657b] Multiplying this by $d\Pi$, we get $\frac{d\Pi}{\cos.^2\Pi}=\frac{d\Pi.(p+q.\cos.2\Pi')}{(p+q).\cos.^2\Pi'}$. Now the first member of this last expression is equal to the differential of the first member of [7657]; hence by

[7657c] substituting the differential of the assumed value in the second member of [7657], we get,

$$dv = \frac{d\Pi'}{\sqrt{p^2-q^2}\cdot\sqrt{1-\frac{b^2 p}{p^2-q^2}-\frac{b^2 q}{p^2-q^2}\cdot\cos.2\Pi'}}. \qquad [7658]$$

This equation being integrated, gives the expression of Π' in v. We shall then have, by § 22, Book II,*

$$\sqrt{\frac{p-q}{p+q}}\cdot\frac{d\Pi'}{\cos.^2\Pi'}=\frac{d\Pi.(p+q.\cos.2\Pi')}{(p+q).\cos.^2\Pi'};\quad\text{consequently,}\quad d\Pi=d\Pi'.\frac{\sqrt{p^2-q^2}}{p+q.\cos.2\Pi'}. \qquad [7657d]$$

From the second and last forms of [7657a] we evidently have $\cos.^2\Pi=\frac{(p+q).\cos.^2\Pi'}{p+q.\cos.2\Pi'}$; [7657$e$]
multiplying by 2, and substituting $2.\cos.^2\Pi=1+\cos.2\Pi$, $2.\cos.^2\Pi'=1+\cos.2\Pi'$, we get,

$$1+\cos.2\Pi=\frac{(p+q)+(p+q).\cos.2\Pi'}{p+q.\cos.2\Pi'};\quad\text{whence}\quad \cos.2\Pi=\frac{q+p.\cos.2\Pi'}{p+q.\cos.2\Pi'}. \qquad [7657e']$$

From this value of $\cos.2\Pi$, we easily deduce [7657f]; and by subtracting from it b^2, we obtain [7657g]. Multiplying the expressions [7657f, g], and extracting the square root of the product, we get [7657h].

$$p-q.\cos.2\Pi=\frac{p^2-q^2}{p+q.\cos.2\Pi'} \qquad [7657f]$$

$$p-b^2-q.\cos.2\Pi=\frac{(p^2-q^2).\left\{1-\frac{b^2p}{p^2-q^2}-\frac{b^2q}{p^2-q^2}.\cos.2\Pi'\right\}}{p+q.\cos.2\Pi'} \qquad [7657g]$$

$$\sqrt{\{(p-q.\cos.2\Pi).(p-b^2-q.\cos.2\Pi)\}}=\frac{(p^2-q^2)}{p+q.\cos.2\Pi'}\sqrt{\left\{1-\frac{b^2p}{p^2-q^2}-\frac{b^2q}{p^2-q^2}.\cos.2\Pi'\right\}}. \qquad [7657h]$$

Now dividing the value of $d\Pi$ [7657d] by the expression [7657h], we obtain the value of dv [7656], in the same form as it is given in [7658].

* (3704) If we substitute in [606 line 3] the values $v=2\Pi$, $u=2\Pi'$, $e=-\frac{q}{p}$, [7658a]
it will become, by a slight reduction, as in [7657]. From the formula we have just mentioned in [606 line 3], or the equivalent expression [660], we have deduced the value of v [665]; and by making in it the preceding changes [7658a], we get,

$$2\Pi=2\Pi'+2\lambda.\sin.2\Pi'+\tfrac{2}{2}.\lambda^2.\sin.4\Pi'+\tfrac{2}{3}.\lambda^3.\sin.6\Pi'+\tfrac{2}{4}.\lambda^4.\sin.8\Pi'+\&c.; \qquad [7658b]$$

λ being a function of e, which is given in [662]; and by substituting in it the preceding value of $e=-\frac{q}{p}$, it becomes, by reduction, $\lambda=\frac{-q}{p+\sqrt{p^2-q^2}}$; or, by using β [7660], [7658c]
$\lambda=-\beta$. Substituting this value of λ in [7658b], and then dividing by 2, we get [7659]. We may observe that the value of β [7660], can be put under a more convenient form for computation, by putting $q=p.\sin.B$; whence,

$$\beta=\frac{q}{p+\sqrt{p^2-q^2}}=\frac{\sin.B}{1+\cos.B}=\frac{2.\sin.\frac{1}{2}B.\cos.\frac{1}{2}B}{2.\cos.^2\frac{1}{2}B}=\tan g.\tfrac{1}{2}B, \qquad [7658d]$$

as is evident from [6, 31] Int.

[7659] $$\Pi = \Pi' - \beta.\sin.2\Pi' + \tfrac{1}{2}.\beta^2.\sin.4\Pi' - \tfrac{1}{3}.\beta^3.\sin.6\Pi' + \&c.;$$

β being determined by the equation,

[7660] $$\beta = \frac{q}{p + \sqrt{p^2 - q^2}}.$$

36. *To apply these formulas to numbers, we must investigate the values of* K and K'. That of K [7621] is easily determined; for the attraction
[7660′] of the sun on Saturn $\frac{S}{D^2}$, is equal to the centrifugal force from the motion of
[7661] Saturn in its orbit, and this force is equal to the square of the velocity
[7662] divided by the radius [54′]. Therefore by putting T' for the time of the sideral revolution of Saturn, and π for the semi-circumference, whose
[7663] radius is unity, we shall have, for the centrifugal force, the expression*
[7664] $\frac{4\pi^2.D}{T'^2}$; and by putting it equal to $\frac{S}{D^2}$ [7660′], we shall obtain,

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[7665a] * (3705) The radius of Saturn's orbit being D [7589i], its circumference is $2\pi D$, and the velocity of the planet is $\frac{2\pi D}{T'}$ [7662]; whose square, divided by the radius D,
[7665b] gives the centrifugal force of Saturn in its revolution about the sun, equal to $\frac{4\pi^2.D}{T'^2}$, as is evident from the rule given in [54′] for estimating this force. Putting this equal to the
[7665c] gravity $\frac{S}{D^2}$ [7660′], and then dividing by D, we get [7665]. The formula [7667], corresponding to Saturn and its satellites, is similar to [7665], changing the sun's mass S, into Saturn's mass 1 [7598]; Saturn's distance from the sun D [7589i], into the satellite's
[7665d] distance from Saturn a; and the periodical time of the planet T' [7662], into that of the
[7665e] satellite T [7666]. Dividing [7665] by [7667], we get $\frac{S.a^3}{D^3} = \frac{T^2}{T'^2}$; multiplying this by $\frac{3}{4}$, we get K [7621], as in [7668]. Lastly, substituting the values of T, T' [7669, 7670], we obtain K [7671]. We may observe that we can obtain the centrifugal force of the
[7665f] satellite, in its revolution about Saturn, from that of Saturn [7665b], by changing the elements, as in [7665c, e], so as to correspond to the satellite; by this means we find that
[7665g] the centrifugal force of the satellite is $\frac{4\pi^2.a}{T^2}$, at the distance a from the satellite; and if
[7665h] we suppose the orbit of the satellite to be circular, the centrifugal force will be equal to the gravity at the same distance a; so that the gravity of the satellite, at the distance a from
[7665i] the planet, will be represented by $\frac{4\pi^2.a}{T^2}$; multiplying this by a^2, we get the force of
[7665k] gravity at the surface of Saturn, equal to $\frac{4\pi^2.a^3}{T^2}$, the radius of the planet being taken for unity, as in [7598′]. This expression of the gravity will be of use in the next note.

$$\frac{S}{D^3} = \frac{4\pi^2}{T'^2}. \qquad [7665]$$

If we put T for the time of the revolution of the outer satellite, we shall [7666]
have, in like manner,

$$\frac{1}{a^3} = \frac{4\pi^2}{T^2}. \qquad [7667]$$

Hence we obtain, as in [7665e],

$$K = \tfrac{3}{4}.\frac{S.a^3}{D^3} = \tfrac{3}{4}.\frac{T^2}{T'^2}. \qquad [7668]$$

Now we have, by observation,

$$T = 79^{\text{days}},3296\,; \qquad [7669]$$

$$T' = 10759^{\text{days}},08\,; \qquad [7670]$$

hence we deduce, Value of K

$$K = 0,0000407739. \qquad [7671]$$

The value of K' [7621] is $K' = \frac{(\rho - \frac{1}{2}\varphi)}{a^2} + \frac{1}{2}B$; the mean radius of the [7672]
body of Saturn being taken for unity [7598']. The oblateness of this planet [7673]
ρ [6044] is unknown, as well as the value of the quantity B, depending upon the masses of the rings and upon the masses of the six interior satellites [7620]; it is therefore impossible to determine accurately the value of K'. We may however determine approximatively the part of this value which depends upon the oblateness of Saturn. For this purpose we shall put t [7674]
for the time of Saturn's rotation upon its axis; and we shall have,*

$$\varphi = \frac{T^2}{t^2.a^3}. \qquad [7675]$$

We have, by observation,

$$t = 0^{\text{days}},428\,; \qquad a = 59,154\,; \qquad [7676]$$

hence we deduce,

$$\varphi = 0,165970. \qquad [7677]$$

We shall suppose that the oblateness of the earth is to the value of φ, which [7678]
corresponds to it, as the oblateness of Saturn is to the corresponding value

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* (3706) We may deduce, from [7665g], the expression of the centrifugal force of a [7675a]
particle on the surface of the spheroid of Saturn, by changing a into 1 [7598'], and T into
t [7674]; by this means it becomes $\frac{4\pi^2}{t^2}$. Dividing this by the expression of the gravity [7675b]
of the same particle towards Saturn, $\frac{4\pi^2.a^3}{T^2}$ [7665k], we get the expression of φ [7675c]
[6044', 7587b], as in [7675]; substituting the values [7670, 7676], we obtain [7677].

of φ.* This principle being found, in [2069], to hold good very nearly for Jupiter compared with the earth. Now for the earth we have $\varphi=\frac{1}{289}$
[7679] [1594a]; and if we suppose the oblateness of the earth to be $\frac{1}{335}$, in conformity with experiments on the pendulum [2048'], we shall have, as in [7674c],

[7680]
$$\rho-\tfrac{1}{2}\varphi=\frac{243}{670}\cdot\frac{T^2}{t^2.a^3}.$$

[7680'] Then by noticing only the part of K' [7621], depending upon this quantity, we shall find,†

Assumed value of K'

[7681]
$$K'=K.\frac{162}{335}\cdot\frac{T'^2}{t^2.a^5}=0{,}4219.K.$$

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* (3707) Using $\varphi=\frac{1}{289}$, $\rho=\frac{1}{335}$ [7679], relative to the earth, we have,

[7679a]
$$\rho-\tfrac{1}{2}\varphi=\frac{1}{335}-\tfrac{1}{2}.\frac{1}{289}=\frac{243}{670\times289}=\frac{243}{670}.\varphi;$$

[7679b] and if we suppose, as in [2069], that the same ratio $\rho-\frac{1}{2}\varphi=\frac{243}{670}.\varphi$, holds good for
[7679c] Saturn, we shall have, from [7675], $\rho-\frac{1}{2}\varphi=\frac{243}{670}\times\frac{T^2}{t^2.a^3}$, as in [7680]. If we suppose the oblateness of the earth to be $\frac{1}{300}$, as in [2056z], instead of $\frac{1}{335}$ [7679], the expression [7679a, c] will become,

[7679d]
$$\rho-\tfrac{1}{2}\varphi=\frac{1}{300}-\tfrac{1}{2}.\frac{1}{289}=\frac{278}{600\times289}=\frac{278}{600}.\varphi=\frac{278}{600}.\frac{T^2}{t^2.a^3},$$

[7679e] which is about a quarter part more than the expression [7680]; and the value of K' [7681] will be increased in the same ratio.

† (3708) Dividing [7680] by a^2, we get the first of the following expressions [7680a], which, by successive reductions, and the substitution of the second value of K [7668], becomes as in [7680b];

[7680a]
$$\frac{\rho-\frac{1}{2}\varphi}{a^2}=\frac{243}{670}.\frac{T^2}{t^2.a^5}=\frac{243}{670}.\frac{1}{a^5}.\left(\tfrac{3}{4}.\frac{T^2}{T'^2}\right).\left(\tfrac{4}{3}.\frac{T'^2}{t^2}\right)=\frac{243}{670}.\frac{1}{a^5}.K.\left(\tfrac{4}{3}.\frac{T'^2}{t^2}\right)$$

[7680b]
$$=\frac{162}{335}.\frac{1}{a^5}.K.\frac{T'^2}{t^2}.$$

This last expression is the same as the first of those in [7681], and by the substitution of
[7680c] the values of a, t, T' [7676, 7670], it becomes $0{,}4219.K$, as in [7681]. This would be increased about one fourth part [7679e], by using the corrected ellipticity $\frac{1}{300}$ [7679d].

We cannot suppose K' to be less than this value, because it is increased by the action of the inner satellites and by that of the ring.* [7681′]

We have by observation $A=33°,3333$; hence this value of K gives,† [7682]

$$\theta=24°,0083; \qquad A-\theta=9°,3250; \quad [7683]$$

$$q=0,03926.K; \qquad p=1,30412.K. \quad [7684]$$

The observations of Bernard in Marseilles, in 1787, give,

$$\lambda=25°,222; \quad [7588] \quad [7685]$$

$$\gamma=13°,593. \quad [7599] \quad [7686]$$

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* (3709) Putting for brevity, for a moment, $h=-2.\sin.\gamma.\cos.\gamma.\sin.(v-\Psi)$, we find [7681a] that the part of $a.\left(\frac{dR}{ds}\right)$ [7602], depending on the ellipticity of Saturn's mass, is $\frac{\rho-\frac{1}{2}\varphi}{a^2}.h$; and that depending on the interior satellite [7617], is $\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.h$. Dividing [7681b] these by h, we get the corresponding parts of K' [7621, 7621b, 7617], namely, $\frac{\rho-\frac{1}{2}\varphi}{a^2}$ and $\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}$; and as this last term is *positive*, it follows that the sum of the [7681c] similar terms, corresponding to the action of all the satellites, or $\frac{1}{2}B$ [7621], must be *positive*, or of the same sign as $\frac{\rho-\frac{1}{2}\varphi}{a^2}$; so that $K'>\frac{(\rho-\frac{1}{2}\varphi)}{a^2}$ [7621]. Now the expression [7681d] of $K'=0,4219.K$ [7681], is computed upon the supposition that B is nothing; hence we have $K'>0,4219.K$, as in [7681′]; observing also that this coefficient 0,4219, will [7681e] be increased about a quarter part, by using the corrected ellipticity $\frac{1}{300}$ [7680c].

† (3710) If we put $c=0,4219$, we shall have $K'=cK$ [7681]; substituting this [7682a] in [7646], we get $\tang.2\theta=\frac{\sin.2A}{c+\cos.2A}$. The value of $A=33°\frac{1}{3}$ [7682], gives [7682b] $\cos.2A=\frac{1}{2}$, $\sin.2A=\frac{1}{2}\sqrt{3}$; hence we obtain $\tang.2\theta=\frac{\sqrt{3}}{2c+1}$; and from this we get θ [7683]. Now if we substitute the preceding values $K'=cK$, $\cos.2A=\frac{1}{2}$, in [7682c] [7649, 7650], we shall get,

$$q=\tfrac{1}{4}K.\{1+c-\sqrt{1+c+c^2}\}; \qquad p=\tfrac{1}{2}K.\{1+c+\sqrt{1+c+c^2}\}-q; \quad [7682d]$$

and by using the value of c [7682a], we obtain the expressions of p, q [7684]. As the expression of c ought to be increased above a quarter part [7681c, 7682a], we shall make a rough estimate of the effect of this correction upon the values of θ, p, q, supposing [7682e] merely for an example that c is increased to $c=0,6$. This gives,

$$\tang.2\theta=\frac{\sqrt{3}}{2,2}=0,787=\tang.42°,46 \text{ [7682b]}; \quad \text{or} \quad \theta=21°,23; \quad [7682f]$$

and $\sqrt{1+c+c^2}=\sqrt{1,96}=1,4$; therefore we have, from [7682d], $q=0,05.K$, [7682g] $p=1,45.K$, instead of the values [7684].

Hence we deduce,*

[7687] $$\Psi = 71^\circ{,}354;$$

[7688] $$\varpi = 16^\circ{,}961;$$

[7689] $$\Pi = 37^\circ{,}789;$$

consequently,†

[7690] $$b^2 = 0{,}00000364437.$$

We have then very nearly, by the preceding formulas,‡

[7687a] * (3711) In the triangle NIH, fig. 98, page 324, we have, by comparing the symbols [7628c–f] with their numerical values [7682–7686], the three angles $NHI=\gamma=13^\circ{,}593$,
[7687b] $HNI=\lambda=25^\circ{,}222$, $NIH=200^\circ-A=166^\circ{,}6667$; to find, by the common rules of spherical trigonometry [1345[53], 1345[15]], the sides $NH=\Psi=71^\circ{,}354$ [7687], and
[7687c] $NI=24^\circ{,}926$. Then in the spherical triangle NIK we have the preceding value of $NI=24^\circ{,}926$; the angle $KNI=\lambda=25^\circ{,}222$ [7685]; and the angle,

[7687d] $$KIN=200^\circ-(A-\theta)=190^\circ{,}6750, \quad [7683];$$

[7687e] to find, by the formula [1345[39]], the side $KI=200^\circ-\Pi=162^\circ{,}211$, or $\Pi=37^\circ{,}789$ [7689]; and then, from [1345[15]], the angle $NKI=\varpi=16^\circ{,}961$ [7688].

[7690a] † (3712) From [7654] we get $b^2=\sin.^2\varpi.(p-q.\cos.2\Pi)$; and by substituting the values of p, q, ϖ, Π [7684, 7688, 7689], it becomes as in [7690].

[7691a] ‡ (3713) If we substitute the value of K [7671] in [7684], we shall see that p is much larger than q, or b^2 [7690]; so that the fractions $\frac{q^2}{p^2}$, $\frac{b^2p}{p^2}$, must be quite small. Again, by multiplying together the two radical expressions in the denominator of [7658],
[7691b] and putting, for brevity, $p^2-q^2-b^2p=m^2$, we get $dv=\frac{d\Pi'}{\sqrt{m^2-b^2q.\cos.2\Pi'}}$. Developing
[7691c] the radical, and multiplying the whole expression by m, we obtain nearly $mdv=d\Pi'+\frac{b^2q}{2m^2}.d\Pi'.\cos.2\Pi'$. Integrating, and adding the constant C, we have
[7691d] $C+mv=\Pi'+\frac{b^2q}{4m^2}.\sin.2\Pi'$; or, by transposition, $\Pi'=C+mv-\frac{b^2q}{4m^2}.\sin.2\Pi'$; and by subtracting $\beta.\sin.2\Pi'$ from both sides of the equation

[7691e] $$\Pi'-\beta.\sin.2\Pi'=C+mv-\left\{\beta+\frac{b^2q}{4m^2}\right\}.\sin.2\Pi'.$$

Moreover β [7660] is of the order $\frac{q}{p}$; and if we neglect terms of the order $\frac{q^2}{p^2}$, we shall get, from [7659], $\Pi=\Pi'-\beta.\sin.2\Pi'$; hence the equation [7691e] becomes,

[7691f] $$\Pi=C+mv-\left\{\beta+\frac{b^2q}{4m^2}\right\}.\sin.2\Pi'.$$

If we neglect terms of the order β, in the value of Π' [7691e], we shall get $\Pi'=C+mv$;

$$\Pi = C + v.\sqrt{p^2-q^2-b^2p} - \left\{ \beta + \frac{b^2q}{4.(p^2-q^2-b^2p)} \right\}.\sin.2.\{C+v.\sqrt{p^2-q^2-b^2p}\};$$ [7691]

Reducing this expression to numbers, and determining the arbitrary constant [7691′]
quantity C, so that Π may be equal to $37°,789$ in 1787, we obtain,* Motion of the node.

$$\Pi = 38°,721 + i.944'',805 - 9937'',7.\sin.2.(38°,721 + i.944'',805);$$ [7692]

i being the number of Julian years elapsed since 1787. [7692′]

These results depend upon the accuracy of the observations above quoted, and particularly upon the ratio of K' to K [7681]. This last quantity depends upon so many and such different elements, which are so difficult to ascertain, that it is almost impossible to determine it *a priori*. We may find it *a posteriori*, after we have determined exactly, by observation, the annual [7693]
motion of the orbit of the satellite upon Saturn's orbit. For if we suppose

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and by substituting it in the last term of Π [7691f], we obtain,

$$\Pi = C + mv - \left\{ \beta + \frac{b^2q}{4m^2} \right\}.\sin.2.(C+mv);$$ [7691g]

which is reduced to the form [7691], by re-substituting the value of m [7691b].

* (3714) The revolution of the outer satellite of Saturn is completed in $79^{\text{days}},3296$, [7669]; therefore the mean value of the arc v, described in i Julian years, is $v = i.\frac{365,2500}{79,3296}.4000000'' = 18416830''.i$; moreover, by using the values of K, p, [7692a]
q, b^2 [7671, 7684, 7690], we get $\sqrt{p^2-q^2-b^2p} = 0,000051294$; hence [7692b]
$v.\sqrt{p^2-q^2-b^2p} = 944'',7.i$. Again, the same values of p, q, b^2, give $\frac{b^2q}{4.(p^2-q^2-b^2p)} = 0,000554$; also, $\frac{q}{p} = \sin.1°,9168 = \sin.B$; whence, [7692c]

$$\beta = \text{tang}.\tfrac{1}{2}B = 0,015055 \text{ [7658}d\text{]}, \quad \text{and} \quad \beta + \frac{b^2q}{4.(p^2-q^2-b^2p)} = 0,015609;$$ [7692d]

multiplying this by the radius in seconds, it becomes $9937'',7$; substituting these numerical values in [7691], we obtain,

$$\Pi = C + 944'',7.i - 9937'',7.\sin.2.(C + 944'',7.i).$$ [7692e]

To find C, we must put, as in [7691′, 7692′], $i = 0$, and $\Pi = 37°,789$; and then [7692f]
the equation [7692e] becomes,

$$37°,789 = C - 9937'',7.\sin.2C; \quad \text{or} \quad C = 37°,789 + 9937'',7.\sin.2C.$$ [7692g]

From this last expression we easily perceive that C must be nearly equal to $38°$, and $9937'',7.\sin.2C$ nearly equal to $9937'',7.\sin.76° = 9240''$; so that

$$C = 37°,789 + 0°,924 = 38°,713 \text{ nearly};$$ [7692h]

and by repeating the calculation with this new value, it becomes $C = 38°,721$. Hence the expression [7692e] becomes very nearly as in [7692]. [7692i]

[7693'] that the fixed plane, to which we have referred the orbit of the satellite, is
[7694] the orbit of Saturn itself, we shall have* $\varpi=\lambda$, $\Gamma=0$; and the preceding analysis will give,

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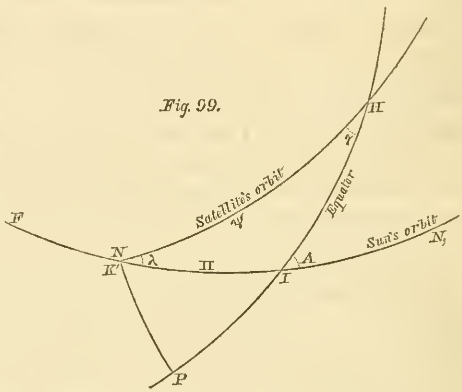

Fig. 99.

* (3715) If we suppose the point K' in fig. 98, page 324, to fall in N, or the point K to be 200° distant from the point N, the
[7695a] arc IK will coincide with the sun's orbit $NIN_{\prime}$, and we shall have the angle $HK'I=$ angle HNI, or $\varpi=\lambda$ [7628e]. Moreover the
[7695b] expression of $\Gamma=200^{\circ}-NK$ [7628d] becomes $\Gamma=200^{\circ}-200^{\circ}=0$ [7695a], as in [7694], corresponding to the annexed fig. 99. Substituting $\varpi=\lambda$, $\Gamma=0$, in [7637, 7637'],
[7695c] and dividing the last of these expressions by $\sin.\lambda$, we get [7695, 7696] respectively; which may be put under the following forms;

[7695d] $$\frac{d\lambda}{dv}=-\frac{K'}{K}.\left\{K.\sin.\gamma.\cos.\gamma.\sin.\Upsilon\right\};$$

[7695e] $$\frac{d\Pi}{dv}=K.\cos.\lambda-\frac{K'}{K}.\left\{K.\frac{\sin.\gamma.\cos.\gamma}{\sin.\lambda}.\cos.\Upsilon\right\}.$$

Multiplying these equations by dv, integrating and neglecting the variations of the
[7695f] coefficients of dv, we get the following values of the variations of λ, Π, which we shall represent by $\delta\lambda$, $\delta\Pi$, respectively;

[7695g] $$\delta\lambda=-\frac{K'}{K}.\left\{K.\sin.\gamma.\cos.\gamma.\sin.\Upsilon\right\}.v;$$

[7695h] $$\delta\Pi=\left\{K.\cos.\lambda-\frac{K'}{K}.\left[K.\frac{\sin.\gamma.\cos.\gamma}{\sin.\lambda}.\cos.\Upsilon\right]\right\}.v.$$

Substituting in these expressions the values K, λ, γ, Υ, v [7671, 7685, 7686, 7687, 7692a], and retaining the factor $\frac{K'}{K}$, they become,

[7695i] $$\delta\lambda=-140'',03.\frac{K'}{K}.i;\qquad -\delta\Pi=\left\{-692'',76+175'',27.\frac{K'}{K}\right\}.i;$$

[7695k] as in [7697, 7699]; and by using the value of $\frac{K'}{K}=0,4219$ [7681], they become $\delta\lambda=-59'',074.i$, $-\delta\Pi=-618'',81.i$, as in [7698, 7700].

[7695l] Some objection having been made to the accuracy of the equation [7696], by M. Plana, in a paper published in the second volume of the Memoirs of the Astronomical Society of London, page 346, etc., another demonstration was given by La Place, in the *Connaisance*

$$\frac{d\lambda}{dv} = -K'.\sin.\gamma.\cos.\gamma.\sin.\Psi\,;$$ [7695]

$$\frac{d\Pi}{dv} = K.\cos.\lambda - K'.\frac{\sin.\gamma.\cos.\gamma}{\sin.\lambda}.\cos.\Psi.$$ [7696]

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des tems for the year 1829, page 248, by means of the formulas [5786*h*, *i*]; which, by [7695*m*]
neglecting terms of the order e^2, and dividing by ndt, become,

$$\frac{d\gamma'}{ndt} = \frac{a}{\sin.\gamma'}.\left(\frac{dR}{d\theta'}\right); \qquad \frac{d\theta'}{ndt} = -\frac{a}{\sin.\gamma}.\left(\frac{dR}{d\gamma'}\right).$$ [7695*n*]

As it will serve for an example for illustrating the use of these formulas, we shall here give the substance of his demonstration. In these formulas, γ' represents the inclination of the satellite's disturbed orbit to the fixed plane, and θ' the longitude of the ascending node of the same orbit upon the same plane, and counted from a fixed point in that plane, as the origin of the longitudes; as is evident from the definitions in [5786*c*—*d*]. Now if we [7695*o*]
neglect the secular motions of the equator and orbit of Saturn, taking the sun's relative orbit [7695*p*]
about Saturn FNI, fig. 99, page 338, for the fixed plane, and F for the fixed point, from which the longitudes are counted, we shall have, according to the notation which is
used in [7695*n*], $\gamma' = \text{angle } HNI$, arc $FN = \theta'$. But we have supposed, in [7628*e*], [7695*q*]
that the angle $HNI = \lambda$; hence $\gamma' = \lambda$. Moreover, if we suppose the longitude of [7695*r*]
the ascending node of the equator upon the fixed plane to be represented by $FI = \alpha$, we [7695*s*]
shall have $FN = FI - NI = \alpha - NI$. Now in fig. 99 we have supposed that the points
K', N coincide, therefore $NI = K'I = \Pi$ [7628*c*]; and by substituting the values of [7695*t*]
FN, NI [7695*q*, *t*] in [7695*s*], we get $\theta' = \alpha - \Pi$, whose differential is $d\theta' = -d\Pi$; [7695*u*]
α being considered as a constant quantity, because the secular equations of the orbit are not
here taken into consideration [7695*p*]. Substituting the values $\gamma' = \lambda$, $d\theta' = -d\Pi$, [7695*v*]
[7695*r*, *u*] in [7695*n*], we obtain,

$$\frac{d\lambda}{ndt} = -\frac{a}{\sin.\lambda}.\left(\frac{dR}{d\Pi}\right); \qquad \frac{d\Pi}{ndt} = \frac{a}{\sin.\lambda}.\left(\frac{dR}{d\lambda}\right).$$ [7695*w*]

If we neglect the terms of R depending on the action of the interior satellites and rings, we may put $B = 0$ [7619]; and then we shall have, in [7621],

$$K = \frac{3S.a^3}{4D^3}, \qquad K' = \frac{\rho - \frac{1}{2}\varphi}{a^2};$$ [7695*x*]

and the parts of R which are to be noticed are, that in [7587] depending on the sun's
action, and that in [7597] depending on the ellipticity of Saturn. After substituting [7596*a*] [7695*y*]
in [7587], and reducing, *we must neglect all the constant quantities because they produce nothing in* [7695*w*]; we must also neglect the terms containing U or its multiples, because these periodical quantities are not here noticed; and if we also neglect the terms of the [7695*z*]
order s, we shall find that the only term of [7596*a*], which requires notice, is the square
of the second term of the second member, namely, $a^2D^2.\sin.^2U.\cos.^2\lambda.\sin.^2v$, which [7696*a*]
can be reduced to the form $\frac{1}{4}.a^2D^2.\cos.^2\lambda + \&c.$, as is evident by substituting $\sin.^2U = \frac{1}{2} - \frac{1}{2}.\cos.2U$; $\sin.^2v = \frac{1}{2} - \frac{1}{2}.\cos.2v$; hence we get from the last term of

[7697] Substituting the preceding values of γ, Υ and λ, we find $140'',03.\frac{K'}{K}$, for
[7698] the annual decrease of λ in 1787; which becomes $-59'',074$, by adopting

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[7696b] R [7587], the following expression, $aR=-\frac{3S.a^3}{8D^3}.\cos.^2\lambda=-\frac{1}{2}K.\cos.^2\lambda$ [7695x],
[7696c] depending upon the sun's action. Again, by substituting $M=1$, $B=1$, $r=a$
[7696d] [7602a], and K' [7695x] in [7597], and neglecting, as in [7695y], the constant part, which produces nothing in [7695w], we get, in aR, the term $aR=K'.v^2$. Substituting
[7696e] the value of v^2 [7601], and neglecting the terms of the order s, then putting $\sin.^2(v-\Upsilon)=\frac{1}{2}-\frac{1}{2}.\cos.2.(v-\Upsilon)$, we get, by retaining as before only the terms which are
[7696f] independent of $v-\Upsilon$, $aR=\frac{1}{2}K'.\sin.^2\gamma=\frac{1}{2}K'-\frac{1}{2}K'.\cos.^2\gamma$. Adding this to the other term of aR [7696b], and neglecting as above the constant term $\frac{1}{2}K'$, we finally obtain,

[7696g] $$aR=-\tfrac{1}{2}K.\cos.^2\lambda-\tfrac{1}{2}K'.\cos.^2\gamma.$$

Substituting [7696g] in the formulas [7695w], we get,

[7696h] $$\frac{d\lambda}{ndt}=\frac{K'.\cos.\gamma}{\sin.\lambda}.\left(\frac{d.\cos.\gamma}{d\Pi}\right);$$

[7696i] $$\frac{d\Pi}{ndt}=K.\cos.\lambda-\frac{K'.\cos.\gamma}{\sin.\lambda}.\left(\frac{d.\cos.\gamma}{d\lambda}\right).$$

In the spherical triangle NHI, fig. 99, page 338, we have, by means of the formula [1345[22]], the equation [7696k]; which, by using the symbols [7628c—f], becomes as in [7696l];

[7696k] $$\cos.NHI=\sin.NIH.\sin.HNI.\cos.NI-\cos.NIH.\cos.HNI;$$

[7696l] $$\cos.\gamma=\sin.A.\sin.\lambda.\cos.\Pi+\cos.A.\cos.\lambda.$$

Substituting the value of $d.\cos.\gamma$ [7696l] in [7696h, i], and putting $ndt=dv$, we get,

[7696m] $$\frac{d\lambda}{dv}=-K'.\cos.\gamma.\sin.A.\sin.\Pi;$$

[7696n] $$\frac{d\Pi}{dv}=K.\cos.\lambda-\frac{K'.\cos.\gamma}{\sin.\lambda}.\{\sin.A.\cos.\lambda.\cos.\Pi-\cos.A.\sin.\lambda\}.$$

[7696o] Now if we suppose the perpendicular arc NP to be let fall upon the arc HIP, so as to form the two rectangular spherical triangles NPI, NPH, we shall have in the first triangle NPI, $\sin.NP=\sin.NIP.\sin.NI=\sin.A.\sin.\Pi$ [1345[28]]; and in like manner in the second triangle NPH, $\sin.NP=\sin.NH.\sin.NHP=\sin.\Upsilon.\sin.\gamma$; hence we
[7696p] have $\sin.A.\sin.\Pi=\sin.\Upsilon.\sin.\gamma$; substituting this in [7696m], it becomes as in [7695]. If we now suppose the arc NP to be drawn perpendicular to HN, so that the angle $PNI=100^\circ-\lambda$, we shall have, in the rectangular spherical triangle HNP,
[7696q] $\cos.NPI=\sin.NHP.\cos.NH=\sin.\gamma.\cos.\Upsilon$ [1345[32]]; and in the oblique spherical triangle NPI, we have $\cos.NPI=\sin.NIP.\sin.PNI.\cos.NI-\cos.NIP.\cos.PNI$ [1345[22]]; or in symbols $\cos.NPI=\sin.A.\cos.\lambda.\cos.\Pi-\cos.A.\sin.\lambda$. Putting these two expressions
[7696r] of $\cos.NPI$ equal to each other, we obtain $\sin.A.\cos.\lambda.\cos.\Pi-\cos.A.\sin.\lambda=\sin.\gamma.\cos.\Upsilon$; substituting this in [7696n], it becomes as in [7696]. Hence we see that the formulas

the preceding ratio of K' to K. Then we find, for the annual motion of the node upon the orbit,

$$-692'',76+175'',27.\frac{K'}{K}; \quad [7695i]$$ [7699]

which gives $-618'',81$ for this motion in the same hypothesis. The observations we now have are not sufficiently accurate to determine, by means of the preceding formula, the ratio of $\frac{K'}{K}$; they serve only to show that the motion of the node of the orbit, upon the orbit of the planet, is really retrograde. [7700]

The ratio $\frac{K'}{K}$, so far as it depends upon the action of the planet Saturn, as we have seen in [7681], is inversely proportional to the fifth power of the semi-axis of the orbit of the satellite,* or of its mean distance from Saturn. [7701]

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given by La Place, in [7695, 7696], agree with those which are deduced from [5786h, i]; and it will be found, upon examination, that M. Plana's calculations lead to the same results, after correcting for *two* small mistakes in his calculation. Now without entering into a minute discussion and explanation of the method used by M. Plana, we shall merely observe, that he deduces his results from the formulas [1337b], with the value of R [7597], and that of v or μ [12860], which is similar to [5344]. But in making the reductions in page 346, line 7, of his memoir, I have found that he accidentally omits a term, connected with the factor $1-\frac{3}{2}.\sin.^2 w$; the corrected value being $1-\frac{3}{2}.\sin.^2 w-\frac{1}{4}.\sin.^2 w.\cos.(2\delta+2\nu)$, according to his notation. This *first* mistake is not particularly noticed, either by La Place or M. Poisson, in their remarks on this subject, in the *Connaisance des tems* for the years 1829, 1831. The *second* mistake of M. Plana was pointed out by M. Poisson, in the *Connaisance des tems* for the year 1831, page 38. It arises from M. Plana's having neglected the effect of the reduction of the longitude of the satellite to the plane of the orbit of the planet; and it is rather singular that this *second* mistake produces a correction in the same factor of exactly the same form and magnitude as the first; so that the true value of this factor becomes, [7696s] [7696t] [7696u] [7696v] [7696w]

$$1-\tfrac{3}{2}.\sin.^2 w-\tfrac{1}{2}.\sin.^2 w.\cos.(2\delta+2\nu), \quad \text{instead of} \quad 1-\tfrac{3}{2}.\sin.^2 w.$$ [7696x]

Finally we may remark, that the calculations of M. Poisson, in the *Connaisance des tems* for 1831, agree with those given by La Place in [7695, 7696], and also with those corrected formulas of M. Plana [7696w, &c.]. [7696y]

* (3716) We have, in [7681], $\frac{K'}{K}=\left(\frac{162}{335}\cdot\frac{T'^2}{t^2}\right)\cdot\frac{1}{a^5}$; and as T', t [7670, 7676] are the same for all the satellites, we shall have $\frac{K'}{K}$ proportional to $\frac{1}{a^5}$. Now for the outer satellite $a=59,154$ [7676], and for the next inferior satellite $a=20,295$ [7702]; [7701a]

Therefore, for the sixth or the outer satellite except one, we must multiply
[7702] the preceding value of $\frac{K'}{K}$ by $\left(\frac{59{,}154}{20{,}295}\right)^5$, to obtain the value of $\frac{K'}{K}$ corresponding to it. Hence we have,

[7703] $$\frac{K'}{K}=88{,}754\,;$$

which gives,

[7704] $$\theta=3088''_c.$$

Therefore the inclination of the intermediate plane [7625] to the equator
[7705] of Saturn, is insensible to us; and as the satellite will move very nearly upon this plane, if b be nothing or very small,* we see that Saturn's

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[7701b] therefore the value of $\frac{K'}{K}$, corresponding to this inferior satellite, will be found by multiplying the value $\frac{K'}{K}=0{,}4219..$ [7681], corresponding to the outer satellite, by
[7701c] $\left(\frac{59{,}154}{20{,}295}\right)^5$; and by this means it becomes as in [7703]. Substituting this in [7645f], also the values of $\cos.2A=\frac{1}{2}$, $\sin.2A=\frac{1}{2}.\sqrt{3}$ [7682b], we get,

[7701d] $$\text{tang.}2\theta=\frac{\frac{1}{2}.\sqrt{3}}{88{,}754+\frac{1}{2}}=0{,}009703=\text{tang.}6177''\,;$$

hence $2\theta=6177''$, or $\theta=3088''$, which differs a little from the calculation of the
[7701e] author, who gives in [7704] $\theta=2933'',6$, which we have altered to $3088''_c$; the mark $_c$ being placed as usual to denote this alteration.

[7705a] * (3717) The angle formed by the fixed plane and the orbit of the satellite, is $\varpi=HKI$, fig. 97, page 324 [7628e], and we have $b=\sqrt{p-q.\cos.2\Pi}.\sin.\varpi$ [7654]; hence if $b=0$, we shall have $\varpi=0$; and if b be small, we shall have ϖ small;
[7705b] observing that for the inner satellites q is much smaller than p, as is evident from [7649, 7650]. For in this case K is very small in comparison with K' [7703, &c.];
[7705c] therefore we have very nearly $\sqrt{(K^2+2KK'.\cos.2A+K'^2)}=K'+K.\cos.2A=K'+\frac{1}{2}K$ [7701c]; substituting this in [7649, 7650], we get $q=\frac{1}{8}K$, $p=K'+\frac{5}{8}K$; whence
[7705d] $\frac{q}{p}=\frac{1}{8}.\frac{K}{K'}$ nearly; and if we use the value of $\frac{K}{K'}$ for the sixth satellite, given in [7703], it becomes $\frac{q}{p}=\frac{1}{710}$ nearly. This must be still further decreased for the inner satellites,
[7705e] in the ratio mentioned in [7701] nearly. Hence we see that for these satellites if b be small, ϖ will also be small [7654]; therefore they must move very nearly upon their intermediate planes [7624, 7626], coinciding almost with the plane of the equator.

If we neglect the forces arising from the action of the satellites and from the oblateness of Saturn, the expression of K' [7621, 7620, 7597] will vanish, and we shall get, from [7646],

action can retain this satellite in very nearly the same plane; and much more so those satellites which are inferior to it, as well as the rings of Saturn. [7705′] This is conformable to what we have demonstrated in [3689].

However, if the mass of the outer satellite be a two hundredth part of that of Saturn, the fixed plane, upon which the orbit of the next inner or sixth satellite moves, will be so much inclined to the plane of the ring that the satellite will vary from this last plane by a sensible quantity. [7706] To prove this, we shall observe that the fixed plane, upon which we suppose the orbit of [7706′] the satellite to move, may be determined by supposing the satellite to move in that plane, and to be retained in it, by the mutual destruction of the forces which tend to draw it from the plane. To prove this we shall resume the expression of s [7623];

$$s = Kv.\sin.\lambda.\cos.\lambda.\cos.v - K'v.\sin.\gamma.\cos.\gamma.\cos.(v-\Psi). \qquad [7707]$$

The fixed plane, upon which the orbit of the outer satellite moves, being [7707′] inclined to the equator of Saturn by the angle θ [7625], if we suppose the [7708] orbit of the satellite to coincide with this plane, we shall have,*

$$\gamma = \theta, \qquad \lambda = A - \theta, \qquad \Psi = 0; \qquad [7709]$$

$\text{tang}.2\theta = \frac{K.\sin.2A}{K.\cos.2A} = \text{tang}.2A$, or $\theta = A$; hence it follows that the fixed plane IK, [7705f] fig. 98, page 324, upon which the orbit of the satellite moves, will then be the orbit of Saturn; and it will have an annual retrograde motion of $-692'',76$ [7699], arising from the sun's disturbing force. As $\frac{K'}{K}$ increases, the fixed plane, which is determined by the [7705g] angle θ [7646], approaches nearer to the equator; and when $\frac{K'}{K}$ is very large, these two planes nearly coincide. Hence we see that the oblateness of Saturn, upon which K' [7705h] [7621] chiefly depends, keeps the orbits of the inner satellites and the rings nearly in the plane of the equator.

* (3718) In this case the arc NHK, fig. 98, page 324, must coincide with IK, and the points N, H, will fall in I; the angle $NHI = N'HK = \gamma$, will become equal to [7709a] $NIK = \theta$; hence $\gamma = \theta$, as in the first of the equations [7709]. Moreover $HNI = \lambda$ [7709b] will become equal to $KIN_{,} = A - \theta$; hence $\lambda = A - \theta$, as in the second of the equations [7709]. Lastly, $NH = \Psi$ [7628c] will vanish, as in the third of the equations [7709]. Substituting these values of γ, λ, Ψ [7709], in the expression of s [7707], it becomes as [7709c] in [7710]. Putting this value of s equal to nothing, and dividing by the common factor $v.\cos.v$, we get the equation [7711]; which is the same as that assumed in [7645] for the [7709d] determination of θ.

therefore by substituting these values of γ, λ, Ψ in [7707], we have,

[7710] $$s = v.\cos.v.\{K.\sin.(A-\theta).\cos.(A-\theta)-K'.\sin.\theta.\cos.\theta\};$$

s will therefore vanish, and the satellite will remain upon the fixed plane, if we have,

[7711] $$K.\sin.(A-\theta).\cos.(A-\theta) = K'.\sin.\theta.\cos.\theta;$$

[7711′] and this is the same as the equation [7645], which we have used in the determination of the inclination θ of the fixed plane to the equator.

[7712] *We shall now consider the sixth satellite, and shall suppose that a, v, s, θ, K, K', correspond to it; and that m', θ', correspond to the seventh or* [7712′] *outer satellite. We shall also put θ' for the inclination of the fixed plane of the seventh satellite to the equator; and we shall suppose the two satellites* [7713] *to move in their fixed planes.* It is evident, by what has been said, that the action of the seventh satellite introduces, into the expression of s, the following term,*

[7714] $$\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.v.\sin.(\theta'-\theta).\cos.(\theta'-\theta).\cos.v;$$

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* (3719) In [7680′, 7681] we have noticed only the chief term of K', arising from the oblateness of Saturn, neglecting those depending on B [7621]; and if we compare [7714a] [7617, 7619], we get $B=\frac{3m'.a^3a'^2}{2.(a^2+a'^2)^{\frac{5}{2}}}$; therefore K' [7621] must be increased by the term $\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}$; and this produces in s [7623 or 7707], the term,

[7714b] $$-\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.v.\sin.\gamma.\cos.\gamma.\cos.(v-\Psi).$$

Now we may remark, that in [7603, &c.] the disturbing satellite m' is *supposed to move in the plane of the equator,* and that $\gamma = NHI = N'HK$; fig. 98, page 324, represents [7714c] the angular inclination of the arc HK of the orbit of m, to the arc HN' of the orbit of m'; but we have supposed, in [7713], that the orbits of the satellites m', m, coincide with their fixed planes respectively; and that these planes are inclined to the equator by [7714d] the angles θ', θ, respectively, passing through the node I. The difference of these angles [7714e] $\theta-\theta'$, represents the inclination of the orbit of m relative to that of m', so that we shall have $\gamma=\theta-\theta'$ [7714b—d]. Substituting this in [7714b], and putting also as in [7709] $\Psi=0$, it becomes,

[7714f] $$-\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.v.\sin.(\theta-\theta').\cos.(\theta-\theta').\cos.v,$$

which is easily reduced to the form [7714].

hence we shall have,*

$$s = v.\cos.v.\left\{K.\sin.(A-\theta).\cos.(A-\theta) - K'.\sin.\theta.\cos.\theta + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\sin.(\theta'-\theta).\cos.(\theta'-\theta)\right\};$$ [7715]

and the fixed plane, relative to the sixth satellite, will therefore be determined by the equation,

$$0 = K.\sin.(A-\theta).\cos.(A-\theta) - K'.\sin.\theta.\cos.\theta + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\sin.(\theta'-\theta).\cos.(\theta'-\theta).$$ [7715']

From this we deduce,†

$$\tang.2\theta = \frac{K.\sin.2A + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\sin.2\theta'}{K' + K.\cos.2A + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\cos.2\theta'}.$$ [7716]

We have by observation, as in [7682, 7702, 7670, &c.],

$$A = 33^{\circ},3333; \qquad a = 20,295; \qquad a' = 59,154;$$
$$T = 15^{\text{days}},9453; \qquad T' = 10759^{\text{days}},08;$$ [7717]

and by putting as in [7683, 7703, 7668],

$$\theta' = 24^{\circ},0083; \qquad \frac{K'}{K} = 88,754;$$ [7718]

$$\tfrac{3}{4}.\frac{T^2}{T'^2} = K;$$ [7719]

we shall have,‡

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* (3720) Connecting together the terms of s [7710, 7714], we obtain the expression [7715]. Putting this equal to nothing, and dividing by $v.\cos.v$, we obtain the equation [7715'], which is similar to [7711]. Substituting the values [7645b, &c.], namely, [7715a]

$$\sin.\theta.\cos.\theta = \tfrac{1}{2}.\sin.2\theta; \qquad \sin.(A-\theta).\cos.(A-\theta) = \tfrac{1}{2}.\sin.(2A-2\theta);$$
$$\sin.(\theta'-\theta).\cos.(\theta'-\theta) = \tfrac{1}{2}.\sin.(2\theta'-2\theta);$$ [7715b]

in [7715'], and then multiplying by 2, we get,

$$0 = K.\sin.(2A-2\theta) - K'.\sin.2\theta + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\sin.(2\theta'-2\theta).$$ [7715c]

† (3721) Developing $\sin.(2A-2\theta)$, $\sin.(2\theta'-2\theta)$, as in [7645d]; substituting the results in [7715c], and arranging separately the terms depending on $\sin.2\theta$, $\cos.2\theta$, we get, [7716a]

$$\left\{K' + K.\cos.2A + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\cos.2\theta'\right\}.\sin.2\theta = \left\{K.\sin.2A + \frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}.\sin.2\theta'\right\}.\cos.2\theta.$$ [7716b]

Substituting $\sin.2\theta = \cos.2\theta.\tang.2\theta$, then rejecting the common factor $\cos.2\theta$, and dividing by the coefficient of $\tang.2\theta$, we get [7716].

‡ (3722) Multiplying the expression [7719] by $\frac{m'.a^3a'^2}{(a^2+a'^2)^{\frac{5}{2}}}\cdot\frac{T'^2}{T^2}$, we get, [7720a]

[7720] $$\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}=13921,21.m'K.$$

[7720′] If we suppose $m'=\frac{1}{200}$, we shall obtain,

[7721] $$\theta=10^\circ,622.$$

[7721′] *This inclination is too great to have escaped the notice of observers, who have*
[7722] *not discovered any sensible deviation in the motion of the sixth satellite from the plane of the ring. We cannot therefore suppose* m' *to have a greater value than* $\frac{1}{200}$ [7720′], *and there is reason to believe that it is even less.* This will appear very probable, when we consider that the mass of the
[7722′] greatest of Jupiter's satellites [7162—7165], is not a ten thousandth part of that of the planet, and that the outer satellite of Saturn is seen, with difficulty, on account of its smallness.

37. *The fixed planes to which we have referred the orbits of the two outer satellites of Saturn, are analogous to those to which we have referred the orbits*
[7723] *of the moon and the satellites of Jupiter in* [5352, 6358]. These planes pass always through the nodes of the equator and the orbit of Saturn, between these two last planes; the orbits of the satellites move upon them, retaining nearly a constant inclination, and their nodes have a retrograde and nearly uniform motion. But these planes are not rigorously fixed; their positions
[7723′] vary by the motions of the equator and orbit of Saturn. We shall now determine these motions, and their influence upon the motions of the orbits of the satellites. We shall use the following symbols:

[7720a′] $$\frac{3m'.a^3a'^2}{4.(a^2+a'^2)^{\frac{5}{2}}}=\left\{\frac{a^3a'^2}{(a^2+a'^2)^{\frac{5}{2}}}\cdot\frac{T'^2}{T^2}\right\}.m'K.$$

Substituting the values [7717] in the coefficient of $m'K$, in the second member of the preceding equation, we find it equal to 13921,21; hence this equation becomes as in [7720]. This calculation may be abridged by putting, for a moment, $\frac{a'}{a}=\text{tang}.C$, or
[7720b] $a'=a.\text{tang}.C$; whence $\frac{a^3a'^2}{(a^2+a'^2)^{\frac{5}{2}}}=\frac{\text{tang}.^2C}{\text{sec}.^5C}=\text{tang}.^2C.\cos.^5C$. Now substituting [7720] in [7716], and dividing the numerator and denominator of the second member by K, putting also $\sin.2A=\sqrt{\frac{3}{4}}$, $\cos.2A=\frac{1}{2}$ [7682b], we get,

[7720c] $$\text{tang}.2\theta=\frac{\sqrt{\frac{3}{4}}+13921,21.m'.\sin.2\theta'}{\frac{K'}{K}+\frac{1}{2}+13921,21.m'.\cos.2\theta'}.$$

Lastly, putting $m'=\frac{1}{200}$ [7720′], and using the values of θ', $\frac{K'}{K}$ [7718], we get 2θ, and then θ, as in [7721].

θ_1 = the inclination of the equator of Saturn to a fixed plane, which is but very little inclined to the orbit of the planet; [7724]

ψ_1 = the distance of the descending node of the equator of Saturn upon this fixed plane, from a fixed axis, drawn from the centre of Saturn in the same plane; the direction of this axis X being more advanced, according to the order of the signs, than that node; [7725]

nt = the angular rotatory motion of the planet Saturn about its axis; [7726]

A, B, C, are the momenta of inertia of the mass of Saturn, relative to the three principal axes. [7726']

Then we shall have, as in [3040, 3041, &c.],*

$$\frac{d\theta_1}{dt} = \frac{(A+B-2C)}{2n.C}.P'; \qquad [7727]$$

$$\frac{d\psi_1}{dt}.\sin.\theta_1 = \frac{(2C-A-B)}{2n.C}.P; \qquad [7728]$$

P and P' being determined by the equations [3016, 3017],

$$P = \frac{3L}{r^5}.\left\{(Y^2-Z^2).\sin.\theta_1.\cos.\theta_1 + YZ.(\cos.^2\theta_1-\sin.^2\theta_1)\right\}; \qquad [7729]$$

$$P' = \frac{3L}{r^5}.\{XY.\sin.\theta_1 + XZ.\cos.\theta_1\}. \qquad [7730]$$

The symbols used in these equations are,

L = the mass of the attracting body [2963]; [7731]

r = the distance of the attracting body from the centre of Saturn; [7732]

X, Y, Z, represent the three co-ordinates of the attracting body [3004, &c.]; [7732'] X, Y, being in the fixed plane, and the axis of X, where the origin of the angle ψ_1 is placed, is drawn from the centre of Saturn towards the *descending node* of the equator of Saturn.

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* (3723) The notation which is used in [7724—7726, &c.], is similar to that in [3006a—b], corresponding to the earth. [7727a] The inclination of the equator and orbit of the earth θ [2907g or 167v], being changed into θ_1 [7724], for the equator and orbit of [7727b] Saturn; and ψ [2907e, &c. or 167v], for the earth, being changed into ψ_1 [7725] for [7727c] Saturn. Then we have, in [3038, 3039], $\Sigma.k.\cos.(it+\varepsilon) = P$; $\Sigma.k'.\sin.(it+\varepsilon) = P'$. Substituting these in [3040, 3041], after accenting θ, ψ, as in [7727b, c], we get the [7727d] expressions [7727, 7728]. The same changes being made in the values of P, P' [3016, 3017], we obtain [7729, 7730]; observing that $r_{\prime}$ [2965] is changed into r [7732], [7727e] being the distance from the attracting to the attracted body. [7727f] Moreover the co-ordinates X, Y, Z [3004—3005], correspond to the similar symbols [7732']; observing that the [7727g] moveable vernal equinox [3004] corresponds to the descending node of the equator [7732'].

We shall also put,*

[7733] $\lambda =$ the inclination of the orbit of the attracting body L to the fixed plane;

[7734] $\Lambda =$ the longitude of the *ascending* node of the orbit of the attracting body L, upon the fixed plane, counted from the *descending node of Saturn's equator*, or axis of X;

[7735] $v =$ the angular distance of the attracting body L, from its ascending node [7734];

[7736] X', Y', Z', are the co-ordinates of the attracting body L, referred to an axis X', which is drawn from the centre of Saturn to the ascending node of that body [7734], and to two other axes perpendicular to it; the axis Y' being in the fixed plane, and the axis Z' perpendicular to the same plane.

Then we shall have,†

[7737] $$X' = r.\cos.v;$$

[7738] $$Y' = r.\cos.\lambda.\sin.v;$$

[7739] $$Z' = r.\sin.\lambda.\sin.v.$$

[7733a] * (3724) For illustration we may refer to the annexed figure 100, where ♈☊$N_{,}$ is the fixed plane; C the centre of Saturn, or the origin of the co-ordinates; C♈ the line drawn from C towards the *descending node*

[7733b] *of Saturn's equator* upon the fixed plane, corresponding to the *vernal equinox* in the theory of the earth's motion; L the place of the attracting satellite. Then the angle L☊$N_{,} = \lambda$ [7733]; the angle ♈C☊ $= \Lambda$

[7733c] [7734]; the angle ☊$CL = v$ [7735]; C♈ is the axis of X; that of Y is perpendicular to it in the fixed plane; that of Z is perpendicular to the fixed plane. C☊ is the

[7733d] axis of X', in the fixed plane; the axis of Y' is in the same plane, and perpendicular to C☊; the axis of Z' is the same as that of Z.

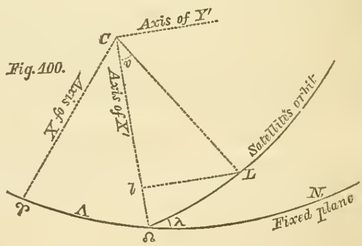

Fig. 100.

[7737a] † (3725) From the place of the attracting satellite L, fig. 100, draw Ll perpendicular to the axis of X', or C☊; then we have $Cl = CL.\cos.LC$☊, or $X' = r.\cos.v$, as

[7737b] in [7737]; also $Ll = CL.\sin.LC$☊ $= r.\sin.v$. Now the line Ll is inclined to the fixed

[7737c] plane by the angle λ [7733]; therefore its projection on the fixed plane, or Y', is

[7737d] $Y' = Ll.\cos.\lambda = r.\cos.\lambda.\sin.v$ [7738]; moreover the distance Z', of the point L from the fixed plane, is $Z' = Ll.\sin.\lambda = r.\sin.\lambda.\sin.v$, as in [7739].

Then we shall have,*

$$X = X'.\cos.\text{A} - Y'.\sin.\text{A}\,;$$ [7740]

$$Y = X'.\sin.\text{A} + Y'.\cos.\text{A}\,;$$ [7741]

$$Z = Z'.$$ [7741']

Therefore,

$$X = r.\cos.\text{A}.\cos.v - r.\cos.\lambda.\sin.\text{A}.\sin.v\,;$$ [7742]

$$Y = r.\sin.\text{A}.\cos.v + r.\cos.\lambda.\cos.\text{A}.\sin.v\,;$$ [7743]

$$Z = r.\sin.\lambda.\sin.v.$$ [7744]

If we neglect the periodical terms depending on the angle v, we shall have,†

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* (3726) In figure 101, L' is the projection of the place of the attracting satellite L, upon the fixed plane; so that $Cl = X'$, $lL' = Y'$; also $CB = X$, $BL' = Y$, and the angle $ACl = \text{A}$. Now draw lA parallel to $L'B$, and lD parallel and equal [7740a]
to AB, then we have $CB = CA - Dl$, $BL' = Al + DL'$. [7740b] But in the rectangular triangle CAl, having $Cl = X'$, and the angle $ACl = \text{A}$, we get, [7740c]

$$CA = X'.\cos.\text{A}, \qquad Al = X'.\sin.\text{A}\,;$$

moreover, in the rectangular triangle $L'Dl$, we have $L'l = Y'$, and the angle $DL'l = \text{A}$; therefore $DL' = Y'.\cos.\text{A}$, $Dl = Y'.\sin.\text{A}$. [7740d]

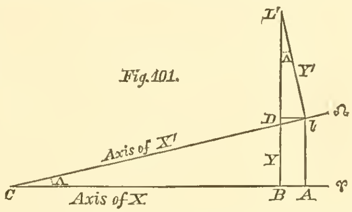

Fig. 101.

Substituting these expressions in the values of $CB = X$, $BL' = Y$ [7740b], we get [7740, 7741]; lastly we have, as in [7733d], [7740e]
$Z = Z'$ [7741']. If we now substitute the values of X', Y', Z' [7737—7739], in those [7740f]
of X, Y, Z [7740—7741'], we shall get [7742—7744].

† (3727) If we retain only the terms which are independent of v, we shall have, by [1, 6, 31] Int.,

$$\sin.^2 v = \tfrac{1}{2}, \qquad \cos.^2 v = \tfrac{1}{2}, \qquad \sin.v.\cos.v = \tfrac{1}{2}.\sin.2v = 0.$$ [7745a]

Substituting the values of X, Y, Z [7742—7744], in the first members of [7745b—f], and using the expressions [7745a], we get the first forms of their second members, which are afterwards reduced;

$$Y^2 = \tfrac{1}{2}r^2.\{\sin.^2\text{A} + \cos.^2\lambda.\cos.^2\text{A}\} = \tfrac{1}{2}r^2.\{\sin.^2\text{A} + (1-\sin.^2\lambda).\cos.^2\text{A}\}$$
$$= \tfrac{1}{2}r^2.\{(\sin.^2\text{A} + \cos.^2\text{A}) - \sin.^2\lambda.\cos.^2\text{A}\} = \tfrac{1}{2}r^2.\{1 - \sin.^2\lambda.\cos.^2\text{A}\}\,;$$ [7745b]

$$Z^2 = \tfrac{1}{2}r^2.\sin.^2\lambda\,;$$ [7745c]

$$XY = \tfrac{1}{2}r^2.(1-\cos.^2\lambda).\sin.\text{A}.\cos.\text{A} = \tfrac{1}{2}r^2.\sin.^2\lambda.\sin.\text{A}.\cos.\text{A}\,;$$ [7745d]

$$XZ = -\tfrac{1}{2}r^2.\sin.\lambda.\cos.\lambda.\sin.\text{A}\,;$$ [7745e]

$$YZ = \tfrac{1}{2}r^2.\sin.\lambda.\cos.\lambda.\cos.\text{A}.$$ [7745f]

The expressions [7745d—f] are the same as those in [7746, 7747, 7748]. Subtracting [7745c] from [7745b], and substituting $1-\sin.^2\lambda = \cos.^2\lambda$, we get [7745].

[7745] $$Y^2-Z^2=\tfrac{1}{2}r^2.\{\cos.^2\lambda-\sin.^2\lambda.\cos.^2\text{A}\};$$

[7746] $$XY=\tfrac{1}{2}r^2.\sin.^2\lambda.\sin.\text{A}.\cos.\text{A};$$

[7747] $$XZ=-\tfrac{1}{2}r^2.\sin.\lambda.\cos.\lambda.\sin.\text{A};$$

[7748] $$YZ=\tfrac{1}{2}r^2.\sin.\lambda.\cos.\lambda.\cos.\text{A}.$$

Hence we obtain,*

[7749] $$\frac{d\theta_1}{dt}=\frac{(A+B-2C)}{C}\cdot\frac{3}{4n}\cdot\frac{L}{r^3}\cdot\{\sin.\theta_1.\sin.^2\lambda.\sin.\text{A}.\cos.\text{A}-\cos.\theta_1.\sin.\lambda.\cos.\lambda.\sin.\text{A}\};$$

[7750] $$\frac{d\Psi_1}{dt}\cdot\sin.\theta_1=\frac{(2C-A-B)}{C}\cdot\frac{3}{4n}\cdot\frac{L}{r^3}\cdot\left\{\begin{array}{l}\sin.\theta_1.\cos.\theta_1.[\cos.^2\lambda-\sin.^2\lambda.\cos.^2\text{A}]\\+[\cos.^2\theta_1-\sin.^2\theta_1].\sin.\lambda.\cos.\lambda.\cos.\text{A}\end{array}\right\}.$$

From these expressions it follows, in the first place, that the satellites, whose orbits are situated in the plane of the equator of Saturn, have not any
[7751] sensible influence upon the values of $\frac{d\theta_1}{dt}$, $\frac{d\Psi_1}{dt}$, because with those bodies we have† $\lambda=\theta_1$, and $\text{A}=200°$, which make these values vanish. The rings may be considered as the aggregate of an infinite number of satellites, and being situated in the plane of the equator, they can have no influence on
[7752] its motions. The equator of Saturn cannot therefore be moved, in a sensible manner, except by the action of the sun, and by that of the outer satellite. Relative to this satellite we have in 1787, by taking the orbit of Saturn at
[7752′] that epoch for the fixed plane,‡

[7749a] * (3728) Substituting the values [7745—7748] in the expressions of P, P' [7729, 7730], and then the results in [7727, 7728], we get [7749, 7750] respectively, without any reductions.

[7751a] † (3729) If the orbit of the satellite NHK, fig. 98, page 324, coincide with the plane of the equator IHN', we shall have the angle $HNI=\lambda$, nearly equal to HIN', or θ_1 [7724]; so that in this case we have very nearly $\lambda=\theta_1$, as in [7751]. Moreover
[7751b] the point I of this figure is the *ascending node* of Saturn's equator, and the *ascending node* of the satellite's orbit; therefore we have, from [7734], $\text{A}=200°$, as in [7751]. Hence we have $\sin.\text{A}=0$, $\cos.\text{A}=-1$, $\lambda=\theta_1$; and by substituting them in the expressions
[7751c] [7749, 7750], they become equal to nothing, as in [7751].

[7753a] ‡ (3730) The expression of λ [7753] is the same as in [7685]. The symbol A [7638] is nearly the same as θ_1 [7724]; hence we have in [7682] the same value of θ_1 as in [7755]. Lastly, we have in [7687c] the arc $NI=24°{,}926$, whose supplement
[7753b] 175°,074 represents the longitude of the point N, or the ascending node of the orbit of the attracting satellite, counted from the descending node of the equator of Saturn in its orbit, being the same as A [7734]. This agrees nearly with [7754].

$$\lambda = \ 25°,222;$$ [7753]

$$\Lambda = 175°,134;$$ [7754]

$$\theta_1 = \ 33°,333.$$ [7755]

Then we have,*

$$\frac{L}{r^3} = L.m^2;$$ [7756]

mt being, in this case, the mean motion of the satellite, and L its mass, that of Saturn being taken for unity. The value of $\frac{2C-A-B}{C}$ is unknown; [7756′]
we shall suppose, as in [6914], that it is, to the corresponding value for the earth, as the ratio of the centrifugal force to gravity, at the surface of the equator of Saturn, is to the similar ratio relative to the earth.† Then we shall suppose that we have for the earth, as in [6917, 6918, 6918b], [7756″]

$$3.(1+\epsilon) = 2,566; \qquad \frac{2C-A-B}{C} = \frac{0,00519323}{1+\epsilon.0,748493} = 0,00582385;$$ [7757]

and as we have by what precedes, for Saturn,

$$\varphi = 0,16597; \quad [7677]$$ [7758]

and for the earth, $\varphi = \frac{1}{289}$ [7679]; we shall have for Saturn, [7759]

$$\frac{2C-A-B}{C} = 0,00582385 \times 289 \times 0,16597 = 0,27934.$$ [7760]

Thus we find, for the annual variation Ψ_1, the following expression; ‡

$$\frac{d\Psi_1}{dt} = 6195''.L.$$ [7761]

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* (3731) The equation [7756], divided by L, gives $\frac{1}{r^3} = m^2$, and this is similar to $\frac{1}{a^3} = n^2$ [6110], relative to Jupiter; a being the mean value of r [6021c], corresponding to [7732]; and n [6022f] corresponds to m [7756′]. [7756a]

† (3732) We have for the earth, as in [6918b], $\frac{(2C-A-B)}{C} = 0,00582385$; and in [7679] the centrifugal force of the earth $\varphi = \frac{1}{289}$. Now the centrifugal force of the earth $\frac{1}{289}$ is to that of Saturn 0,16597 [7677], as the preceding value 0,00582385, relative to the earth, is to the corresponding value relative to Saturn [7760]; upon the principles assumed in [7756″]. [7757a] [7757b]

‡ (3733) The time of the revolution of the outer satellite, $T = 79^{\text{days}},3296$ [7669], gives for its annual motion [7756′], $m = \frac{365,2500}{79,3296} \cdot 400° = 1841°,68$. The symbol n [7761a]

[7762] We have seen, in [7722], that L is less than $\frac{1}{200}$; therefore $\frac{d\Psi_1}{dt}$ is, at the most, $31''$; and there is every reason to believe that it is much less, and that it does not exceed two or three centesimal seconds.

[7763] The value of $\frac{d\Psi_1}{dt}$, arising from the sun's action, is very nearly equal to $0'',878$;* consequently it is insensible.

[7761b] [3015] represents, in the present case, the annual angular rotatory motion of Saturn about its axis; which, by using the value $t=0^{\text{days}},428$ [7676], becomes,

[7761c]
$$n=\frac{365{,}250}{0{,}428}\,.\,400^{\circ}=341355^{\circ}.$$

Substituting these values and that of $\frac{L}{r^3}$ [7756], in the first member of [7761d], and then making successive reductions, using [7760], we get,

[7761d]
$$\frac{(2C-A-B)}{C}\cdot\frac{3}{4n}\cdot\frac{L}{r^3}=\frac{(2C-A-B)}{C}\cdot\frac{3}{4n}\,.\,L.m^2=20817''.L.$$

[7761e] The value of $\theta_1=33^{\circ},333$ [7755], gives $\sin.\theta_1=\frac{1}{2}$, $\cos.\theta_1=\frac{1}{2}.\sqrt{3}$; hence $\sin.\theta_1.\cos.\theta_1=\frac{1}{4}.\sqrt{3}$, $\cos.^2\theta_1-\sin.^2\theta_1=\frac{1}{2}$; substituting these quantities and the last expression of [7761d], in [7750], then multiplying the result by 2, we get [7761f]. The three terms of this formula are computed as in [7761g], using the values of λ, Λ [7753, 7754]; and the final result is as in [7761];

[7761f]
$$\frac{d\Psi_1}{dt}=20817''.L.\{\tfrac{1}{2}.\sqrt{3}.[\cos.^2\lambda-\sin.^2\lambda.\cos.^2\Lambda]+\sin.\lambda.\cos.\lambda.\cos.\Lambda\}$$

[7761g]
$$=L.\{15343''-2296''-6852''\}=6195''.L.$$

[7761h] If we put $L=\frac{1}{200}$ in [7761], it becomes $\frac{d\Psi_1}{dt}=30'',97$; and as we have seen, in [7722], that $L<\frac{1}{200}$, the expression of $\frac{d\Psi_1}{dt}$ must be less than $31''$, as in [7762].

[7763a] * (3734) If the attracting body be supposed to be the sun, moving in its relative orbit, the angle of inclination $KNI=\lambda$, fig. 98, page 324, will become $\lambda=0$. Substituting this in [7750], and then dividing by $\sin.\theta_1$, we get,

[7763b]
$$\frac{d\Psi_1}{dt}=\frac{(2C-A-B)}{C}\cdot\frac{3}{4n}\cdot\frac{L}{r^3}\,.\cos.\theta_1.$$

In this case L represents the sun's mass, and r the distance of Saturn from the sun; so

[7763c] that if we put $M't$ for the mean motion of Saturn about the sun, we shall have, in like manner as in [6105], $\frac{L}{r^3}=M'^2$. The value of M' is easily deduced from the expression

[7763d] of T' [7670], which gives $M'=\frac{365{,}25}{10759{,}08}\,.\,4000000''=135792''$; hence

[7763e] $\frac{L}{r^3}=M'^2=(135792'')^2$. Substituting this, with the values of n, θ_1 [7761c, 7755], and

Hence it follows that the motion of the equator of Saturn upon the orbit of this planet, is much slower than that of the orbit of the outer or seventh satellite; and it is easy to prove, by the formulas of the second and seventh books, that the motion of Saturn's orbit, referred to the equator of this planet, is also much less than that of the orbit of the outer satellite.* This being supposed, we shall resume the equation [7653], [7764] [7765]

$$\frac{d\varpi.\cos.\varpi}{\sin.\varpi} = -\frac{q\,d\Pi.\sin.2\Pi}{p-q.\cos.2\Pi}. \qquad [7766]$$

This equation gives, by neglecting the square of q,†

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the factor [7757a], in [7763b], it becomes $\frac{d\Psi_1}{dt} = 0'',98$, which differs a little from the calculation of the author in [7763], but is, like his result, insensible; being much less than the quantity $6195''.L$ [7761], corresponding to the satellite, as is observed in [7764]. [7763f]

* (3735) The motion of the node of any satellite, arising from the sun's disturbing force, is easily deduced from that of the moon $\frac{3}{4}m^2.v$ [4800b], by merely taking the symbols to correspond to the proposed planet and satellite. In other words, we must suppose, as in [5117a], that v represents the mean motion of the satellite, and mv the mean motion of the planet, so that m represents the ratio of the mean angular motion of the satellite, in comparison with that of the planet. Then the motion of the node of the satellite, arising from the sun's action, will be as above $\frac{3}{4}m^2.v$. In this case we have $m=\frac{T}{T'}$, consequently $\frac{3}{4}m^2=\frac{3}{4}.\frac{T^2}{T'^2}=K$ [7668]; hence the motion of the node is represented by $Kv=0.0000407739.v$ [7671]. Substituting, for v, the annual motion of the satellite 18416800″ [7761a], we find that the annual motion of the node becomes 751″. This quantity is much greater than the annual motion of the node of Saturn's orbit $\frac{d\theta^{v}}{dt}$ [4518 line 3], which is about -9^s, or $-27''$; agreeing with what is stated in [7765]. What is here said relative to $\frac{d\Psi_1}{dt}$ is also true for $\frac{d\theta_1}{dt}$, as is evident from the comparison of the formulas [7749, 7750], and the similar ones relative to the other bodies in [4518, &c.]. [7765a] [7765b] [7765c] [7765d] [7765e] [7765f]

† (3736) If we neglect terms of the order q^2, we may change the divisor $p-q.\cos.2\Pi$, in the second member of [7766], into p; then putting $d\varpi.\cos.\varpi = d.\sin.\varpi$, $d\Pi.\sin.2\Pi = -\frac{1}{2}.d.\cos.2\Pi$, we find that [7766] may be put under the form $\frac{d.\sin.\varpi}{\sin.\varpi} = \frac{1}{2}.\frac{q}{p}.d.\cos.2\Pi$; and its integral is, [7766a] [7766b]

[7767] $$\sin.\varpi = b.\left\{1+\frac{q}{2p}.\cos.2\Pi - \tfrac{1}{2}.\int.\cos.2\Pi.d.\frac{q}{p}\right\};$$

[7768] b being an arbitrary constant quantity. Thus, by neglecting the periodical quantities depending on the angle 2Π, we shall find that *the inclination* ϖ *of the orbit of the outer satellite, upon the intermediate plane between the orbit*
[7769] *and equator of Saturn, remains always the same,** *notwithstanding the variations of this plane. This is conformable to what we have found for the moon, in* [4803]. *Saturn's equator carries with it, in its motion, the*
[7770] *intermediate plane and the orbit of the satellite, which preserves always, upon this plane, the same mean inclination; with a retrograde and nearly uniform motion,*†

[7766c] $$\log.\sin.\varpi - \log.b = \frac{q}{2p}.\cos.2\Pi - \tfrac{1}{2}.\int.\cos.2\Pi.d.\frac{q}{p};$$

as we may easily perceive by taking its differential; b being an arbitrary constant quantity. If we neglect terms of the order q^2, in the second member of this equation, it may be reduced to the form [7766d], as is evident from [58] Int.;

[7766d] $$\log.\sin.\varpi - \log.b = \log.\left\{1+\frac{q}{2p}.\cos.2\Pi - \tfrac{1}{2}.\int.\cos.2\Pi.d.\frac{q}{p}\right\}.$$

Reducing this equation to natural numbers, we get,

[7766e] $$\frac{\sin.\varpi}{b} = 1+\frac{q}{2p}.\cos.2\Pi - \tfrac{1}{2}.\int.\cos.2\Pi.d.\frac{q}{p};$$

and by multiplying by b, we obtain [7767].

[7769a] * (3737) From what is said in [7764—7766], it follows that the angle NKI, fig. 98, page 324, and the point K, vary with much greater rapidity than the angle $IHN_{,}$ and the point I; or, in other words, the variation of the angle ϖ is much greater than that of
[7769b] A, and the variation of the longitude Π, is much greater than that of the equinox I. Now as the symbols K, K' [7621] are nearly constant, the values of q, p [7649, 7650] will vary only in consequence of the slow variations of the angle A; and as the variations
[7769c] of the angle 2Π are much more rapid than those of A, we may consider $\frac{q}{2p}$ as nearly constant, during a revolution of the angle 2Π; and the mean value of $\sin.\varpi$, during this
[7769d] time, will be nearly equal to b; so that the mean value of ϖ will be nearly constant. Thus, notwithstanding the variations which the changes in the value of the angle A produce in the value of θ, and therefore also in the situation of the plane IK, yet the
[7769e] angle $\varpi = NKI$, or the inclination of the satellite's orbit to the intermediate plane IK, will be very nearly constant; but liable to small periodical inequalities.

[7770a] † (3738) We have seen in [7769a, &c.] that the motion of the node K, fig. 98, page 324, is much more rapid than that of the equinox I; and if we neglect the motion of I, we shall find that the motion of K is measured by the variations of $IK = 200^{\circ} - \Pi$

varying however a little on account of the variations of the respective inclinations of the equator and the orbit of Saturn. [7770′]

[7628c], so that $d.IK = -d\Pi$. Substituting this in [7652], we get,

$$\frac{d.IK}{dv} = -p.\cos.\varpi + q.\cos.\varpi.\cos.2\Pi;$$ [7770b]

and as q is much smaller than p [7649, 7650, 7684], we have very nearly,

$$\frac{d.IK}{dv} = -p.\cos.\varpi;$$ [7770c]

so that the variation of IK is very nearly proportional to $-p.\cos.\varpi$. Now as ϖ [7769] is nearly constant, and p [7650] varies only in consequence of the slow variations of A, the motion must be very nearly *uniform;* and as the sign is negative it must be retrograde, [7770d] as in [7770]. The variations of this motion depend on those of A, the angle of inclination of the equator to the orbit of Saturn, as is observed in [7770′].

CHAPTER XVIII.

ON THE SATELLITES OF URANUS.

38. WITH respect to the satellites of Uranus, we have still less
[7771] information than on those of Saturn. Herschel is the only person who has particularly observed them; and it follows, from his observations, that they all move in nearly the same plane, which is almost perpendicular to that of the orbit of the planet; it is therefore the only phenomenon which we have to explain.

If we apply to these bodies the formulas of the preceding chapter, we
[7772] shall see that the action of the planet alone is not sufficient to maintain the orbit of the outer satellite in the plane of the other orbits. We do not know the time of the rotation of Uranus, but it is not probably much less than that
[7773] of Jupiter or Saturn. If we suppose it to be the same as that of Saturn, we shall have, as in [7681],

[7774]
$$K' = K.\frac{162}{335}.\frac{T'^2}{t^2.a^5}.$$

[7775] We have, in this case, $T' = 30689^{\text{days}}$, and according to Herschel, $a = 91,008$; hence we deduce,*

[7776]
$$K' = 0,39824.K.$$

The plane of the equator of Uranus is supposed to be very nearly perpendicular to its orbit, and as A denotes the inclination of these planes
[7777] to each other [7638], we shall put $A' = \frac{1}{2}\pi - A$; π being the
[7778] semi-circumference, whose radius is unity; so that A' will be a very small
[7779] angle. Then if we put $\theta' = \frac{1}{2}\pi - \theta$ in the equation [7646], namely,

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[7776a] * (3739) Using the values [7775] and that of t [7676], we find that the value of K' [7774] becomes as in [7776].

$$\text{tang.}2\delta = \frac{K.\sin.2A}{K'+K.\cos.2A};$$ [7780]

we shall get, very nearly,*

$$\delta' = \frac{KA'}{K-K'}; \quad \text{or} \quad \delta'-A' = \frac{K'A'}{K-K'}.$$ [7781]

Now $A-\delta$ [7628f], or $\delta'-A'$, being the inclination of the plane, upon which the orbit of the satellite moves, to the orbit of the planet, this inclination will be,† [7782]

$$A-\delta = \frac{K'A'}{K-K'} = \frac{0{,}39824}{0{,}60176}.A' = 0{,}6618.A';$$ [7783]

therefore it will be very small, if we notice only the action of the sun and Uranus. The fixed plane will then coincide very nearly with the orbit of the planet, and the outer satellite will finally cease to move in the plane [7784] of the equator of Uranus, in which the orbits of the other satellites are supposed to be situated. But the outer satellite may be retained, in the plane of the equator, by the action of the inferior satellites. To prove this, we shall observe that, if we put α for the ratio of the radius of the outermost satellite but one m', to that of the outermost satellite m, the value of K' [7785] [7621] will be increased, by the action of the inferior satellite m', by the quantity $\frac{1}{4}m'.\alpha.b_{\frac{3}{2}}^{(1)}$;‡ m' being the mass of the inferior satellite, in parts of [7786]

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* (3740) Substituting the values [7777, 7779] in [7780], we get,

$$\text{tang.}(\pi-2\delta') = \frac{K.\sin.(\pi-2A')}{K'+K.\cos.(\pi-2A')}.$$ [7781a]

Now as δ', A', are very small, we have very nearly,

$$\text{tang.}(\pi-2\delta') = -2\delta'; \qquad \sin.(\pi-2A') = 2A'; \qquad \cos.(\pi-2A') = -1.$$ [7781b]

Substituting these expressions in [7781a], we get $-2\delta' = \frac{K.2A'}{K'-K}$; dividing it by -2, [7781c] we obtain δ' [7781]; and by subtracting A', we get $\delta'-A'$ [7781].

† (3741) Subtracting the value of A' [7777] from that of δ' [7779], gives $A-\delta=\delta'-A'$; and by using the value of $\delta'-A'$ [7781], we get the first expression in [7783a] [7783]. The second and third forms in [7783], are deduced from the first, by the substitution of the value of K' [7776].

‡ (3742) The angle γ [7599] represents, in the present case, the inclination of the primitive orbit of the outer satellite to the plane of the equator of Uranus; and if we neglect [7785a] γ^2 on account of its smallness, we may put $\cos.\gamma = 1$; hence the coefficient of $-2aa'$, [7785b] in the denominator of [7615], becomes,

that of Uranus, taken for unity; and $b_{\frac{3}{2}}^{(1)}$ being determined by the formulas

[7787] of § 49, Book II. According to Herschel's observations, we have very nearly $\alpha = \frac{1}{2}$; hence we have nearly,*

[7788] $$b_{\frac{3}{2}}^{(1)} = \frac{31}{12};$$

[7785c] $$\cos.(v-\Upsilon).\cos.v' + \sin.(v-\Upsilon).\sin.v' = \cos.(v-v'-\Upsilon) \quad [24] \text{ Int.}$$

In the present notation a corresponds to the outer satellite, and a' to the next interior

[7785d] satellite, so that $a > a'$, which is the reverse of what is supposed in [6304]; so that this

[7785e] formula now becomes $\alpha = \frac{a'}{a}$, as in [7785]. Substituting the values [7785b, c] in [7615], then dividing the numerator and denominator of the second member by a^3, using α [7785e], we get the expression [7785f]; and by developing the denominator, as in [1005], it becomes as in [7785g].

[7785f] $$a.\left(\frac{dR}{ds}\right) = -\frac{m'.\alpha.\sin.\gamma.\sin.v'}{\{1-2\alpha.\cos.(v-v'-\Upsilon)+\alpha^2\}^{\frac{3}{2}}}$$

[7785g] $$= -m'.\alpha.\sin.\gamma.\sin.v'.\{\tfrac{1}{2}b_{\frac{3}{2}}^{(0)} + b_{\frac{3}{2}}^{(1)}.\cos.(v-v'-\Upsilon) + \&c.\}.$$

The only term of this last expression, producing the angle $v-\Upsilon$, which is retained in

[7785h] [7619], is that depending on $b_{\frac{3}{2}}^{(1)}$; observing that the factor $\sin.v'.\cos.(v-v'-\Upsilon)$, in this term, produces the quantity $\frac{1}{2}.\sin.(v-\Upsilon)$; hence the expression [7785g] contains the following term;

[7785i] $$a.\left(\frac{dR}{ds}\right) = -\tfrac{1}{2}.m'.\alpha.b_{\frac{3}{2}}^{(1)}.\sin.\gamma.\sin.(v-\Upsilon).$$

Putting this equal to the assumed value in [7619], we obtain,

[7785k] $$-B.\sin.\gamma.\cos.\gamma.\sin.(v-\Upsilon) = -\tfrac{1}{2}.m'.\alpha.b_{\frac{3}{2}}^{(1)}.\sin.\gamma.\sin.(v-\Upsilon).$$

Dividing this equation by $-2.\sin.\gamma.\sin.(v-\Upsilon)$, and then putting $\cos.\gamma = 1$ [7785b], we

[7785l] get the value of $\frac{1}{2}B$, which represents the increment of K' [7621], arising from the action of the inner satellite m' upon the outer satellite m; namely,

[7785m] $$\tfrac{1}{2}B = \tfrac{1}{4}.m'.\alpha.b_{\frac{3}{2}}^{(1)} = \text{increment of } K'.$$

[7785n] This differs from the result given by the author in the original work in [7786], namely, increment of $K' = \frac{1}{4}.m'.\alpha^2.b_{\frac{3}{2}}^{(1)}$. We have given the corrected values in [7786, &c.].

* (3743) Substituting the value of $b_{-\frac{1}{2}}^{(1)}$ [989] in $b_{\frac{3}{2}}^{(1)}$ [992], we get,

[7788a] $$b_{\frac{3}{2}}^{(1)} = \frac{3\alpha}{(1-\alpha^2)^2}.\left\{1 - \frac{1.1}{2.4}.\alpha^2 - \&c.\right\}.$$

If we retain only the first two terms of the series, and put $\alpha = \frac{1}{2}$ [7787], we get

[7788b] $b_{\frac{3}{2}}^{(1)} = \frac{31}{12}$, as in [7788]; hence the corresponding increment of K' [7785m] becomes

therefore the value of K' is increased by the quantity $\frac{31.m'}{96}$. If we suppose [7789]
$K=\frac{31.m'}{96}$, we shall have K for the increment of K'; and by adding this [7789']
to the part in [7776], we get,

$$K' = 1,39824.K;$$ [7790]

consequently,*

$$\theta = \frac{A'}{0,39824}:$$ [7791]

therefore the inclination of the orbit of the outermost satellite to the equator of Uranus, will be very small. [7792]

The duration of the sideral revolution of this satellite, is $107^{\text{days}},6944$; [7793]
hence we have, for Uranus,† $K=0,000009235987$; which, by supposing [7794]

$$\frac{31.m'}{96} = K,$$ [7795]

$\frac{31.m'}{96}$, as in [7789]; observing that in the original work it is given $\frac{31.m'}{192}$, in consequence of the mistake mentioned in [7785n]. This quantity $\frac{31.m'}{96}$ being supposed equal to K, [7788c]
as in [7789'], we find, in the subsequent part of the calculation [7792], that this renders the inclination of the orbit of the outer satellite to the equator of Saturn very small, and the probability of this assumed value is examined in [7796, &c.]. [7788d]

* (3744) We have, in [7777], $2A=\pi-2A'$; whence $\sin.2A=\sin.2A'=2A'$ nearly; also $\cos.2A=-1$ nearly. Substituting these values and that of K' [7790] in [7780], then dividing the numerator and denominator of the second member by K, we get [7791a]
$\text{tang}.2\theta=\frac{K.2A'}{K'-K}=\frac{2A'}{0,39824}$. Now as A' is very small [7778], we can satisfy this equation [7791b]
by supposing 2θ to be very small, and equal to $\frac{2A'}{0,39824}$; whence we get the value of θ [7791c]
[7791], and as this is small, we may infer, as we have done in [7705a, &c.] for Saturn's satellites, that the inclination of the orbit of the outermost satellite to the equator of Uranus, is small [7792].

† (3745) We have in [7668] $K=\frac{3}{4}.\frac{T^2}{T'^2}$; substituting $T=107^{\text{days}},6944$ [7793],
$T'=30689^{\text{days}}$ [7775], we get $K=0,0000092359..$ [7794]; putting this equal to the [7795a]
assumed expression of K [7789'], we get $m'=\frac{96}{31}\times 0,0000092359=0,0000286$; being [7795b]
about half the quantity given by the author in the original work [7796], where he puts
$m'=0,0000572035$; which we have decreased, by correcting for the error mentioned in [7795c]

gives,

[7796] $$m' = 0,0000286_c.$$

Now this mass of the satellite, and even a greater mass, can be supposed
[7797] probable [7795d]; therefore the orbit of the outer satellite may be retained in the plane of the equator of the planet, by the action of the inferior satellites. As to the orbits of the other satellites, the action alone of Uranus suffices to maintain them in the plane of the equator of this planet. For the
[7798] ratio of K' to K varies reciprocally as the fifth power of the radius of the orbit [7701]; and it is, for the outermost satellite but one, thirty-two times greater than that relative to the outermost satellite;* so that we then have,

[7799] $$K' = 12,74368.K;$$

which gives,

[7799'] $$\theta = \frac{A'}{11,74368};$$

thus θ is very small and insensible.

[7785n, &c.]. This decrease of the mass serves to confirm the reasoning in [7797, &c.],
[7795d] namely, that the mass m' [7796] is not greater than may be supposed probable; observing that it is nearly of the same order relative to the mass of Saturn, as the masses of Jupiter's satellites [7162, &c.] are relative to that of Jupiter.

[7799a] * (3746) The ratio of the radii being $\frac{1}{2}$ [7787], that of the values of K' [7774] will be inversely as $(\frac{1}{2})^5$, or directly as 32 to 1, as is shown in [7701, 7701a]. Therefore if
[7799b] we multiply the expression of K' [7776], corresponding to the outer satellite, by 32, we shall get the value of K' [7776], corresponding to the interior satellite, namely, $K' = 32 \times 0,39824.K = 12,74368.K$, as in [7799]. Substituting this in $\text{tang.}2\theta = \frac{K.2A'}{K'-K}$
[7799c] [7791b], it becomes $\text{tang.}2\theta = \frac{2A'}{11,74368}$; whence we easily deduce the value of θ [7799'].

NINTH BOOK.

THEORY OF COMETS.

The greatness of the excentricities of the orbits of the comets, and of their inclinations to the ecliptic, do not permit us to apply, to the perturbations of these bodies, the formulas which are used for the planets, in the second and sixth books. It is impossible, in the present state of analysis, to express the perturbations of the comets by analytical formulas, which include, like those of the planets, an indefinite number of revolutions; we can only determine them by parts, and by means of mechanical quadratures. The [7800]
method of Chapter VIII. Book II. is peculiarly appropriate to this object, because it gives, by simple quadratures, the variations in each element of the orbit, supposing it to be a variable ellipsis. Then to obtain, at any instant, the situation of the comet, it is only necessary to substitute, in the usual formulas of the elliptical motion, the elements augmented by these variations. We shall now proceed to develop this method, so that those who wish to apply it to the motion of a comet will find no other difficulty than what arises from the great labor of the numerical process.

CHAPTER I.

GENERAL THEORY OF THE PERTURBATIONS OF COMETS.

1. We shall suppose, as in [916′, &c.], that

$x, y, z,$ represent the rectangular co-ordinates of the comet m, referred to the sun's centre as their origin; [7801]

$x', y', z',$ the rectangular co-ordinates of the disturbing planet m', referred to the same origin; [7801′]

and we shall have, as in [913, 914, &c.],*

[7802] R
$$R=\frac{m'.(xx'+yy'+zz')}{r'^3}-\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}};$$

[7803] r, r', *being the radii vectores of* m, m'. We shall also suppose *the sun's*
[7804] *mass equal to unity*, so that $\mu=1+m$ [914']. We then have, as in [915],

Fundamental equations. First form.

[7805]
$$0=\frac{ddx}{dt^2}+\frac{\mu.x}{r^3}+\left(\frac{dR}{dx}\right);$$

[7806]
$$0=\frac{ddy}{dt^2}+\frac{\mu.y}{r^3}+\left(\frac{dR}{dy}\right);$$

[7807]
$$0=\frac{ddz}{dt^2}+\frac{\mu.z}{r^3}+\left(\frac{dR}{dz}\right).$$

[7807'] *In the case of* $R=0$, *these equations correspond to an elliptical orbit*,† as we
[7808] have seen in the second book [545—555', &c.]; and as the value of R is
Symbols δx, δy, δz.
very small, we may put δx, δy, δz, for the alterations it produces in the values of x, y, z, corresponding to the elliptical orbit, neglecting the squares and products of those quantities. Then the three equations [7805—7807]
[7809] can be reduced to the following forms;

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* (3747) The expression of R [7802] is the same as that in [913, 914], or in [949],
[7802a] retaining only the two bodies m, m'; whose radii vectores are r, r', respectively [914']. Moreover we have, in [914'], $\mu=M+m$; and by putting, as in [7803], the mass of the
[7802b] sun $M=1$, it becomes $\mu=1+m$, as in [7804]. The equations [915] are the same as in [7805—7807].

[7810a] † (3748) If there be no disturbing body we may put $m'=0$ in [7802]; then we shall
[7810b] have $R=0$, as in [7807']; and the equation [7805] will become $0=\frac{ddx}{dt^2}+\frac{\mu.x}{r^3}$, corresponding to the system [545] and to the elliptical orbit [555']. In the disturbed orbit, x changes into $x+\delta x$, r into $r+\delta r$ [7808]; and then the equation [7805] becomes,

[7810c]
$$0=\frac{dd.(x+\delta x)}{dt^2}+\frac{\mu.(x+\delta x)}{(r+\delta r)^3}+\left(\frac{dR}{dx}\right);$$

so that by subtracting the equation [7810b], we get,

[7810d]
$$0=\frac{dd.\delta x}{dt^2}+\frac{\mu.\delta x}{(r+\delta r)^3}+\mu x.\left\{\frac{1}{(r+\delta r)^3}-\frac{1}{r^3}\right\}+\left(\frac{dR}{dx}\right).$$

[7810e] Now by development we have $(r+\delta r)^{-3}=r^{-3}-3r^{-4}.\delta r$, neglecting terms of the order δr^2. Substituting this, with $\mu=1+m$ [7804], in [7810d], and neglecting the terms containing $m\delta x$, $m\delta r$, which are of the order of the square of the disturbing forces, it becomes as in
[7810f] [7810]. In like manner, from [7806, 7807], we obtain [7811, 7812] respectively; or they may be derived from [7810], by changing successively x into y or z, and the contrary.

Fundamental equations. Second form.

$$0=\frac{dd.\delta x}{dt^2}+\frac{\delta x}{r^3}-\frac{3x.\delta r}{r^4}+\left(\frac{dR}{dx}\right); \quad [7810]$$
$$0=\frac{dd.\delta y}{dt^2}+\frac{\delta y}{r^3}-\frac{3y.\delta r}{r^4}+\left(\frac{dR}{dy}\right); \quad (A) \quad [7811]$$
$$0=\frac{dd.\delta z}{dt^2}+\frac{\delta z}{r^3}-\frac{3z.\delta r}{r^4}+\left(\frac{dR}{dz}\right). \quad [7812]$$

It is only necessary to find values of δx, δy, δz, *which will satisfy the* [7813] *equations* [7810—7812]; for by adding these values respectively, to those of x, y, z, corresponding to the elliptical motion, *containing six arbitrary* [7813′] *constant quantities* [602″], we shall obtain the complete integrals of the three primitive differential equations of the comet's motion [7805—7807].

2. *We shall consider the value of* R *in the two extreme limits of the comet's distance from the sun. When the ratio* $\frac{r}{r'}$ *of the radius vector of* [7814] *the comet to that of the planet, is a very small fraction, the value of* $\left(\frac{dR}{dx}\right)$ *will be very small in comparison with* $\frac{\mu x}{r^3}$, and the ratio of the first of these quantities to the second, will be of the order $\frac{m'.r^3}{r'^3}$.* In this case, we may [7815]

Comet far within the planet's orbit.

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* (3749) If we suppose, as in [7870a], that f represents the distance of the comet from the planet m', or the distance of the extremities of the two radii r, r'; and γ the [7815a] angle included by these two radii; we shall have as in [1432a, h], by changing a, b, c, r, R, into x', y', z', r', r, respectively,

$$f^2=(x'-x)^2+(y'-y)^2+(z'-z)^2=r'^2-2rr'.\cos.\gamma+r^2. \quad [7815b]$$

Now the partial differential of R [7802], relative to x, gives the first expression in [7815c]; and by substituting [7815b], then developing and reducing, we finally obtain [7815d];

$$\left(\frac{dR}{dx}\right)=\frac{m'.x'}{r'^3}+\frac{m'.(x-x')}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}=\frac{m'.x'}{r'^3}+\frac{m'.(x-x')}{\{r'^2-2rr'.\cos.\gamma+r^2\}^{\frac{3}{2}}} \quad [7815c]$$
$$=\frac{m'.x'}{r'^3}+\frac{m'.(x-x')}{r'^3}.\left\{1-\frac{2r}{r'}.\cos.\gamma+\frac{r^2}{r'^2}\right\}^{-\frac{3}{2}}=\frac{m'.x'}{r'^3}+\frac{m'.(x-x')}{r'^3}.\left\{1+\frac{3r}{r'}.\cos.\gamma+\&c.\right\}. \quad [7815d]$$

The first and third terms in this last expression mutually destroy each other, and the remaining terms are evidently of the order $\frac{m'.x}{r'^3}$; and this is to the term $\frac{\mu x}{r^3}$ [7805], as [7815e] $\frac{m'.r^3}{r'^3}$ to 1; μ being nearly equal to unity [7804]. This agrees with [7815]; and as the terms of R are so very small, we may neglect them, and then the orbit will become elliptical, as in [7815′].

[7815′] consider R as being nearly equal to nothing, and the motion of the comet as elliptical.

[7816] *If $\frac{r}{r'}$ be a great number, or, in other words, if the comet be much farther off from the sun than the planet is, we may develop R in a series, according*

[7816′] *to the powers of $\frac{r'}{r}$; and by neglecting, in this series, the terms of the order* $\frac{m'.r'}{r^4}$, we shall have,*

Comet far without the planet's orbit.

[7817] $$\left(\frac{dR}{dx}\right)=\frac{m'.(x-x')}{r^3}+\frac{m'.x'}{r'^3}+3m'.(xx'+yy'+zz').\frac{x}{r^5};$$

then the differential equation in δx [7810] becomes,†

[7818] $$0=\frac{dd.\delta x}{dt^2}+\frac{\delta x}{r^3}-\frac{3x.(x\delta x+y\delta y+z\delta z)}{r^5}+\frac{m'.(x-x')}{r^3}+\frac{m'.x'}{r'^3}+3m'.(xx'+yy'+zz').\frac{x}{r^5}.$$

[7818′] The differential equations in δy, δz [7811, 7812], evidently produce similar equations. We shall now suppose,

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* (3750) We have, in [914′], the values of r^2, r'^2 [7817b]. Substituting them in the development of the first expression of

[7817a] $$f^2=(x'^2+y'^2+z'^2)-2.(xx'+yy'+zz')+(x^2+y^2+z^2) \quad [7815b],$$

and then comparing the result with the second expression of f^2 [7815b], we get [7817c];

[7817b] $$r^2=x^2+y^2+z^2; \qquad r'^2=x'^2+y'^2+z'^2;$$

[7817c] $$rr'.\cos.\gamma=xx'+yy'+zz'.$$

Instead of developing the radical $\{r'^2-2rr'.\cos.\gamma+r^2\}^{-\frac{3}{2}}$ [7815c], according to the powers

[7817d] of $\frac{r}{r'}$, we shall now develop it according to the powers of $\frac{r'}{r}$; by this means we shall find, that the last formula in [7815d] becomes, by changing the order of the terms,

[7817e] $$\left(\frac{dR}{dx}\right)=\frac{m'.x'}{r'^3}+\frac{m'.(x-x')}{r^3}.\left\{1+\frac{3r'}{r}.\cos.\gamma+\&c.\right\}=\frac{m'.(x-x')}{r^3}+\frac{m'.x'}{r'^3}+\frac{3m'.(x-x').rr'.\cos.\gamma}{r^5}+\&c.$$

Substituting, in the last term of this expression, the value of $rr'.\cos.\gamma$ [7817c], and neglecting the small terms of the second order in r', x', y', z', it becomes as in [7817].

[7818a] † (3751) The differential of r^2 [7817b], relative to the characteristic δ, being divided by $2r$, gives $\delta r=\frac{x\delta x+y\delta y+z\delta z}{r}$. Substituting this in the third term of [7810], and

[7818b] using the value of $\left(\frac{dR}{dx}\right)$ [7817], we obtain [7818]. If we change reciprocally x into y, we shall get a similar equation in y, corresponding to [7811]; and by changing reciprocally x into z, we get an equation in z corresponding to [7812].

$$\delta x = Ax + A'x' ;$$ [7819]

$$\delta y = Ay + A'y' ;$$ [7820]

$$\delta z = Az + A'z' ;$$ [7821]

and by observing that we have very nearly, as in [7810b, 7804],

$$\frac{ddx}{dt^2} = -\frac{x}{r^3}; \qquad \frac{ddx'}{dt^2} = -\frac{x'}{r'^3};$$ [7822]

we shall find, that the differential equation in δx [7818], will give,*

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* (3752) The object in view, in [7813], is merely to satisfy the equation [7818], and the similar ones in y, z. For this purpose the values [7819—7821] are assumed, with the arbitrary constant quantities A, A'; and it is finally shown, in [7824], that the constant values of A, A' may be found, which will satisfy the differential equation [7818], and the similar ones in y, z. To prove this, we shall first show that the substitution of $\delta x = Ax + A'x'$ [7819] in [7810, or 7818], produces the equation [7823]. For if we substitute this value of δx in the two first terms of [7818], $\frac{dd.\delta x}{dt^2} + \frac{\delta x}{r^3}$, they become, [7823a] [7823b] [7823c]

$$A.\left(\frac{ddx}{dt^2} + \frac{x}{r^3}\right) + A'.\left(\frac{ddx'}{dt^2} + \frac{x'}{r^3}\right);$$ [7823c']

and by using the values [7822], we find that this expression is reduced to the form $A'.\left(-\frac{x'}{r'^3} + \frac{x'}{r^3}\right)$, being the same as the first and second terms of [7823 line 1]. Again, by substituting the values [7819—7821] in the first member of [7823d], we get the second member of the same expression; and by using the value of r^2 [7817b], it becomes as in [7823e];

$$x\delta x + y\delta y + z\delta z = A.(x^2+y^2+z^2) + A'.(xx'+yy'+zz');$$ [7823d]

$$x\delta x + y\delta y + z\delta z = A.r^2 + A'.(xx'+yy'+zz').$$ [7823e]

Multiplying [7823e] by $-\frac{3x}{r^5}$, we get, for the third term of [7818] $-\frac{3x.(x\delta x+y\delta y+z\delta z)}{r^5}$, the expression $-\frac{3A.x}{r^3} - \frac{3A'.x.(xx'+yy'+zz')}{r^5}$, as in the third and fourth terms of [7823 line 1]. Lastly, the three remaining terms of [7818], are the same as those in [7823 line 2]; observing that the first of these terms of [7818], $\frac{m'.x}{r^3}$, is placed the third in order in [7823]. Now substituting $A = \frac{1}{3}m'$, $A' = m'$ [7824], in [7823 line 1], we find that the terms in this line are destroyed respectively by the terms immediately below them, in [7823 line 2]; therefore the equation [7823] is satisfied by these values of A, A'; and by using them we obtain, from [7819—7821], the expressions [7825—7827], which satisfy the equation [7818 or 7810]. The same results will be obtained if we use the equation [7811], changing reciprocally the ordinates x, x', into y, y'; or if we use the equation [7812], changing reciprocally x, x', into z, z'. Hence it appears that the values of δx, δy, δz [7825—7827], satisfy the system of equations [7810—7812], according to the requirements in [7823a]. [7823f] [7823g] [7823h] [7823i] [7823k]

[7823]
$$0=\frac{A'.x'}{r^3}-\frac{A'.x'}{r'^3}-\frac{3A.x}{r^3}-\frac{3A'.x.(xx'+yy'+zz')}{r^5} \qquad 1$$
$$-\frac{m'.x'}{r^3}+\frac{m'.x'}{r'^3}+\frac{m'.x}{r^3}+\frac{3m'.x.(xx'+yy'+zz')}{r^5}. \qquad 2$$

We may satisfy this equation by putting,

[7824] $$A=\tfrac{1}{3}m', \qquad A'=m';$$

which give,

[7825] $$\delta x=\tfrac{1}{3}m'.x+m'.x';$$

[7826] $$\delta y=\tfrac{1}{3}m'.y+m'.y';$$

[7827] $$\delta z=\tfrac{1}{3}m'.z+m'.z'.$$

Therefore these equations satisfy the differential equation in δx [7810]; and it is evident that they also satisfy the differential equations in δy and δz, [7811, 7812].

[7828] *The preceding result is a very simple corollary to the theorem which we have given in* [451″, &c.]. *For it follows, from this theorem, that when the comet is at a great distance from the sun, it may be considered as being attracted towards the common centre of gravity of the sun and planet by a mass equal to the sum of these three bodies;* therefore it will describe, very nearly, an ellipsis about that point; and the attractive force which causes it

[7829] to be described, will be $\frac{1+m+m'}{(r+\delta r)^2}$;* supposing that the radius vector of this last ellipsis is $r+\delta r$, and the corresponding rectangular co-ordinates

[7830] $x+\delta x$, $y+\delta y$, $z+\delta z$. We may suppose this ellipsis to be entirely similar to that whose co-ordinates are x, y, z; and to be described in the

[7831] same time. For this purpose, it is only necessary for the attractive forces in

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[7829a] * (3753) The comet is supposed to be at a great distance from the centre of gravity of the sun and planet; so that it will be acted upon in nearly the same manner as if the mass $1+m'$, of the sun and planet, were placed in the common centre of gravity of these two

[7829b] bodies. Now this mass $1+m'$ acts upon the mass m of the comet, placed at the distance $r+\delta r$, with a force which is represented by the sum of these masses $1+m+m'$, divided by

[7829c] $(r+\delta r)^2$, as in [7829]; that is, by the force $\frac{1+m+m'}{(r+\delta r)^2}$; supposing, as in [7829 or 7834′],

[7829d] that $r+\delta r$ is the radius vector of the comet, referred to the centre of gravity of the sun and planet. If we neglect the planet's mass m', and put r for the radius vector referred to

[7829e] the sun's centre, the force becomes $\frac{1+m}{r^2}$, as in [7832].

the corresponding points to be as $r+\delta r$ to r,* which gives, [7831′]

$$\frac{1+m+m'}{(r+\delta r)^2} : \frac{1+m}{r^2} :: r+\delta r : r.$$ [7832]

Hence we deduce,

$$\delta r = \tfrac{1}{3} m'.r ; \quad [7831f]$$ [7833]

consequently the co-ordinates of this last ellipsis are,†

$$(1+\tfrac{1}{3}m').x ; \qquad (1+\tfrac{1}{3}m').y ; \qquad (1+\tfrac{1}{3}m').z.$$ [7834]

These co-ordinates are referred to the common centre of gravity of the sun and planet. To reduce them to the sun's centre, we must add the co-ordinates of this centre of gravity, referred to the sun's centre. Now these last co-ordinates are evidently represented by $m'x'$, $m'y'$, $m'z'$; therefore the co-ordinates of the comet, referred to the sun's centre, will be, [7834′] [7835]

$$(1+\tfrac{1}{3}m').x+m'x' ; \qquad (1+\tfrac{1}{3}m').y+m'y' ; \qquad (1+\tfrac{1}{3}m').z+m'z' ;$$ [7836]

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* (3754) If two bodies revolve about an attracting point in similar ellipses, and in similar situations, with the same periodical time of revolution, and corresponding radii vectores $r+\delta r$ and r, it is evident that their tangential velocities in their orbits will be as the radii $r+\delta r$ and r; and the deflections from the tangents, in the direction of the attracting force, and in a given time dt, will be in the same ratio $r+\delta r$ to r; because the *orbits are supposed to be similar in form and position.* Now these deflections are evidently proportional to the attractive forces; therefore these forces must be to each other as $r+\delta r$ to r, as in [7831′]; and if we take, for the forces, the values given in [7829c, e], we shall obtain the analogy in [7832]. From this we easily deduce the equation, [7831a] [7831b] [7831c] [7831d]

$$(r+\delta r)^3 = r^3.\left(\frac{1+m+m'}{1+m}\right) = r^3.\left(1+\frac{m'}{1+m}\right) = r^3.(1+m'), \text{ nearly.}$$ [7831e]

Extracting the cube root of this last expression, we get, very nearly, $r+\delta r = r.(1+\frac{1}{3}m')$; whence $\delta r = \frac{1}{3}m'.r$, as in [7833]. [7831f]

† (3755) From the similarity of the forms and positions of the orbits [7831c], it follows that the co-ordinates x, y, z, must have the same relation to the co-ordinates $x+\delta x$, $y+\delta y$, $z+\delta z$, respectively, as the radii r to $r+\delta r$, or 1 to $1+\frac{1}{3}m'$ [7833]; hence we easily obtain the expressions of $x+\delta x$, $y+\delta y$, $z+\delta z$ [7834]. To these we must add the parts $m'x'$, $m'y'$, $m'z'$ [7835], respectively corresponding to X, Y, Z [126, 127]; observing that $X = \frac{\Sigma.mx}{\Sigma.m}$ [126], becomes in this case $\frac{m'.x'}{1+m'}$, or $m'x'$ nearly; because the value of x relative to the sun's centre is 0, and the sun's mass is 1 [7804]. In like manner we get $Y = m'y'$, $Z = m'z'$, from [127]. These parts being added respectively to those in [7834], we obtain the co-ordinates $x+\delta x$, $y+\delta y$, $z+\delta z$ [7836], referred to the sun's centre. [7834a] [7834b] [7834c] [7834d]

which agree with those in [7825—7827]; and as these co-ordinates contain six arbitrary quantities [7813′], they satisfy completely the differential
[7836′] equations of the comet's motion, when we suppose R to be equal to the following function $R_{,}$;*

[7837] $$R_{,}=-\frac{m'}{r}-m'.(xx'+yy'+zz').\left(\frac{1}{r^3}-\frac{1}{r'^3}\right).$$

This being premised we shall suppose,

[7838] $$R'=R+\frac{m'}{r}+m'.(xx'+yy'+zz').\left(\frac{1}{r^3}-\frac{1}{r'^3}\right);$$

[7839] $$\delta x=\tfrac{1}{3}m'.x+m'x'+\delta x_1;$$

[7840] $$\delta y=\tfrac{1}{3}m'.y+m'y'+\delta y_1;$$

[7841] $$\delta z=\tfrac{1}{3}m'.z+m'z'+\delta z_1;$$

and the differential equations in δx, δy, δz [7810—7812] will give,†

[7837a] * (3756) From $r=\sqrt{x^2+y^2+z^2}$ [7817b], we obtain $\left(\frac{dr}{dx}\right)=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$; substituting this in the partial differential of the assumed value of R [7837]; *which, for the*
[7837a′] *sake of distinctness, we shall denote by* $R_{,}$, *though it is not so accented by the author;* we get,

[7837b] $$\left(\frac{dR_{,}}{dx}\right)=\frac{m'.x}{r^3}-m'x'.\left(\frac{1}{r^3}-\frac{1}{r'^3}\right)+3m'.(xx'+yy'+zz').\frac{x}{r^5},$$

being the same as the value of $\left(\frac{dR}{dx}\right)$, which is used in [7817, 7818]; producing the parts of the co-ordinates of the comet's orbit contained in [7825—7827], or those in [7836].
[7837c] The expression of $R_{,}$ [7837] does not contain the whole of the function R [7802], on
[7837d] account of the terms which are neglected in [7816′]; *and if we suppose that these small and neglected terms of* R *are equal to* R', we shall have,

[7837e] $$R=R_{,}+R',\qquad\text{or}\qquad R'=R-R_{,}.$$

Substituting $R_{,}$ [7837] in this value of R', we get [7838]. If we now suppose that the
[7837f] quantity R' has the effect to augment the values of δx, δy, δz, by the terms δx_1, δy_1, δz_1, respectively, their complete values will be as in [7839—7841].

† (3757) Substituting $R=R_{,}+R'$ [7837e] in [7810], we get,

[7842a] $$0=\frac{dd.\delta x}{dt^2}+\frac{\delta x}{r^3}-\frac{3x.\delta r}{r^4}+\left(\frac{dR_{,}}{dx}\right)+\left(\frac{dR'}{dx}\right).$$

If we substitute the values of δx, δy, δz [7839–7841], also $\delta r+\delta r_1$ [7818a, 7845] for δr, we shall find that the parts which are independent of δx_1, δy_1, δz_1, are destroyed by the
[7842b] terms arising from $\left(\frac{dR_{,}}{dx}\right)$, as is evident from the calculations in [7818—7827]; therefore these terms may be neglected; and the remaining quantities, depending on δx_1, δy_1, δz_1,

Fundamental equations. Third form.

$$\left.\begin{array}{l} 0=\frac{dd.\delta x_1}{dt^2}+\frac{\delta x_1}{r^3}-\frac{3x.\delta r_1}{r^4}+\left(\frac{dR'}{dx}\right); \\ 0=\frac{dd.\delta y_1}{dt^2}+\frac{\delta y_1}{r^3}-\frac{3y.\delta r_1}{r^4}+\left(\frac{dR'}{dy}\right); \\ 0=\frac{dd.\delta z_1}{dt^2}+\frac{\delta z_1}{r^3}-\frac{3z.\delta r_1}{r^4}+\left(\frac{dR'}{dz}\right). \end{array}\right\} (B)$$ [7842] [7843] [7844]

In these equations δr_1 is what δr [7818a] becomes, when we change δx, δy, δz, into δx_1, δy_1, δz_1. These equations differ from those in [7810—7812] only by the change of R into R'. *They may be used with advantage in the calculation of the perturbations of the comet, in the superior part of its orbit; because R' is then very small* [7816′, 7837d]. [7845]

Variations of the elements of the comet's orbit.

3. *We shall now consider the variations of the elements of the orbit. We shall take for the fixed plane the primitive orbit of the comet, which permits us* [7846] *to neglect the square of z, being of the order of the square of the disturbing* [7846′] *force.* If we suppose, as in [1022, 1010′],

$$h=e.\sin.\varpi; \qquad l=e.\cos.\varpi;$$ [7847]

$e=$ the ratio of the excentricity of the orbit to the semi-major axis; [7848]

$\varpi=$ the longitude of the perihelion, counted from the axis of x; [7848′]

we shall have, as in [1176, &c.],*

$$dh=dx.\left\{x.\left(\frac{dR}{dy}\right)-y.\left(\frac{dR}{dx}\right)\right\}+(xdy-ydx).\left(\frac{dR}{dx}\right);$$ [7849]

$$dl=dy.\left\{y.\left(\frac{dR}{dx}\right)-x.\left(\frac{dR}{dy}\right)\right\}+(ydx-xdy).\left(\frac{dR}{dy}\right).$$ [7850]

δr_1, R', will produce the equation [7842]. In like manner we may deduce [7843, 7844] from [7811, 7812]; or they may be more easily derived from [7842], by changing [7842c] reciprocally x into y, or x into z.

* (3758) Neglecting the square of the disturbing forces, as in [1253′], we shall have, from [1251], $\mu e.\sin.\varpi=f'$, $\mu e.\cos.\varpi=f$; substituting the values [7847], we get [7849a] $\mu h=f'$, $\mu l=f$; whose differentials are $\mu dh=df'$, $\mu dl=df$; and since df', df [7849b] [1257], or dh, dl, are of the order of the disturbing forces R, we may, by neglecting the square of these forces, put $\mu=1$ [7804]; and then the differential equations [7849b] [7849c] become $dh=df'$, $dl=df$. Substituting now the values of df', df [1176], and neglecting the terms containing z or dz, multiplied by the partial differentials of R, [7849d] because they are of the order of the square of the disturbing masses, they become as in [7849e] [7849, 7850].

These two equations give the values of de, $d\varpi$; for we have,*

[7851] $$de = dh.\sin.\varpi + dl.\cos.\varpi;$$

[7852] $$ed\varpi = dh.\cos.\varpi - dl.\sin.\varpi;$$

[7852′] *and if, for greater simplicity, we take the line of apsides for the axis of* x, we shall obtain,

[7853] $$de = dl; \qquad ed\varpi = dh.$$

The equations of the elliptical motion give, as in [605′, 606, &c.],†

Elliptical motion.

[7854] [7855] [7856] [7857]
$$\left.\begin{array}{c} fndt + \varepsilon - \varpi = u - e.\sin.u; \\ r = a.(1 - e.\cos.u); \\ \operatorname{tang}.\tfrac{1}{2}.(v - \varpi) = \sqrt{\frac{1+e}{1-e}}.\operatorname{tang}.\tfrac{1}{2}u; \\ n = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}. \end{array}\right\} (O)$$

Supposing the symbols to be represented by,

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* (3759) The differentials of [7847] give,

[7851a] $$dh = de.\sin.\varpi + ed\varpi.\cos.\varpi; \qquad dl = de.\cos.\varpi - ed\varpi.\sin.\varpi.$$

Multiplying this value of dh by $\sin.\varpi$, and that of dl by $\cos.\varpi$, then taking the sum of
[7851b] the products, we get, by a slight reduction, the value of de [7851]. Again, multiplying dh, dl [7851a], by $\cos.\varpi$, $-\sin.\varpi$, respectively, and taking the sum of the products,
[7851c] we get the expression of $ed\varpi$ [7852]. Now in [1188′] the longitude of the perihelion is ϖ, and the longitude of the disturbed body m is v; both being counted from the same axis of
[7851d] x, as in [500″, &c. 7846]; so that if we suppose this axis to coincide with the perihelion,
[7851e] as in [7852′], we shall have $\varpi = 0$; then the expressions [7851, 7852] become as in
[7851f] [7853]; and [7847] changes into $h=0$, $l=e$. Lastly, the integrals of [7853] give $\delta e = \delta l$, $e\delta\varpi = \delta h$; which are used in [7978].

[7854a] † (3760) The equations [7854—7857] are the same as in [605′, 606], changing v into $v - \varpi$, and nt into $fndt + \varepsilon - \varpi$, to conform to the present notation; as is evident by comparing the notation in [602‴, &c.] with that in [7858, 7859]; observing that the
[7854b] substitution of the value of n [605′] in [601], multiplied by n, gives $ndt = du.(1 - e.\cos.u)$, whose integral, considering n as variable, is as in [7854]; $\varepsilon - \varpi$ being added to complete the integral, being equivalent to nT in the equation [602], where n is considered as
[7854c] constant. Comparing the results of the present notation with those in [606, &c.], we perceive the correctness of the definitions [7859, &c.].

$fndt+\varepsilon =$ the mean longitude of the comet; [7858]

$fndt+\varepsilon-\varpi =$ the mean anomaly of the comet, counted from the perihelion $= V$ [7903']; [7858']

$v-\varpi =$ the true anomaly of the comet, counted from the perihelion; [7859]

$u =$ the excentric anomaly of the comet, counted from the perihelion; [7860]

x, y, are the co-ordinates of the comet x; the axis of x being the line drawn from the focus to the perihelion; [7860']

then we shall have,*

$$x = r.\cos.(v-\varpi); \qquad y = r.\sin.(v-\varpi). \qquad [7861]$$

Hence the equations [7855, 7856] give,†

$$x = a.(\cos.u-e), \qquad \text{or} \qquad \cos.u = e + \frac{x}{a}; \qquad [7862]$$

$$y = a.\sqrt{1-e^2}.\sin.u, \qquad \text{or} \qquad \sin.u = \frac{y}{a.(1-e^2)^{\frac{1}{2}}}. \qquad [7862']$$

We shall suppose that,

$\lambda =$ the inclination of the orbit of the planet m', to that of the comet; [7863]

$\gamma =$ the longitude of the ascending node of the orbit of the planet m' upon the comet's orbit, counted from the perihelion; [7864]

$v' =$ the angle which the radius vector r' of the disturbing planet makes with the line of nodes; [7865]

then we have,‡

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* (3761) The radius vector r forms the angle $v-\varpi$ with the axis of x, as is evident from [7859, 7860']; hence we easily deduce the values of x, y [7861]. [7861a]

† (3762) The equations [7855, 7856] are equivalent to those of the elliptical motion in [603, 604, 605]; from which we have deduced the values of $\cos.v$, $\sin.v$ [735]; and if we change v into $v-\varpi$, to conform to the present notation [7854a], we shall have, from [735], by multiplying by r, [7862a]

$$r.\cos.(v-\varpi) = a.\cos.u - ae; \qquad r.\sin.(v-\varpi) = a.\sqrt{1-e^2}.\sin.u; \qquad [7862b]$$

substituting these expressions in [7861], we get [7862, 7862'].

‡ (3763) The expressions of x' y', z' [7866—7868], are found like those in [7742—7744], and they may be derived from these last formulas by changing v, r, X, Y, Z, Λ, into v', r', x', y', z', γ, respectively; λ, X', Y', Z', being unchanged. [7866a]

[7866] $$x' = r'.\cos.\gamma.\cos.v' - r'.\cos.\lambda.\sin.\gamma.\sin.v';$$

[7867] $$y' = r'.\sin.\gamma.\cos.v' + r'.\cos.\lambda.\cos.\gamma.\sin.v';$$

[7868] $$z' = r'.\sin.\lambda.\sin.v'.$$

The value of R [7802] gives,

[7869] $$\left(\frac{dR}{dx}\right) = m'.\left\{\frac{x'}{r'^3} - \frac{(x'-x)}{f^3}\right\};$$

[7870] putting,* $f = \sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$. In like manner we have,

[7871] $$\left(\frac{dR}{dy}\right) = m'.\left\{\frac{y'}{r'^3} - \frac{(y'-y)}{f^3}\right\}.$$

Variations of the elements. This being premised, we find that the value of dl [7850] gives,†

de

[7872] $$de = -m'.a.\sqrt{1-e^2}.du.\cos.u.\{xy'-yx'\}.\left\{\frac{1}{r'^3} - \frac{1}{f^3}\right\}$$
$$-m'.a^2.\sqrt{1-e^2}.du.(1-e.\cos.u).\left\{\frac{y'}{r'^3} - \frac{(y'-y)}{f^3}\right\};$$

in like manner, the value of dh [7849] gives,

$d\varpi$

[7873] $$ed\varpi = -m'.adu.\sin.u.(xy'-yx').\left(\frac{1}{r'^3} - \frac{1}{f^3}\right)$$
$$+m'.a^2\sqrt{1-e^2}.du.(1-e.\cos.u).\left\{\frac{x'}{r'^3} - \frac{(x'-x)}{f^3}\right\}.$$

[7874] We have as in [1177], in the variable ellipsis, observing that $\mu = 1$ *very nearly* [7304],

[7875] $$d.\frac{1}{a} = 2\text{d}R;$$

the differential characteristic d refers only to the co-ordinates of m [916'].

[7870a] * (3764) It is evident, from [949', 7801, 7801'], that $f = \sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$ expresses the distance of the comet m from the disturbing planet m'.

† (3765) Multiplying [7869] by $-y$, and [7871] by x, then taking the sum of the products, we get, by neglecting the terms which destroy each other,

[7872a] $$x.\left(\frac{dR}{dy}\right) - y.\left(\frac{dR}{dx}\right) = m'.\{xy'-yx'\}.\left\{\frac{1}{r'^3} - \frac{1}{f^3}\right\}.$$

[7872b] The differentials of x, y [7862, 7862'], give $dx = -adu.\sin.u$; $dy = a.\sqrt{1-e^2}.du.\cos.u$; substituting all these values, we get,

[7872c] $$xdy - ydx = a^2.\sqrt{1-e^2}.du.\{(\cos.u-e).\cos.u + \sin.^2u\} = a^2.\sqrt{1-e^2}.du.\{1-e.\cos.u\}.$$

[7872d] Substituting [7872a, b, c, 7871] in $dl = de$ [7850, 7853], gives [7872]; and in like manner, by substituting [7872a, b, c 7869] in $dh = ed\varpi$ [7849, 7853], we get [7873].

If we neglect the square of z, we shall have,*

$$\mathrm{d}R = m'.\left\{ \frac{(x'dx+y'dy)}{r'^3} - \frac{\{(x'-x).dx+(y'-y).dy\}}{f^3} \right\}; \qquad [7876]$$

$$= m'.\left\{ \left(\frac{x'}{r'^3} - \frac{(x'-x)}{f^3}\right).dx + \left(\frac{y'}{r'^3} - \frac{(y'-y)}{f^3}\right).dy \right\}; \qquad [7876']$$

consequently,

$$\mathrm{d}R = -m'.adu.\sin.u.\left\{ \frac{x'}{r'^3} - \frac{(x'-x)}{f^3} \right\} + m'.a.\sqrt{1-e^2}.du.\cos.u.\left\{ \frac{y'}{r'^3} - \frac{(y'-y)}{f^3} \right\}. \qquad [7877]$$

Hence we deduce, as in [7876b],

$$da = 2m'.a^3du.\sin.u.\left\{ \frac{x'}{r'^3} - \frac{(x'-x)}{f^3} \right\} - 2m'.a^3.\sqrt{1-e^2}.du.\cos.u.\left\{ \frac{y'}{r'^3} - \frac{(y'-y)}{f^3} \right\}. \qquad [7878]$$

Then we have, as in [1181], by putting $\mu = 1$ [7874],

$$dn = 3an.\mathrm{d}R; \qquad [7879]$$

consequently,†

$$\int ndt = Nt + 3\int(ndt.\int a\mathrm{d}R); \qquad [7880]$$

N being a constant quantity. *Therefore we shall have, by the preceding formulas, the variations of the excentricity and of the perihelion of the orbit; also those of the greater axis, and of the mean motion of the comet.* [7881]

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* (3766) The differential of R [7802] relative to the characteristic d, neglecting z^2, zdz, as in [7846'], gives the expression [7876]; whence we get [7876'], by merely changing the arrangement of the terms. [7876a] Substituting in [7876'] the values of dx, dy [7872b], we obtain [7877]. Developing the first member of [7875] it becomes $-a^{-2}da = 2.\mathrm{d}R$; multiplying this by $-a^2$, and substituting the value of $\mathrm{d}R$ [7877], we get the expression of da [7878]. [7876b]

† (3767) The integral of [7879] is $n = \mathcal{N} + 3\int an.\mathrm{d}R$; the constant quantity $\mathcal{N}$ being added to complete the integral. Multiplying this by dt, and again integrating, we [7880a] get $\int ndt = \mathcal{N}t + 3\int dt.\int an.\mathrm{d}R$. Now a, n differ from their mean values by quantities of [7880b] the order of the disturbing forces, and R is of the same order; so that by neglecting [7880c] quantities of the order of the square of these forces, we may bring n from under the sign of integration, and put the preceding expression [7880b] under the form [7880]. The [7880d] differential of this expression being divided by dt, gives $n = \mathcal{N} + 3n\int a.\mathrm{d}R$; and by [7880d'] neglecting terms of the order of the square of the disturbing force, it may be put under the form $n = \mathcal{N}.\{1 + 3\int a.\mathrm{d}R\}$, which is used in [7901]. [7880e]

To obtain the variation of the epoch ε, we shall observe that, if the ellipsis be invariable, the equation [7854] will give, by taking its differential,

[7882] $$n dt = du.(1-e.\cos.u).$$

This equation holds good in the variable ellipsis [1167″, &c.]; therefore we have,*

[7883] $$d\varepsilon - d\varpi = du.(1-e.\cos.u) - de.\sin.u;$$

[7883′] *supposing* u *to vary in this last equation only by reason of the variations of the elements* e *and* ϖ [7885], whilst in the former case [7882] it varies only with the variation of the time t. The equation [7856] gives, by supposing ε, ϖ, to be the only variable quantities,†

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* (3768) It appears from [7856] that u is a function of v, e, ϖ; and in the invariable
[7883a] ellipsis v, t, are considered as the variable quantities; but in the variable ellipsis we must also consider e, ϖ, as variable; so that u will vary with e, ϖ, as in [7883′]. Now taking
[7883b] the differential of [7854], considering all as variable, and then subtracting from the result the expression [7882], we get [7883], in which du varies only with e, ϖ, as in [7883′, 7885].

† (3769) The equation [7884] is the double of the differential of [7856], considering u, e, ϖ, as variable. Adding 1 to the square of [7856], and substituting

[7884a] $$1+\mathrm{tang.}^2\tfrac{1}{2}.(v-\varpi) = \frac{1}{\cos.^2\frac{1}{2}.(v-\varpi)}, \qquad 1+\mathrm{tang.}^2\tfrac{1}{2}u = \frac{1}{\cos.^2\frac{1}{2}u} \quad [34''' \text{ Int.}],$$

we get,

[7884b] $$\frac{1}{\cos.^2\frac{1}{2}.(v-\varpi)} = \frac{1}{\cos.^2\frac{1}{2}u} + \frac{2e}{1-e}.\mathrm{tang.}^2\tfrac{1}{2}u.$$

Substituting this in [7884], then multiplying by $\cos.^2\frac{1}{2}u$, and reducing by means of the expressions [34′, 31, 1] Int., namely,

[7884c] $\cos.^2\frac{1}{2}u.\mathrm{tang.}\frac{1}{2}u = \cos.\frac{1}{2}u.\sin.\frac{1}{2}u = \frac{1}{2}.\sin.u$; $\quad(\cos.\frac{1}{2}u.\mathrm{tang.}\frac{1}{2}u)^2 = (\sin.\frac{1}{2}u)^2 = \frac{1}{2}.(1-\cos.u)$;

we obtain,

[7884d] $$-d\varpi.\left\{1+\frac{e.(1-\cos.u)}{1-e}\right\} = du.\sqrt{\frac{1+e}{1-e}} + \frac{de.\sin.u}{(1-e).\sqrt{1-e^2}}.$$

[7884e] The first member of this equation is easily reduced to the form $-d\varpi.\left\{\frac{1-e.\cos.u}{1-e}\right\}$; substituting this, and then multiplying by $\sqrt{\frac{1-e}{1+e}}$, we obtain the value of du [7885]. Finally, substituting this expression of du in [7883], we get,

[7884f] $$d\varepsilon - d\varpi = -\frac{d\varpi.(1-e.\cos.u)^2}{\sqrt{1-e^2}} - \frac{de.\sin.u.(1-e.\cos.u)}{1-e^2} - de.\sin.u;$$

which is easily reduced to the form [7886]; and as $d\varpi$ is known from [7873], we shall get $d\varepsilon$ from [7886].

$$-\frac{d\varpi}{\cos.^2\frac{1}{2}.(v-\varpi)} = \frac{du}{\cos.^2\frac{1}{2}u}.\sqrt{\frac{1+e}{1-e}} + \frac{2de.\text{tang}.\frac{1}{2}u}{(1-e).\sqrt{1-e^2}}.$$ [7884]

Substituting for $\cos.^2\frac{1}{2}.(v-\varpi)$ its value, given by the same equation, we shall have,

$$du = -\frac{d\varpi.(1-e.\cos.u)}{\sqrt{1-e^2}} - \frac{de.\sin.u}{1-e^2}.$$ [7885]

Hence we deduce,

$$d\varepsilon - d\varpi = -\frac{d\varpi.(1-e.\cos.u)^2}{\sqrt{1-e^2}} - \frac{de.\sin.u.(2-e^2-e.\cos.u)}{1-e^2};$$ $d\varepsilon$ [7886]

this equation determines $d\varepsilon - d\varpi$, *consequently also the value of* $d\varepsilon$. [7886′]

If we integrate, by quadratures, the differentials of e, ϖ, a, n, ε, [7872, 7873, 7878, 7879, 7886], *we shall have, for any time whatever, all the elements of the comet's motion in its orbit; whence we may obtain the position of the comet in its orbit by means of the equations* [7854—7857]. *It will then only remain to determine the situation of the plane of this orbit, relative to the ecliptic. For this purpose we shall resume the equations* [1173]; [7886″]

$$dc = dt.\left\{ y.\left(\frac{dR}{dx}\right) - x.\left(\frac{dR}{dy}\right) \right\};$$ [7887]

$$dc' = dt.\left\{ z.\left(\frac{dR}{dx}\right) - x.\left(\frac{dR}{dz}\right) \right\};$$ [7888]

$$dc'' = dt.\left\{ z.\left(\frac{dR}{dy}\right) - y.\left(\frac{dR}{dz}\right) \right\}.$$ [7889]

If we put,

φ = the inclination of the comet's orbit to the plane of x, y [1173′]; [7890]

θ = the longitude of the ascending node of the comet's orbit upon the plane of x, y [1173′]; [7890′]

we shall have, as in [1174, 1175], by using $\mu = 1$ [7874],

$$\text{tang}.\varphi = \frac{\sqrt{c'^2+c''^2}}{c};$$ φ [7891]

$$\text{tang}.\theta = \frac{c''}{c'};$$ θ [7891′]

$$a.(1-e^2) = c^2+c'^2+c''^2.$$ [7892]

When we take, for the fixed plane of xy, *the primitive orbit of the comet,* c' *and* c'', *as well as* z, *will be of the order of the disturbing forces; therefore by neglecting the square of these forces, and substituting for* R *its* [7893]

*value, we shall have,**

[7894] $$\frac{dc'}{dt} = -m'.a.\{\cos.u-e\}.z'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right);$$

[7895] $$\frac{dc''}{dt} = -m'.a.\sqrt{1-e^2}.\sin.u.z'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right);$$

[7896] $$c = \sqrt{a.(1-ce)}.$$

Values of c, c', c''.

Now we have, as in [7882, 7857, 7874],

[7897] $$ndt = du.(1-e.\cos.u);$$

[7897'] $$n^2 = \frac{1}{a^3};$$

therefore by substitution we get,†

[7894a] * (3770) The functions $\left(\frac{dR}{dx}\right)$, $\left(\frac{dR}{dy}\right)$, $\left(\frac{dR}{dz}\right)$, are of the order m' [7802]; hence dc', dc'' [7888, 7889] are of the order m'; and their integrals c', c'', must also be of the order m', as is evident from the equation [7891]; and taking into view that, as the primitive
[7894b] orbit of the comet is assumed for the plane of xy, the angle φ [7890 or 7891] must be of the order m', as well as the ordinate z. Hence it is evident that if we neglect terms of the order m'^2, the equations [7888, 7889] will become,

[7894c] $$\frac{dc'}{dt} = -x.\left(\frac{dR}{dz}\right); \qquad \frac{dc''}{dt} = -y.\left(\frac{dR}{dz}\right).$$

[7894d] Now from [7802] we have $\left(\frac{dR}{dz}\right) = \frac{m'z'}{r'^3} - \frac{m'.(z'-z)}{f^3}$, using f [7870]; and by neglecting
[7894e] the term $m'z$, which is of the second order, it becomes $\left(\frac{dR}{dz}\right) = m'.z'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right)$; substituting this in [7894c], we get,

[7894f] $$\frac{dc'}{dt} = -m'.xz'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right); \qquad \frac{dc''}{dt} = -m'.yz'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right);$$

then substituting the values of x, y [7862, 7862'] in [7894f], we get [7894, 7895].
[7894g] Lastly, as c'^2, c''^2, are of the order of the square of the disturbing force [7894a], they may be neglected in [7892]; and then taking its square root, we get c [7896].

† (3771) From the equation [7897'] we get $n = a^{-\frac{3}{2}}$; substituting this in [7897], and
[7897a] then multiplying by $a^{\frac{3}{2}}$, we obtain $dt = a^{\frac{3}{2}}.du.(1-e.\cos.u)$; dividing this by c [7896], we obtain $\frac{dt}{c} = \frac{adu.(1-e.\cos.u)}{\sqrt{1-e^2}}$. Multiplying successively the equations [7894, 7895] by
[7897b] this last expression of $\frac{dt}{c}$, we get the equations [7898, 7899] respectively. Integrating
[7897c] these equations, we obtain the values of c', c''; and by substitution in [7891, 7891'], we get the inclination φ, and the longitude of the node θ.

$$\frac{dc'}{c} = -\frac{m'.a^2.du}{\sqrt{1-e^2}}.(1-e.\cos.u).(\cos.u-e).z'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right); \qquad [7898]$$

$$\frac{dc''}{c} = -m'.a^2.du.(1-e.\cos.u).\sin.u.z'.\left(\frac{1}{r'^3}-\frac{1}{f^3}\right). \qquad [7898']$$

Integrating these two equations, we can determine, for any instant of time whatever, the inclination of the orbit to the fixed plane, and the position of its nodes. [7899]

4. *The most important point in the theory of the perturbations of the motion of a comet, is the difference in the times of two successive returns to the perihelion. We shall now show how this can be determined, by taking for an example the comet of* 1682, which was at its perihelion in 1759, and putting, [7899'] Return to the perihelion. Symbols T, N.

$T =$ the interval between the times of passing the perihelion in 1682 and 1759. [7900]

Then we may determine N by putting,

$NT = 2\pi =$ the circumference of a circle whose radius is unity [7903a] ; [7900']

and we have, as in [7880e],

$$n = N.\{1+3a.f\mathrm{d}R\}. \qquad [7901]$$

If we commence the integral $f\mathrm{d}R$ *at the time of passing the perihelion in* 1682, *which we shall take for the epoch or origin of the time* t, *we may suppose,* [7901'] First epoch of t, 1682.

$$n = N.\{1+\delta q+3a.f_0^t\mathrm{d}R\}; \qquad \left[\text{General value of } n \text{ at the time } t.\right] \qquad [7902]$$

δq *being an arbitrary constant quantity.** Now we have, as in [7858'], [7902']

$$V = f_0^t ndt + \varepsilon - \varpi; \qquad [7903]$$

V *being the comet's mean anomaly.* Hence we obtain, [7903']

$$V = Nt.(1+\delta q) + 3a.f_0^t.(Ndt.f_0^t\mathrm{d}R) + \varepsilon - \varpi + \delta\varepsilon - \delta\varpi; \qquad [7904]$$

$\delta\varepsilon$, $\delta\varpi$, *being the variations of* ε *and* ϖ, *from the time of passing the perihelion* [7905]
in 1682; ε, ϖ, *correspond to that epoch when* $\varepsilon - \varpi = 0$; *since by hypothesis* [7906]

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* (3772) By means of this constant quantity δq [7902'], we are enabled to satisfy the assumed equation $NT = 2\pi$ [7900'], as will be seen in [7908, &c.]. Substituting the [7903a]
value of n [7902] in V [7903], it becomes as in [7904]; the terms $\delta\varepsilon - \delta\varpi$ being added [7903b]
on account of the variation of $\varepsilon - \varpi$, since the commencement of the epoch. If we suppose that when $t = T$, n becomes equal to N', as in [7910], the expression [7902] [7903c]
will become,

$$N' = N.\{1+\delta q+3a.f_0^T.\mathrm{d}R\}; \qquad [7903d]$$

which is used in [7917f].

[7907] *we then have* $V=0$ [7901′, 7903′]. Moreover we have supposed, in [7900, 7900′], that t being equal to T, $V=2\pi$, and $NT=2\pi$; therefore we shall have,*

[7908] $$0=NT.\delta q+3a.\textstyle\int_0^T.(Ndt.\int_0^T \mathrm{d}R)+\delta\varepsilon-\delta\varpi;$$

[7909] the variations $\delta\varepsilon$, $\delta\varpi$, as well as the double integral, being taken between the limits $t=0$, and $t=T$. *This equation gives the value of the constant quantity* δq; *therefore we shall have, at any time whatever, the value of* n, [7902]. *This value will give that of the semi-major axis of the orbit, by*

[7909′] *means of the equation* $n^2=\frac{1}{a^3}$ [7897′].

N' We shall now put,

[7910] $N'=$ the value of n, at the time of passing the perihelion in 1759;

[7910′] $a_1=$ the semi-major axis of the comet's orbit, at the time of the perihelion in 1759;

a_1

Second epoch of t, 1759. *Then taking the time of passing the perihelion in* 1759 *for the epoch or origin of the time* t, *we shall have*,†

[7911] $$V=N't+\delta\varepsilon-\delta\varpi+3a_1.\textstyle\int(N'dt.\int \mathrm{d}R);$$

[7912] $\delta\varepsilon$, $\delta\varpi$, *as well as the integrals, being supposed to commence at the time of passing the perihelion in* 1759. The values of e, ϖ, ε, will be determined by the observations of the comet, made at the same epoch; for

[7912′] a_1 being known by what precedes,‡ the perihelion distance in 1759 will give the corresponding value of e. We shall now put,

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[7904a] * (3773) We have supposed in [7900′], that when $t=T$ we shall have $NT=2\pi$; and at that time the mean anomaly becomes $V=2\pi$ [7900, 7903′]; hence $V=NT$ [7900′]. Substituting this and $t=T$ in [7904], it becomes,

[7904b] $$NT=NT.(1+\delta q)+3a.\textstyle\int_0^T.(Ndt.\int_0^T \mathrm{d}R)+\varepsilon-\varpi+\delta\varepsilon-\delta\varpi.$$

[7904c] Rejecting NT, which occurs in both members of the equation, and putting $\varepsilon-\varpi=0$, as in [7906], it becomes as in [7908]; and from this we deduce the value of δq, which is to be used in [7902] in finding n; and then a is obtained from the equation [7909′].

[7911a] † (3774) The equation [7911] is similar to [7904]; changing N into N', a into a_1, and supposing the integrals to commence at the time of the perihelion in 1759, using also

[7911b] for this epoch $\varepsilon-\varpi=0$, as in [7906]. In this case the constant δq is to be neglected, because there is no assumed equation like that in [7900′] which we must satisfy.

[7913a] ‡ (3775) The value of a_1 is found for the time T, or 1759, by the method pointed out in [7909′ or 7904c, &c.].

$T'=$ the unknown interval between the time of passing the perihelion in 1759, and the next time of passing it; and at this last time we shall have, [7913]
$V=2\pi$. Substituting these in [7911], we get, [7914]

$$2\pi=N'T'+\delta\varepsilon-\delta\varpi+3a_1.\int_0^{T'}(N'dt.\int_0^{T'}\mathrm{d}R). \qquad [7915]$$

the values of $\delta\varepsilon$, $\delta\varpi$, as well as the integrals, are taken from $t=0$ to $t=T'$. This equation will determine T'. [7916]

We may avoid the double signs of integration, which occur in these expressions, by observing that,*

$$3a_1.\int(N'dt.\int\mathrm{d}R)=3N't.\int a_1\mathrm{d}R-3a_1.\int N't.\mathrm{d}R; \qquad [7917]$$

therefore by marking, with a horizontal line placed above these quantities, whose limits are taken from the perihelion in 1682 *to that in* 1759, *and by a double line those which extend from the perihelion in* 1759 *to the following perihelion*, we shall find that the preceding expression of n will give, as in [7917f], [7917'] [7918]

$$N'T=2\pi-\overline{\delta\varepsilon}+\overline{\delta\varpi}+\overline{3a.\int_0^T Nt.\mathrm{d}R}. \qquad [7919]$$

This equation will determine N', and therefore a_1. Then we shall have,†

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* (3776) Integrating $\int dt.\int\mathrm{d}R$ by parts, as in [1716a], we get $\int dt.\int\mathrm{d}R=t.\int\mathrm{d}R-\int t.\mathrm{d}R$, as is easily proved by taking the differential of both members, and reducing. Multiplying this by $3a_1N'$, it becomes as in [7917], observing that N', a_1 [7910, 7910'] are constant, therefore they may be placed under the sign $\int$. Substituting this in [7915] it becomes, by supposing $t=0$ in the perihelion of 1759, and $t=T'$ in the next perihelion [7913], and using the notation in [7917', &c.], [7917a] [7917b]

$$2\pi=N'T'+\overline{\overline{\delta\varepsilon}}-\overline{\overline{\delta\varpi}}+3N'T'.\int_0^{T'}a_1\mathrm{d}R-3a_1.\int_0^{T'}N't.\mathrm{d}R. \qquad [7917c]$$

Again, if we multiply [7917a] by $3aN$, we obtain $3a\overset{N}{\wedge}\int dt.\int\mathrm{d}R=3Nt.\int a.\mathrm{d}R-3a.\int Nt.\mathrm{d}R$; [7917d]
substituting this and $\varepsilon-\varpi=0$ [7904c] in the second member of [7904b], also $NT=2\pi$ [7900'] in its first member, we get, when $t=T$,

$$2\pi=NT.\{1+\delta q+3a.\int_0^T\mathrm{d}R\}-3a.\int_0^T Nt.\mathrm{d}R+\overline{\delta\varepsilon}-\overline{\delta\varpi}. \qquad [7917e]$$

Multiplying [7903d] by T, and substituting the resulting product in the second member of the preceding equation, we get $2\pi=N'T-3a.\int_0^T Nt.\mathrm{d}R+\overline{\delta\varepsilon}-\overline{\delta\varpi}$; whence, by transposition, we deduce the value of $N'T$ [7919]. From [7919] we deduce the value of N', and then the value of a_1 from the equation $N'^2=\frac{1}{a_1^{\,3}}$, similar to [7909']. [7917f] [7917g]

† (3777) From [7917c] we get,

$$N'T'=2\pi-\overline{\overline{\delta\varepsilon}}+\overline{\overline{\delta\varpi}}-3N'T'.\int_0^{T'}a_1.\mathrm{d}R+3a_1.\int_0^{T'}N't.\mathrm{d}R; \qquad [7920a]$$

subtracting from this the value of $N'T$ [7919], we get [7920]; the lines being placed above the quantities, according to the notation adopted in [7917', &c.]. We may observe

Anomalistical revolution.

[7920] $$N'.(T'-T) = \overline{\delta\varepsilon} - \overline{\overline{\delta\varepsilon}} - \overline{\delta\varpi} + \overline{\overline{\delta\varpi}} - 3N'T'.\overline{\overline{\textstyle\int_0^{T'} a_1.\mathrm{d}R}}$$
$$-3a.\overline{\textstyle\int_0^{T} Nt.\mathrm{d}R} + 3a_1.\overline{\overline{\textstyle\int_0^{T'} N't.\mathrm{d}R}}\,;$$

[7921] *and this equation will determine the difference* $T'-T$, *of the two anomalistical revolutions of the comet.*

5. *The whole difficulty of the problem is now reduced to that of determining numerically the changes in the elements of the orbit.* We have already observed that we can only find them by means of mechanical quadratures, and for this purpose we have several methods furnished by [7922] analysis. We shall here explain the process which appears to be the most convenient and simple. For this purpose we shall give, in a few words, *the principle of the theory of generating functions.*

Principle of generating functions.

[7923] We shall suppose that u is a function of t, and that by developing it according to the powers of t, we shall have, as in [607a],

Generating function u.

[7924] $$u = y^{(0)} + y^{(1)}.t + y^{(2)}.t^2 + y^{(3)}.t^3 + \&c.\,;$$

[7924′] *then* u *is the generating function of the different coefficients* $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, &c. It is evident that $y^{(i)}$ being the coefficient of t^i in the development of u, it is also the coefficient independent of t, in the development of $\frac{u}{t^i}$. Now we have,

[7925] $$\frac{u}{t^i} = u.\left(1+\frac{1}{t}-1\right)^i; \quad \text{or} \qquad (i)$$

[7925′] $$\frac{u}{t^i} = u.\left\{1+i.\left(\frac{1}{t}-1\right)+\frac{i.(i-1)}{1.2}.\left(\frac{1}{t}-1\right)^2+\&c.\right\};$$

[7926] the coefficient independent of t, in $u.\left(\frac{1}{t}-1\right)$, is evidently $y^{(1)}-y^{(0)}$, or $\Delta.y^{(0)}$; *the characteristic* Δ *being that of finite integrals.** It is also

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[7920b] that the limits of the integrals 0, T, and 0, T', are not inserted in the original work in connection with the sign $\int$; and by introducing them here, it becomes unnecessary to place these lines above the integrals, because their values are sufficiently designated by these limits; but being in the original work we have retained them.

[7926a] * (3778) The expression of u [7924] being substituted in the first member of [7926a'], neglecting the negative powers of t, produces the second member of that equation, which is easily reduced to the form [7926b], by using the characteristic Δ of finite differences;

evident, by considering $u.\left(\frac{1}{t}-1\right)$ as a new generating function, and then developing it; neglecting the negative powers of t; we shall have, as in [7926b], [7926′]

$$u.\left(\frac{1}{t}-1\right)=\Delta.y^{(0)}+\Delta.y^{(1)}.t+\Delta.y^{(2)}.t^2+\Delta.y^{(3)}.t^3+\&c. \qquad [7927]$$

Hence it follows, that the coefficient independent of t, in the development of $u.\left(\frac{1}{t}-1\right).\left(\frac{1}{t}-1\right)$, or $u.\left(\frac{1}{t}-1\right)^2$, is $\Delta.y^{(1)}-\Delta.y^{(0)}$, or $\Delta^2.y^{(0)}$, [7926d]. Following the same method of reasoning, we see that the coefficient independent of t, in the development of $u.\left(\frac{1}{t}-1\right)^3$, is $\Delta^3.y^{(0)}$, [7926f], and so on for other cases. Hence it appears that the equation [7925′] will give, by passing from the generating functions to the [7928] [7928′]

$$u.\left(\frac{1}{t}-1\right)=(y^{(1)}-y^{(0)})+(y^{(2)}-y^{(1)}).t+(y^{(3)}-y^{(2)}).t^2+\&c. \qquad [7926a']$$

$$u.\left(\frac{1}{t}-1\right)=\Delta.y^{(0)}+\Delta.y^{(1)}.t+\Delta.y^{(2)}.t^2+\&c. \qquad [7926b]$$

Multiplying this last equation by $\left(\frac{1}{t}-1\right)$, we get [7926c]; and by a similar process it is reduced to the form [7926d], neglecting the negative powers of t;

$$u.\left(\frac{1}{t}-1\right)^2=(\Delta.y^{(1)}-\Delta.y^{(0)})+(\Delta.y^{(2)}-\Delta.y^{(1)}).t+(\Delta.y^{(3)}-\Delta.y^{(2)})t^2+\&c. \qquad [7926c]$$

$$u.\left(\frac{1}{t}-1\right)^2=\Delta^2.y^{(0)}+\Delta^2.y^{(1)}.t+\Delta^2.y^{(2)}.t^2+\&c. \qquad [7926d]$$

Now multiplying [7926d] by $\left(\frac{1}{t}-1\right)$, reducing, and neglecting the negative powers of t, we get, in like manner,

$$u.\left(\frac{1}{t}-1\right)^3=(\Delta^2.y^{(1)}-\Delta^2.y^{(0)})+(\Delta^2.y^{(2)}-\Delta^2.y^{(1)}).t+(\Delta^2.y^{(3)}-\Delta^2.y^{(2)}).t^2+\&c. \qquad [7926e]$$

$$u.\left(\frac{1}{t}-1\right)^3=\Delta^3.y^{(0)}+\Delta^3.y^{(1)}.t+\Delta^3.y^{(2)}.t^2+\&c. \qquad [7926f]$$

Proceeding in the same way we finally obtain,

$$u.\left(\frac{1}{t}-1\right)^r=\Delta^r.y^{(0)}+\Delta^r.y^{(1)}.t+\Delta^r.y^{(2)}.t^2+\&c. \qquad [7926g]$$

This agrees with the results in [7926—7928′]. Before closing this note we may observe, that the principles of this calculation of generating functions were first given by La Place in the *Mémoires de l'Académie Royale des Sciences*, for the year 1797. [7926h]

coefficients,*

[7929] General formula.
$$y^{(i)} = y^{(0)} + i.\Delta.y^{(0)} + \frac{i.(i-1)}{2}.\Delta^2.y^{(0)} + \frac{i.(i-1).(i-2)}{1.2.3}.\Delta^3.y^{(0)} + \&c.$$

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* (3779) Substituting the results [7924′, 7926b, d, f, g] in the second member of [7925′], we find that the term independent of t becomes like the second member of [7929],

[7929a] which represents, as in the first member of [7925′], the term of $\frac{u}{t^i}$, which is independent of t; and this in [7924] is represented by $y^{(i)}$; hence we get [7929].

Convenient integral formulas. We can arrange the formula [7929] according to the powers of i, which tends to simplify the results when the intervals are small, as we shall see in [7929w, &c.]. For this purpose we shall put, for brevity,

[7929b]
$$A_1 = \Delta.y^{(0)} - \tfrac{1}{2}.\Delta^2.y^{(0)} + \tfrac{1}{3}.\Delta^3.y^{(0)} - \tfrac{1}{4}.\Delta^4.y^{(0)} + \&c.\,;$$

[7929c]
$$A_2 = \tfrac{1}{2}.\Delta^2.y^{(0)} - \tfrac{1}{2}.\Delta^3.y^{(0)} + \tfrac{11}{24}.\Delta^4.y^{(0)} - \&c.\,;$$

[7929d]
$$A_3 = \tfrac{1}{6}.\Delta^3.y^{(0)} - \tfrac{1}{4}.\Delta^4.y^{(0)} + \&c.\,;$$
$$A_4 = \tfrac{1}{24}.\Delta^4.y^{(0)} + \&c.$$

And by substituting these expressions in [7929], arranged according to the powers of i, we shall get,

[7929e]
$$y^{(i)} = y^{(0)} + A_1.i + A_2.i^2 + A_3.i^3 + A_4.i^4 + \&c.$$

Multiplying this by di, and integrating, we obtain,

[7929f]
$$\int y^{(i)}.di = y^{(0)}.i + \tfrac{1}{2}.A_1.i^2 + \tfrac{1}{3}.A_2.i^3 + \tfrac{1}{4}.A_3.i^4 + \tfrac{1}{5}.A_4.i^5 + \&c. + \text{constant}.$$

[7929g] Taking the constant quantity, so that the expression may vanish when $i = -\frac{1}{2}$, and then putting $i = +\frac{1}{2}$, we get by successive reductions, and re-substituting A_2, A_4, &c. [7929c, d, &c.],

[7929h]
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} y^{(i)}.di = y^{(0)} + \tfrac{1}{12}.A_2 + \tfrac{1}{80}.A_4 + \&c.\,;$$

[7929i]
$$= y^{(0)} + \tfrac{1}{12}.\{\tfrac{1}{2}.\Delta^2.y^{(0)} - \tfrac{1}{2}.\Delta^3.y^{(0)} + \tfrac{11}{24}.\Delta^4.y^{(0)} - \&c.\} + \tfrac{1}{80}.\tfrac{1}{24}.\Delta^4.y^{(0)} + \&c.$$

[7929k]
$$= y^{(0)} + \tfrac{1}{24}.\Delta^2.y^{(0)} - \tfrac{1}{24}.\Delta^3.y^{(0)} + \tfrac{223}{5760}.\Delta^4.y^{(0)} + \&c.$$

This may be reduced to a more simple form, by introducing the finite differences of the term $y^{(-1)}$, which precedes $y^{(0)}$ in the order of the series of terms,

[7929l]
$$\ldots . y^{(-2)},\quad y^{(-1)},\quad y^{(0)},\quad y^{(1)},\quad y^{(2)},\quad y^{(3)} \ldots\ldots y^{(i)}, \&c.$$

[7929l'] For by the usual principle of finite differences, we have $y^{(0)} = y^{(-1)} + \Delta.y^{(-1)}$, &c. If we substitute the finite differences of this expression in [7929k], we shall obtain, by successive reductions,

[7929m]
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} y^{(i)}.di = y^{(0)} + \tfrac{1}{24}.\{\Delta^2.y^{(-1)} + \Delta^3.y^{(-1)}\} - \tfrac{1}{24}.\{\Delta^3.y^{(-1)} + \Delta^4.y^{(-1)}\} + \tfrac{223}{5760}.\Delta^4.y^{(-1)} + \&c.$$

[7929n]
$$= y^{(0)} + \tfrac{1}{24}.\Delta^2.y^{(-1)} - \tfrac{17}{5760}.\Delta^4.y^{(-1)} + \&c.$$

If we increase the limits of the integral by unity, making them $+\frac{1}{2}$, $+\frac{3}{2}$, the effect will be to increase the indices of y in the second member by unity; hence we obtain [7929o], and by a similar process we get [7929p, q, &c.];

Although this expression of $y^{(i)}$ has been found upon the supposition that i is a positive integral number, yet it may be extended to any value of i. Then [7930]

$$\int_{\frac{1}{2}}^{\frac{3}{2}} y^{(i)}.di = y^{(1)} + \tfrac{1}{24}.\Delta^2.y^{(0)} - \tfrac{17}{5760}.\Delta^4.y^{(0)} + \&c.\,; \qquad [7929o]$$

$$\int_{\frac{3}{2}}^{\frac{5}{2}} y^{(i)}.di = y^{(2)} + \tfrac{1}{24}.\Delta^2.y^{(1)} - \tfrac{17}{5760}.\Delta^4.y^{(1)} + \&c.\,; \qquad [7929p]$$

$$\int_{\frac{5}{2}}^{\frac{7}{2}} y^{(i)}.di = y^{(3)} + \tfrac{1}{24}.\Delta^2.y^{(2)} - \tfrac{17}{5760}.\Delta^4.y^{(2)} + \&c.\,; \qquad [7929q]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} y^{(i)}.di = y^{(i)} + \tfrac{1}{24}.\Delta^2.y^{(i-1)} - \tfrac{17}{5760}.\Delta^4.y^{(i-1)} + \&c. \qquad [7929r]$$

The sum of all the integrals [7929n—r] gives,

$$\begin{aligned} \int_{-\frac{1}{2}}^{i+\frac{1}{2}} y^{(i)}.di = {} & y^{(0)} + y^{(1)} + y^{(2)} + y^{(3)} \ldots\ldots\ldots\ldots + y^{(i)} \\ & + \tfrac{1}{24}.\{\Delta^2.y^{(-1)} + \Delta^2.y^{(0)} + \Delta^2.y^{(1)} + \Delta^2.y^{(2)} \ldots + \Delta^2.y^{(i-1)}\} \\ & - \tfrac{17}{5760}.\{\Delta^4.y^{(-1)} + \Delta^4.y^{(0)} + \Delta^4.y^{(1)} + \Delta^4.y^{(2)} \ldots + \Delta^4.y^{(i-1)}\}. \\ & + \&c. \end{aligned} \qquad [7929s]$$

If the interval of time, which is taken for unity, is so small that the fourth differences, and those of a higher order, can be neglected, the preceding expression becomes, [7929s′]

$$\begin{aligned} \int_{-\frac{1}{2}}^{i+\frac{1}{2}} y^{(i)}.di = {} & y^{(0)} + y^{(1)} + y^{(2)} + y^{(3)} \ldots\ldots + y^{(i)} \\ & + \tfrac{1}{24}.\{\Delta^2.y^{(-1)} + \Delta^2.y^{(0)} + \Delta^2.y^{(1)} \ldots + \Delta^2.y^{(i-1)}\}. \end{aligned} \qquad [7929t]$$

Now the finite difference of $y^{(0)}$ [7929l'] is $\Delta.y^{(0)} = \Delta.y^{(-1)} + \Delta^2.y^{(-1)}$; whence $\Delta^2.y^{(-1)} = \Delta.y^{(0)} - \Delta.y^{(-1)}$; and in like manner, by merely increasing the indices on y, we have,

$$\Delta^2.y^{(0)} = \Delta.y^{(1)} - \Delta.y^{(0)}\,; \qquad \Delta^2.y^{(1)} = \Delta.y^{(2)} - \Delta.y^{(1)},\ \&c.\,; \qquad \Delta^2.y^{(i-1)} = \Delta.y^{(i)} - \Delta.y^{(i-1)}. \qquad [7929u]$$

Adding these expressions together, and neglecting the terms which mutually destroy each other, we obtain,

$$\Delta^2.y^{(-1)} + \Delta^2.y^{(0)} + \Delta^2.y^{(1)} \ldots + \Delta^2.y^{(i-1)} = \Delta.y^{(i)} - \Delta.y^{(-1)}. \qquad [7929v]$$

Substituting this in [7929t], we finally get,

Useful formula for integrating by quadratures.

$$\begin{aligned} \int_{-\frac{1}{2}}^{i+\frac{1}{2}} .y^{(i)}.di = {} & y^{(0)} + y^{(1)} + y^{(2)} + y^{(3)} \ldots\ldots + y^{(i)} \\ & + \tfrac{1}{24}.\{\Delta.y^{(i)} - \Delta.y^{(-1)}\} \end{aligned} \qquad [7929w]$$

$$\begin{aligned} & = \text{the sum of all the calculated quantities } y^{(0)},\ y^{(1)} \ldots y^{(i)} \\ & + \tfrac{1}{24}.\{\text{first difference following the last term} - \text{first difference preceding the first term.}\}. \end{aligned} \qquad [7929x]$$

We may observe that, in calculating the differences of the series of terms $y^{(0)}$, $y^{(1)}$, $y^{(2)} \ldots y^{(i)}$, the first term is $\Delta.y^{(0)}$, and the last $\Delta.y^{(i-1)}$; and by means of the other differences [7941, &c.], we can estimate from $\Delta.y^{(0)}$ the value of $\Delta.y^{(-1)}$; also from $\Delta.y^{(i-1)}$, the value of $\Delta.y^{(i)}$; which are required in the formula [7929x]. We may finally remark that the ordinates $y^{(0)}$, $y^{(1)}$, $y^{(2)} \ldots y^{(i)}$, which occur in the formula [7929w or x], [7929y]
correspond to the middle of each of the successive intervals of time, which is taken for unity; [7929z]
thus $y^{(0)}$ corresponds to the middle time between $i = -\frac{1}{2}$ and $i = \frac{1}{2}$; $y^{(1)}$ to the middle

[7930′] *$y^{(i)}$ is the ordinate of a parabolic curve whose absciss is i; this curve being supposed to pass through the extremities of the equidistant ordinates $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, &c.; the interval which separates them being taken for unity.* Whatever be the nature of the curve which is taken into consideration, we know that each of its very small arcs may be taken for a parabolic arc, whose ordinate
[7931] $y^{(i)}$ is expressed by a series of successive powers of the absciss, counted from the origin of the arc. The coefficients of these powers must be determined, so that the curve may pass through the extremities of the adjacent ordinates $y^{(0)}$, $y^{(1)}$, &c.; and then we shall evidently have the
[7932] preceding expression of $y^{(i)}$ [7929]. Multiplying it by di, and integrating from $i=0$ to $i=1$, we obtain,*

[7933] General integral. $$\textstyle\int_0^1 y^{(i)}.di = y^{(0)} + \frac{1}{2}.\Delta.y^{(0)} - \frac{1}{12}.\Delta^2.y^{(0)} + \frac{1}{24}.\Delta^3.y^{(0)} - \frac{19}{720}.\Delta^4.y^{(0)} + \frac{3}{160}.\Delta^5.y^{(0)} - \frac{863}{60480}.\Delta^6.y^{(0)} + \&c.$$

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time between $i=\frac{1}{2}$ and $i=\frac{3}{2}$; $y^{(2)}$ to the middle time between $i=\frac{3}{2}$ and $i=\frac{5}{2}$;
[7930a] and so on to $y^{(i)}$, corresponding to the middle time between $i-\frac{1}{2}$ and $i+\frac{1}{2}$, that is to say, to i. In finding the variations of any one of the elements by the method of
[7930b] quadratures, it is usual to take for the epoch of the time t, the time of the commencement of the integral, when $i=-\frac{1}{2}$ [7929w]; so that we shall have generally $t=i+\frac{1}{2}$; and
[7930c] by using the values of i [7929z, &c.], we find that $y^{(0)}$ corresponds to the middle interval between $t=0$ and $t=1$, that is to say, to $t=\frac{1}{2}$; $y^{(1)}$ corresponds to the middle
[7930d] interval between $t=1$ and $t=2$, or $t=\frac{3}{2}$; $y^{(2)}$ corresponds to the middle interval
[7930e] between $t=2$ and $t=3$, or $t=\frac{5}{2}$; and so on to $y^{(i)}$, *which corresponds to the middle interval between* $t=i$ *and* $t=i+1$, or to $t=i+\frac{1}{2}$, which is used hereafter in finding the variation of the epoch ε [8014l].

* (3780) Multiplying [7929] by di, and then integrating, we get,

[7933a] $$\textstyle\int y^{(i)}.di = y^{(0)}.\int di + \Delta.y^{(0)}.\int di.i + \frac{1}{2}.\Delta^2.y^{(0)}.\int di.i.(i-1) + \frac{1}{6}.\Delta^3.y^{(0)}.\int di.i.(i-1).(i-2) + \&c.$$

If we suppose the integrals to commence when $i=0$, and then put $i=1$, we shall get,

[7933b] $\int di = i = 1$; $\quad \int di.i = \frac{1}{2}.i^2 = \frac{1}{2}$; $\quad \int di.i.(i-1) = \frac{1}{3}.i^3 - \frac{1}{2}.i^2 = -\frac{1}{6}$;

[7933c] $\int di.i.(i-1).(i-2) = \frac{1}{4}.i^4 - i^3 + i^2 = \frac{1}{4}$;

[7933d] $\int di.i.(i-1).(i-2).(i-3) = \frac{1}{5}.i^5 - \frac{3}{2}.i^4 + \frac{11}{3}.i^3 - 3i^2 = -\frac{19}{30}$;

[7933e] $\int di.i.(i-1).(i-2).(i-3).(i-4) = \frac{1}{6}.i^6 - 2i^5 + \frac{35}{4}.i^4 - \frac{50}{3}.i^3 + 12i^2 = \frac{9}{4}$;

[7933f] $\int di.i.(i-1).(i-2).(i-3).(i-4).(i-5) = \frac{1}{7}.i^7 - \frac{5}{2}.i^6 + 17i^5 - \frac{225}{4}.i^4 + \frac{274}{3}.i^3 - 60i^2 = -\frac{863}{84}$, &c.

Substituting these in [7933a] we get [7933], which represents the area of the parabolic
[7933g] curve included between $y^{(0)}$ and $y^{(1)}$; and by changing $y^{(0)}$ into $y^{(1)}$, we get [7934], representing the space included between the co-ordinates $y^{(1)}$ and $y^{(2)}$; and so on for
[7933h] others, to the term $y^{(n)}$. Adding all these quantities together, we get the whole expression of the area $\int y^{(i)}.di$, included between the co-ordinates $y^{(0)}$, $y^{(n)}$, as in [7935].

This will be the area of the curve included between $y^{(0)}$ *and* $y^{(1)}$. In like manner, the area included between the co-ordinates $y^{(1)}$, $y^{(2)}$, will be,

$$\int_1^2 y^{(i)}.di = y^{(1)} + \tfrac{1}{2}.\Delta.y^{(1)} - \tfrac{1}{12}.\Delta^2.y^{(1)} + \&c. \qquad [7934]$$

and so on for others. Therefore if we represent, by $\int_0^n y^{(i)}.di$, the whole area included between the ordinates $y^{(0)}$, $y^{(n)}$, we shall have, as in [7933h],

$$\begin{array}{llr} \int_0^n y^{(i)}.di = & y^{(0)}+y^{(1)}+y^{(2)}+ \ldots\ldots\ldots +y^{(n-1)} & 1 \\ & +\tfrac{1}{2}.\{\Delta.y^{(0)}+\Delta.y^{(1)}+\Delta.y^{(2)} \ldots\ldots +\Delta.y^{(n-1)}\} & 2 \\ & -\tfrac{1}{12}.\{\Delta^2.y^{(0)}+\Delta^2.y^{(1)}+\Delta^2.y^{(2)} \ldots +\Delta^2.y^{(n-1)}\} & 3 \\ & +\tfrac{1}{24}.\{\Delta^3.y^{(0)}+\Delta^3.y^{(1)}+\Delta^3.y^{(2)} \ldots +\Delta^3.y^{(n-1)}\} & 4 \\ & +\&c. & 5 \end{array} \qquad [7935]$$

Now we have,*

$$\Delta.y^{(0)}+\Delta.y^{(1)} \ldots\ldots +\Delta.y^{(n-1)} = (y^{(1)}-y^{(0)})+(y^{(2)}-y^{(1)})+\ldots+(y^{(n)}-y^{(n-1)})\text{; or,} \qquad [7936]$$

$$\Delta.y^{(0)}+\Delta.y^{(1)} \ldots\ldots +\Delta.y^{(n-1)} = y^{(n)}-y^{(0)}. \qquad [7936']$$

In like manner we have,

$$\Delta^2.y^{(0)}+\Delta^2.y^{(1)} \ldots\ldots +\Delta^2.y^{(n-1)} = \Delta.y^{(n)}-\Delta.y^{(0)}. \qquad [7937]$$

and so on; therefore,

$$\begin{array}{llr} \int_0^n y^{(i)}.di = & \tfrac{1}{2}.y^{(0)}+y^{(1)}+y^{(2)} \ldots\ldots\ldots y^{(n-1)}+\tfrac{1}{2}.y^{(n)} & 1 \\ & -\ \tfrac{1}{12}\ .\{\Delta.y^{(n)}-\Delta.y^{(0)}\} & 2 \\ & +\ \tfrac{1}{24}\ .\{\Delta^2.y^{(n)}-\Delta^2.y^{(0)}\} & 3 \\ & -\ \tfrac{19}{720}\ .\{\Delta^3.y^{(n)}-\Delta^3.y^{(0)}\} & 4 \\ & +\ \tfrac{3}{160}\ .\{\Delta^4.y^{(n)}-\Delta^4.y^{(0)}\} & 5 \\ & -\tfrac{863}{60480}.\{\Delta^5.y^{(n)}-\Delta^5.y^{(0)}\} & 6 \\ & +\&c. & 7 \end{array} \qquad [7938]$$

The values $\Delta.y^{(n)}$, $\Delta^2.y^{(n)}$, &c., depend on $y^{(n+1)}$, $y^{(n+2)}$,† &c.; and it is

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* (3781) The second member of [7936] is easily deduced from the first, by substituting $\Delta.y^{0}=y^{(1)}-y^{(0)}$, $\Delta.y^{(1)}=y^{(2)}-y^{(1)}$, $\Delta.y^{(2)}=y^{(3)}-y^{(2)}$, &c. Neglecting the terms which [7938a] mutually destroy each other, it becomes simply $y^{(n)}-y^{(0)}$, as in [7936']. Substituting this [7938b] in [7935 lines 1, 2], and making a slight reduction, we get [7938 line 1]. The differential of [7936'], relative to the characteristic Δ, gives [7937]; substituting this in [7935 line 3], we get [7938 line 2]. The differential of [7937] relative to Δ, being substituted in [7935 line 4], gives [7938 line 3]; and in like manner we obtain the other lines of [7938]. [7938c]

† (3782) This is evident from the system of equations [755], changing the characteristic of finite differences δ into Δ, and putting $\beta=y^{(n)}$, $\beta'=y^{(n+1)}$, $\beta''=y^{(n+2)}$, &c., in order to conform to the present notation. We may incidentally observe that the formula [7939a] [7929] can be easily deduced from that in [756].

[7938′] supposed that we have computed only the values $y^{(0)}$, $y^{(1)} \ldots y^{(n)}$. To avoid this difficulty, we shall observe that the coefficient of t^n, in the development of the function $u.\left(\frac{1}{t}-1\right)^r$, is $\Delta^r.y^{(n)}$ [7926g]; now we have,*

[7939] $$u.\left(\frac{1}{t}-1\right)^r = u.(1-t)^r.\{1-(1-t)\}^{-r} = u.(1-t)^r.\left\{1+r.(1-t)+\frac{r.(r+1)}{1.2}.(1-t)^2+\&c\right\}$$

[7939′] $$u.\left(\frac{1}{t}-1\right)^r = u.\left\{(1-t)^r+r.(1-t)^{r+1}+\frac{r.(r+1)}{1.2}.(1-t)^{r+2}+\&c.\right\}.$$

[7940] The coefficient of t^n, in the development of $u.(1-t)^q$, is generally† $\Delta^q.y^{(n-q)}$. Therefore the equation [7939′] gives, by passing from the generating functions to the coefficients,

[7941] $$\Delta^r.y^{(n)} = \Delta^r.y^{(n-r)}+r.\Delta^{r+1}.y^{(n-r-1)}+\frac{r.(r+1)}{1.2}.\Delta^{r+2}.y^{n-r-2}+\&c.$$

Putting successively $r=1$, $r=2$, $r=3$, &c., we obtain the values of $\Delta.y^{(n)}$, $\Delta^2.y^{(n)}$, &c., which depend only upon the co-ordinates $y^{(n)}$, $y^{(n-1)}$, &c. Substituting them in the expression of $\int_0^n y^{(i)}.di$ [7938], we get,‡

[7940a] * (3783) We have identically $\left(\frac{1}{t}-1\right)=(1-t).\{1-(1-t)\}^{-1}$. Involving this to the power r, then multiplying by u, and developing by the binomial theorem, we get successively [7939, 7939′].

† (3784) If we multiply [7926g] by t^r, then change r into q, we shall get, as in [7940],

[7941a] $$u.(1-t)^q = \Delta^q.y^{(0)}.t^q+\Delta^q.y^{(1)}.t^{q+1}+\Delta^q.y^{(2)}.t^{q+2}+\ldots+\Delta^q.y^{(n-q)}.t^n+\&c.$$

Now putting successively $q=r$, $q=r+1$, $q=r+2$, &c. in [7941a], we shall get,

[7941b] from the general term $\Delta^q.y^{(n-q)}.t^n$, the coefficients of t^n in $u.(1-t)^r$, $u.(1-t)^{r+1}$, $u.(1-t)^{r+2}$, &c.; and by substituting them in the second member of [7939′], we shall get the second member of [7941]; its first member being the corresponding term of [7926g].

‡ (3785) Putting successively $r=1$, $r=2$, $r=3$, &c. in [7941], and substituting the resulting values in [7938], we get,

[7942a] $$\begin{aligned}\int_0^n y^{(i)}.di = {} & \tfrac{1}{2}.y^{(0)}+y^{(1)}+y^{(2)}+\ldots\ldots+y^{(n-1)}+\tfrac{1}{2}.y^{(n)} \\ & -\tfrac{1}{12}.\{\Delta.y^{(n-1)}+\Delta^2.y^{(n-2)}+\Delta^3.y^{(n-3)}+\Delta^4.y^{(n-4)}+\Delta^5.y^{(n-5)}+\&c.-\Delta.y^{(0)}\} \\ & +\tfrac{1}{24}.\{\ldots+\Delta^2.y^{(n-2)}+2\Delta^3.y^{(n-3)}+3\Delta^4.y^{(n-4)}+4\Delta^5.y^{(n-5)}+\&c.-\Delta^2.y^{(0)}\} \\ & -\tfrac{19}{720}.\{\ldots\ldots\ldots+\Delta^3.y^{(n-3)}+3\Delta^4.y^{(n-4)}+6\Delta^5.y^{(n-5)}+\&c.-\Delta^3.y^{(0)}\} \\ & +\tfrac{3}{160}.\{\ldots\ldots\ldots\ldots\ldots+\Delta^4.y^{(n-4)}+4\Delta^5.y^{(n-5)}+\&c.-\Delta^4.y^{(0)}\} \\ & -\tfrac{863}{60480}.\{\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\Delta^5.y^{(n-5)}+\&c.-\Delta^5.y^{(0)}\} \\ & +\&c.\end{aligned}$$

Reducing this expression, it becomes as in [7942].

Integral formula.

$$\left.\begin{aligned}\textstyle\int_0^n y^{(i)}.di = \tfrac{1}{2}.y^{(0)}+y^{(1)}+y^{(2)} \ldots\ldots\ldots +y^{(n-1)}+\tfrac{1}{2}.y^{(n)}\\ -\tfrac{1}{12}.\{\Delta.y^{(n-1)}-\Delta.y^{(0)}\}\\ -\tfrac{1}{24}.\{\Delta^2.y^{(n-2)}+\Delta^2.y^{(0)}\}\\ -\tfrac{19}{720}.\{\Delta^3.y^{(n-3)}-\Delta^3.y^{(0)}\}\\ -\tfrac{3}{160}.\{\Delta^4.y^{(n-4)}+\Delta^4.y^{(0)}\}\\ -\tfrac{863}{60480}.\{\Delta^5.y^{(n-5)}-\Delta^5.y^{(0)}\}\\ -\&c.\end{aligned}\right\}(P)$$ [7942]

6. *To apply the formula* [7942] *to the variations of the elements of the orbit of the comet, we shall take, for the absciss, the excentric anomaly of the comet, which we have denoted by* u [7860]; and we shall represent, by Qdu, the differential variation of one of the elements of the orbit; then taking u *from degree to degree*, we must determine the corresponding values of Q. If we denote them by $Q^{(0)}$, $Q^{(1)} \ldots\ldots Q^{(n)}$, the formula [7942] will give the value of $\int Qdu$, or the variation of the element of the orbit, corresponding to the supposed variation in the arc of the excentric anomaly. *It will most frequently happen that the first finite difference will be the only term necessary to be noticed in this formula; but near those parts where the planet is at its minimum of distance from the disturbing planet, which renders the values of* $\frac{1}{f^3}$ *and of course those of* Q *very large,* we must notice the other differences. It will then be useful to decrease the interval between the equidistant co-ordinates, by taking the excentric anomaly for every half degree.*

[7943] [7944] [7945] [7945′] [7946] [7947]

7. *We have seen, in* [7836], *that the most sensible part of the perturbations of a comet may be expressed analytically when the comet is at a considerable distance from the disturbing planet, or in the superior part of its orbit; which gives an accurate and simple method of computing these perturbations. We shall develop, by this means, the corresponding variations of the elements of the orbit.*

Variations of the elements depending on $R_{,}$. [7947′]

We shall resume the expression of dh [7849],

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* (3786) The symbol f represents the distance of the comet from the disturbing planet m' [7870a]; and when this is small, the term $\frac{1}{f^3}$ becomes great, and it will therefore produce, in the expressions of de, $ed\varpi$, dR, da, dn, $d\varepsilon - d\varpi$, [7872, 7873, 7877, 7878, 7879, 7883], some terms of the form Qdu, in which Q is quite large.

[7947a] [7947b]

[7948] $$dh = dx.\left\{ x.\left(\frac{dR}{dy}\right) - y.\left(\frac{dR}{dx}\right) \right\} + (xdy - ydx).\left(\frac{dR}{dx}\right).$$

Putting for R its value [7838], namely,

[7949] $$R = R' - \frac{m'}{r} - m'.(xx' + yy' + zz').\left(\frac{1}{r^3} - \frac{1}{r'^3}\right);$$

we find, as in [7837d], that R' is small in comparison with the other

[7950] part of R, or $R_{,}$ [7837], when the radius vector of the comet r is much greater than that of the disturbing planet; *and in this case the perturbations of the comet depending upon* $R_{,}$, *or the greatest part of* R, *are represented, as in* [7825—7827], *in the following manner;*

[7951] $$\delta x = m'.(\tfrac{1}{3}x + x'); \qquad \delta y = m'.(\tfrac{1}{3}y + y'); \qquad \delta z = m'.(\tfrac{1}{3}z + z').$$

This being premised, we have, by neglecting the square of z in the equation [1171 line 3],*

[7952] $$0 = h + y.\left\{ \frac{1}{r} - \frac{dx^2}{dt^2} \right\} + \frac{xdx.dy}{dt^2}.$$

Taking the variation of this equation, relative to the characteristic δ, we get,†

[7953] $$0 = \delta h + \delta y.\left\{ \frac{1}{r} - \frac{dx^2}{dt^2} \right\} - y.\left\{ \frac{\delta r}{r^2} + \frac{2dx.d\delta x}{dt^2} \right\} + \frac{xdx.d\delta y}{dt^2} + \frac{xdy.d\delta x}{dt^2} + \frac{dxdy.\delta x}{dt^2}.$$

If we substitute the preceding values of δx, δy, we obtain,‡

[7952a] * (3787) We have in [7849b], $f' = \mu h$, or by [7874], $f' = h$ nearly. Substituting this in the fifth of the equations [1171], and neglecting terms of the order z^2,

[7952b] $\frac{zdz}{dt}$, $\frac{dz^2}{dt^2}$, $m'z$, &c., as in [7846], it becomes as in [7952]. In like manner, by substituting $f = l$ [7849b, 7874] in the fourth of the equations [1171], and neglecting similar quantities, we get,

[7952c] $$0 = l + x.\left\{ \frac{1}{r} - \frac{dy^2}{dt^2} \right\} + \frac{ydy.dx}{dt^2};$$

which will be of use hereafter.

[7953a] † (3788) The variation of [7952] being taken relative to the characteristic δ, gives, without any reduction, the expression [7953], dt being constant.

‡ (3789) The first expression of δr [7954b] is the same as that in [7818a]; and by

[7954a] substituting the approximate values of δx, δy, δz [7951], it becomes as in the second form of [7954b], which can be reduced to the form [7954c], by using r^2 [7817b], and neglecting $\frac{m'}{r}.zz'$, as in [7952b]. Moreover the differentials of δx, δy [7951], give $d\delta x$, $d\delta y$ [7954d];

$$\delta h = m'.\left\{y.\frac{dx^2}{dt^2} - \frac{xdxdy}{dt^2}\right\} - m'y'.\left\{\frac{1}{r} - \frac{dx^2}{dt^2}\right\} + m'y.\frac{(xx'+yy')}{r^3} + \frac{2m'y.dxdx'}{dt^2} - \frac{m'.xdxdy'}{dt^2} - \frac{m'.xdydx'}{dt^2} - \frac{m'.x'.dxdy}{dt^2}.$$ [7954]

This value of δh, being augmented by an arbitrary constant quantity, expresses the alteration of h arising from the part of R, which is represented by $R_{\prime}$ in [7837]; it must therefore result from the integration of the preceding expression of dh [7948], by substituting for R the function $R_{\prime}$ [7837], namely, [7955]

$$R_{\prime} = -\frac{m'}{r} - m'.(xx'+yy'+zz').\left(\frac{1}{r^3} - \frac{1}{r'^3}\right);$$ [7956]

which may be confirmed *a posteriori* by calculation [7958d], by observing that we may here suppose $\frac{m'x}{r^3} = -m'.\frac{ddx}{dt^2}$, $\frac{m'x'}{r'^3} = -m'.\frac{ddx'}{dt^2}$, &c. [7822]. [7956′]

If we substitute, in this value of δh [7954], $h + \frac{y}{r}$ instead of

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$$\delta r = \frac{x\delta x + y\delta y + z\delta z}{r} = \frac{m'}{r}.\{\tfrac{1}{3}.(x^2+y^2+z^2)+(xx'+yy'+zz')\}$$ [7954b]

$$= \frac{m'}{r}.\{\tfrac{1}{3}r^2+(xx'+yy')\} = \tfrac{1}{3}m'r + \frac{m'}{r}.(xx'+yy');$$ [7954c]

$$d\delta x = m'.\{\tfrac{1}{3}dx + dx'\}; \qquad d\delta y = m'.\{\tfrac{1}{3}dy + dy'\}.$$ [7954d]

Substituting the values of δx, δy, δr, $d\delta x$, $d\delta y$ [7951, 7954c, d], in [7953], and transposing the terms so as to get the value of δh, we obtain, without any reduction,

$$\delta h = -\tfrac{1}{3}m'y.\left\{\frac{1}{r} - \frac{dr^2}{dt^2}\right\} - m'y'.\left\{\frac{1}{r} - \frac{dx^2}{dt^2}\right\} + m'y.\left\{\frac{1}{3r} + \frac{xx'+yy'}{r^3} + \frac{2dx.(\frac{1}{3}dx+dx')}{dt^2}\right\} - \frac{m'.xdx.(\frac{1}{3}dy+dy')}{dt^2} - \frac{m'.xdy.(\frac{1}{3}dx+dx')}{dt^2} - \frac{m'.dxdy.(\frac{1}{3}x+x')}{dt^2}.$$ [7954e]

This expression consists of fifteen terms, and if we compare it with [7954] we shall find that the third to the tenth terms of [7954] inclusively correspond respectively to the terms 3, 4, 6, 7, 9, 11, 13, 15 of [7954e]. The first and fifth terms of [7954e] destroy each other. The second and eighth are $\frac{1}{3}m'y.\frac{dx^2}{dt^2} + \frac{2}{3}m'y.\frac{dx^2}{dt^2} = m'y.\frac{dx^2}{dt^2}$, as in the first of [7954]. The three remaining terms of [7954e], namely, the tenth, twelfth and fourteenth, being added together, give $-m'.\frac{xdx.dy}{dt^2}$, as in the second and only remaining terms of [7954]; therefore the value of δh [7954] is equal to that in [7954e]. [7954f] [7954g]

[7957] $y.\frac{dx^2}{dt^2}-x.\frac{dxdy}{dt^2}$ [7952], we shall get,*

Part of δh [7958] depending on $R_{,}$.

$$\delta h = m'.\left(h+\frac{y}{r}\right)-m'x.\frac{(xy'-x'y)}{r^3}-m'.\frac{dx'}{dt}.\frac{(xdy-ydx)}{dt}$$
$$-m'.\frac{dx}{dt}.\frac{(xdy'-y'dx+x'dy-ydx')}{dt}.$$

[7959] *Hence it follows, that to obtain the variation of* h, *from a given point of the orbit to another point, so far as it depends on* $R_{,}$ [7956], *or on the part of* R *which is independent of* R', *it is only necessary to subtract the value of the second member of the preceding equation* [7958], *in the first point, from its value in the second point.*

[7960] If we change, in the equation [7958], h into l, x into y, x' into y', and the contrary, we shall obtain the variation of l arising from $R_{,}$, or from the part of R which is independent of R'; hence we get,†

Part of δl [7961] depending on $R_{,}$.

$$\delta l = m'.\left(l+\frac{x}{r}\right)+m'y.\frac{(xy'-yx')}{r^3}+m'.\frac{dy'}{dt}.\frac{(xdy-ydx)}{dt}$$
$$+m'.\frac{dy}{dt}.\frac{(xdy'-y'dx+x'dy-ydx')}{dt}.$$

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* (3790) Substituting in [7954] the value of $y.\frac{dx^2}{dt^2}-x.\frac{dxdy}{dt^2}$ [7957], also

[7958a] $\frac{1}{r}=\frac{r^2}{r^3}=\frac{xx+yy}{r^3}$, neglecting $\frac{z^2}{r^3}$, as in [7846], it becomes,

[7958b]

$$\delta h = m'.\left\{h+\frac{y}{r}\right\}-m'y'.\left\{\frac{x^2+y^2}{r^3}-\frac{dx^2}{dt^2}\right\}+m'y.\frac{(xx'+yy')}{r^3}+\frac{2m'.ydxdx'}{dt^2}$$
$$-\frac{m'.xdxdy'}{dt^2}-\frac{m'.xdydx'}{dt^2}-\frac{m'.x'.dxdy}{dt^2};$$

[7958c] which, by a different arrangement of the terms, becomes as in [7958]. This value of δh is deduced from the general expression of δh [7953], by substituting the values of δx, δy, δz [7951], which were computed upon the supposition that $R'=0$, or $R=R_{,}$ [7837e]; it must therefore necessarily follow, that if we substitute $R=R_{,}$ in [7849], we shall, by integration, get the same value of δh as in [7958], as is observed in [7956']. This is so

[7958d] evident that we have not thought it to be necessary to repeat the calculations, by the method pointed out in [7956', &c.].

† (3791) If we change x, x', y, y', into y, y', x, x', respectively, we shall find that

[7961a] the expression of dh [7849] changes into that of dl [7850], and h [7952] into l [7952c]; also δx into δy, and δy into δx [7951]. Hence it is evident that if we make the same

[7961b] changes in the expression of δh [7958], we shall get the value of δl, as in [7961]; the arrangement of the terms being varied a little so as to make the factors appear in the same forms as in [7958].

Subtracting the value of the second member of this equation in a given point of the orbit, from its value in another point, we shall have in that interval the variation of l depending on $R_,$ [7956], or on the part of R [7949], which is independent of R'. The variations of h and l give those of e and ϖ, observing that we have,* [7962]

$$e\delta e = h\delta h + l\delta l; \qquad e^2\delta\varpi = l\delta h - h\delta l.$$ [7963]

We have, in [1171 line 5],†

$$\frac{1}{a} = \frac{2}{r} - \frac{(dx^2+dy^2)}{dt^2};$$ [7964]

which gives,

$$\frac{\delta a}{a^2} = \frac{2\delta r}{r^2} + \frac{2dx.d\delta x + 2dy.d\delta y}{dt^2}.$$ [7965]

Substituting the values of δr, $d\delta x$, $d\delta y$, which are given in [7954c, d], we obtain, [7965′]

$$\frac{\delta a}{a^2} = \tfrac{2}{3}.\frac{m'}{r} + 2m'.\frac{(xx'+yy')}{r^3} + \tfrac{2}{3}.m'.\frac{(dx^2+dy^2)}{dt^2} + 2m'.\frac{(dxdx'+dydy')}{dt^2}.$$ [7966]

If we substitute in this equation for $\frac{dx^2+dy^2}{dt^2}$ its value $\frac{2}{r}-\frac{1}{a}$ [7964], and multiply by a^2, we get, [7966′]

Part of δa depending on $R_,$.

$$\delta a = \frac{2m'.a^2}{r} - \tfrac{2}{3}.m'a + 2m'.a^2.\frac{(xx'+yy')}{r^3} + 2m'.a^2.\frac{(dxdx'+dydy')}{dt^2}.$$ [7967]

Hence we may find δn, by means of the equation $n^2 = \frac{1}{a^3}$ [7897′], which gives, as in [6662], [7968]

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* (3792) The variations of the expressions of h, l [7847], relative to the characteristic δ, give, in like manner as in [7851, 7852],

$$\delta e = \delta h.\sin.\varpi + \delta l.\cos.\varpi; \qquad e\delta\varpi = \delta h.\cos.\varpi - \delta l.\sin.\varpi.$$ [7963a]

Multiplying these by e, and substituting the values of $e.\sin.\varpi$, $e.\cos.\varpi$ [7847], we get [7963].

† (3793) Putting $\mu = 1$ [7874] in the last of the equations [1171], and neglecting dz^2, as in [7952b], we get the expression [7964]. Its variation relative to the characteristic δ, gives [7965], by changing its signs; and we may observe that terms of the order m'^2 are neglected in this expression of $\frac{1}{a}$. Substituting in [7965] the differentials of the values of δx, δy [7954d], also $\delta r = \frac{1}{3}.m'r + \frac{m'}{r}.(xx'+yy')$ [7954e], we get [7966]. Multiplying this by a^2, and substituting $\frac{2}{3}.m'a^2.\frac{(dx^2+dy^2)}{dt^2} = \frac{4}{3}.\frac{m'a^2}{r} - \frac{2}{3}m'a$ [7966′], we get, by a slight reduction, the expression [7967]. [7965a] [7965b] [7965c]

[7969] $$\frac{\delta n}{n} = -\tfrac{3}{2}.\frac{\delta a}{a};$$

Part of δn depending on $R_{\prime}$. substituting [7967] in [7969], and multiplying by n, we get,

[7970] $$\delta n = -\frac{3m'.an}{r} + m'n - 3m'.an.\frac{(xx'+yy')}{r^3} - 3m'.an.\frac{(dxdx'+dydy')}{dt^2}.$$

Subtracting the values of δa, δn [7967, 7970], at a given point of the orbit,

[7971] from their values at another point, we shall obtain the variations of a and n in that interval, arising from $R_{\prime}$, or the part of R which is independent of R'.

[7972] To obtain the variation of the mean anomaly, depending upon the part of R, which we have represented by $R_{\prime}$ [7837], we shall observe that this

$\overline{\delta n}$ [7973] variation is equal to* $\int\delta n.dt+\delta\varepsilon-\delta\varpi$. We shall put $\overline{\delta n}$ *for the whole value of* δn, *at the point of the orbit where we begin to consider separately this part of* R; that is, the value of δn which results from the preceding

[7973′] perturbations. *We shall have, by supposing* t *to commence at this point,*

[7974] $$\int\delta n.dt+\delta\varepsilon-\delta\varpi = \overline{\delta n}.t+\int\delta' n.dt+\delta\varepsilon-\delta\varpi; \quad [7972d]$$

$\delta' n$ [7975] $\delta' n$ *being the variation of* n, *counted from the point mentioned in* [7973′], *and depending upon* $R_{\prime}$ [7837], *or upon the part of* R *which is independent of* R'. We have, as in [7976c],

[7976] $$\int\delta' n.dt+\delta\varepsilon-\delta\varpi = \int\left\{\delta' n.dt - \frac{d\varpi.(1-e.\cos.u)^2}{\sqrt{1-e^2}} - \frac{de.\sin.u.(2-e^2-e.\cos.u)}{1-e^2}\right\}.$$

Integrating by parts the second member of this equation, we obtain,†

[7972a] * (3794) The mean anomaly is expressed by $V=\int ndt+\varepsilon-\varpi$ [7858′]. Its variation relative to the characteristic δ, is $\delta V=\int\delta n.dt+\delta\varepsilon-\delta\varpi$. The variations being supposed

[7972b] to commence at the time $t=0$, when δn is equal to $\overline{\delta n}$ [7973]; and the general value

[7972c] of δn is $\delta n=\overline{\delta n}+\delta' n$ [7973, 7975]; substituting this in $\int\delta n.dt$, we get, $\int\delta n.dt=\overline{\delta n}.t+\int\delta' n.dt$; hence the value of δV [7972b] becomes as in [7974]; the

[7972d] integrals being supposed to commence at the time when δn is equal to $\overline{\delta n}$.

† (3795) We shall put for brevity,

[7976a] $$\varphi(u) = \frac{(1-e.\cos.u)^2}{\sqrt{1-e^2}}; \qquad \Psi(u) = \frac{\sin.u.\{2-e^2-e.\cos.u\}}{1-e^2};$$

and by substituting these symbols in [7886], we get,

[7976b] $$d\varepsilon - d\varpi = -d\varpi.\varphi(u) - de.\Psi u.$$

Adding $\delta' n.dt$ to both members of this equation, then integrating and putting $\int d\varepsilon=\delta\varepsilon$,

[7976c] $\int d\varpi=\delta\varpi$, $\int de=\delta e$, we get [7976]. Now if we integrate [7976b] by parts, as in [1716a], we obtain,

$$\int\delta'n.dt+\delta\varepsilon-\delta\varpi=\text{constant}-\frac{\delta\varpi.(1-e.\cos.u)^2}{\sqrt{1-e^2}}-\frac{\delta e.\sin.u.(2-e^2-e.\cos.u)}{1-e^2} \quad 1$$ [7977]

$$+\int\left\{\frac{\delta'n}{n}.du.(1-e.\cos.u)+2e\delta\varpi.\frac{du.\sin.u.(1-e.\cos.u)}{\sqrt{1-e^2}}+\frac{\delta e.du.(1-e.\cos.u)(e+2.\cos.u)}{1-e^2}\right\}; \quad 2$$

ndt being, as in [7882], equal to $du.(1-e.\cos.u)$. We have, as in [7851e,f],

$$h=0;\qquad \delta h=e\delta\varpi;\qquad l=e;\qquad \delta l=\delta e.$$ [7978]

We shall put $m'nq$ for the value of the preceding expression of δn, at the new origin we have assigned to the time t [7973]; we shall then have for the value of $\delta'n$ [7975], the following expression, [7979]

$$\delta'n=\delta n-m'nq;$$ [7980]

and by substituting the preceding values of δh, δl, we shall find,* [7980']

$$\delta\varepsilon-\delta\varpi=\text{constant}-\delta\varpi.\varphi(u)-\delta e.\Psi(u)+\int\delta\varpi.\left(\frac{d.\varphi(u)}{du}\right).du+\int\delta e.\left(\frac{d.\Psi(u)}{du}\right).du;$$ [7976d]

considering $\varphi(u)$, $\Psi(u)$, as functions of u only, without noticing the variableness of e, because it introduces only terms of the order m'^2, which are neglected; moreover from [7882] we have $\int\delta'n.dt=\int\frac{\delta'n}{n}.ndt=\int\frac{\delta'n}{n}.du.(1-e.\cos.u)$; adding this to [7976$d$], we get, [7976$e$]

$$\int\delta'n.dt+\delta\varepsilon-\delta\varpi=\text{constant}-\delta\varpi.\varphi(u)-\delta e.\Psi(u) \quad 1$$

$$+\int\left\{\frac{\delta'n}{n}.du.(1-e.\cos.u)+\delta\varpi.\left(\frac{d.\varphi(u)}{du}\right).du+\delta e.\left(\frac{d.\Psi(u)}{du}\right).du\right\} \quad 2$$ [7976f]

The first line of this expression is the same as the first line of [7977]; and the integral in the second line of [7976f] is the same as that in [7977 line 2]; observing that, in finding the differential of the expression $\Psi(u)$ [7976a], we get, by successive reductions, [7976g]

$$d.\{\sin.u.(2-e^2-e.\cos.u)\}=du.\{\cos.u.(2-e^2-e.\cos.u)+e.\sin.^2u\}=du.\{\cos.u.(2-e^2-e.\cos.u)+e.(1-\cos^2u)\}$$ [7976h]

$$=du.\{2.\cos.u+e.(1-2.\cos.^2u)-e^2.\cos.u\}=du.(1-e.\cos.u).(e+2.\cos.u).$$ [7976i]

* (3796) Substituting $\delta\varpi=\frac{\delta h}{e}$, $\delta e=\delta l$ [7978], in the terms in the first line of the second member of [7977], we get the terms of the second member of [7981 line 1]. Again if we substitute the expression of $\delta'n=-m'nq+\delta n$ [7980], in the first term of [7977 line 2], we obtain the expression [7981b]; and by using the value of ndt [7882], in the first term of its second member, it is reduced to the form [7981c]; [7981a]

$$\int\frac{\delta'n}{n}.du.(1-e.\cos.u)=-\int m'q.du.(1-e.\cos.u)+\int\frac{\delta n}{n}.du.(1-e.\cos.u)$$ [7981b]

$$=-m'nq.t+\int\frac{\delta n}{n}.du.(1-e.\cos.u).$$ [7981c]

If we now suppose the integral expression in [7977 line 2] to be represented by $-m'nq.t+V_{,}$, we shall find, by the substitution of [7981c] in [7977 line 2],

$$V_{,}=\int du.(1-e.\cos.u).\left\{\frac{\delta n}{n}+e\delta\varpi.\frac{2.\sin.u}{\sqrt{1-e^2}}+\delta e.\frac{(e+2.\cos.u)}{1-e^2}\right\};$$ [7981d]

Variation of the mean anomaly [7981] depending on $R_{,}$.

$$\int \delta' n.dt + \delta\varepsilon - \delta\varpi = \text{constant} - \frac{\delta h.(1-e.\cos.u)^2}{e.\sqrt{1-e^2}} - \frac{\delta l.\sin.u.(2-e^2-e.\cos.u)}{1-e^2} \quad 1$$
$$-m'nq.t + \frac{m'.(xy'-x'y)}{a^2.\sqrt{1-e^2}}. \quad 2$$

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and the author observes in [7980'] that if we use the values of δh, δl, δn [7958, 7961, 7970], this integral expression will become of the form in the last term of [7981 line 2], making

[7981e] $V_{,} = \frac{m'.(xy'-x'y)}{a^2.\sqrt{1-e^2}}$; and it now remains to be proved that the differentials of these two

[7981f] expressions of $V_{,}$, are identical; or by using the values $e\delta\varpi = \delta h$, $\delta e = \delta l$ [7978], we must prove that the first member of the following equation [7981g] can be reduced to the same form as its second member, by substituting the values of δh, δl, δn [7958, 7961, 7970];

[7981g] $$du.(1-e.\cos.u).\left\{\frac{\delta n}{n} + \delta h.\frac{2.\sin.u}{\sqrt{1-e^2}} + \delta l.\frac{(e+2.\cos.u)}{1-e^2}\right\} = \frac{m'}{a^2.\sqrt{1-e^2}}.\{xdy'-ydx'+y'dx-x'dy\}.$$

The two first terms of δh, δl, δn, depending on the unaccented letters x, y, being substituted in the first member of [7981g], mutually destroy each other. For if we retain

[7981h] only these two terms, and use also the values $h=0$, $l=e$ [7978], we shall have from [7970, 7958, 7961], by neglecting for brevity the factor m', which is common to all the terms,

[7981i] $$\frac{\delta n}{n} = -\frac{3a}{r} + 1; \qquad \delta h = \frac{y}{r}; \qquad \delta l = e + \frac{x}{r};$$

and if we substitute the values of r, x, y [7855, 7862, 7862'], they will become,

[7981k] $$\frac{\delta n}{n} = -\frac{3}{1-e.\cos.u} + 1 = \frac{(-2-e.\cos.u)}{1-e.\cos.u};$$

[7981l] $$\delta h = \frac{\sqrt{1-e^2}.\sin.u}{1-e.\cos.u}; \qquad \delta l = e + \frac{\cos.u-e}{1-e.\cos.u} = \frac{(1-e^2).\cos.u}{1-e.\cos.u}.$$

Substituting these in the first member of [7981g], we get, without any reduction,

[7981m] $$du.\{(-2-e.\cos.u)+2.\sin.^2u+\cos.u.(e+2.\cos.u)\};$$

and by putting $2.\sin.^2u+2.\cos.^2u=2$, we find that the terms mutually destroy each

[7981n] other; so that we shall now have to notice only those terms of δn, δh, δl, which contain the accented letters x', dx', y', dy', which we shall successively compute, taking them in the same order.

First. Noticing only the terms which contain explicitly the finite quantity x', neglecting

[7981o] for a moment, for brevity, the factor $m'x'$, common to all these terms, and which will be re-substituted in [7981z], we shall get, from [7970, 7958, 7960], the following expressions;

[7981p] $$\frac{\delta n}{n} = -\frac{3ax}{r^3}; \qquad \delta h = \frac{xy}{r^3} - \frac{dx}{dt}.\frac{dy}{dt}; \qquad \delta l = -\frac{y^2}{r^3} + \frac{dy^2}{dt^2}.$$

Now we have, in [7862, 7862'], the following values of x, y, whose differentials are as in [7981r];

[7981q] $$x = a.\{\cos.u-e\}; \qquad y = a.\sqrt{1-e^2}.\sin.u;$$

[7981r] $$dx = -adu.\sin.u; \qquad dy = adu.\sqrt{1-e^2}.\cos.u;$$

If we subtract the value of the second member of this equation, at the new origin of t [7973'], from its value at another point of the orbit, we shall [7981']

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and if we substitute $1-e.\cos.u=\frac{r}{a}$ [7855], in the expression of ndt [7882], we shall [7981s]

get, by using [7897'], $du=\frac{an.dt}{r}=\frac{dt}{a^{\frac{1}{2}}r}$. Substituting this value of du in [7981r], and [7981s']

dividing by dt, we obtain,

$$\frac{dx}{dt}=-\frac{a^{\frac{1}{2}}}{r}.\sin.u;\qquad \frac{dy}{dt}=\frac{a^{\frac{1}{2}}.\sqrt{1-e^2}}{r}.\cos.u. \qquad [7981t]$$

Substituting the values [7981q, t] in [7981p], we get, by a very slight reduction, the first expressions in [7981u, v, w]. The second forms are deduced from the first, respectively, by substituting in the terms between the braces, in [7981v, w], the value of $\frac{r}{a}$ [7981s];

$$\frac{\delta n}{n}=\frac{3a^2}{r^3}.(e-\cos.u); \qquad [7981u]$$

$$\delta h=\frac{a^2.\sqrt{1-e^2}.\sin.u}{r^3}.\left\{(\cos.u-e)+\frac{r}{a}.\cos.u\right\}=\frac{a^2.\sqrt{1-e^2}.\sin.u}{r^3}.\{2.\cos.u-e-e.\cos.^2u\} \qquad [7981v]$$

$$\delta l=\frac{a^2.(1-e^2)}{r^3}.\left\{-\sin.^2u+\frac{r}{a}.\cos.^2u\right\}=\frac{a^2.(1-e^2)}{r^3}.\{-\sin.^2u+\cos.^2u-e.\cos.^3u\}. \qquad [7981w]$$

Substituting the expressions [7981u, v, w] in the first member of [7981g], it becomes as in [7981x]; putting $\sin.^2u=1-\cos.^2u$, and arranging according to the powers of $\cos.u$, we get the first form of [7981y]; and by successive reductions, using the value of $\frac{r}{a}$ [7981s], and that of dy [7981r], we finally obtain the last expression in [7981y];

$$du.(1-e.\cos.u).\frac{a^2}{r^3}.\{3.(e-\cos.u)+2.\sin.^2u.(2.\cos.u-e-e.\cos.^2u)+(e+2.\cos.u).(-\sin.^2u+\cos.^2u-e.\cos.^3u)\} \qquad [7981x]$$

$$=du.(1-e.\cos.u).\frac{a^2}{r^3}.\{-\cos.u.(1-2e.\cos.u+e^2.\cos.^2u)\}=-du.(1-e.\cos.u)^3.\frac{a^2}{r^3}.\cos.u$$
$$=-du.\left(\frac{r}{a}\right)^3.\frac{a^2}{r^3}.\cos.u=-du.\frac{\cos.u}{a}=-\frac{dy}{a^2.\sqrt{1-e^2}}. \qquad [7981y]$$

Connecting this last expression with the factor $m'x'$, which was neglected in [7981o], we finally get $-\frac{dy}{a^2.\sqrt{1-e^2}}.m'x'$, for the term in the first member of [7981g], depending on x'; being the same as that which is given by the author, as in the second member of [7981g]. [7981z]

Second. We shall now compute the terms depending on dx'. If we retain only these terms, and neglect for brevity the factor $m'dx'$, which is common to all of them, taking care to re-substitute it at the end of the calculation, in [7982g], we shall get from [7970, 7958, 7960], [7982a]

$$\frac{\delta n}{n}=-\frac{3a.dx}{dt^2};\qquad \delta h=-\frac{(xdy-ydx)}{dt^2}+\frac{ydx}{dt^2};\qquad \delta l=-\frac{ydy}{dt^2}. \qquad [7982b]$$

[7981''] have the variation of the mean anomaly during that interval, arising from $R_{,}$ [7837], or from the part of R which is independent of R'.

Now from the values of x, y, dx, dy [7981q, r, t], we obtain by successive reductions, using [7981s],

[7982c]
$$xdy-ydx=a^2.du.\sqrt{1-e^2}.\{\cos.u.(\cos.u-e)+\sin.^2u\}$$
$$=a^2.du.\sqrt{1-e^2}.\{1-e.\cos.u\}=ar.du.\sqrt{1-e^2};$$

[7982c'] substituting this, and $dt^2=ar^2.du^2$ [7981s'], together with the values of dx, dy [7981r], we get, from [7982b],

[7982d]
$$\frac{\delta n}{n}=\frac{3a.\sin.u}{r^2.du};\qquad \delta h=-\frac{a.\sqrt{1-e^2}}{r^2.du}.\left\{\frac{r}{a}+\sin.^2u\right\};\qquad \delta l=-\frac{a.(1-e^2).\sin.u.\cos.u}{r^2.du}.$$

Substituting these last expressions in the first member of [7981g], it becomes as in [7982e]. This is successively reduced to the form [7982f] by putting, in the factor between the braces, $\sin.^2u+\cos.^2u=1$, and $1-e.\cos.u=\frac{r}{a}$ [7981s]; using also y [7981q];

[7982e]
$$(1-e.\cos.u).\frac{a.\sin.u}{r^2}.\left\{3-2.\left(\frac{r}{a}+\sin.^2u\right)-\cos.u.(e+2.\cos.u)\right\}=(1-e.\cos.u).\frac{a.\sin.u}{r^2}.\left\{1-e.\cos.u-2.\frac{r}{a}\right\}$$

[7982f]
$$=(1-e.\cos.u).\frac{a.\sin.u}{r^2}.\left\{-\frac{r}{a}\right\}=-\frac{r}{a}.\frac{a.\sin.u}{r^2}.\frac{r}{a}=-\frac{\sin.u}{a}=-\frac{y}{a^2.\sqrt{1-e^2}},\quad [7981q].$$

Multiplying this last expression by the factor $m'dx'$, which was neglected in [7982a], we

[7982g] get $-\frac{m'y.dx'}{a^2.\sqrt{1-e^2}}$, for the part of the first member of [7981g] depending on dx'; being the same as that given by the author in the second member of [7981g].

Third. We shall now take into consideration the terms which contain y' explicitly.

[7982h] Noticing only these terms, and omitting, as above, the common factor $m'y'$, which is re-substituted in [7982p], we shall get from [7970, 7958, 7961],

[7982i]
$$\frac{\delta n}{n}=-\frac{3ay}{r^3};\qquad \delta h=-\frac{x^2}{r^3}+\frac{dx^2}{dt^2};\qquad \delta l=\frac{xy}{r^3}-\frac{dx}{dt}.\frac{dy}{dt};$$

substituting the values of $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ [7981q, t], they become by successive reductions, using $\frac{r}{a}$ [7981s],

[7982k]
$$\frac{\delta n}{n}=-\frac{3a^2.\sqrt{1-e^2}}{r^3}.\sin.u;$$

[7982l]
$$\delta h=\frac{a^2}{r^3}.\left\{-(\cos.u-e)^2+\frac{r}{a}.\sin.^2u\right\}=\frac{a^2}{r^3}.\{-(\cos.u-e)^2+(1-e.\cos.u).(1-\cos.^2u)\}$$
$$=\frac{a^2}{r^3}.\{1-2.\cos.^2u+e.\cos.u+e.\cos.^3u-e^2\};$$

[7982m]
$$\delta l=\frac{a^2.\sqrt{1-e^2}.\sin.u}{r^3}.\left\{(\cos.u-e)+\frac{r}{a}.\cos.u\right\}=\frac{a^2.\sqrt{1-e^2}.\sin.u}{r^3}.\{2.\cos.u-e-e.\cos.^2u\}.$$

Substituting the values [7982k, l, m] in the first member of [7981g], it becomes as in [7982n]; and the terms between the braces may be reduced to

$$-1+2e.\cos.u-e^2.\cos.^2u=-(1-e.\cos.u)^2;$$

To obtain the variations of the inclination of the orbit and of the node, depending upon the same part of R, we must observe that we have, in [1171 line 1], [7981‴]

$$c' = \frac{xdz-zdx}{dt}; \qquad c'' = \frac{ydz-zdy}{dt}. \qquad [7982]$$

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hence we get [7982o], using the value of $\frac{r}{a}$ [7981s], and that of dx [7981r];

$$du.(1-e.\cos.u).\frac{a^2.\sin.u}{r^3.\sqrt{1-e^2}}.\left\{\begin{matrix}-3.(1-e^2)+2.(1-2.\cos.^2u+e.\cos.u+e.\cos.^3u-e^2)\\+(e+2.\cos.u).(2.\cos.u-e-e.\cos.^2u)\end{matrix}\right\} \qquad [7982n]$$

$$= du.(1-e.\cos.u).\frac{a^2.\sin.u}{r^3.\sqrt{1-e^2}}.\{-(1-e.\cos.u)^2\} = -du.(1-e.\cos.u)^3.\frac{a^2.\sin.u}{r^3.\sqrt{1-e^2}}$$

$$= -du.\left(\frac{r}{a}\right)^3.\frac{a^2.\sin.u}{r^3.\sqrt{1-e^2}} = -\frac{adu.\sin.u}{a^2.\sqrt{1-e^2}} = \frac{dx}{a^2.\sqrt{1-e^2}}. \qquad [7982o]$$

Connecting this last expression with the common factor $m'y'$ [7982h], we get $\frac{m'y'.dx}{a^2.\sqrt{1-e^2}}$; [7982p]
as in the third term of the second member of [7981g].

Fourth. The terms of [7970, 7958, 7961] depending on dy', give, by neglecting in like manner as above, the common factor $m'dy'$, and re-substituting it in [7982v], [7982q]

$$\frac{\delta n}{n} = -\frac{3a.dy}{dt^2}; \qquad \delta h = -\frac{xdx}{dt^2}; \qquad \delta l = \frac{(xdy-ydx)}{dt^2} + \frac{xdy}{dt^2}. \qquad [7982q']$$

Substituting the values [7981q, r, 7982c, c′], they become as in [7982r, s], by using $\frac{r}{a}$ [7981s] in [7982s];

$$\frac{\delta n}{n} = -\frac{3a.\sqrt{1-e^2}.\cos.u}{r^2.du}; \qquad \delta h = \frac{a.\sin.u.\{\cos.u-e\}}{r^2.du}; \qquad [7982r]$$

$$\delta l = \frac{a.\sqrt{1-e^2}}{r^2.du}.\left\{\frac{r}{a}+\cos.u.(\cos.u-e)\right\} = \frac{a.\sqrt{1-e^2}}{r^2.du}.\{1-2e.\cos.u+\cos.^2u\}. \qquad [7982s]$$

Hence the first member of [7981g], depending on these terms, becomes as in [7982t]; and by putting $\sin.^2u = 1-\cos.^2u$, we get the first expression [7982u]; then by successive reductions, using x, r [7981q, s], we get the last form of [7982u];

$$du.(1-e.\cos.u).\frac{a}{r^2.du\sqrt{1-e^2}}.\left\{\begin{matrix}-3.(1-e^2).\cos.u+2.\sin.^2u.(\cos.u-e)\\+(e+2.\cos.u).(1-2e.\cos.u+\cos.^2u)\end{matrix}\right\} \qquad [7982t]$$

$$= (1-e.\cos.u).\frac{a}{r^2.\sqrt{1-e^2}}.\{(1-e.\cos.u).(\cos.u-e)\} = (1-e.\cos.u)^2.\frac{x}{r^2.\sqrt{1-e^2}} = \frac{x}{a^2.\sqrt{1-e^2}}. \qquad [7982u]$$

Connecting this last expression with the factor $m'dy'$ [7982q], it becomes $\frac{m'x.dy'}{a^2.\sqrt{1-e^2}}$, [7982v]
as in the second member of [7981g]. From what has been said it appears that the second member of [7981g] is equivalent to the first member of the same equation. Hence it follows that the equation [7981] is equivalent to that in [7977]. [7982w]

Hence we get,*

[7983] $$\delta c' = \frac{x.d\delta z + \delta x.dz - z.d\delta x - \delta z.dx}{dt};$$

[7984] $$\delta c'' = \frac{y.d\delta z + \delta y.dz - z.d\delta y - \delta z.dy}{dt}.$$

[7985] If we substitute the values of δx, δy, δz, given in [7825—7827], we shall have,†

[7986] $$\delta c' = \tfrac{2}{3} m'c' + m'.\frac{(xdz' + x'dz - zdx' - z'dx)}{dt};$$

[7987] $$\delta c'' = \tfrac{2}{3} m'c'' + m'.\frac{(ydz' + y'dz - zdy' - z'dy)}{dt}.$$

We shall now observe that z, $\frac{dz}{dt}$, c' and c'', either vanish, or are of the

[7988] order of the disturbing forces [7893]; therefore, by neglecting the square of these forces, we shall have,‡

[7989] $$\delta c' = m'.\frac{(xdz' - z'dx)}{dt};$$

[7990] $$\delta c'' = m'.\frac{(ydz' - z'dy)}{dt}.$$

Variations of the inclination and node depending on $R_{,}$.

From these equations we may deduce, by means of the formulas [7891, 7891′],§

[7983a] * (3797) The differentials of the equations [7982], taken relative to the characteristic δ, give the expressions [7983, 7984].

† (3798) Substituting the values [7825, 7827] in the numerator of [7983], and reducing by means of the value of c' [7982], it becomes,

[7986a]
$$\begin{aligned} &\tfrac{1}{3} m'.(xdz + xdz - zdx - zdx) + m'.(xdz' + x'dz - zdx' - z'dx) \\ &= \tfrac{2}{3} m'.(xdz - zdx) + m'.(xdz' + x'dz - zdx' - z'dx) \\ &= \tfrac{2}{3} m'.c'dt + m'.(xdz' + x'dz - zdx' - z'dx); \end{aligned}$$

[7986b] substituting this in [7983] we get [7986]. In like manner we get from [7984], by using [7826, 7827], the value of $\delta c''$ [7987]; or it is more easily derived from [7986] by

[7986c] changing x into y and y into x; for by this means $\delta c'$ [7983] changes into $\delta c''$ [7984], and $\delta c'$ [7986] into $\delta c''$ [7987].

[7988a] ‡ (3799) The quantities z, $\frac{dz}{dt}$ [7846′], are of the order m'; therefore c', c'' [7982] are of the same order; and by neglecting terms of the order m'^2, we may reject

[7988b] $m'c'$, $m'c''$, $m'.x'dz$, $m'.zdx'$, $m'.y'dz$, $m'.zdy'$, from [7986, 7987]; by this means they will become as in [7989, 7990] respectively.

§ (3800) The values of $\delta c'$, $\delta c''$ [7989, 7990], being substituted for c', c'', respectively,

[7990a] in [7891, 7891′], give the corresponding values of φ, θ; referred to the plane of the orbit at the time of the commencement of the integral.

the variations of the inclinations of the orbit and the node, arising from $R_{,}$, in the part of the orbit now under consideration. [7990′]

8. We can obtain the variations of the elements of the orbit, relative to the part R' of R, by the formulas in [7872—7920], by changing R into R' [7838], in the expressions of dh, dl, $d.\frac{1}{a}$, dc', dc'', [7849, 7850, 7875, 7888, 7889], and integrating them by means of quadratures. In the upper portion of the orbit, R' being very small, the values of these integrals will also be very small; but in this portion, where it is so advantageous to divide R into two parts, we may determine without quadratures, by means of converging series, the variations of the elements of the orbit corresponding to R'. For this purpose we shall resume the expression of R' [7838]; and by developing it in a series, we shall have,* [7991] [7992]

$$R'=\frac{m'.r'^2}{2r^3}-\tfrac{3}{2}m'.\frac{(xx'+yy'+zz'-\frac{1}{2}r'^2)^2}{r^5}-\tfrac{5}{2}m'.\frac{(xx'+yy'+zz'-\frac{1}{2}r'^2)^3}{r^7}-\&c. \qquad [7993]$$

Now we have,†

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* (3801) If we put for brevity for a moment, $w=rr'.\cos.\gamma-\frac{1}{2}r'^2=xx'+yy'+zz'-\frac{1}{2}r'^2$ [7993a] [7817c], we shall have, from [7815b],

$$f^2=(x'-x)^2+(y'-y)^2+(z'-z)^2=r^2-2w. \qquad [7993b]$$

Substituting this in the second term of R [7802], we obtain [7993c]; and by development by the binomial theorem, it becomes as in [7993d]. If we substitute in the second term of this development $\frac{w}{r^2}$, the second expression of w [7993a], we shall get, by making a slight reduction, the expression of R [7993e];

$$R=\frac{m'.(xx'+yy'+zz')}{r'^3}-m'.(r^2-2w)^{-\frac{1}{2}} \qquad [7993c]$$

$$=\frac{m'.(xx'+yy'+zz')}{r'^3}-\frac{m'}{r}.\left\{1+\frac{w}{r^2}+\tfrac{3}{2}.\frac{w^2}{r^4}+\tfrac{5}{2}.\frac{w^3}{r^6}+\&c.\right\} \qquad [7993d]$$

$$=-\frac{m'}{r}-m'.(xx'+yy'+zz').\left(\frac{1}{r^3}-\frac{1}{r'^3}\right)+\frac{m'.r'^2}{2r^3}-\frac{m'}{r}.\left\{\tfrac{3}{2}.\frac{w^2}{r^4}+\tfrac{5}{2}.\frac{w^3}{r^6}+\&c.\right\}. \qquad [7993e]$$

Substituting this last expression of R in the value of R' [7838], we easily obtain,

$$R'=\frac{m'.r'^2}{2r^3}-\frac{m'}{r}.\left\{\tfrac{3}{2}.\frac{w^2}{r^4}+\tfrac{5}{2}.\frac{w^3}{r^6}+\&c.\right\}; \qquad [7993f]$$

and by re-substituting the second value of w [7993a], it becomes as in [7993].

† (3802) If we put, as in [7851e], $\varpi=0$, the expression of x, y [7861] will become as in [7994]; we may also put $z=0$, by neglecting the square of z, as in [7846′]. The [7995a]

[7994] $$x = r.\cos.v\,; \qquad y = r.\sin.v\,; \qquad z = 0\,;$$

[7995] $$r = \frac{a.(1-e^2)}{1+e.\cos.(v-\varpi)}\,;$$

we have also x', y', z', in functions of sines and cosines of v' and its multiples. We must substitute R' for R, in the differential expressions of the elements of the orbit, and then develop them, using the following values of r^2dv, r'^2dv', which are easily deduced from [1057], by putting $\mu = 1$ [7874];

[7996] $$r^2dv = dt.\sqrt{a.(1-e^2)}\,;$$

[7997] $$r'^2dv' = dt.\sqrt{a'.(1-e'^2)}\,;$$

and by this means we shall find that the part of each of these differential expressions, corresponding to R', will be expressed by a series of terms of the form,*

[7998] $$H.dv'.\cos.(iv + i'v' + \text{A})\,;$$

[7999] *i, i', being integral numbers, positive or negative; and H, A, constant quantities.* Integrating the term [7998] by parts, we obtain, as in [7998g],

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[7995b] value of r [7995] is the same as that in [378 or 603]. These values are to be substituted
[7995c] in [7993]; also the values of x', y', z' [7866—7868], and $r' = \frac{a'.(1-e'^2)}{1+e'.\cos.(v'-\varpi')}$, which is similar to that in [7995].

[7998a] * (3803) If we substitute R' [7993] for R, in [7849], we shall get for dh an expression of the form $dh = dx.\varphi(x, y, z, r, x', y', z', r') + dy.\Psi(x, y, z, r, x', y', z', r')$;
[7998b] φ, Ψ, being symbols of functions. Now if we substitute in the values of x, y [7994], the expression of r [7995], we shall obtain by development x, y, in functions of the sines and
[7998c] cosines of the angle v and its multiples; so that dx, dy, will be represented by dv, multiplied by similar functions. In like manner x', y', z', r', may be expressed by means of the sines and cosines of the angle v' and its multiples. Using these values we find that
[7998d] the expression of dh [7998a] becomes of the form $dh = dv.\text{function}(v, v')$; or, by using the value of dv [8001], $dh = dv'.\text{function}(v, v')$, corresponding to the form assumed in
[7998e] [7998], which is of the usual form given in [957, &c.], i, i', being integral numbers, positive or negative, as in [957' or 1011']. Similar forms may be given to the other
[7998f] quantities dl, $d.\frac{1}{a}$, dc', dc'' [7991]. The reason for using dv' instead of dv, will be seen in [8003, &c.], where we shall find that this enables us to obtain the chief part of the integral, under a finite form, in [8005]; the other part [8002] being much smaller, is
[7998g] integrated by approximation, as in [8009]. Now integrating [7998] by parts, it becomes as in [8000], as is easily proved by taking the differential of the last expression.

$$H.\textstyle\int dv'.\cos.(iv+i'v'+A) = \text{constant} + \frac{H}{i'}.\sin.(iv+i'v'+A) - H.\frac{i}{i'}.\int dv.\cos.(iv+i'v'+A).$$ [8000]

If in this last term we substitute the following value of dv,*

$$dv = \frac{r'^2.dv'}{r^2}.\frac{\sqrt{a.(1-e^2)}}{\sqrt{a'.(1-e'^2)}};$$ [8001]

we shall get,

$$-H.\frac{i}{i'}.\textstyle\int dv.\cos.(iv+i'v'+A) = -H.\frac{i}{i'}.\frac{\sqrt{a.(1-e^2)}}{\sqrt{a'.(1-e'^2)}}.\int\frac{r'^2.dv'}{r^2}.\cos.(iv+i'v'+A).$$ [8002]

This term is much less than the integral,

$$H.\textstyle\int dv'.\cos.(iv+i'v'+A), \quad [7998]$$ [8003]

when $\frac{r'}{r}$ is a small fraction; it is also diminished considerably by the factor $\frac{\sqrt{a.(1-e^2)}}{\sqrt{a'.(1-e'^2)}}$; for $a.(1-e)$ is the perihelion distance of the comet [8004] [681″], and this distance is much less than the values of a', corresponding to the three superior planets. The integral expression [7998] is therefore very nearly represented by,

$$H.\textstyle\int dv'.\cos.(iv+i'v'+A) = \text{constant} + \frac{H}{i'}.\sin.(iv+i'v'+A).$$ [8005]

To obtain a more correct value of this integral, we must add to its first value [8006]
the integral expression [8002]. Substituting in [8002], for $\frac{r'^2}{r^2}$, its value,† [8007]

$$\frac{r'^2}{r^2} = \frac{a'^2.(1-e'^2)^2.\{1+e.\cos.(v-\varpi)\}^2}{a^2.(1-e^2)^2.\{1+e'.\cos.(v'-\varpi')\}^2},$$ [8008]

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* (3804) Dividing the expression [7996] by that in [7997], and then multiplying the quotient by $\frac{r'^2.dv'}{r^2}$, we get [8001]; substituting this in the last term of [8000], it becomes [8001a] as in [8002]. This is very small, because $\frac{r'^2}{r^2}$ is supposed to be quite small. Moreover [8001b] the factor $\frac{\sqrt{a.(1-e^2)}}{\sqrt{a'.(1-e'^2)}} = \frac{\sqrt{a.(1-e)}}{\sqrt{a'.(1-e')}}.\sqrt{\frac{1+e}{1+e'}}$, is small, since e' is small, and e nearly [8001c] equal to 1; so that this factor becomes nearly equal to $\frac{\sqrt{a.(1-e)}}{\sqrt{a'}}.\sqrt{2}$; which is very [8001d] small, because the perihelion distance of the comet $a.(1-e)$, is supposed to be much less than the mean distance a' of the disturbing planet from the sun. Hence it is evident that if we neglect the integral [8002] we shall obtain the expression [8005] for an approximate value of the integral of the function [7998]. [8001e]

† (3805) Dividing r' [7995c] by r [7995], and squaring the quotient, we get [8008]. [8008a]

we can, on account of the smallness of e', develop the integral expression [8002] in a series of terms of the form,

[8009] $$H'.\int dv'.\sin.(sv+s'v'+\text{A});$$

and we may integrate each of these terms by the method we have just explained. Thus we shall have, in a very converging manner, the value of

[8010] $$H.\int dv'.\sin.(iv+i'v'+\text{A}); \quad [7998]$$

therefore we shall have, by analytical formulas, the values of the elements in the superior part of the orbit.

9. We can, by the preceding formulas, calculate the perturbations which
[8010′] the comet of 1759 suffers in its successive revolutions, and predict its next appearance.* To do this, we must proceed in the following manner. We

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* (3806) This was published by the author in 1805; and the calculations which he has proposed were afterwards made by several persons; particularly by Damoiseau, Burckhardt, Rosenberger, Pontécoulant and Lubbock. The work of Damoiseau is given
[8009a] in detail, in the *Memorie della reale Accademia delle Scienze di Torino*, Tomo xxiv, 1820; the co-ordinates of the comet, and the perturbations relative to Jupiter, being made for every degree of the value of u, as in [7945]. In calculating the action of Saturn, the intervals in the value of u are 2°; in those of Uranus 6°. The method used by him is similar to that in § 3, 9, of this book. In the *Connaissance des tems* for 1832, page
[8009b] 25, &c., he gives an abstract of this method, with an additional calculation of the action of the earth upon the comet in the year 1759, when it passed quite near to the earth, so as to affect the mean motion and the time of its revolution. Any person engaged in calculations of this kind will do well to look over these memoirs, to see the manner in which he has arranged the numerical process, which may serve as a model for such computations. Similar calculations have also been made by Pontécoulant, who gained the
[8009c] prize proposed by *l'Académie des Sciences* of Paris, on this subject, in the year 1829. The results of his labors are given, in a very abridged form, in the *Connaissance des tems* for 1833, page 104, &c.; and afterwards, in the volume for the year 1837, page 102. In this last calculation he has altered the mass of Jupiter, to conform to the late observations of Professor Airy and others [6787b, &c.], and has also computed the action of the earth upon the comet, before passing the perihelion in 1759. Rosenberger has likewise calculated the elements of this orbit at the time of its former appearances, and has given the results of
[8009d] his calculations in vol. 8, 9, of Schumacher's *Astronomische Nachrichten*. Lubbock has also made some calculations on its elements, in the Memoirs of the Astronomical Society of London, vol. 4, &c. These are the most important works upon this comet; but the restricted limits of the present commentary prevent us from inserting a minute account of them; and we shall merely give the final results, or the elements of the orbit of the comet at the time of the perihelion in 1835, according to the calculations of Pontécoulant,

must begin with a new and very careful discussion of the observations of the comet, at the times of its appearance in 1682 and 1759; and must then [8011]
determine the elements of the orbit at these two epochs, supposing it to

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Damoiseau and Lubbock. The comparison of these results will show the degree of correctness which has been obtained in the computation of the elements of this orbit.

Elements of Halley's comet.

	Pontécoulant.	*Damoiseau.*	*Lubbock.*	
Passage of the perihelion in 1835, mean time at Paris,	November 12^{day},6	November 4^{day},32	Oct'r 30^{day},1993	
Place of the perihelion *in the orbit*, ...	304^d 31^m 43^s	304^d 27^m 24^s	304^d 23^m 39^s	
Longitude of the ascending node, ...	55 30 00	55 09 07	55 03 59	[8009e]
Inclination of the orbit, ...	17 44 24	17 41 05	17 42 50	
Ratio of the excentricity to the semi-major axis, . .	0,9675212	0,9673055	0,967348	
Semi-major axis, ...	17,99755	17,9852	17,98355	

Since writing the preceding part of this note, the comet has again appeared; and, *at the time of printing this page, is visible in the heavens*, not far distant from the place corresponding to the elements of Mr. Pontécoulant; but the time of its appearance has not [8009e']
yet been sufficiently long to enable astronomers to estimate correctly its present path, so as to compare it with the preceding elements.

Methods of computing the variations of the elements of the orbit of a comet or planet.

The method of calculating the variations of the elements of the orbit of the disturbed body, and then using these corrected elements in finding its place at any given time, has become very important, within a few years, on account of its great use in computing the places of the very small planets Ceres, Juno, Pallas, Vesta, and the comets of Encke and Biela; the inclinations and excentricities of these orbits being so great that the development [8009f]
in a series, according to the powers and products of these quantities, which is used with the other planets, and the tabular forms of the corrections of their mean places, cannot be successfully applied to these small bodies. The methods of making such calculations may be seen in the works mentioned in the former part of this chapter; also in the treatise of Bessel, on the comet of 1807; in that of Hansen, on the theory of Jupiter and Saturn; or [8009g]
in that of Airy, in the Nautical Almanac for 1837, &c. Instead of computing the variations of the elements by means of integral formulas $\int Qdu$ [7945'], for every degree of the [8009h]
excentric anomaly u, as La Place has directed in this chapter, it has been found convenient to compute these integrals for *equal intervals of time*; as, for example, 10, 20 or 30 days, &c. Thus, in the abovementioned work of Bessel, he uses the interval of 30 days, which [8009i]
is taken as the *unit* of time. In this case the formulas for the computation of da, de, &c. require different developments from those which have been given in this chapter; and on account of the importance of the subject we have here given these formulas, with the necessary explanations.

To avoid references to other parts of the work for a definition of the symbols, we shall here observe that,

[8011'] move in an ellipsis, whose greater axis corresponds to the duration of the revolution from 1682 to 1759. Then using the elements of 1682, we may

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Symbols.

[8009k] $x, y, z,$ represent the rectangular co-ordinates of the disturbed body m; the axis of x being the line drawn in the invariable ecliptic from the sun towards the fixed first point of Aries; the axis of y is perpendicular to that of x, and is drawn in the invariable ecliptic towards the first point of Cancer. The axis of z is drawn through the sun, perpendicular to the ecliptic, and towards the north pole of the ecliptic;

[8009l] x', y', z'; x'', y'', z'', &c. represent respectively the rectangular co-ordinates of the disturbing bodies m', m'', &c. corresponding to the same fixed axes;

[8009m] $r\ r', r''$, &c. are the radii vectores of the bodies m, m', m'', &c. respectively;

[8009n] f, f', f'', &c. the distances of the planets m', m'', m''', &c. respectively from m;

[8009o] $v =$ the true longitude of the disturbed body m, which is supposed to be measured from the first point of Aries, along the fixed ecliptic to the node, by the quantity θ, and then along the orbit of the planet m by the quantity $v-\theta$;

[8009p] $a =$ the mean distance of the body m from the sun;

[8009q] $e =$ the excentricity of the orbit of the body m, in terms of a as the unit;

[8009r] $\varpi =$ the longitude of the perihelion of the body m, counted from the fixed first point of Aries, along the fixed ecliptic to the node, by the quantity θ, and then along the orbit of the planet by the quantity $\varpi-\theta$;

[8009s] $\theta =$ the longitude of the node of the body m, on the fixed ecliptic;

[8009t] $\gamma =$ the inclination of the orbit of the body m, to the fixed ecliptic;

[8009u] $n =$ the mean sideral motion of the body m, in a unit of time;

[8009v] $\varepsilon =$ the longitude of the epoch, counted upon the fixed ecliptic; hence,

[8009w] $\int ndt+\varepsilon$ corresponds, in the variable ellipsis, to what is called the *mean longitude* in the invariable ellipsis. This longitude is supposed to be measured on the ecliptic from the first point of Aries to the node, and then upon the plane of the orbit.

[8009x] Now if we have computed, for any time, the values of the variable elements $a, e, \varpi, \theta, \gamma, n, \varepsilon$, we can find, by them, the place of the body m, supposing the mean longitude in the
[8009y] variable ellipsis to be $\int ndt+\varepsilon$; and using the variable or computed values of n, ε, without altering the time t.

The formulas for computing the variations of the elements have already been given in
[8009z] [5751, 5786, 5787, 5773b, 5786h, i, 7879], supposing the mass m of the disturbed planet to be so very small, in comparison with that of the sun $M=1$, that it may be neglected;
[8010a] and then $\mu = M+m = 1+m$ [530iv], may be put equal to unity; as can always be done in computing the place of either of the newly discovered planets, or of the periodical
[8010b] comets. In other words, we may more accurately express the masses of the disturbing bodies m', m'', &c. in parts of the *sum of the masses* $M+m$, taken as unity. For convenience of reference we shall here give the formulas we have just mentioned, together

determine, by the preceding method, the alterations in the elements and in the mean anomaly, in the three first quadrants of the excentric anomaly, or [8011'']

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with the expressions of the abridged symbols A, B, C, A', B', C'; which will be used in the course of this note.

Symbols and formulas.

$R = m'.\frac{(xx'+yy'+zz')}{r'^3} - \frac{m'}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}}$; [7802] [8010c]

$\mu = na^{\frac{3}{2}} = 1$; whence $\sqrt{a} = a^2 n$; [7857, 8010a] [8010d]

$f = \sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}$; [7870] $f' = \sqrt{\{(x''-x)^2+(y''-y)^2+(z''-z)^2\}}$; &c. [8010e]

$da = -2a^2.\mathrm{d}R$; [5786] [8010f]

$dn = \quad 3an.\mathrm{d}R$; [7879] [8010g]

$d\varepsilon = -\frac{andt.\sqrt{1-e^2}}{e}.\{1-\sqrt{1-e^2}\}.\left(\frac{dR}{de}\right) + 2a^2.ndt.\left(\frac{dR}{da}\right)$; [5787] [8010h]

$de = \quad \frac{andt.\sqrt{1-e^2}}{e}.\left(\frac{dR}{dv}\right) - \frac{a.(1-e^2)}{e}.\mathrm{d}R$; [5751] [8010i]

$d\varpi = -\frac{andt}{e.\sqrt{1-e^2}}.\sin.(v-\varpi).\{2+e.\cos.(v-\varpi)\}.\left(\frac{dR}{dv}\right) + \frac{a^2.ndt.\sqrt{1-e^2}}{e}.\cos.(v-\varpi).\left(\frac{dR}{dr}\right)$; [8010k]
[5773b]

$d\gamma = \quad \frac{andt}{\sqrt{1-e^2}.\sin.\gamma}.\left(\frac{dR}{d\theta}\right)$; [5786h] [8010l]

$d\theta = -\frac{andt}{\sqrt{1-e^2}.\sin.\gamma}.\left(\frac{dR}{d\gamma}\right)$; [5786i] [8010m]

$A = \left(\frac{dR}{dx}\right) = \frac{m'x'}{r'^3} + \frac{m'.(x-x')}{f^3}$; [8010n]

$B = \left(\frac{dR}{dy}\right) = \frac{m'y'}{r'^3} + \frac{m'.(y-y')}{f^3}$; [8010o]

$C = \left(\frac{dR}{dz}\right) = \frac{m'z'}{r'^3} + \frac{m'.(z-z')}{f^3}$; [8010p]

$A_{,} = \cos.\theta.\sin.(v-\theta) + \sin.\theta.\cos.\gamma.\cos.(v-\theta)$;
$B_{,} = \sin.\theta.\sin.(v-\theta) - \cos.\theta.\cos.\gamma.\cos.(v-\theta)$; $\quad C_{,} = -\sin.\gamma.\cos.(v-\theta)$; [8010p']

$A' = Ax + By + Cz$; [8010q]

$B' = \quad A.\{\cos.\theta.\sin.(v-\theta)+\sin.\theta.\cos.\gamma.\cos.(v-\theta\}$
$\quad + B.\{\sin.\theta.\sin.(v-\theta)-\cos.\theta.\cos.\gamma.\cos.(v-\theta\}$
$\quad - C.\sin.\gamma.\cos.(v-\theta)$ [8010r]

$\quad = AA_{,} + BB_{,} + CC_{,}$ [8010r']

$\quad = A.\frac{\cos.\theta}{\cos.\Upsilon}.\sin.(v-\theta+\Upsilon) + B.\frac{\sin.\theta}{\cos.\chi}.\sin.(v-\theta-\chi) - C.\cos.(v-\theta).\sin.\gamma$; [8010s]

$C' = -A.\sin.\theta.\sin.\gamma + B.\cos.\theta.\sin.\gamma - C.\cos.\gamma$; [8010t]

$\mathrm{tang}.\Upsilon = \mathrm{tang}.\theta.\cos.\gamma$; [8010u]

$\mathrm{tang}.\chi = \mathrm{cotang}.\theta.\cos.\gamma$. [8010v]

[8011‴] from $u=0°$ to $u=300°$. In the last quadrant it is preferable to count backwards, from the epoch of 1759 to the commencement of that quadrant,

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The values of x, y, z, which are to be used in the preceding formulas, may easily be deduced from those of x', y', z' [7866—7868], by changing the elements λ, γ, v', r'

[8010w] [7863—7865] of the planet m', into the corresponding quantities γ, θ, $v-\theta$, r,

$x, y, z.$ [8009t, s, o, m] relative to the planet m; hence we have,

[8010x] $$x=r.\{\cos.\theta.\cos.(v-\theta)-\sin.\theta.\cos.\gamma.\sin.(v-\theta)\};$$

[8010y] $$y=r.\{\sin.\theta.\cos.(v-\theta)+\cos.\theta.\cos.\gamma.\sin.(v-\theta)\};$$

[8010z] $$z=r.\sin.\gamma.\sin.(v-\theta).$$

Taking the differentials of these expressions in the invariable ellipsis, where θ, γ, are constant, we get, by substituting the values of the coefficients of dr, deduced from

dx, dy, dz. [8010x, y, z],

[8011a] $$dx=dr.\frac{x}{r}+rdv.\{-\cos.\theta.\sin.(v-\theta)-\sin.\theta.\cos.\gamma.\cos.(v-\theta)\};$$

[8011b] $$dy=dr.\frac{y}{r}+rdv.\{-\sin.\theta.\sin.(v-\theta)+\cos.\theta.\cos.\gamma.\cos.(v-\theta)\};$$

[8011c] $$dz=dr.\frac{z}{r}+rdv.\sin.\gamma.\cos.(v-\theta).$$

These differentials being of the first order, are also satisfied in the variable ellipsis

[8011d] [1167″]. In like manner the general differential of R [8010e], *relative to the co-ordinates of the body* m, which is usually denoted by the characteristic d, is as in [8011f], and being of the first order of differentials, *holds good in the variable ellipsis* [1167″].

[8011e] Substituting in [8011f] the values [8010n, o, p], it becomes as in [8011g]; and by using the values of dx, dy, dz [8011a, b, c], it becomes as in [8011h]; the symbols A', B' [8010q, r], being introduced for abridgment;

$\mathrm{d}R$

[8011f] $$\mathrm{d}R=\left(\frac{dR}{dx}\right).dx+\left(\frac{dR}{dy}\right).dy+\left(\frac{dR}{dz}\right).dz$$

[8011g] $$=Adx+Bdy+Cdz$$

[8011h] $$=A'.\frac{dr}{r}-B'.rdv.$$

From [7996] we get dv [8011k], by using $\sqrt{a}=a^2n$ [8010d]. Taking the differential

[8011i] of $\log.r$ [7995], we get the first expression [8011l]; and by substituting the value of $1+e.\cos.(v-\varpi)$ [7995], and dv [8011k], it becomes as in the second form [8011l]. Substituting the values [8011k, l], containing dt, in [8011h], we get [8011m];

[8011k] $$dv=\frac{dt.\sqrt{a.(1-e^2)}}{r^2}=\frac{a^2n.dt.\sqrt{1-e^2}}{r^2};$$

[8011l] $$\frac{dr}{r}=\frac{edv.\sin.(v-\varpi)}{1+e.\cos.(v-\varpi)}=\frac{anedt.\sin.(v-\varpi)}{r.\sqrt{(1-e^2)}};$$

[8011m] $$\mathrm{d}R=\frac{ane}{\sqrt{1-e^2}}.\frac{\sin.(v-\varpi)}{r}.A'dt-a^2n.\sqrt{1-e^2}.\frac{B'}{r}.dt.$$

which is the same thing as to fix the origin of the angle u at the perihelion of 1759, and to count backwards towards 1682, making u negative, and [8011''''']

Substituting this value of dR in [8010f, g], we finally obtain,

da

$$da = -\frac{2a^3ne}{\sqrt{1-e^2}} \cdot \frac{\sin.(v-\varpi)}{r} \cdot A'dt + 2a^4 n.\sqrt{1-e^2} \cdot \frac{B'}{r} \cdot dt ; \qquad [8011n]$$

dn

$$dn = \frac{3a^2n^2e}{\sqrt{1-e^2}} \cdot \frac{\sin.(v-\varpi)}{r} \cdot A'dt - 3a^3n^2.\sqrt{1-e^2} \cdot \frac{B'}{r} \cdot dt. \qquad [8011o]$$

From the general differential of R, relative to the co-ordinates of the body m [8011g], we can easily deduce its partial differentials relative to θ, γ, a, e, ϖ, which occur in the formulas [8010h—m], by noticing, in the second member of [8011g], the parts of dx, dy, dz, depending on the quantity which is considered as variable. Thus if γ be the proposed variable quantity, the expression [8011g] will give, [8011p]

$$\left(\frac{dR}{d\gamma}\right) = A.\left(\frac{dx}{d\gamma}\right) + B.\left(\frac{dy}{d\gamma}\right) + C.\left(\frac{dz}{d\gamma}\right). \qquad [8011q]$$

Taking the partial differentials of x, y, z [8010x, y, z], relative to γ, we obtain,

$$\left(\frac{dx}{d\gamma}\right) = r.\sin.\theta.\sin.\gamma.\sin.(v-\theta) ; \quad \left(\frac{dy}{d\gamma}\right) = -r.\cos.\theta.\sin.\gamma.\sin.(v-\theta) ; \quad \left(\frac{dz}{d\gamma}\right) = r.\cos.\gamma.\sin.(v-\theta); \qquad [8011r]$$

substituting these in [8011q], and using the abridged symbol C' [8010t], we get,

$$\left(\frac{dR}{d\gamma}\right) = -C'r.\sin.(v-\theta) ; \qquad [8011s]$$

hence the expression of $d\theta$ [8010m] becomes,

$d\theta$

$$d\theta = \frac{andt}{\sqrt{1-e^2}.\sin.\gamma} \cdot C'r.\sin.(v-\theta). \qquad [8011t]$$

If we take the partial differential of R, relative to v, we get, in like manner as in [8011q],

$$\left(\frac{dR}{dv}\right) = A.\left(\frac{dx}{dv}\right) + B.\left(\frac{dy}{dv}\right) + C.\left(\frac{dz}{dv}\right). \qquad [8011u]$$

The partial differentials of the expressions [8010x, y, z], relative to v, become, by using the abridged symbols $A_{,}$, $B_{,}$, $C_{,}$ [8010p'],

$$\left(\frac{dx}{dv}\right) = -A_{,}.r ; \qquad \left(\frac{dy}{dv}\right) = -B_{,}.r ; \qquad \left(\frac{dz}{dv}\right) = -C_{,}.r. \qquad [8011v]$$

Substituting these in [8011u], and using the abridged value of B' [8010r'], we obtain,

$$\left(\frac{dR}{dv}\right) = -(AA_{,} + BB_{,} + CC_{,}).r = -B'r. \qquad [8011w]$$

If we use the expressions [8011w, m], we find that de [8010i] becomes as in [8011x], and by reduction as in [8011y];

$$de = -\frac{andt.\sqrt{1-e^2}}{e} \cdot B'r - \frac{a.(1-e^2)}{e} \cdot \left\{ \frac{ane}{\sqrt{1-e^2}} \cdot \frac{\sin.(v-\varpi)}{r} \cdot A'dt - a^2n.\sqrt{1-e^2} \cdot \frac{B'}{r} \cdot dt \right\} \qquad [8011x]$$

de

$$= -a^2n.\sqrt{1-e^2} \cdot \frac{\sin.(v-\varpi)}{r} \cdot A'dt + \frac{an.\sqrt{1-e^2}}{e} \cdot \frac{B'}{r} \cdot \left\{ a^2.(1-e^2) - r^2 \right\} .dt. \qquad [8011y]$$

Substituting in $d\varpi$ [8010k], the values $\left(\frac{dR}{dv}\right) = -B'r$ [8011w], and $\left(\frac{dR}{dr}\right) = A' \cdot \frac{1}{r}$, [8011h], we get,

[8012] using the elements of the orbit and epoch which were observed in 1759. In the first and last quadrants of the ellipsis, the comet is nearest to the

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$d\varpi$

[8011z] $$d\varpi = \frac{a^2ndt.\sqrt{1-e^2}}{e} \cdot \frac{\cos.(v-\varpi)}{r} . A' + \frac{andt}{e.\sqrt{1-e^2}} . r.\sin.(v-\varpi).\{2+e.\cos.(v-\varpi)\}.B'.$$

Putting this expression of $d\varpi$ equal to that in [5783], namely, $d\varpi = -\frac{andt.\sqrt{1-e^2}}{e}.\left(\frac{dR}{de}\right)$, and then dividing by $-\frac{andt.\sqrt{1-e^2}}{e}$, we obtain,

[8012a] $$\left(\frac{dR}{de}\right) = -a.\frac{\cos.(v-\varpi)}{r}.A' - \frac{r}{1-e^2}.\sin.(v-\varpi).\{2+e.\cos.(v-\varpi)\}.B'.$$

The partial differential of R, relative to a, is easily deduced from [8011h], which gives

[8012b] $\left(\frac{dR}{da}\right) = \frac{A'}{r}.\left(\frac{dr}{da}\right)$, because a depends on r only; but from [7995] we have,

[8012c] $$\left(\frac{dr}{da}\right) = \frac{1-e^2}{1+e.\cos.(v-\varpi)} = \frac{r}{a}; \quad \text{hence} \quad \left(\frac{dR}{da}\right) = \frac{A'}{r}.\frac{r}{a} = \frac{A'}{a}.$$

Substituting this value and that of $\left(\frac{dR}{de}\right)$ [8012a], we find that the formula [8010h] becomes as in [8012c'], or by reduction as in [8012d];

[8012c'] $$d\epsilon = \frac{andt.\sqrt{1-e^2}}{e}.\{1-\sqrt{1-e^2}\}.\left\{a.\frac{\cos.(v-\varpi)}{r}.A' + \frac{r}{1-e^2}.\sin.(v-\varpi).[2+e.\cos.(v-\varpi)].B'\right\} + 2andt.A'$$

[8012d] $$= \left\{\frac{a^2ndt.(1-e^2)}{e}.\left(\frac{1}{\sqrt{1-e^2}}-1\right).\frac{\cos.(v-\varpi)}{r} + 2andt\right\}.A' + \frac{andt}{e}.\left(\frac{1}{\sqrt{1-e^2}}-1\right).r.\sin.(v-\varpi).\{2+e.\cos.(v-\varpi)\}.B'.$$

We may deduce the expression of $\left(\frac{dR}{d\theta}\right)$ from that of $\left(\frac{dR}{d\gamma}\right)$ [8011s]. For if we divide the value of $d\gamma$ [8010l] by that of $d\theta$ [8010m], and multiply the result by $-\left(\frac{dR}{d\gamma}\right)$, we shall get,

[8012e] $$\left(\frac{dR}{d\theta}\right) = -\frac{d\gamma}{d\theta}.\left(\frac{dR}{d\gamma}\right).$$

Now without altering the values of $a, e, v-\varpi, r, x, y, z$, we can suppose the plane of the orbit of the body m to revolve on the radius vector r, as an axis, so as to change θ
[8012f] into $\theta+d\theta$, and γ into $\gamma+d\gamma$; observing that the planes of the two successive differential triangles, described by the radius vector in the two equal and successive elements
[8012g] of time dt, intersect each other in the line r, or radius vector corresponding to the middle of these two times. Then the ratio of $d\theta$ to $d\gamma$ can be found, *by putting the differentials*
[8012h] *of any of the co-ordinates* x, y, z [8010x, y, z] *equal to nothing*, considering $\theta, \gamma, v-\theta$, as the only variable quantities. If we put for brevity $v-\theta = \mathrm{v}$, we can determine $d\mathrm{v}$
[8012i] by putting, as in [8012h], $d.\left(\frac{z}{r}\right) = d.(\sin.\gamma.\sin.\mathrm{v}) = 0$ [8010z], which gives

disturbing planets, particularly Jupiter the most considerable of all of them. [8012′]
It is farthest off in the second and third quadrants. Therefore it is important

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$d\gamma.\cos.\gamma.\sin.\mathrm{v}+d\mathrm{v}.\sin.\gamma.\cos.\mathrm{v}=0$; whence $d\mathrm{v}=-d\gamma.\mathrm{cotang}.\gamma.\mathrm{tang}.\mathrm{v}$. This is to [8012k]
be substituted in the differential of $\frac{x}{r}=\cos.\theta.\cos.\mathrm{v}-\sin.\theta.\cos.\gamma.\sin.\mathrm{v}$ [8010x], and the [8012l]
result put equal to nothing [8012h]; hence we have, without reduction,

$$0=-d\theta.(\sin.\theta.\cos.\mathrm{v}+\cos.\theta.\cos.\gamma.\sin.\mathrm{v})+d\mathrm{v}.(-\cos.\theta.\sin.\mathrm{v}-\sin.\theta.\cos.\gamma.\cos.\mathrm{v}) +d\gamma.\sin.\theta.\sin.\gamma.\sin.\mathrm{v}. \qquad [8012m]$$

Substituting the value of $d\mathrm{v}$ [8012k], we find that the term depending on $d\gamma$ becomes,

$$d\gamma.\mathrm{cotang}.\gamma.\mathrm{tang}.\mathrm{v}.\left\{\cos.\theta.\sin.\mathrm{v}+\sin.\theta.\cos.\gamma.\cos.\mathrm{v}+\sin.\theta.\frac{\sin.^2\gamma}{\cos.\gamma}.\cos.\mathrm{v}\right\}; \qquad [8012n]$$

or by reduction,

$$d\gamma.\mathrm{cotang}.\gamma.\mathrm{tang}.\mathrm{v}.\left\{\cos.\theta.\sin.\mathrm{v}+\frac{\sin.\theta.\cos.\mathrm{v}}{\cos.\gamma}\right\}=d\gamma.\frac{\mathrm{tang}.\mathrm{v}}{\sin.\gamma}.\{\cos.\theta.\cos.\gamma.\sin.\mathrm{v}+\sin.\theta.\cos.\mathrm{v}\}. \qquad [8012o]$$

Substituting this in [8012m], and then dividing by the factor $\sin.\theta.\cos.\mathrm{v}+\cos.\theta.\cos.\gamma.\sin.\mathrm{v}$, which occurs in both terms, we get,

$$0=-d\theta+d\gamma.\frac{\mathrm{tang}.\mathrm{v}}{\sin.\gamma}, \qquad \text{or} \qquad \frac{d\gamma}{d\theta}=\frac{\sin.\gamma}{\mathrm{tang}.\mathrm{v}}. \qquad [8012p]$$

Substituting this in [8012e], we finally obtain, by re-substituting $\mathrm{v}=v-\theta$ [8012i], and using [8011s],

$$\left(\frac{dR}{d\theta}\right)=-\frac{\sin.\gamma}{\mathrm{tang}.(v-\theta)}.\left(\frac{dR}{d\gamma}\right)=C'r.\sin.\gamma.\cos.(v-\theta); \qquad [8012p']$$

hence the expression of $d\gamma$ [8010l] becomes,

$$d\gamma=\frac{andt}{\sqrt{1-e^2}}.C'r.\cos.(v-\theta). \qquad d\gamma \; [8012p'']$$

We can put the mean longitude in the variable ellipsis $\int ndt+\varepsilon$ [8009y], under a form which is more convenient for the present purpose, by substituting $\varepsilon=\varepsilon_{,}+\int tdn$; whence [8012q]
we have $\int ndt+\varepsilon=\int(ndt+tdn)+\varepsilon_{,}=nt+\varepsilon_{,}$; and this last expression is of the same [8012r]
form as in the fixed ellipsis, *changing the constant elements* n, ε, *of the invariable ellipsis, into the variable elements* n, $\varepsilon_{,}$, corresponding respectively to the time t, for which these elements have been computed. Taking the differential of the expression of $\varepsilon=\varepsilon_{,}+\int tdn$ [8012s]
[8012q], and transposing tdn, we get $d\varepsilon_{,}=d\varepsilon-tdn$; and by substituting the value of [8012t]
dn [8011o], we obtain the value of $d\varepsilon_{,}$ [8012v], relative to the variable ellipsis, to be used with the variable value of n, in forming $nt+\varepsilon_{,}$, or the mean longitude in the variable [8012u]
ellipsis; hence we have,

$$d\varepsilon_{,}=d\varepsilon-\frac{3a^2n^2e}{\sqrt{1-e^2}}.\sin.(v-\varpi).\frac{t}{r}.A'dt+3a^3n^2.\sqrt{1-e^2}.\frac{t}{r}.B'dt. \qquad [8012v]$$

There is yet another correction to be made in the expression of $d\varepsilon_{,}$, $d\varpi$, arising from [8012w]
the peculiar manner in which the angles v, ϖ, are counted; *the part* θ *of these angles*

[8012″] to obtain as accurately as possible, in the first and last quadrants, the position

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being measured on the fixed plane, and the remaining parts $v-\theta$, $\varpi-\theta$, *upon the*
[8012w′] *plane of the variable orbit* [8009o, r]. To estimate this correction we shall take the differential of $v=\theta+\mathrm{v}$ [8012i], and by substituting successively the values of $d\mathrm{v}$, $d\gamma$, $d\theta$,
[8012x] [8012k, p, 8011t], $1-\cos.\gamma=2.\sin.^2\frac{1}{2}\gamma$, $\sin.\gamma=2.\sin.\frac{1}{2}\gamma.\cos.\frac{1}{2}\gamma$ [1, 31] Int., we get,

[8012y]
$$dv=d\theta+d\mathrm{v}=d\theta-d\gamma.\frac{\cos.\gamma}{\sin.\gamma}.\mathrm{tang.v}=d\theta-d\theta.\cos.\gamma=2d\theta.\sin.^2\tfrac{1}{2}\gamma$$
$$=\frac{andt}{\sqrt{1-e^2}.\sin.\gamma}.2.\sin.^2\tfrac{1}{2}\gamma.C'r.\sin.(v-\theta)$$

[8012z]
$$=\frac{andt}{\sqrt{1-e^2}}.\mathrm{tang.}\tfrac{1}{2}\gamma.C'r.\sin.(v-\theta).$$

Hence it appears that the correction [8012z] must be added to the angles dv, $d\varpi$, and
[8013a] as this does not affect the true anomaly $v-\varpi$, because in noticing this term only we have $dv-d\varpi=0$, it will not affect the mean anomaly; consequently we may apply the correction to the epoch $\varepsilon_{,}$, instead of applying it to v. We must therefore add together
[8013b] the quantities [8012v, z], to obtain the complete expression of $d\varepsilon_{,}$; and must also add the expressions [8011z, 8012z], to get the complete value of $d\varpi$; so that we shall finally have the values of $d\varepsilon$, $d\varpi$, contained in the following table [8013f, h], where we have collected, for convenience of reference, the complete differentials of the elements; and to preserve
[8013c] the symmetry in the notation, we have omitted the mark placed below the symbol $d\varepsilon_{,}$ [8012r, &c.], and have given its complete value in [8013f], without the accent. The formulas in this table, taking them in succession as they occur, correspond respectively to those in [8011n, o], [8012d, v, z], [8011y], [8011z, 8012z], [8012p″], [8011t];

Differentials of the elements.

[8013d]
$$da=-\frac{2a^3ne}{\sqrt{1-e^2}}.\frac{\sin.(v-\varpi)}{r}.A'dt+2a^4n.\sqrt{1-e^2}.\frac{B'}{r}.dt;$$

[8013e]
$$dn=\frac{3a^2n^2e}{\sqrt{1-e^2}}.\frac{\sin.(v-\varpi)}{r}.A'dt-3a^3n^2.\sqrt{1-e^2}.\frac{B'}{r}.dt;$$

[8013f]
$$d\varepsilon=\left\{\frac{a^2n.(1-e^2)}{e}.\left(\frac{1}{\sqrt{1-e^2}}-1\right).\frac{\cos.(v-\varpi)}{r}+2an-\frac{3a^2n^2e}{\sqrt{1-e^2}}.\sin.(v-\varpi).\frac{t}{r}\right\}.A'dt$$
$$+\left\{\frac{an}{e}.\left(\frac{1}{\sqrt{1-e^2}}-1\right).r.\sin.(v-\varpi).[2+e.\cos.(v-\varpi)]+3a^3n^2.\sqrt{1-e^2}.\frac{t}{r}\right\}.B'dt$$
$$+\frac{an}{\sqrt{1-e^2}}.\mathrm{tang.}\tfrac{1}{2}\gamma.r.\sin.(v-\theta).C'dt;$$

[8013g]
$$de=-a^2n.\sqrt{1-e^2}.\frac{\sin.(v-\varpi)}{r}.A'dt+\frac{an.\sqrt{1-e^2}}{e}.\left\{\frac{a^2.(1-e^2)-r^2}{r}\right\}.B'dt;$$

[8013h]
$$d\varpi=\frac{a^2n.\sqrt{1-e^2}}{e}.\frac{\cos.(v-\varpi)}{r}.A'dt+\frac{an}{e.\sqrt{1-e^2}}.r.\sin.(v-\varpi).\{2+e.\cos.(v-\varpi)\}.B'dt$$
$$+\frac{an}{\sqrt{1-e^2}}.\mathrm{tang.}\tfrac{1}{2}\gamma.r.\sin.(v-\theta).C'dt;$$

[8013i]
$$d\gamma=\frac{an}{\sqrt{1-e^2}}.r.\cos.(v-\theta).C'dt;$$

[8013k]
$$d\theta=\frac{an}{\sqrt{1-e^2}.\sin.\gamma}.r.\sin.(v-\theta).C'dt.$$

and distance of the comet from those planets whose attractions may produce an alteration of several degrees in their elongations from the comet. To [8012‴]

The expressions [8013d—k] are to be integrated by the method of quadratures, which is particularly explained in [7929a—7930e]. The limits of these integrals are from $t=0$ [8013l] to $t=t$. Now if we suppose that the elements a, n, ε, e, ϖ, γ, θ, correspond to the [8013m] time $t=0$, and the same symbols accented correspond to the time $t=t$, we shall [8013n] obtain the value of any one of the elements a', n', ε', &c.; as, for example, a' by a formula similar to $a'=a+\int_0^t a_{,}dt$; $a_{,}$ being used for brevity to represent the coefficient [8013o] of dt in the second member of da [8013d].

In finding the integrals of the second members of [8013d—k], as in [8013n, &c.], we ought in strictness to use the variable elements a, e, &c.; but as the quantities da, dn, [8013p] $d\varepsilon$, &c. are of the same order as the disturbing forces, we can suppose the elements a, e, &c. to be constant, by neglecting terms of the order of the square of the disturbing forces, and can then use the values of these elements corresponding to $t=0$. After having calculated [8013q] the variations of the elements, at the end of any time $t=T$, if there should be found any essential change in their values, we can use, for the intervals which follow the epoch T, [8013r] the elements corresponding to that epoch, instead of the original elements corresponding to $t=0$.

We shall, as in [4079], *take the sun's mean distance from the earth for the unit of linear measure, and shall suppose the excentricity* e *of any body to be expressed in parts of the* [8013s] *mean distance of that body from the sun.* The circular arcs n, dn, $d\varepsilon$, $d\varpi$, $d\gamma$, $d\theta$, are expressed, in the preceding formulas [8013d—o], in terms of the radius, as the unit of [8013t] measure; but for practical purposes it is more convenient to denote them by seconds. We shall therefore suppose that the arcs [8013t] are expressed in seconds, in the formulas [8013d—k]; and to render the formulas homogeneous we shall change n into $n.\sin.1''$, [8013u] when necessary, as is usual in such cases; n *being the number of seconds in the planet's mean sideral motion, during one of the intervals which is taken for unity.* [8013v]

It will be found convenient to use the symbol $e=\sin.\varphi$ [5985 line 1], in the numerical operations; and from this we easily deduce the following expressions, which occur in [8013d—k];

$$e=\sin.\varphi; \qquad \sqrt{1-e^2}=\cos.\varphi; \qquad \frac{e}{\sqrt{1-e^2}}=\text{tang}.\varphi; \qquad \frac{\sqrt{1-e^2}}{e}=\text{cotang}.\varphi; \qquad [8013w]$$

$$\frac{1}{\sqrt{1-e^2}}-1=\frac{1}{\cos.\varphi}-1=\frac{1-\cos.\varphi}{\cos.\varphi}=\frac{2.\sin.^2\frac{1}{2}\varphi}{\cos.\varphi}=\frac{(2.\sin.\frac{1}{2}\varphi.\cos.\frac{1}{2}\varphi).\sin.\frac{1}{2}\varphi}{\cos.\varphi.\cos.\frac{1}{2}\varphi}=\frac{\sin.\varphi.\sin.\frac{1}{2}\varphi}{\cos.\varphi.\cos.\frac{1}{2}\varphi} \qquad [8013x]$$

$$=\text{tang}.\varphi.\text{tang}.\tfrac{1}{2}\varphi;$$

$$\frac{(1-e^2)}{e}.\left(\frac{1}{\sqrt{1-e^2}}-1\right)=\frac{\cos.^2\varphi}{\sin.\varphi}.\text{tang}.\varphi.\text{tang}.\tfrac{1}{2}\varphi=\cos.\varphi.\text{tang}.\tfrac{1}{2}\varphi. \qquad [8013y]$$

We shall now represent the coefficients of dt, in the expressions of da, dn, $d\varepsilon$, de, $d\varpi$, $d\gamma$, $d\theta$ [8013d—k], by $\delta'a$, $\delta'n$, $\delta'\varepsilon$, $\delta'e$, $\delta'\varpi$, $\delta'\gamma$, $\delta'\theta$, respectively; and by substituting [8013z]

[8012iv] obtain a greater degree of accuracy, we may again calculate the alterations of the elements and of the mean anomaly from 1682, using the greater axis

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[8014a] in these formulas the symbols [8013w, x, y], also $t=i+\frac{1}{2}$ [7930e], they become as in [8014i—p] respectively. Finally, to obtain at one view all the formulas which are used in these calculations, we have inserted, in [8014b—h], the values of A, B, C, A', B', C', Υ, χ [8010n—v]; the symbols being the same as in [8009k, &c.], and the terms which are between the braces are considered as constant in the numerical computations of the formulas [8014i—p].

Table of formulas.

[8014b] $$A=m'.\left\{\frac{x'}{r'^3}+\frac{(x-x')}{f^3}\right\}+m''.\left\{\frac{x''}{r''^3}+\frac{(x-x'')}{f'^3}\right\}+\&c.;$$

[8014c] $$B=m'.\left\{\frac{y'}{r'^3}+\frac{(y-y')}{f^3}\right\}+m''.\left\{\frac{y''}{r''^3}+\frac{(y-y'')}{f'^3}\right\}+\&c.;$$

[8014d] $$C=m'.\left\{\frac{z'}{r'^3}+\frac{(z-z')}{f^3}\right\}+m''.\left\{\frac{z''}{r''^3}+\frac{(z-z'')}{f'^3}\right\}+\&c.;$$

[8014e] $$\text{tang.}\Upsilon=\text{tang.}\theta.\cos.\gamma; \qquad \text{tang.}\chi=\text{cotang.}\theta.\cos.\gamma;$$

[8014f] $$A'=Ax+By+Cz;$$

[8014g] $$B'=A.\frac{\cos.\theta}{\cos.\Upsilon}.\sin.(v-\theta+\Upsilon)+B.\frac{\sin.\theta}{\cos.\chi}.\sin.(v-\theta-\chi)-C.\cos.(v-\theta).\sin.\gamma;$$

[8014h] $$C'=-A.\sin.\theta.\sin.\gamma+B.\cos.\theta.\sin.\gamma-C.\cos.\gamma;$$

[8014i] $$\delta'a=-\{2a^3.\text{tang}.\varphi.n.\sin.1''\}.\frac{\sin.(v-\varpi)}{r}.A'+\{2a^4.\cos.\varphi.n.\sin.1''\}.\frac{B'}{r};$$

[8014k] $$\delta'n=\quad\{3a^2.\text{tang}.\varphi.n^2.\sin.1''\}.\frac{\sin.(v-\varpi)}{r}.A'-\{3a^3.\cos.\varphi.n^2.\sin.1''\}.\frac{B'}{r};$$

[8014l] $$\begin{aligned}\delta'\varepsilon=&\left\{(a^2.\cos.\varphi.\text{tang}.\tfrac{1}{2}\varphi.n).\frac{\cos.(v-\varpi)}{r}+(2an)-(3a^2.\text{tang}.\varphi.n^2.\sin.1'').(i+\tfrac{1}{2}).\frac{\sin.(v-\varpi)}{r}\right\}.A'\\&+\left\{\begin{array}{l}\left(2a.\frac{\text{tang}.\frac{1}{2}\varphi}{\cos.\varphi}.n\right).r.\sin.(v-\varpi)+(a.\text{tang}.\varphi.\text{tang}.\tfrac{1}{2}\varphi.n).r.\sin.(v-\varpi).\cos.(v-\varpi)\\+(3a^3.\cos.\varphi.n^2.\sin.1'').\frac{(i+\frac{1}{2})}{r}\end{array}\right\}.B'\\&+\frac{an.\text{tang}.\frac{1}{2}\gamma}{\cos.\varphi}.r.\sin.(v-\theta).C';\end{aligned}$$

[8014m] $$\begin{aligned}\delta'e=&-\{a^2.\cos.\varphi.n.\sin.1''\}.\frac{\sin.(v-\varpi)}{r}.A'\\&+\left\{a^3.\frac{\cos.^3\varphi}{\sin.\varphi}.n.\sin.1''\right\}.\frac{B'}{r}-\{a.\text{cotang}.\varphi.n.\sin.1''\}.B'r;\end{aligned}$$

[8014n] $$\begin{aligned}\delta'\varpi=&\{a^2.\cot.\varphi.n\}.\frac{\cos.(v-\varpi)}{r}.A'+\left\{\frac{2an}{\sin.\varphi.\cos.\varphi}\right\}.r.\sin.(v-\varpi).B'\\&+\left\{\frac{an}{\cos.\varphi}\right\}.r.\sin.(v-\varpi).\cos.(v-\varpi).B'+\left\{\frac{an.\text{tang}.\frac{1}{2}\gamma}{\cos.\varphi}\right\}.r.\sin.(v-\theta).C';\end{aligned}$$

[8014o] $$\delta'\gamma=\left\{\frac{an}{\cos.\varphi}\right\}.r.\cos.(v-\theta).C';$$

[8014p] $$\delta'\theta=\left\{\frac{an}{\cos.\varphi.\sin.\gamma}\right\}.r.\sin.(v-\theta).C'.$$

corresponding to that epoch, which will be known by means of the preceding approximation. We may then, at 25 degrees of excentric anomaly, use the [8012^v]

The values of $\delta' a$, $\delta' n$, $\delta' \varepsilon$, $\delta' e$, $\delta' \varpi$, $\delta' \gamma$, $\delta' \theta$, represent respectively the increments of a, n, ε, e, ϖ, γ, θ, *in one of the intervals of time which is taken for unity*, using, in their computation, the values of r, v, x, y, z, &c., *which correspond to the middle of the proposed interval*, according to the directions in [7929z]; so that for the first interval we must put $t=\frac{1}{2}$; for the second interval we must put $t=\frac{3}{2}$; and generally for the interval $i+1$, we must put $t=i+\frac{1}{2}$. As the variations which are to be computed are very small, it will not be necessary to find the values of r, v, x, y, z, &c. to any great degree of accuracy; the angles v, v', &c. may be taken to the nearest minute, and the distances to four or five places of decimals; and we may remark that the epochs being supposed to commence at noon, at the meridians of Greenwich, Berlin, &c., most of these quantities can be found by inspection in the Ephemeris, computed for these meridians; and this is one of the important advantages of this method of computing the variations of the elements. The increment $\delta' a$ is expressed in parts of the mean distance of the earth from the sun, taken as unity. The increment $\delta' e$ is expressed in parts of the mean distance of the planet from the sun, taken as unity. The other quantities, $\delta' n$, $\delta' \varepsilon$, $\delta' \varpi$, $\delta' \gamma$, $\delta' \theta$, are expressed in seconds. When we have calculated any one of these increments, as $\delta' a$, for any number of successive intervals, the whole variation of a through the whole of these intervals, which we shall represent by δa, can be found by means of the formula [7929x], which gives, [8014q] [8014r] [8014s] [8014t]

Integrating by quadratures.

$\delta a =$ sum of all the calculated quantities $\delta' a$

$+\frac{1}{24}$.{first difference following the last term—first difference preceding the first term}. [8014u]

We have supposed, in computing this formula, that the interval or unit of time is so small that the fourth differences of $\delta' a$ can be neglected [7929s']. The selection of an appropriate value of this unit or interval can easily be made, in any particular case, by estimation, from the results of an actual calculation. Thus if we take, in the first instance, 32 days for the unit of time, we may compute the variations of any one of the elements; as, for example, $\delta' a$ for $t=16^{\text{days}}$, $t=48^{\text{days}}$, $t=80^{\text{days}}$, &c.; then if we find that the fourth differences are too important to be neglected, we may compute the values of $\delta' a$ for the intermediate times, $t=24^{\text{days}}$, $t=40^{\text{days}}$, &c., also for $t=8^{\text{days}}$; by this means the unit of the interval is reduced to 16 days. In the same way the intervals may be reduced to 8 days, if the fourth differences, with the interval of 16 days, are too great to be neglected. The length of these intervals must depend very much upon the positions of the large disturbing planets, particularly Jupiter; and when this planet approaches near to the body whose orbit is required, the intervals must be taken shorter than in other parts of the orbit. Now as it is convenient to retain the same interval throughout the whole calculation, we can use the smallest of these selected intervals as the unit of time; and [8014v] [8014w] [8014x] [8014y]

Methods of computing the variations of the elements for any interval.

[8012vi] elements of the new ellipsis corresponding to that anomaly, and by means of it compute the alterations it suffers from 25° to 50° of anomaly. We must

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[8014z] when the disturbing planet is at a great distance from the body, we may calculate the value of $\delta' a$ for every second or third interval, and then compute the intermediate values by the usual rules of interpolation.

As an example we may take the elements of the orbit of Vesta [4079i] for the epoch 1831, July 23$^{\text{days}}$, 0^{h}, mean time at Berlin. These give, at the time of that epoch,

[8015a]
$$a = 2{,}361484\,; \qquad e = 0{,}0885601\,; \qquad \varepsilon = 84^{d}\ 47^{m}\ 03^{s};$$
$$\varpi = 249^{d}\ 11^{m}\ 37^{s}; \qquad \theta = 103^{d}\ 20^{m}\ 28^{s}; \qquad \gamma = 7^{d}\ 07^{m}\ 57^{s}.$$

If we suppose the unit of time to be 8 days, the sideral motion of the body during that interval will be represented by $n = 8 \times 977^{s}{,}7554$ [4079i line 7]. We shall now suppose

[8015b] that it is required to find the variations of these elements, in the 48 days immediately following the epoch; or, in other words, in six of the intervals which are taken for unity. In this case, the times of the commencement of the successive intervals are July 23, 31; August 8, 16, 24; September 1, 9; corresponding respectively to $t = 0$, $t = 1$, $t = 2$,

[8015c] $t = 3$, $t = 4$, $t = 5$, $t = 6$. The times of the middle of these intervals are July 27; August 4, 12, 20, 28; September 5; corresponding, in the formula [7929w], to the values

[8015d] $i = 0$, $i = 1$, $i = 2$, $i = 3$, $i = 4$, $i = 5$, and to the co-ordinates $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$, $y^{(5)}$, respectively; then the formula [7929w] becomes,

[8015e]
$$\int_{-\frac{1}{2}}^{\frac{11}{2}} y^{(i)}.di = y^{(0)} + y^{(1)} + y^{(2)} + y^{(3)} + y^{(4)} + y^{(5)} + \tfrac{1}{24}.\{\Delta.y^{(5)} - \Delta.y^{(-1)}\}.$$

By means of this formula we can compute the whole variation of any one of the elements [8015a], as, for example, that of a, which is represented by δa, during the proposed period of 48 days, from July 23 to September 9. For if we suppose that the values of $\delta' a$

[8015f] in the successive intervals are represented by $\delta' a^{(0)}$, $\delta' a^{(1)}$, $\delta' a^{(2)}$, $\delta' a^{(3)}$, $\delta' a^{(4)}$, $\delta' a^{(5)}$, we may substitute them in [8015e], instead of $y^{(0)}$, $y^{(1)}$, &c.; and then that formula will become,

[8015g]
$$\delta a = \int_{-\frac{1}{2}}^{\frac{11}{2}} \delta a^{(i)}.di = \delta' a^{(0)} + \delta' a^{(1)} + \delta' a^{(2)} + \delta' a^{(3)} + \delta' a^{(4)} + \delta' a^{(5)} + \tfrac{1}{24}.\{\Delta.\delta' a^{(5)} - \Delta.\delta' a^{(-1)}\}.$$

We here see that the first differences of $\delta' a^{(0)}$, $\delta' a^{(1)}$, &c., give explicitly the following series of quantities;

[8015h]
$$\Delta.\delta' a^{(0)}, \qquad \Delta.\delta' a^{(1)}, \qquad \Delta.\delta' a^{(2)}, \qquad \Delta.\delta' a^{(3)}, \qquad \Delta.\delta' a^{(4)}.$$

This series does not expressly include $\Delta.\delta' a^{(5)}$, $\Delta.\delta' a^{(-1)}$, which occur in [8015g]; but they may be obtained, by taking the second or a higher order of differences of the series of terms [8015h], and then continuing the series one term further, at its commencement and termination, by the usual methods of differences, as in [7938′, &c.]. If we change

[8015i] successively the element a into n, ε, e, ϖ, γ, θ, in the formula [8015g], we shall get the whole variations of these elements, which we shall represent by δn, $\delta \varepsilon$, δe, $\delta \varpi$, $\delta \gamma$, $\delta \theta$,

rectify the ellipsis again at this epoch, and calculate in the ellipsis, thus [8012vii]
rectified, the perturbations from 50° to 100°. In like manner we must

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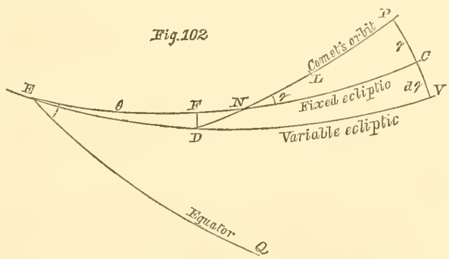
respectively; hence the corrected elements at the end of the interval of $t=43$ days, [8015k]
namely a', n', ε', &c., will become,

$$a'=a+\delta a, \qquad n'=n+\delta n, \qquad \varepsilon'=\varepsilon+\delta\varepsilon, \qquad e'=e+\delta e;$$
$$\varpi'=\varpi+\delta\varpi, \qquad \gamma'=\gamma+\delta\gamma, \qquad \theta'=\theta+\delta\theta. \qquad [8015l]$$

Having computed, in this manner, the corrected elements corresponding to the time t, if we wish to find the planet's place at that time, we must calculate it as in [8009y], in the same manner as if the body were moving *in an undisturbed elliptical orbit*, whose elements [8015m]
are a', n', ε', e', ϖ', γ', θ', with the mean longitude $n't+\varepsilon'$; this last quantity being the same *as if, at the beginning of the time, its mean longitude had been* ε', *and its mean sideral motion, from the beginning of the time, had been* n', *in every interval.* Finally we may [8015n]
remark that, as these elements a', n', &c. vary slowly, we may use them in computing very nearly the place of the body m, for a limited period of time t', before and after the [8015o]
time t; taking care however that the time t' be not so large as to produce any essential change in the values of these elements. This is conformable to what we have already observed in [4079g—h].

In the preceding calculations the longitudes v, ϖ, are referred to the fixed ecliptic and to the fixed equinox. To reduce them to the variable equinox we must, as usual, increase [8015p]
them by the precession of the equinoxes since the epoch. The correction for the decrease in the obliquity of the ecliptic E, requires a small calculation, because [8015q]
the part θ is counted on the ecliptic, and the remainder, $v-\theta$ or $\varpi-\theta$, upon the orbit of the planet, [8009o, r]. The method of making this reduction is easily obtained by referring to the annexed figure, where ENC is the fixed ecliptic, EDV the variable ecliptic, DNP the comet's orbit, EQ the equator.

Fig. 102



Then in the present notation we have $EN=\theta$, angle $PNC=\gamma$, angle $NEQ=E$. If we take the arc $NP=$ arc $NC=90°$, and [8015r]
continue the arc PC to V, drawing also DF perpendicular to EN, we shall have, by neglecting infinitely small quantities of the second order, arc $PC=\gamma$, arc $CV=d\gamma$. [8015s]
Now the longitude of any point L of the orbit, when the fixed ecliptic is used, is represented by the sum of the arcs $EN+NL$; but in the variable ecliptic it is represented by $ED+DL$, or $EF+DL$, neglecting infinitely small quantities of the second order. [8015t]
Subtracting the first of these expressions from the last, we get the increment of the longitude dv or $d\varpi$, arising from the variation of the obliquity of the ecliptic, namely,

rectify the fundamental ellipsis from 100° to 200°, and we must determine the perturbations as far as 300° of excentric anomaly. Then using the elements and epoch of 1759, and rectifying the ellipsis at -25°, -50°, -70°, and -100°, we shall obtain the alterations in the last quadrant of the excentric anomaly. Thus we shall obtain with much accuracy, by a second approximation, the perturbations of the comet from 1582 to 1759. [8013] We must make the same calculation from 1759 to the time of its next passage through the perihelion; but as the time of this passage is unknown, when we shall have computed as far as 300°, we must rectify the ellipsis for each 25° to 400°. These calculations being carefully made, must give, within a few days, the time when the comet will next pass its perihelion. The only point, on which there remains any degree of doubt, is that relative to the mass of the planet Uranus; and the observations of this passage will be one of the best methods to ascertain it.

[8015u] $dv = d\varpi = (DL - NL) - (EN - EF) = DN - FN = FN.\sec.\gamma - FN = FN.\frac{(1-\cos.\gamma)}{\cos.\gamma}.$

In the differential triangles FED, FND, we have,

[8015v] $$FD = \text{angle } FED \times \sin.ED = \text{angle } FED \times \sin.\theta\,;$$

[8015w] $$FN = FD \times \text{cotang}.\gamma = \text{angle } FED \times \sin.\theta \times \text{cotang}.\gamma\,;$$

Substituting this in [8015u], we get, by using [42'] Int.,

[8015x] $$dv = d\varpi = \text{angle } FED \times \sin.\theta \times \text{cotang}.\gamma \times \frac{(1-\cos.\gamma)}{\cos.\gamma}$$

[8015x'] $$= \text{angle } FED \times \sin.\theta \times \frac{(1-\cos.\gamma)}{\sin.\gamma} = \text{angle } FED \times \sin.\theta \times \text{tang}.\tfrac{1}{2}\gamma\,;$$

so that the increment of the longitude of the planet, or the increment of the longitude of the perihelion, depending on the diminution of the obliquity FED, is represented in the following manner;

[8015y] $$\text{Increment of longitude} = \text{the diminution of the obliquity} \times \sin.\theta \times \text{tang}.\tfrac{1}{2}\gamma.$$

Again, in the differential triangle CEV, we have $EC = EN + 90^d = \theta + 90^d$, and

[8015z] $$CV = \text{angle } CEV \times \sin.EC = CEV \times \cos.\theta = \text{diminution of the obliquity} \times \cos.\theta,$$

which represents the *increment* of the inclination γ.

CHAPTER II.

ON THE PERTURBATIONS OF THE MOTION OF A COMET WHEN IT APPROACHES VERY NEAR TO A PLANET.

10. *We shall now consider the case in which a comet approaches very near to a disturbing planet. If the planet be Jupiter, its attraction upon the comet may exceed that of the sun, and this action can entirely change the elements of its orbit. This singular case, which appears to have taken place with the first comet of the year* 1770, *deserves particular attention.* [8013′]

We have, by the preceding chapter, the six following equations;*

Fundamental differential equations.

$$0=\frac{ddx}{dt^2}+(1+m).\frac{x}{r^3}+\frac{m'x'}{r'^3}+\frac{m'.(x-x')}{f^3}; \qquad [8014]$$

$$0=\frac{ddy}{dt^2}+(1+m).\frac{y}{r^3}+\frac{m'y'}{r'^3}+\frac{m'.(y-y')}{f^3}; \qquad [8015]$$

$$0=\frac{ddz}{dt^2}+(1+m).\frac{z}{r^3}+\frac{m'z'}{r'^3}+\frac{m'.(z-z')}{f^3}; \qquad [8016]$$

$$0=\frac{ddx'}{dt^2}+(1+m').\frac{x'}{r'^3}+\frac{mx}{r^3}-\frac{m.(x-x')}{f^3}; \qquad [8017]$$

$$0=\frac{ddy'}{dt^2}+(1+m').\frac{y'}{r'^3}+\frac{my}{r^3}-\frac{m.(y-y')}{f^3}; \qquad [8018]$$

$$0=\frac{ddz'}{dt^2}+(1+m').\frac{z'}{r'^3}+\frac{mz}{r^3}-\frac{m.(z-z')}{f^3}. \qquad [8019]$$

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* (3807) Substituting the values $\left(\frac{dR}{dx}\right)$, $\left(\frac{dR}{dy}\right)$, $\left(\frac{dR}{dz}\right)$ [7869, 7871, 7894d], and $\mu=1+m$ [7804] in the differential equations of the comet's motion [7805—7807], they become as in [8014—8016]; and by changing m, r, x, y, z, into m', r', x', y', z', respectively, and the contrary, we obtain the corresponding differential equations of the planet's motion, as in [8017—8019]. [8014a] [8014b]

Symbols x_1, y_1, z_1. If we suppose,

[8020] $$x-x'=x_1;\qquad y-y'=y_1;\qquad z-z'=z_1;$$

Fundamental equations. we shall obtain, from [8014—8019], the three following equations;*

Second form.

$$\left.\begin{array}{ll} [8021] & 0=\dfrac{ddx_1}{dt^2}+x_1.\left\{\dfrac{m+m'}{f^3}+\dfrac{1}{r^3}\right\}+x'.\left(\dfrac{1}{r^3}-\dfrac{1}{r'^3}\right); \\ [8022] & 0=\dfrac{ddy_1}{dt^2}+y_1.\left\{\dfrac{m+m'}{f^3}+\dfrac{1}{r^3}\right\}+y'.\left(\dfrac{1}{r^3}-\dfrac{1}{r'^3}\right); \\ [8023] & 0=\dfrac{ddz_1}{dt^2}+z_1.\left\{\dfrac{m+m'}{f^3}+\dfrac{1}{r^3}\right\}+z'.\left(\dfrac{1}{r^3}-\dfrac{1}{r'^3}\right). \end{array}\right\}\ (Q)$$

[8024] In these equations, x_1, y_1, z_1, *are the co-ordinates of the comet, referred to*
[8025] *the centre of gravity of the planet, and* f *is the distance of these bodies from each other.* If we suppose f [7870] to be so small that $\frac{m+m'}{f^3}$ exceeds
[8026] considerably the terms depending upon the sun's action, we may, at least in a first approximation, neglect these last terms, and then the three preceding equations will give the elliptical motion of m about m'.† The difference

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† (3808) Subtracting the equation [8017] from [8014], we get,

[8021a] $$0=\frac{dd.(x-x')}{dt^2}+(x-x').\frac{(m+m').}{f^3}+\frac{x}{r^3}-\frac{x'}{r'^3};$$

and by substituting $x=x'+x_1$ [8020], it becomes,

[8021b] $$0=\frac{ddx_1}{dt^2}+x_1.\frac{(m+m')}{f^3}+\frac{x'+x_1}{r^3}-\frac{x'}{r'^3};$$

which is easily reduced to the form [8021], by merely altering the arrangement of the terms. In like manner, by subtracting [8018] from [8015], we get [8022]; and by subtracting [8019] from [8016], we get [8023]; using the values of y_1, z_1 [8020].

† (3809) When r' differs from r, by a very small quantity of the order x_1, the factor
[8027a] $\frac{1}{r^3}-\frac{1}{r'^3}$ will become of the order $\frac{1}{r^3}$ multiplied by a term of the order $\frac{x_1}{r'}$ or $\frac{x_1}{x'}$; so that $x'.\left(\frac{1}{r^3}-\frac{1}{r'^3}\right)$ [8021], may be considered as of the order $x_1.\frac{1}{r^3}$; and when $\frac{1}{r^3}$ is
[8027b] extremely small, in comparison with $\frac{m+m'}{f^3}$, we may neglect the former terms, and the equation [8021] will become $0=\frac{ddx_1}{dt^2}+x_1.\frac{(m+m')}{f^3}$. In like manner the equations
[8027c] [8022, 8023] become $0=\frac{ddy}{dt^2}+y_1.\frac{(m+m')}{f^3}$; $0=\frac{ddz_1}{dt^2}+z_1.\frac{(m+m')}{f^3}$. These equations are similar to those in [7810b, 7805—7807], putting $\mu=m+m'$, $r=f$, also
[8027d] $R=0$, so as to correspond to an elliptical orbit, as is observed in [7807']. The

of the sun's action upon the comet and planet is a force which disturbs that motion. This disturbing force is to the action $\frac{m'}{f^2}$, of the planet upon the comet, of the order $\frac{f^3}{m'.r^3}$. Whenever this last quantity is small, we may, without any sensible error, suppose the relative motion of the comet about the planet to be elliptical. On the contrary, when this quantity is very great we may neglect $\frac{m'}{f^3}$ in comparison with $\frac{1}{r^3}$, and consider the motion of the comet about the sun as elliptical. It is only between these two situations that there can be any degree of uncertainty; but on account of the rapidity of the comet's motion, the interval of time which separates these positions is so small, that we may, without any sensible error, suppose that the comet's motion is either elliptical about the planet, or elliptical about the sun. Now to ascertain, with some degree of precision, this *limit* of the distance of the comet from the planet; so that, when the comet is at a less distance than this limit, we may consider its motion to be elliptical about the planet, and when at a greater distance from the planet to be elliptical about the sun; we shall suppose that the comet is situated between the planet and the sun; then the sun's action upon the comet will be $\frac{1}{r^2}$, and that of the planet upon the comet $\frac{m'}{(r'-r)^2}$. Hence it is evident that when the comet is beyond the limit of the sphere of activity of the planet, the quantity $\frac{1}{r^2}$ must greatly exceed $\frac{m'}{(r'-r)^2}$. The difference of the action of the sun upon the planet and comet, is $\frac{1}{r^2}-\frac{1}{r'^2}$, or very nearly $\frac{2.(r'-r)}{r^3}$; * within

[8027] [8028] [8029] [8030] [8031] [8032] [8033]

quantities we have neglected are of the order $x_1.\frac{1}{r^3}$ [8027b], and those retained are of the order $x_1.\frac{(m+m')}{f^3}$, or $x_1.\frac{m'}{f^3}$; m being small in comparison with m'. Dividing the first of these forces $x_1.\frac{1}{r^3}$, by the second $x_1.\frac{m'}{f^3}$, we get, as in [8028], $\frac{f^3}{m'.r^3}$ for the ratio of the disturbing force of the sun, to the absolute force of the planet upon the comet.

[8027e] [8027f]

* (3810) The quantity $\frac{1}{r^2}-\frac{1}{r'^2}$ [8033], is easily reduced to the form,

$$\frac{r'^2-r^2}{r^2r'^2}=\frac{(r'-r)}{r^2r'}\cdot\frac{(r'+r)}{r'};\qquad [8033a]$$

the limit of the sphere of activity of the planet upon the comet, this last
[8033′] quantity must be very small in comparison with $\frac{m'}{(r'-r)^2}$. *We may satisfy these two conditions by supposing* $\frac{m'}{(r'-r)^2}$ *to be a mean proportional between*
[8034] $\frac{1}{r^2}$ *and* $2.\frac{(r'-r)}{r^3}$; which gives, for the radius $r'-r$ of the sphere of activity of the planet,

[8035] $$r'-r=r.\sqrt[5]{\tfrac{1}{2}m'^2}.$$

[8035′] The error will be so much the less as the planet's mass is decreased.* The radius of this sphere of activity may in fact be increased without any sensible error. For if we resume the equation [8021],†

[8036] $$0=\frac{ddx_1}{dt^2}+\frac{(m+m').x_1}{f^3}+\frac{x}{r^3}-\frac{x'}{r'^3};$$

we shall see that the term $\frac{x}{r^3}-\frac{x'}{r'^3}$ adds to the value of x_1 only the double

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[8033b] and when r' is nearly equal to r, or $\frac{r'+r}{r}$ nearly equal to 2, it becomes $\frac{2.(r'-r)}{r^2r'}$ or $\frac{2.(r'-r)}{r^3}$, as in [8033]. The conditions required in [8032, 8033′] are, that $\frac{1}{r^2}>\frac{m'}{(r'-r)^2}$, and $\frac{m'}{(r'-r)^2}>\frac{2.(r'-r)}{r^3}$. Both these conditions are satisfied by supposing, as in [8034], that
[8033c] $\frac{1}{r^2}:\frac{m'}{(r'-r)^2}::\frac{m'}{(r'-r)^2}:\frac{2.(r'-r)}{r^3}$; because if the first of these terms is much greater than the second, the third will be much greater than the fourth. From this proportion we get
D'Alembert's method. $(r'-r)^5=r^5.\frac{1}{2}m'^2$; whence we easily deduce [8035]. *We may remark, that this method*
[8033d] *of determining the perturbations of the motions of a comet, when approaching very near to a planet, was first proposed by D'Alembert, in his Opuscules,* Tome vi, page 304.

[8035a] * (3811) The disturbing force of the sun $\frac{2.(r'-r)}{r^3}$ [8033], being divided by the absolute force of the planet $\frac{m'}{(r'-r)^2}$ [8032], gives $\frac{2.(r'-r)^3}{m'.r^3}$ for the ratio of this disturbing force to
[8035b] the planet's action; but from [8035] we have $2.(r'-r)^3=2r^3.m'^{\frac{6}{5}}.(\frac{1}{2})^{\frac{3}{5}}=\frac{4}{3}r^3.m'^{\frac{6}{5}}$ nearly; hence the preceding ratio becomes $\frac{4}{3}m'^{\frac{1}{5}}$, which is decreased by decreasing the mass m', as in [8035′].

[8036a] † (3812) The term of [8021] having the divisor r^3, is $\frac{x_1+x'}{r^3}=\frac{x}{r^3}$ [8020]; substituting this in [8021], it becomes as in [8036].

integral* $\iint dt.\left(\frac{x'}{r'^3}-\frac{x}{r^3}\right)$. Now this double integral is very small, when it [8037]
is limited to a small value of t; for the function $\frac{x'}{r'^3}-\frac{x}{r^3}$ is very small, x' and r' differing but very little from x and r respectively. *Therefore we may, in the calculation of the perturbations of a comet which approaches very near to a planet, suppose the planet to have a sphere of activity, in which the relative motion of the comet is affected only by the planet's attraction; and that beyond this point the absolute motion of the comet about the sun is performed in exactly the same manner as if the sun alone acted upon it.* [8038] [8038']

Formulas when the comet is within the sphere of activity of the planet.

11. *We shall now develop this hypothesis, and determine the new elements of the comet's orbit at the time it passes from the sphere of the planet's attraction.* For this purpose we shall commence with the investigation of the relative orbit of the comet about the planet, while within the sphere of that attraction. We have, in [572], the six following equations;† [8039]

$$c_1=\frac{x_1.dy_1-y_1.dx_1}{dt};\qquad c'_1=\frac{x_1.dz_1-z_1.dx_1}{dt};\qquad c_1''=\frac{y_1.dz_1-z_1.dy_1}{dt}; \qquad [8040]$$

$$h_1=-\frac{m'y_1}{f}+\frac{(dx_1^2+dz_1^2)}{dt^2}-\frac{x_1.dx_1.dy_1}{dt^2}-\frac{z_1.dz_1.dy_1}{dt^2}; \qquad [8041]$$

$$l_1=-\frac{m'x_1}{f}+\frac{(dy_1^2+dz_1^2)}{dt^2}-\frac{y_1.dy_1.dx_1}{dt^2}-\frac{z_1.dz_1.dx_1}{dt^2}; \qquad [8042]$$

$$\frac{m'}{a_1}=\frac{2m'}{f}-\frac{(dx_1^2+dy_1^2+dz_1^2)}{dt^2}; \qquad [8043]$$

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* (3813) If we put for brevity $\frac{(m+m').x_1}{f^3}=T$, $\frac{x'}{r'^3}-\frac{x}{r^3}=T'$, the equation [8036] [8037a]
may be reduced to the form $0=\frac{ddx_1}{dt^2}+T-T'$; T' being much smaller than T, and [8037b]
both being considered as functions of the very small portion of time t, and of constant quantities. Multiplying the preceding equation by dt^2, we may put it under the following integral form, neglecting the constant quantities introduced by these integrations; $x_1=-\iint T.dt^2+\iint T'.dt^2$; so that the part of x_1 depending on T', is proportional to the very small quantity $\iint T'.dt^2$, being the same as that in [8037]; remarking, however, that the introduction of the small disturbing quantity T', into the equation [8037b], affects also the terms composing the value of T, by terms of the same small order, and by this means produces other similar variations in the value of x_1. [8037c] [8037d] [8037e]

† (3814) We have, in [7849b], $f'=\mu h$, $f=\mu l$; and in the present notation $\mu=m+m'$ [530iv], or simply $\mu=m'$; because the mass of the comet is small in [8040a]

[8043'] c_1, c_1', c_1'', h_1, l_1, a_1, being arbitrary constant quantities.

[8044] $\theta =$ the longitude of the ascending node of the relative orbit of the comet, counted from the axis of x_1 [585''''];

[8045] $\varphi =$ the inclination of the relative orbit of the comet upon the plane of x_1, y_1 [585$^{\text{iv}}$].

Then we shall have, as in [591],

[8046]
$$\text{tang.}\theta = \frac{c_1''}{c_1'}; \qquad \text{tang.}\varphi = \frac{\sqrt{c_1'^2+c_1''^2}}{c_1};$$

[8047] c_1, c_1', c_1'', being given, by means of the equations [8040], in functions of the values of x_1, y_1, z_1, $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dz_1}{dt}$, at the time of the entrance of the comet into the sphere of activity of the planet; and as these values are supposed to be known, we shall have also the values of θ and φ, at that time.

[8048] If we put I *for the longitude of the projection of the perihelion* [593'], we shall have, as in [594', &c.],*

[8049]
$$\text{tang.}I = \frac{h_1}{l_1}.$$

[8050] *The semi-major axis* a_1 is given by means of [8043]. Then we have, as in [714, &c.].†

[8051]
$$m'.a_1.(1-e_1^2) = 2m'f - \frac{m'f^2}{a_1} - \frac{f^2.df^2}{dt^2};$$

[8052] *which gives the excentricity* e_1. Thus we shall have all the elements of the relative orbit.

[8053] We shall now refer the co-ordinates x_1 and y_1 to the line of nodes. If we suppose x_1', y_1', z_1', to be these new co-ordinates, we shall have,‡

[8040b] comparison with that of the planet; so that we may put $f'=m'h$, $f=m'l$; or, for
[8040c] brevity, $f'=h_1$, $f=l_1$. Substituting these values in the equations [572 lines 1, 3, 2, 5], we get those in [8040, 8041, 8042, 8043] respectively. The accents being placed below
[8040d] x, y, z, c, c', c'', a, e, in order to conform to the present notation; also changing r into f, and making some slight reductions. In the same manner the equations [591] change into [8046].

[8049a] * (3815) Substituting the values $f'=h_1$, $f=l_1$ [8040c], in [594'], we get [8049].

[8051a] † (3816) We have, in [714], $\mu a.(1-e^2) = 2\mu r - \frac{\mu r^2}{a} - \frac{r^2dr^2}{dt^2}$. Substituting $\mu = m'$, $r=f$ [8040a, d], and placing the accents below a, e, as in [8040c], we get [8051].

[8054a] ‡ (3817) The expressions of x_1', y_1' [8054, 8055], are found in the same manner as those of x', y' [586]. Now multiplying [8054] by $\cos.\theta$, and [8055] by $-\sin.\theta$, then

$$x_1' = x_1.\cos.\theta + y_1.\sin.\theta;$$ [8054]

$$y_1' = -x_1.\sin.\theta + y_1.\cos.\theta;$$ [8055]

$$z_1' = z_1.$$ [8056]

We shall now refer the co-ordinates x_1', y_1', to the plane of the relative orbit; putting x_1'', y_1'', for these new co-ordinates; and we shall have,* [8057]

$$x_1' = x_1'';$$ [8058]

$$y_1' = y_1''.\cos.\varphi;$$ [8059]

$$z_1 = y_1''.\sin.\varphi.$$ [8060]

Lastly, we shall refer the co-ordinates x_1'', y_1'' to the major axis, and shall put ϖ *for the longitude of the perihelion counted from the line of nodes.* [8060']
We shall have, by putting x_1''', y_1''', for these new co-ordinates; the axis of x_1''' being the line drawn from the centre of the planet to its perihelion; † [8061]

$$x_1''' = x_1''.\cos.\varpi + y_1''.\sin.\varpi;$$ [8062]

$$y_1''' = -x_1''.\sin.\varpi + y_1''.\cos.\varpi.$$ [8063]

These different equations give, ‡

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taking the sum of the products, we get x_1 [8054b]. In like manner, by multiplying [8054] by $\sin.\theta$, and [8055] by $\cos.\theta$, then taking the sum of the products, we get y_1 [8054c]; these values of x_1, y_1, will be of use hereafter;

$$x_1 = x_1'.\cos.\theta - y_1'.\sin.\theta; \qquad y_1 = x_1'.\sin.\theta + y_1'.\cos.\theta.$$ [8054b]

* (3818) Here the axis of x_1'' or x_1' is not changed; but the ordinate y_1'' is taken in the relative orbit, perpendicular to the line of nodes, and is then projected upon the plane [8059a]
of x_1', y_1', into the ordinate $y_1' = y_1''.\cos.\varphi$, as in [8059]; and the distance z_1 of its extremity, from the plane of x_1', y_1', is evidently represented by $z_1 = y_1''.\sin.\varphi$, as [8059b]
in [8060].

† (3819) The transformation, here treated of, consists in changing the rectangular axes x_1'', y_1'', into the rectangular axes x_1''', y_1''', in the same plane; so that the angle [8062a]
formed by the axes x_1'', x_1''', or by the axes y_1'', y_1''', may be represented by ϖ. In this case the reduction is made in the same manner as in [8054, 8055], by changing θ into [8062b]
ϖ, and adding two accents to the symbols x_1, y_1, x_1', y_1'; whence we obtain [8062, 8063].

‡ (3820) Multiplying [8062, 8063] by $\cos.\varphi$, and substituting the values [8058, 8059], we get,

$$x_1'''.\cos.\varphi = x_1'.\cos.\varpi.\cos.\varphi + y_1'.\sin.\varpi;$$ [8065a]

$$y_1'''.\cos.\varphi = -x_1'.\sin.\varpi.\cos.\varphi + y_1'.\cos.\varpi.$$ [8065b]

Substituting the values of x_1', y_1' [8054, 8055], in [8065a, b], we get [8064, 8065] respectively.

[8064] $$x_1'''.\cos.\varphi = x_1.\{-\sin.\varpi.\sin.\theta+\cos.\varpi.\cos.\theta.\cos.\varphi\} +y_1.\{\sin.\varpi.\cos.\theta+\cos.\varpi.\sin.\theta.\cos.\varphi\};$$

[8065] $$y_1'''.\cos.\varphi = x_1.\{-\cos.\varpi.\sin.\theta-\sin.\varpi.\cos.\theta.\cos.\varphi\} +y_1.\{\cos.\varpi.\cos.\theta-\sin.\varpi.\sin.\theta.\cos.\varphi\}.$$

[8066] Therefore we shall have the values of x_1''', y_1''', corresponding to the time of the entrance of the comet into the sphere of the planet's activity. We shall also have, by taking the differentials of these equations, the values of $\frac{dx_1'''}{dt}$, $\frac{dy_1'''}{dt}$, relative to that point.

The preceding equations give,*

Co-ordinates.

[8067] [8068] [8069] $$\left.\begin{aligned} x_1 &= x_1'''.\{\cos.\varpi.\cos.\theta-\sin.\varpi.\sin.\theta.\cos.\varphi\} \\ &\quad +y_1'''.\{-\sin.\varpi.\cos.\theta-\cos.\varpi.\sin.\theta.\cos.\varphi\}; \\ y_1 &= x_1'''.\{\cos.\varpi.\sin.\theta+\sin.\varpi.\cos.\theta.\cos.\varphi\} \\ &\quad +y_1'''.\{-\sin.\varpi.\sin.\theta+\cos.\varpi.\cos.\theta.\cos.\varphi\}; \\ z_1 &= x_1'''.\sin.\varpi.\sin.\varphi+y_1'''.\cos.\varpi.\sin.\varphi. \end{aligned}\right\} (S)$$

[8070] If we denote by $\overline{x_1}$, $\overline{y_1}$, $\overline{x_1'''}$, &c. the values of x_1, y_1, x_1''', &c. at the time of the entrance of the comet into the sphere of activity of the planet, and by the same letters, with two lines at the top, their values when quitting it, we shall evidently have,†

[8067a] * (3821) Multiplying [8062] by $\cos.\varpi$, and [8063] by $-\sin.\varpi$, then adding the products, we get, by making some small reductions, the value of x_1'' [8067b], or that of x_1' [8058]. In like manner, by multiplying [8062] by $\sin.\varpi$, and [8063] by $\cos.\varpi$, then adding the products, we get the value of y_1'' [8067c]; multiplying this by $\cos.\varphi$, and substituting the result in [8059], we get y_1' [8067d];

[8067b] $$x_1''=x_1'''.\cos.\varpi-y_1'''.\sin.\varpi=x_1'; \quad [8058]$$

[8067c] $$y_1''=x_1'''.\sin.\varpi+y_1'''.\cos.\varpi;$$

[8067d] $$y_1'=x_1'''.\sin.\varpi.\cos.\varphi+y_1'''.\cos.\varpi.\cos.\varphi.$$

[8067e] Multiplying the value of x_1' [8067b] by $\cos.\theta$, and that of y_1' [8067d] by $-\sin.\theta$, then taking the sum of the products, we get the value of x_1 [8054b], as in [8067]. In like manner, multiplying x_1' [8067b] by $\sin.\theta$, and y_1' [8067d] by $\cos.\theta$, then taking the sum of the products, we get y_1 [8054b], as in [8068]. Finally multiplying [8067c] by

[8067f] $\sin.\varphi$, and substituting the product in z_1 [8060], we get [8069].

[8071a] † (3822) We shall suppose, in the annexed figure 103, that C is the centre of gravity of the planet; DAA' a circle described about the centre C, with the radius $r.\sqrt[5]{\tfrac{1}{2}m'^2}$ [8035], and representing the limit of the sphere of activity of the planet. Within this

[8071b] limit the comet is supposed, as in [8038], to move in an undisturbed orbit $ABPA'B'$, about

$$\overline{x_1'''} = \overline{\overline{x_1'''}}; \qquad \overline{y_1'''} = -\overline{\overline{y_1'''}}; \qquad [8071]$$

$$\frac{\overline{dx_1'''}}{dt} = -\frac{\overline{\overline{dx_1'''}}}{dt}; \qquad \frac{\overline{dy_1'''}}{dt} = \frac{\overline{\overline{dy_1'''}}}{dt}. \qquad [8072]$$

By means of these equations, we can find, in the first place, the values of

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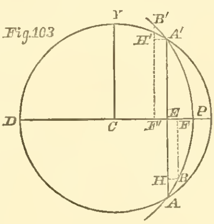
the planet; and if P be the least distance of the comet from the planet, we shall have [8071c]
the line drawn through CP for the axis of x_1''' [8061]; and the line CY, which is perpendicular to it, for the axis of y_1'''. The comet enters the sphere of activity at A, and quits it at A'. For both these points the value of x_1''' is CE; which is represented [8071d]
by $\overline{x_1'''}$ for the point A, and $\overline{\overline{x_1'''}}$ for the point A'; hence we have $\overline{x_1'''} = \overline{\overline{x_1'''}}$, as in [8071]. Moreover for the first of these points we have $AE = -\overline{y_1'''}$; and for the second [8071e]
point $A'E = \overline{\overline{y_1'''}}$; and as $AE = A'E$, we get $\overline{y_1'''} = -\overline{\overline{y_1'''}}$, as in [8071]. We shall now suppose that the comet, upon entering the sphere of activity, can describe the arc AB in the time dt; and upon [8071f]
quitting the sphere of activity can describe the equal arc $A'B'$, in the same time dt. [8071g] Then the co-ordinates of the point B are $CF = \overline{x_1'''} + \overline{dx_1'''}$; $BF = -\overline{y_1'''} + \overline{dy_1'''}$; and those of [8071h]
the point B' are $CF' = \overline{\overline{x_1'''}} - \overline{\overline{dx_1'''}}$; $B'F' = \overline{\overline{y_1'''}} + \overline{\overline{dy_1'''}}$. [8071i]
Subtracting respectively from these four expressions the values given in [8071d, &c.], namely $CE = \overline{x_1'''}$, $AE = -\overline{y_1'''}$,

Fig. 103



$CE = \overline{\overline{x_1'''}}$, $A'E = \overline{\overline{y_1'''}}$, we get the increments of these co-ordinates respectively; and [8071k]
if we draw the lines BH, $A'H'$, parallel to CP, we shall evidently have, for these increments, the following expressions;

$$EF = HB = \overline{dx_1'''}; \quad AH = \overline{dy_1'''}; \quad EF' = A'H' = -\overline{\overline{dx_1'''}}; \quad H'B' = \overline{\overline{dy_1'''}}. \qquad [8071l]$$

But by construction we evidently have the triangles AHB, $A'H'B'$, similar and equal;
hence we have $HB = A'H'$, $AH = H'B'$; or in symbols [8071l], $\overline{dx_1'''} = -\overline{\overline{dx_1'''}}$; [8071m]
$\overline{dy_1'''} = \overline{\overline{dy_1'''}}$; dividing these by dt, we get the equations [8072]. Now if we have [8071n]
x_1, y_1, z_1, and their differentials, at the time the comet enters into the sphere of activity
of the planet, we may thence obtain $\overline{x_1'''}$, $\overline{y_1'''}$, and their differentials, by means of [8071o]
[8064, 8065]; and then $\overline{\overline{x_1'''}}$, $\overline{\overline{y_1'''}}$, and their differentials, from [8071, 8072]. Substituting these last expressions in the formulas [8067—8069], and in their differentials, [8071p]
we obtain x_1, y_1, z_1, and their differentials, at the time the comet quits the sphere of activity of the planet [8074]. Thence we obtain the elements of the elliptical motion about the sun, after quitting the sphere of activity of the planet, as in [8074′, &c.].

$\overline{\overline{x_1'''}}$, $\overline{\overline{y_1'''}}$, $\overline{\overline{\frac{dx_1'''}{dt}}}$, $\overline{\overline{\frac{dy_1'''}{dt}}}$, in terms of the values of x_1, y_1, z_1, $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dz_1}{dt}$, at the time of entrance of the comet into the sphere of activity of the planet; and we may thence deduce, by means of the equations [8067–8069],

[8073] and of their differentials, the values of x_1, y_1, z_1, $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dz_1}{dt}$, upon quitting that sphere, in functions of their values at the time of the entrance into it. Adding to these values those of x', y', z', $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, $\frac{dz'}{dt}$,

[8074] corresponding to the time when the comet quits the sphere of activity, we shall have the corresponding values of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ [8020, &c.]; consequently, by means of the formulas [572—597′], we can obtain the new elements of the comet's orbit. To obtain the values of x', y', z', and their

[8074′] differentials, upon quitting the sphere of activity, we must find the time required to traverse that sphere, which may be easily done by the formulas

[8075] of the elliptical motion, explained in the third chapter of the second book [606, &c.].

12. *When the variations* δx, δy, δz, *are very small, as is the case relative to the motion of the comet of* 1770, *disturbed by the earth, it will be much*

[8076] *more simple to calculate the alterations of the elements of the orbit, by the formulas of the preceding chapter. We shall consider the most important of these variations, namely, that of the comet's mean motion.* We have, by what has been said,*

[8077] $$dn = 3an.\mathrm{d}R = 3m'.an.\frac{(x'dx+y'dy+z'dz)}{r'^3} - 3m'.an.\frac{\{(x'-x).dx+(y'-y).dy+(z'-z).dz\}}{f^3}.$$

[8077a] * (3823) Substituting in dn [7879] the value of R [7802], we get [8077], using f [7870]. If we retain only the term containing f^3, and substitute the differentials of [8078], namely $dx=\alpha dt$, $dy=\beta dt$, $dz=\gamma dt$, we shall get,

[8077b] $$dn = -3m'.andt.\frac{\{(x'-x).\alpha+(y'-y).\beta+(z'-z).\gamma\}}{f^3}.$$

Now by using the values [8078, 8079, 7870], and the abridged symbols [8081—8085], we obtain successively,

[8077c] $$(x'-x).\alpha+(y'-y).\beta+(z'-z).\gamma = \{(A'-A).\alpha+(B'-B).\beta+(C'-C).\gamma\}+\{(\alpha'-\alpha).\alpha+(\beta'-\beta).\beta+(\gamma'-\gamma).\gamma\}.t$$

[8077d] $$= F+Ht;$$

[8077e] $$f^2=(x'-x)^2+(y'-y)^2+(z'-z)^2=\{(A'-A)+(\alpha'-\alpha).t\}^2+\{(B'-B)+(\beta'-\beta).t\}^2+\{(C'-C)+(\gamma'-\gamma).t\}^2$$

[8077f] $$= M+2Nt+Lt^2.$$

Substituting [8077d, f] in dn [8077b], we get [8080]; and by integrating, [8086].

In the interval of time during which the earth's action is sensible, we may consider the motions of the planet and comet as being rectilinear. Therefore if we put,

$$x = A + \alpha t; \qquad y = B + \beta t; \qquad z = C + \gamma t;$$ [8078]

$$x' = A' + \alpha' t; \qquad y' = B' + \beta' t; \qquad z' = C' + \gamma' t;$$ [8079]

we shall have, by noticing only the terms divided by f^3, which is the only one that can become sensible on account of the smallness of f,

$$dn = -\frac{3m'.an.(F+Ht).dt}{(M+2Nt+Lt^2)^{\frac{3}{2}}};$$ [8080]

the following symbols being used in this equation;

$$F = (A'-A).\alpha + (B'-B).\beta + (C'-C).\gamma;$$ [8081]

$$H = (\alpha'-\alpha).\alpha + (\beta'-\beta).\beta + (\gamma'-\gamma).\gamma;$$ [8082]

$$M = (A'-A)^2 + (B'-B)^2 + (C'-C)^2;$$ [8083]

$$N = (A'-A).(\alpha'-\alpha) + (B'-B).(\beta'-\beta) + (C'-C).(\gamma'-\gamma);$$ [8084]

$$L = (\alpha'-\alpha)^2 + (\beta'-\beta)^2 + (\gamma'-\gamma)^2.$$ [8085]

Hence we shall have, by integrating [8080],

$$\delta n = -3m'.an.\int \frac{dt.(F+Ht)}{(M+2Nt+Lt^2)^{\frac{3}{2}}}.$$ [8086]

The integral must be taken during the whole time in which the action of the planet upon the comet is sensible. Before and after this time the distance $\sqrt{M+2Nt+Lt^2}$, of the comet from the planet, is considerable; [8087] and then the elements of the preceding integral become insensible, so that it may be taken from $t = -\infty$ to $t = \infty$, which gives,* [8087']

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* (3824) We may easily prove the correctness of the following integral [8088b], by taking the differential of its second member; then reducing all the terms to the common denominator $(LM-N^2).(M+2Nt+Lt^2)^{\frac{3}{2}}$; since by this means the coefficient of t^2, in [8088a] the numerator, will vanish; and the other terms of the numerator will become $(LM-N^2).(F+Ht).dt$.

$$\int \frac{dt.(F+Ht)}{(M+2Nt+Lt^2)^{\frac{3}{2}}} = \frac{(FN-HM)-(HN-FL).t}{(LM-N^2).(M+2Nt+Lt^2)^{\frac{1}{2}}} + \text{constant}.$$ [8088b]

Multiplying [8088b] by $-3m'.an$, we get the expression of δn [8086]; and if we substitute $\sqrt{(M+2Nt+Lt^2)} = f$ [8077f]; also for brevity $P = \frac{FN-HM}{LM-N^2}$, [8088c] $Q = \frac{HN-FL}{LM-N^2}$; we shall get,

$$\delta n = 3m'.an. \left\{ -\frac{P}{f} + Q.\frac{t}{f} + \text{constant.} \right\}.$$ [8088d]

The symbol f represents the distance of the comet from the planet [8025], *which is* [8088d']

[8088] $$\delta n = \frac{6m'.an.\{HN-FL\}}{(LM-N^2).\sqrt{L}}.$$

[8089] If we put f' *for the shortest distance of the planet from the comet, we shall have*,*

[8090] $$f'^2.L = LM-N^2;$$

therefore,

[8091] $$\delta n = \frac{6m'.an.(HN-FL)}{f'^2.L.\sqrt{L}};$$

[8092] we may here observe that $\sqrt{L}$ *represents the relative velocity of the comet.*†

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always positive, while t varies from $-\infty$ to $+\infty$; and at each of these limits the quantity $-\frac{P}{f}$ vanishes from the integral [8088d]; so that it becomes,

[8088e] $$\delta n = 3m'.an.\left\{Q.\frac{t}{f} + \text{constant}\right\}.$$

Now when f is very large, the terms depending on M, Nt [8077f], become very small

[8088f] in comparison with Lt^2; so that we shall have approximately $f=\sqrt{Lt^2}$, or $f=\mp t.\sqrt{L}$ [8077f]; the *upper* sign being used when t is *negative;* the *lower* sign

[8088g] when t is *positive* [8088d']; so that when $t=-\infty$, we shall have f infinite, and

[8088h] $\frac{t}{f}=-\frac{1}{\sqrt{L}}$. Substituting this in δn [8088e], and taking the constant quantity so as to make the integral vanish at this limit $t=-\infty$, we shall have constant $=Q.\frac{1}{\sqrt{L}}$;

[8088i] whence $\delta n=3m'.an.Q.\left\{\frac{t}{f}+\frac{1}{\sqrt{L}}\right\}$ [8088e]. At the other limit $t=+\infty$, we have

[8088k] $f=t.\sqrt{L}$ [8088f], or $\frac{t}{f}=\frac{1}{\sqrt{L}}$; substituting this in δn [8088i], we finally obtain $\delta n=6m'.an.Q.\frac{1}{\sqrt{L}}$; which is the same as in [8088], using Q [8088c].

* (3825) The least value of f is found by putting the differential of f^2 [8077f]

[8090a] equal to nothing, which gives $0=2Ndt+2Lt.dt$; whence $t=-\frac{N}{L}$. Substituting this in f^2 [8077f], and then multiplying by L, we get [8090]. Lastly, by substituting this value of $LM-N^2$ in the denominator of [8088], we get [8091].

† (3826) The velocity of the comet, reduced to a direction parallel to the axis of x,

[8092a] is $\frac{dx}{dt}=\alpha$ [8078]; that of the planet is $\frac{dx'}{dt}=\alpha'$ [8079]; their difference, $\alpha'-\alpha$, is the

[8092b] relative velocity of the bodies reduced to the same direction. In like manner the relative

[8092c] velocity, in a direction parallel to the axis of y, is $\beta'-\beta$; and in a direction parallel to the axis of z, is $\gamma'-\gamma$. *The whole relative velocity is equal* to the square root of the

[8092d] sum of the squares of these three expressions [39'], or $\sqrt{(\alpha'-\alpha)^2+(\beta'-\beta)^2+(\gamma'-\gamma)^2}$, which is represented by $\sqrt{L}$ in [8085].

13. *We shall apply these methods to the motion of the first comet of* 1770, *which was disturbed by the action of the earth and Jupiter.* Astronomers had made many abortive attempts to reduce its observed motion to the laws of the parabolic theory. At length Lexell discovered that it described an ellipsis, in which the duration of the revolution was only $5\frac{2}{3}$ years; and he found that this orbit satisfied all the observations of the comet. So remarkable a result could not be admitted but after the most incontestible proof; and to obtain it, the subject was proposed, as a prize question, by the National Institute, in order to determine the theory of the comet by a new discussion of the observations, and an examination of the positions of the stars with which it had been compared. This has been done by Burckhardt with the greatest care, in his paper which gained the prize; and his researches have produced nearly the same result as those of Lexell; so that we cannot now have any doubt relative to this point. A comet whose revolution is so short, ought frequently to re-appear; but it was not seen before 1770, and has never been seen since. To explain this phenomenon Lexell remarked, that in 1767 and 1779 this comet approached very near to Jupiter, whose great attraction could change the perihelion distance of the comet so as to render it visible in 1770, instead of being invisible as it was before, and afterwards render it invisible in 1779. But before we can admit of this explanation we must prove that the same elements of the orbit, which satisfy the first condition, will also agree equally well with the second; or, at least, that it is only necessary to make some very slight alterations, and such as can be comprised within the limits of those which the attraction of the planets may have produced, during the interval between 1767 and 1779. Burckhardt, at my request, has willingly undertaken to apply the preceding formulas to the computation of Jupiter's action upon the comet of 1767; he supposed the elements of its orbit, at the moment of quitting the sphere of activity of Jupiter, to be as follows. [*Mem. Acad.* 1806, page 20].

Remarkable comet of 1770.

[8093]

[8094]

[8095]

Time of passing the perihelion 1770, August 14^{day},0348; the day being supposed to begin at midnight.		1
Place of the ascending node upon the ecliptic in 1770, . .	146°,5327;	2
Inclination of the orbit, ...	1°,7377;	3
Place of the perihelion in 1770, ...	395°,8525;	4
Ratio of the excentricity to the semi-major axis, ...	0,785604;	5
Duration of the sideral revolution, ...	2050^{days},095.	6

[8096]

Elements of the orbit of the comet, after quitting the sphere of activity of Jupiter in 1767.

[8096′] He then fixed upon the 9th of May, 1767, at mid-day, for the time when the comet quitted the sphere of Jupiter's attraction. Using these *data*, then [8097] taking for the axis of x the radius vector of Jupiter at that epoch; the sun's mean distance from the earth for the unity of distance, and *one day for* [8097′] *the element of the time* dt; he has found, for the moment of the comet's quitting that sphere of activity [8096′], the following numerical values;*

[8098]
$$x_1 = \quad 0{,}086953; \qquad y_1 = -0{,}2144740; \qquad z_1 = -0{,}0271989; \quad 1$$
$$dx_1 = -0{,}001286; \qquad dy_1 = \quad 0{,}0036553; \qquad dz_1 = -0{,}00004212. \quad 2$$

These results give the following elements of the relative orbit of the comet about Jupiter.†

[8098a] * (3827) Having the elements of the comet [8096], and also those of the planet Jupiter, we may, from them, compute the longitudes, latitudes and distances of those bodies from the sun, on the 9th of May, 1767. We may then reduce these quantities to [8098b] rectangular co-ordinates x, y, z, for the comet, and x', y', z', for the planet; and by this means these quantities will be obtained at that epoch. Then computing the values of these [8098c] co-ordinates on the 10th of May, 1767, namely, $x+dx$, $y+dy$, $z+dz$; $x'+dx'$, $y'+dy'$, $z'+dz'$, we get the values of these quantities; and by subtracting from them the [8098d] corresponding values found for May 9th, 1767, we obtain the values of dx, dy, dz, [8098e] dx', dy', dz'. With these values, and the equations [8020], we get $x_1 = x - x'$; $y_1 = y - y'$; $z_1 = z - z'$; $dx_1 = dx - dx'$; $dy_1 = dy - dy'$; $dz_1 = dz - dz'$; as in [8098].

[8099a] † (3828) Substituting the expressions [8020] in the value of f^2 [7870], we get $f^2 = x_1^2 + y_1^2 + z_1^2$, and its differential gives $fdf = x_1 dx_1 + y_1 dy_1 + z_1 dz_1$. Substituting [8099b] the numerical expressions [8098], we get f^2 and fdf. Using these, and dt [8097′, 5987(8)], [8099c] we obtain, from [8040—8043], the values c_1, c_1', c_1'', h_1, l_1, $\frac{m'}{a_1}$; and as m' is known, we have also the value of a_1, which is negative, and corresponds to a hyperbolic orbit [8099d] about Jupiter. Then from [8046] we get $\theta = 313°{,}6573$, $\varphi = 77°{,}7185$; and from [8099e] [8049] we obtain $I = 248°{,}6321$; from [8051] we get $m'.a_1.(1-e_1^2)$; and as m', a_1, are known, we may thence deduce the value of e_1. Having thus obtained all the elements [8099f] of the relative orbit [8099 lines 1—5], we can then compute the time when the comet [8099g] entered the sphere of activity of Jupiter in 1767, or was at the distance $r.\sqrt[5]{\tfrac{1}{2}m'^2}$ [8035] from that planet, as in [8099 line 6]; also the values of x_1''', y_1''', at that time, and [8099h] their variations dx_1''', dy_1''', for one day. Thence we deduce x_1, y_1, z_1, and their [8099i] differentials, by means of the formulas [8067—8069], as in [8100]. From the tables of Jupiter's motion we obtain, at the same time, the values of x', y', z', dx', dy', dz'; and then we get, from [8020], the values of x, y, z, and their differentials, as in [8101]; also [8099k] $r = \sqrt{x^2+y^2+z^2}$. From r, x, y, z, dx, dy, dz, we deduce the corresponding elements of

Hyperbolic elements of the orbit about Jupiter in 1767.

Ascending node upon Jupiter's orbit, ... 313°,6573 ; 1
Inclination, ... 77°,7185 ; 2
Semi-major axis, ... —0,0220462 ; 3 [8099]
Ratio of the excentricity to the semi-major axis, ... 1,86220 ; 4
Place of the perijove, ... 248°,6321 ; 5
Entrance into the sphere of Jupiter's attraction 1767, Jan. 18^{day},358. 6

Hence he deduced the values of x_1, y_1, z_1, x, y, z, and their differentials, at the time of entering into the sphere of Jupiter's attraction, namely ;

$$x_1 = -0{,}106206\,; \qquad y_1 = 0{,}101175\,; \qquad z_1 = -0{,}181074\,; \qquad 1$$
$$dx_1 = -0{,}00169912\,; \qquad dy_1 = 0{,}00122295\,; \qquad dz_1 = -0{,}00326065\,; \qquad 2$$ [8100]

which give, at the time of entering into the sphere of Jupiter's attraction,

$$x = \ \ 5{,}263124\,; \qquad y = -0{,}696215\,; \qquad z = -0{,}181074\,; \qquad 1$$
$$dx = -0{,}002949\,; \qquad dy = -0{,}008356\,; \qquad dz = -0{,}00326065. \qquad 2$$ [8101]

Comet invisible before 1767.

With these values he has determined the ellipsis, which the comet described before its entrance into that sphere of attraction, in 1767, *and he has found its semi-major axis equal to* 13,293 ; *and the ratio of the excentricity to the semi-major axis equal to* 0,61772 ; *consequently the perihelion distance was* 5,0826. [8102] *At this distance the comet would be invisible, and it would in fact disappear at a much shorter distance.* [8103]

To obtain the effect of Jupiter's action upon the comet in 1779, Burckhardt supposed the elements of its orbit, at the time of its entrance into the sphere of attraction, to be as follows ; [8103′]

Elements of the comet before entering the sphere of activity of Jupiter in 1779.

Time of passing the perihelion in 1770, August 14^{day},0261 ; 1
Place of the ascending node upon the ecliptic in 1770, 146°,5722 ; 2
Inclination to the ecliptic, ... 1°,7503 ; 3
Place of the perihelion in 1770, ... 395°,8367 ; 4 [8104]
Ratio of the excentricity to the semi-major axis, ... 0,785474 ; 5
Duration of the sideral revolution, ... 2042^{days},682. 6

These elements differ but very little from the preceding [8096] ; the difference being within the limits of the variations arising from the attraction of the planets ; since the action of the earth alone is sufficient to produce a

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the orbit of the comet about the sun, by means of the formulas [572, 591, 594, &c.]. The last of the equations [572], gives the mean distance a [8102]. With this, and the formulas [595, 596, 596′], we get the excentricity e [8102]. [8099l]

considerable part of them. The time of the comet's entrance into the sphere of activity of Jupiter, is supposed to be the 20th of June, 1779, at mid-day; and by taking, for the axis of x, the radius vector of Jupiter at that epoch, he found that,*

[8105]
$$x_1 = 0{,}066007\,; \qquad y_1 = 0{,}227497\,; \qquad z_1 = -0{,}0095839\,; \qquad 1$$
$$dx_1 = -0{,}001319\,; \qquad dy_1 = -0{,}00375765\,; \qquad dz_1 = 0{,}00004690. \qquad 2$$

These values give the following elements of the relative orbit of the comet about Jupiter;

Hyperbolic orbit of the comet about Jupiter in 1779.

[8106]

Ascending node upon the orbit of Jupiter, ...	76°,9126;	1
Inclination, ...	30°,6056;	2
Semi-major axis, ...	—0,0205086;	3
Ratio of the excentricity to the semi-major axis, . . .	1,26586;	4
Place of the perijove, ...	36°,3407;	5
Time of quitting Jupiter's sphere of activity 1779, October	3^{day},9320.	6

Hence the following values of x, y, z, and their differentials, at the time of passing out from the sphere of Jupiter's activity, are obtained;

[8107]
$$x = 5{,}617747\,; \qquad y = 0{,}729731\,; \qquad z = 0{,}1072202\,; \qquad 1$$
$$dx = 0{,}00266133\,; \qquad dy = 0{,}00692084\,; \qquad dz = 0{,}00177469. \qquad 2$$

Comet invisible after the year 1779.

[8108] From these values the ellipsis which the comet described about the sun, at the time of quitting the sphere of Jupiter's activity, can be determined; and the semi-major axis is found equal to 6,388, also the ratio of the excentricity to the semi-major axis 0,47797, which gives the perihelion distance equal to 3,3346. *With such a perihelion distance the comet will always be invisible. Therefore we see that Jupiter's attraction could render this planet visible in* 1770, *instead of being invisible as it was before, and then*

[8109] *render it invisible again after* 1779; *and we may conceive of an infinite number of other variations in the elements, in which the action of the planets may produce similar results.* It appears to me therefore, that we must attribute to the attraction of Jupiter the remarkable phenomenon we have here undertaken to explain.

[8110] Of all the known comets this one approached nearest to the earth; it must therefore have experienced a sensible action from it. We shall now

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* (3829) The values [8105—8108] are obtained from the elements [8104], in the

[8105a] same way as [8098, 8099, 8101 8102] are deduced from [8096]; by the process which is explained in the two preceding notes.

determine, by the formulas of the preceding article, the alterations which this action has produced in the duration of its sideral revolution. We shall use the last elements we have given of this comet, and shall take July $2^{\text{day}},0567$, for the epoch or origin of the time t; this being about the time of its nearest approach to the earth. Lastly, we shall take *one day for the unit of time;* and the earth's radius vector, at the origin of the time t, for the axis of x; we shall then have,* [8111] [8112]

$$A'=0; \qquad B'=0; \qquad C'=0; \qquad 1$$
$$A=0,004890; \qquad B=0,0031249; \qquad C=0,0146097; \qquad 2$$
$$\alpha'=0,000150; \qquad \beta'=0,0169135; \qquad \gamma'=0; \qquad 3$$
$$\alpha=0,012153; \qquad \beta=0,0186114; \qquad \gamma=-0,006110. \qquad 4$$
[8113]

These values give,

$$F=-0,000028325; \qquad H=-0,000214805; \qquad L=0,000184287; \qquad 1$$
$$M=\ 0,000247121; \qquad N=-0,000025265. \qquad 2$$
[8114]

Hence we deduce, as in [8118f],

$$\frac{\delta n}{n}=104,791.m'.a. \qquad [8115]$$

The semi-major axis of the earth's orbit a', being taken for unity, we have $\frac{a}{a'}=a$; moreover, $\frac{n'^2}{n^2}=\frac{a^3}{a'^3}=a^3$ [6143]; or $a=\frac{n'^{\frac{2}{3}}}{n^{\frac{2}{3}}}$; hence [8115] becomes, [8116] [8116']

$$\frac{\delta n}{n}=104,791.m'.\frac{n'^{\frac{2}{3}}}{n^{\frac{2}{3}}}. \qquad [8117]$$

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* (3830) Taking the plane of the earth's orbit for that of x', y', we shall have $z'=0$; and then the last of the equations [8079] gives $C'=0$, $\gamma'=0$, as in [8113]. [8118a] Taking the earth's centre, at the epoch of the time t [8111], for the origin of the co-ordinates, we shall get $A'=0$, $B'=0$ [8079], as in [8113]. Then the formulas [8078, 8079] become, [8118b]

$$x=A+\alpha t; \qquad y=B+\beta t, \qquad z=C+\gamma t; \qquad x'=\alpha' t, \qquad y'=\beta' t, \qquad z'=0. \qquad [8118c]$$

From the three first of these equations it appears that A, B, C, represent the values of x, y, z, respectively, corresponding to the epoch, or $t=0$. Moreover, from the equations [8118c], we find that α, β, γ, α', β', represent the values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, at the same epoch; so that they may be computed from the tables, as in [8113, 8098c, d]. From these we obtain F, H, &c. [8114], by means of the formulas [8081—8085], and then $\frac{\delta n}{n}$ from [8091, 8090]. [8118d] [8118e] [8118f]

[8118] If we put T for the time of the revolution of the comet, and δT for the corresponding variation of δn, we shall have,*

[8119] $$nT = 400^{\circ} = (n + \delta n).(T + \delta T);$$

hence we deduce,

[8120] $$\frac{\delta n}{n} = -\frac{\delta T}{T}.$$

[8121] Putting the sideral year equal to T', we shall have,

[8122] $$\frac{n}{n'} = \frac{T'}{T};$$

therefore we have, as in [8119f],

[8123] $$\delta T = -104{,}791.m'.\left(\frac{T}{T'}\right)^{\frac{2}{3}}.T.$$

Supposing, as in [4061],

[8124] $$m' = \frac{1}{329630};$$

[8124′] and putting $T = 2042^{\text{days}},682$ [8104], we obtain,

[8125] $$\delta T = -2^{\text{days}},046;$$

Diminution of the comet's revolution

which is the diminution of the duration of the comet's revolution by means of the earth's attraction.

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* (3831) The mean motion of the comet m, in the time T, is represented by nT; moreover as T is the time of the sideral revolution of the comet, we shall have, as in [8119a] [7273′], $nT = 400^{\circ}$. If the action of the planet changes n into $n + \delta n$, and T into $T + \delta T$, we shall have, in like manner, $(n+\delta n).(T+\delta T) = 400^{\circ}$. Hence we obtain, as [8119b] in [8119], $nT = (n+\delta n).(T+\delta T) = nT + n.\delta T + T.\delta n + \delta n.\delta T$; and if we neglect the term $\delta n.\delta T$ on account of its smallness, rejecting also the term nT, which occurs in both members, we get $0 = n.\delta T + T.\delta n$; which is easily reduced to the form [8120], by [8119c] dividing by nT. Again, we obtain, as in [8119a], the motion of the earth in a sideral [8119d] year [8121], $n'T' = 400^{\circ}$; hence $nT = n'T'$ [8119a]; which may be reduced to the [8119e] form [8122], or $\frac{n'^{\frac{2}{3}}}{n^{\frac{2}{3}}} = \left(\frac{T}{T'}\right)^{\frac{2}{3}}$; substituting this in [8117], we get,

[8119f] $$\frac{\delta n}{n} = 104{,}791.m'.\left(\frac{T}{T'}\right)^{\frac{2}{3}} = -\frac{\delta T}{T} \quad [8120];$$

multiplying this last expression by $-T$, we obtain [8123]. Now substituting in [8123] [8119g] the values of m', T [8124, 8124′], also $T' =$ one sideral year $= 365^{\text{days}},256384$ [4068], we get δT [8125].

CHAPTER III.

ON THE MASSES OF THE COMETS, AND THEIR ACTION UPON THE PLANETS.

14. THE comets suffer great perturbations from the action of the planets; they must therefore re-act upon the planets, so as to trouble their motions; and we can determine, by the formulas of the two preceding chapters, the variations in the elements of the orbits of the planets, arising from the action of the comets. Fortunately this action is insensible, and the mutual attractions of the planets upon each other is sufficient to account for all the [8126]
known inequalities in the motions of the planets and their satellites. In fact the observations on the motions of the planets are represented with so great a degree of accuracy, by merely taking into consideration their action upon each other, that we cannot refuse our assent to the supposition, that the masses of the comets are excessively small. Of all the known comets, that which appears to have passed the nearest to the earth is the first comet of 1770. We have seen, in the preceding chapter, that the earth's action decreased the time of the sideral revolution of the comet $2^{\text{days}},046$ [8125]; now we have, as in [1208],*

Smallness of the masses of the comets.

$$\delta n' = -\frac{m.\sqrt{a}}{m'.\sqrt{a'}}\cdot\delta n; \qquad [8127]$$

* (3832) The equation [8127] is the same as [1208], changing ζ into δn, and ζ' into $\delta n'$, in order to conform to the present notation; dividing this by n', we get [8128]; [8127a]
and by substituting the value of $\frac{\delta n}{n}$ [8117], it becomes $\frac{\delta n'}{n'} = -104{,}791.m.\frac{\sqrt{a}}{\sqrt{a'}}\cdot\frac{n^{\frac{1}{3}}}{n'^{\frac{1}{3}}}$; [8127b]
but from [8116'], we have $\frac{\sqrt{a}}{\sqrt{a'}} = \frac{n'^{\frac{1}{3}}}{n^{\frac{1}{3}}}$; substituting this in the preceding equation, it becomes as in [8129']. Now we find, in like manner as in [8120], $\frac{\delta n'}{n'} = -\frac{\delta T'}{T'}$; [8127c]
substituting this in [8129'], and then multiplying by $-T'$, we get [8130]. Finally,

consequently,

[8128]
$$\frac{\delta n'}{n'} = -\frac{m.\sqrt{a}}{m'.\sqrt{a'}} \cdot \frac{n}{n'} \cdot \frac{\delta n}{n}.$$

[8129] Substituting $\frac{n'^{\frac{1}{3}}}{n^{\frac{1}{3}}}$ for $\sqrt{\frac{a}{a'}}$ [8116], and for $\frac{\delta n}{n}$ its value [8117], we obtain,

[8129′]
$$\frac{\delta n'}{n'} = -104,791.m\,;$$

hence we deduce,

[8130]
$$\delta T' = 104,791.mT'.$$

[8130′] If we suppose the mass of the comet m to be equal to the earth's mass m', [8124], we shall find, for the increment of the sideral year $\delta T'$,

[8131]
$$\delta T' = 0^{\text{day}},11612.$$

[8132] We are certain, from all the observations that have been made, particularly from the numerous comparisons of Maskelyne's observations, which were used by Delambre in constructing his solar tables, that the comet of 1770 has not altered the sideral year $2'',8$; thus we are sure that the mass of this comet is not $\frac{1}{5000}$ part of that of the earth.*

[8127d] putting $m = \frac{1}{329630}$ [8130′, 8124], $T' = 365^{\text{days}},256384$ [8119g], we find that it becomes as in [8131].

[8127e] We may also remark, that the expression of $\delta n'$ [8127], may be computed by finding $\delta n'$ from a formula similar to [8091]; that is, by changing $m, n, a, x, y, z, A, B, C, \alpha, \beta, \gamma$, into $m', n', a', x', y', z', A', B', C', \alpha', \beta', \gamma'$, respectively, and the contrary. This does not change the value of f' [8089, 7870], or of M, N, L [8083—8085]; but $F, H, \delta n$ [8081, 8082, 8091], become respectively,

[8127f]
$$F' = (A-A').\alpha' + (B-B').\beta' + (C-C').\gamma' = -F-N\,; \quad [8081, 8084]$$

[8127g]
$$H' = (\alpha-\alpha').\alpha' + (\beta-\beta').\beta' + (\gamma-\gamma').\gamma' = -H-L\,; \quad [8082, 8085]$$

[8127h]
$$\delta n' = \frac{6m.a'n'.(H'N-F'L)}{f'^2.L.\sqrt{L}}.$$

[8127i] Now from [8127f, g], we have $H'N\text{-}F'L = (\text{-}H\text{-}L).N + (F+N).L = -(HN\text{-}FL)$; substituting this in [8127h], and then dividing it by the expression of δn [8091], we

[8127k] get $\frac{\delta n'}{\delta n} = -\frac{m.a'n'}{m'.an}$; which is easily reduced to the form [8127], by means of [8116′].

[8132a] * (3833) If we suppose the mass m to be represented by $m = m'.\mu'$, instead of $m = m'$ [8130′], the expression [8130], instead of being of the form [8131], will become $\delta T' = 0^{\text{day}},11612.\mu'$; or, by using the centesimal division of the day, $\delta T' = 11612''.\mu'$.

It follows, from the calculations of the preceding chapter, that this comet passed directly through the space where Jupiter and his satellites were then [8133]
situated; and yet it does not appear that the comet produced the slightest alteration in the motions of these bodies. [8133′]

The fall of a comet upon a planet would not sensibly affect the motion of the planet.

It not only happens that the comets do not trouble the motions of the planets and satellites, by their attractions; but if, in the immensity of past ages, some of the comets have encountered them, which is very probable, it does not seem that the shock can have had much influence on the motions of the planets and satellites. It is difficult not to admit that the orbits of the [8134]
planets and satellites were nearly circular at their origin, and that the smallness of their ellipticity, as well as their common direction from west to east, depend upon the primitive state of the planetary system. The action of the comets, and their impact upon those bodies, have not varied these phenomena; yet if one of them, with a mass equal to that of the moon, should encounter the moon, or a satellite of Jupiter, there is not the least doubt that it would render the orbit of the satellite very excentric. [8135]
Astronomy also presents to us two other phenomena, which seem to date their origin from that of the planetary system, and which would have been altered by a very small shock. We here allude to the equality in the rotatory motions of the moon, and the librations of the three inner satellites of Jupiter. It is evident, from the formulas explained in the fifth book and in the preceding book, that the shock of a comet, whose mass was only $\frac{1}{1000}$ [8136]
part of that of the moon, would be sufficient to give a very sensible value to the actual libration of the moon, and to that of the satellites. We may therefore rest assured relative to the influence of the comets, and astronomers have no reason to fear that their action can impair the accuracy of astronomical tables.

Putting this equal to $2'',8$ [8132], we get $\mu' = \frac{2,8}{11612} = \frac{1}{4147}$; consequently $m = \frac{m'}{4147}$; [8132b]
hence it follows that, if the sideral year is not altered $2'',8$ by the comet's attraction, its mass *must be less than* $\frac{m'}{4147}$; agreeing nearly with the remark in [8132]. [8132c]

TENTH BOOK.

ON SEVERAL SUBJECTS RELATIVE TO THE SYSTEM OF THE WORLD.

In the preface to this work we have stated, that we should examine several questions relative to the system of the world, and we shall devote this book to that object. To complete our plan, there will then remain nothing more than to give an historical account of the discoveries which have elevated the science of Physical Astronomy to the rank it now holds.

CHAPTER I.

ON ASTRONOMICAL REFRACTIONS.

[8137] 1. The motion of light, in the mediums through which it passes, particularly in the atmosphere,* is one of the most important objects of

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[8137a] * (3834) The author here adopts, as the basis of his calculations, the Newtonian hypothesis of the emission of light; and it would seem, at first view, that this computation of the refraction of the heavenly bodies depends wholly on that manner of considering the subject; but upon examination we very easily perceive, that this hypothesis is introduced in § 1, 2, chiefly for the purpose of demonstrating, by means of it, the usual laws of
[8137b] reflexion and refraction of light [8205, 8188]. Now we can assume these laws *as being the results of observation*, and can found on them alone the theory of refraction of the heavenly bodies. By this means we can obtain, as in [8262c—t], the fundamental
[8137c] differential equation of refraction [8262], without bringing into view either Newton's theory of emission, or the wave theory of Huygens; neglecting, however, some very small quantities, which are within the limits of the observations, and such as are usually neglected by La Place. We have here introduced this subject, because many important observations and discoveries have been made relative to the properties of light, since the publication of
[8137d] this volume, particularly that of the *polarization*, by Malus; the *principle of interferences*,

astronomy; whether we consider it in relation to the theory, or to its effect [8137′]
upon every astronomical observation. We view the heavenly bodies through

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by Young; and a multitude of valuable results, by Fresnel, Brewster, Biot, Arago, &c.
The tendency of these observations has manifestly been to render doubtful the theory of [8137e]
Newton, and to bring forward, with much approbation, the undulatory or wave theory, as
the true system of nature. In this theory of Huygens, it is supposed that all space is filled
with a subtle elastic medium, or ether, which pervades all bodies, occupying the intervals
between their particles. By the agitation of the luminous body, the particles of the ether [8137f]
are supposed to be put in motion, and thus waves are formed in it, similar to those which
are produced in the air, water, &c., when disturbed. These vibrations of the ether produce,
on the eye, the sensation of light; in a manner analogous to that of the production of sound, [8137g]
in the ear, by the vibrations of the air. This analogy may be considered as a circumstance
in favor of the wave theory of light, since we find, in the economy of nature, that these [8137h]
vibrations are frequently used in air, water, &c. in communicating motion with great
rapidity, and with but very little actual displacing of the particles; as, for example, in the [8137i]
case of the motion of air, which produces sound, moving at the rate of nearly 800 miles in
an hour; or in that of the motion of the wave of the ocean, producing the tide, which is
not much less rapid; since Dr. Whewell has lately shown, that the *cotidal line*, or wave of [8137k]
the sea, corresponding to the time of high water, passes, in succession, through the whole
extent of the Atlantic ocean, from the Cape of Good Hope to the shores of North America, [8137l]
a distance exceeding 6000 miles, in about 10 hours; being at the rate of above 600 miles
in an hour; and this rapid velocity is attended with so little displacement in the particles of
the fluid, that the motion of the wave is hardly perceived by those who are borne on it.
In bodies having a great refractive power, the ether is supposed to exist in a state of less [8137m]
elasticity than in those bodies where the refractive power is less; and upon this principle
it must follow that the motions are propagated with the *least velocities in the most* [8137n]
refrangible bodies, which is directly contrary to the results of the Newtonian theory,
[8189], where the velocity of the ray is supposed to *increase in passing from a vacuum*
into a refracting medium. Some experiments made on this point seem to be in favor of the [8137o]
wave theory. Upon the principles here mentioned, it follows *that the velocity of light* [8137p]
continues the same, in the same body, and under similar circumstances; that it decreases in [8137q]
passing into a body of a greater refrangible power, but increases in entering a less [8137r]
refrangible body. The direction of the ray, as in the case of the motion of air producing
sound, is in a line perpendicular to the surface of the wave; and when the motion is [8137s]
propagated in an uniform ether, the wave is bounded by spherical surfaces, and the direction [8137t]
of the ray is from the centre of these spherical waves; so that in this theory the rays of
light move in a right line, in a uniform medium. When the medium through which the
vibrations are communicated is not uniformly elastic, the wave will make unequal progress [8137u]
in different directions, according to the law of the elasticity. In this case the figure of the

Wave theory of light of Huygens.

The ray has the least velocity in the most refrangible body.

[8137″] a transparent medium, which, by inflecting their rays, changes their apparent positions, and makes them appear in different places from those which they

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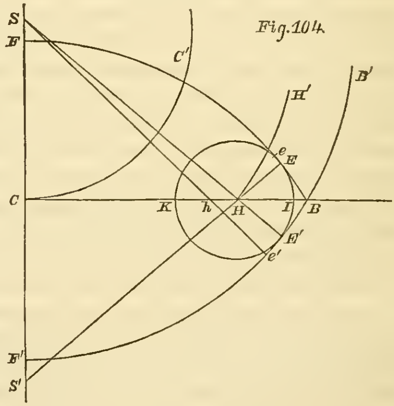
curve will not be spherical, and a double refraction may be produced, as in the case of Iceland crystal.

The limits of the present note will not allow us to enter into a full explanation of the
[8137v] wave theory, though we may possibly do it hereafter, in an appendix; but as the doctrine of the reflexion and refraction of light is investigated, in this chapter, by means of the Newtonian theory, it seems proper that we should show that the same can be done by
[8137w] means of the wave theory. We shall therefore insert, in this place, the demonstrations of the laws of reflexion and refraction, in the elementary manner in which it was first
[8137x] explained by Huygens, without attempting to enter into a minute examination of the corpuscular action of the particles of the ether, in producing these motions, according to the methods of Poisson, Cauchy, Lamé, &c.

Reflexion of light, in the wave theory.



It is supposed that every vibrating portion of the ether, however it may have acquired its motion, whether by reflexion or refraction, *can be regarded as a centre of vibration, from which a system of*
[8137y] *secondary waves emanates in all directions, according to the laws of the propagation of waves in the medium.* Upon this principle we shall now demonstrate the law of the reflexion of light, or *the*
[8137z] *equality of the angles of incidence and reflexion.* We shall suppose CB to be the plane of reflexion; S the luminous objects, from which proceed successively the concentric spherical waves $C'C$, $H'H$, $B'B$, &c. To simplify the
[8138a] demonstration we shall only consider a section of these waves, by a plane passing through S, perpendicular to the reflecting plane, and forming the plane of the annexed figure; the
[8138b] circular arc $B'B$ being continued below the plane or line CB, in the part $BE'e'F'$. Draw the line SCS' perpendicular to the line CB, and make $CS'=CS$; through any
[8138c] point whatever H of the line CB, draw the line $SHE'=SF'=SB$; continue the line $S'H$ to E, making $HE=HE'$, or $S'E=SE'$. Then the point E, above the
[8138d] reflecting plane, is situated, relative to the points B and S, in an exactly similar position to
[8138e] that of the point E', below the same plane, relative to the points B and S'; and from this construction it will follow, that if a circular arc BF be described about the centre S',
[8138f] it will pass through the point E, in like manner as the similarly situated arc BF' passes

really occupy. It is therefore important to determine the law of this inflexion, so as to obtain the real situations of these bodies. [8137‴]

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through E', making *the angles* SHC, $S'HC$, BHE, BHE', *equal to each other.* [8138g]
Moreover the circle $EIE'K$, described about H as a centre, with the radius HE, will touch the circular arcs $BE'F'$, BEF, in the points E', E, respectively. These results [8138h]
hold good, whatever be the situation of the point H upon the line CB. Thus if the point H fall in C, the lines SH, $S'H$, will coincide with SCS'; so that if a circle be [8138i]
described about the centre C, with the radius CF, it will touch the arcs BF, BF' in [8138k]
the points F, F', respectively. Moreover, if the point h be taken infinitely near to h, the point E' will change into the infinitely near point e', of the arc $BE'F'$; and the point E will change into the infinitely near point e of the arc BEF; and it will be evident that the arc $E'e'$ is common to the circles $BE'F'$, $EIE'K$; therefore $E'e'$ is perpendicular to $E'H$ or $E'S$; in like manner EH, or ES', *is perpendicular to* Ee. [8138l]
Now if the direct motion of the waves $C'C$, $H'H$, &c. were not interrupted by the [8138m]
plane CB, the particles of the wave would arrive at the points F', E', B, at the same moment of time; but when the particle C of the wave $C'C$, arrives at the point C, it is reflected back with the same velocity [8137q], and a *secondary wave* emanates from C [8138n]
as a centre. This secondary wave arrives at the situation F, at the same moment of time in which the direct wave would arrive at F', if its motion had not been intercepted; moreover the *direction* of the motion of this secondary wave, at the point F, is in the line $S'F$, as if it proceeded from S'. In like manner, when the direct wave has arrived into [8138o]
the situation $H'H$, the particle at H is reflected; and a secondary wave, with the same velocity, is formed about H as a centre; and as $HE=HE'$, the point E of this [8138p]
secondary wave will arrive at E, at the same time that the direct wave would be at E', if it were not interrupted. Moreover the motion of this secondary wave, at the point E, is perpendicular to the wave Ee [8138l], or in the direction $S'E$, as if it proceeded from [8138q]
S'; and the same is true wherever the point H of the plane CB, or the corresponding point E' of the arc $BE'F'$, may be situated. Hence the effect of the reflected waves will be to make the object S appear at S', when viewed by an observer at E; and then [8138r]
we shall have, as in [8138g], the angle $SHC=$ the angle BHE; consequently *the angle of incidence is equal to the angle of reflexion*; which is the well known law of the reflexion [8138s]
of light, and is the same as is deduced from the Newtonian theory in [8205, &c.].

The demonstration of the law of refraction is made in a somewhat similar manner, using the figure 105, page 442, which resembles figure 104, page 440, and is marked with the [8138t]
same letters; CB being the plane of refraction; S the luminous object; $C'C$, $H'H$, [8138u]
$B'BE'e'F'$, successive portions of the spherical waves emanating from S. Then when the point H of the wave $H'H$, meets the surface of CB, of a medium of *greater* Refraction of light, in the wave theory.
refractive power, the velocity of the wave is decreased [8137n], in a ratio which we shall [8138v]
represent by

DIFFERENTIAL EQUATION OF THE MOTION OF LIGHT.

[8137iv] We shall consider the trajectory described by a ray of light which passes through the atmosphere, *supposing all its strata to be spherical, and varying*

[8138w] $$\frac{m}{n} = \frac{\text{velocity of the refracted ray}}{\text{velocity of the incident ray}};$$

[8138w′] *n being greater than m*; so that while the wave would pass over the distance HE', with its original velocity, it will describe only the diminished distance $\frac{m}{n} \times HE'$, which we shall make equal to HE, and then we

[8138x] shall have $HE = \frac{m}{n} \times HE'$; lastly we shall suppose that a circle is described about H as a centre, with this radius

[8138y] HE. In like manner, if we take a point h, in the line CB, infinitely near to H, we may suppose another circle to be drawn about h as a centre, with a radius

[8138z] $he = \frac{m}{n} \times he'$. Then the infinitely small arc Ee, which is a common tangent to these two small circles, will represent the situation of the point of the refracted

[8139a] wave, corresponding to $E'e'$. In this case the luminous body will be seen, in the direction EHS', by the eye at E;

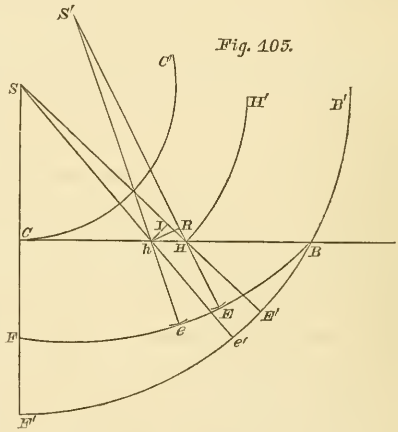


Fig. 105.

[8139b] and in the direction ehS', by the eye at e; so that the position of the luminous body will be changed, by the refraction, from S to S'. We may, in the same way, find the other

[8139c] parts of the refracted wave $BEeF$. From this construction we can easily deduce the law of the refraction of light. For if we let fall from h, upon $E'HS$, the perpendicular hI, and then upon EHS' the perpendicular hR, we shall have, by using the values of he, HE [8138z, x],

[8139d] $$HI = he' - HE'; \qquad HR = he - HE = \frac{m}{n} . he' - \frac{m}{n} . HE'.$$

Multiplying the first of these equations by $\frac{m}{n}$, we find that the second member of the

[8139e] product becomes equal to the second member of the expression of HR [8139d]; hence we have $\frac{m}{n} \times HI = HR$. Dividing this by Hh, we obtain $\frac{m}{n} \times \frac{HI}{Hh} = \frac{HR}{Hh}$; and by

[8139f] referring to the rectangular differential triangles HIh, HRh, we finally obtain

in density according to a function of their height. If we suppose the ray to return, from the eye of the observer, towards the heavenly body, it will evidently describe the same curve as that which it traced in coming from the body to the observer. [8137v]

We shall put,

$\frac{m}{n} \times \sin.HhI = \sin.IhR$. Now HhI is evidently equal to the angle of incidence of the direct ray SH, upon the plane of refraction BC; and IhR is equal to the angle of refraction; hence we finally obtain, Law of refraction.

$$\frac{m}{n} \times \text{sine of angle of incidence} = \text{sine of angle of refraction}; \qquad [8139g]$$

being the same as that which is deduced from the Newtonian theory, in [8188, &c.].

Before closing this article we shall observe, that the method of demonstration of the laws of reflexion and refraction, depending on the principle of the least action, which is given in Vol. I, page 40, note (19*a*), is not applicable to the wave theory; because the wave theory depends on the oscillations of many particles of matter, instead of being limited, as in the principle of the least action [49′, &c.] to the motion of one material part. However, we shall have, in both theories, by referring to figure 105, supposing n to m to express the ratio of the sine of incidence to refraction, and the points S, E, to be given, [8139h] [8139i]

$$m \times SR + n \times RE = \text{a minimum}; \qquad [8139k]$$

as has been shown in the note (19*a*). Now if we represent the velocity of the incident ray by n, that of the refracted ray, in the wave theory, will be m, as is evident from [8138*w*]. Substituting these values in [8139*k*], after dividing it by the constant quantity mn, we get, [8139l]

$$\frac{\text{incident ray } SR}{\text{velocity of incident ray}} + \frac{\text{refracted ray } RE}{\text{velocity of the refracted ray}} = \text{a minimum}. \qquad [8139m]$$

The first member of this last equation evidently expresses the sum of the times of describing the lines SR, RE, or the whole time of describing the space included between *the two given points* S, E. Hence it appears that *this time is a minimum, in the wave theory*; and this circumstance is mentioned by Huygens, in Vol. II, page 34, of his works. [8139n]

Symbols.

[8138] $r =$ the radius drawn from the earth's centre to any part of that trajectory;

[8138′] $a =$ the vertical line drawn from the observer at the surface, towards the centre of the earth, supposing it to be spherical;

[8139] $v =$ the angle which the radius r makes with the vertical line a drawn from the observer towards the centre of the earth, supposing it to be spherical;

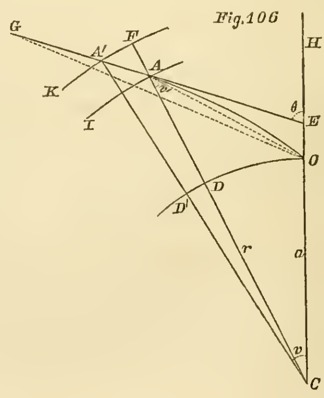
[8139′] $\varphi =$ the force which attracts the ray of light from its rectilinear course. This force evidently draws the ray towards the centre of the stratum, or towards the centre of the earth; since there is no reason why it should deviate on the one side rather than on the other. We shall consider this force as a function of r;

[8140] $\theta =$ the angle which the tangent to the curve, at any point, makes with the vertical line a drawn through the place of the observer; or, in other words, θ represents the *apparent* zenith distance of the extremity of that tangent, supposing it to be continued infinitely towards the heavens;

[8140′] $\Theta =$ the value of θ, or the apparent zenith distance, at the place of the observer;

[8140″] $v' =$ the angle formed by the radius r and the tangent of the curve.

Then we get, from the equation [376],*



[8141a] * (3835) In the annexed figure 106, C is the centre of the earth; ODD' its surface; O the place of the observer; $OAA'G$ the path of the ray of light, coming from the body at G; AE the tangent at the point A; AA' a small arc of this trajectory; $A'F$ is the perpendicular, let fall upon $CDAF$. Then, according to the above notation, we have $CA=r$; $AF=dr$;

[8141b] $OCA=v$; $ACA'=dv$; $A'F=rdv$; $HEA=\theta$; $EAC=v'$. If we suppose the constant quantity $-q^2$, to be connected with the integral

[8141c] $2\int\varphi dr$ [375, 376], the last of these formulas will be as in [8141], which is easily reduced to the form [8141′]. Again, by [369′] we have

[8141d] $xdy-ydx=cdt$; comparing this with the second of the equations [372], we obtain [8143].

Differential equations of the motion of a ray of light.

$$dv = \frac{cdr}{r.\sqrt{q^2r^2 - c^2 - 2r^2.\int\varphi dr}}$$ [8141]

$$= \frac{cdr}{r^2.\sqrt{q^2 - \frac{c^2}{r^2} - 2\int\varphi dr}}; \quad (1)$$ [8141′]

q^2 being a constant quantity, which is added to the integral $2\int\varphi dr$. Moreover, if we put the element of the time equal to dt, we shall have, as in [369′, 372], [8142]

$$r^2dv = cdt.$$ [8143]

Then we have,*

$$v + v' = \theta;$$ [8144]

$$\text{tang.}v' = \frac{rdv}{dr}$$ [8145]

$$= \frac{c}{\sqrt{q^2r^2 - c^2 - 2r^2.\int\varphi dr}}$$ [8146]

$$= \frac{c}{r.\sqrt{q^2 - \frac{c^2}{r^2} - 2\int\varphi dr}}$$ [8146′]

hence we easily deduce,†

General value of the refraction $d\theta$.

$$d\theta = \frac{\frac{c}{r}.\varphi dr}{(q^2 - 2\int\varphi dr).\sqrt{q^2 - \frac{c^2}{r^2} - 2\int\varphi dr}}. \quad (2)$$ [8147]

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* (3836) In the triangle ACE, we evidently have,

$$\text{angle } ACE + \text{angle } EAC = \text{angle } HEA;$$ [8145a]

and by using the symbols [8141b], it becomes as in [8144]. In the differential triangle FAA', we have $\text{tang.}FAA' = \frac{A'F}{AF} = \frac{rdv}{dr}$, as in [8145]; and by substituting the value of dv [8141 or 8141′], it becomes as in [8146 or 8146′]. [8145b]

† (3837) Putting for brevity $w = q^2r^2 - c^2 - 2r^2.\int\varphi dr$, we get $dv = \frac{cdr}{rw^{\frac{1}{2}}}$ [8141], [8146a]
$\text{tang.}v' = cw^{-\frac{1}{2}}$ [8146]. The differential of this last expression is $dv'.(\cos.v')^{-2} = -\frac{1}{2}cw^{-\frac{3}{2}}dw$; [8146b]
and since $(\cos.v')^{-2} = 1 + \text{tang.}^2v' = 1 + c^2w^{-1}$ [34‴] Int., we have,

$$dv' = \frac{-\frac{1}{2}cw^{-\frac{3}{2}}dw}{1 + c^2w^{-1}} = \frac{-\frac{1}{2}cdw}{w^{\frac{1}{2}}.(c^2 + w)}.$$ [8146c]

Adding this to the expression of dv [8146a], we get the value of $dv + dv' = d\theta$ [8144]; [8146c′]
hence we have, by successive reductions, and the substitution of

$$dw = 2q^2rdr - 4rdr.\int\varphi dr - 2\varphi r^2dr = \frac{(c^2 + w)}{r}.2dr - 2\varphi r^2dr \quad [8146a],$$ [8146d]

[8148] The angle θ at the origin of the curve is the complement of the apparent altitude of the body. At the other extremity it denotes the complement of its true altitude. Strictly speaking, the complement of this last altitude is the angle formed by the vertical line passing through the observer, and by a line drawn from the body to the observer. But on account of the limited height of the atmosphere and the smallness of the astronomical refraction, this right line may be supposed to coincide with the tangent drawn to the curve, described by the ray of light, at the point where it enters into the
[8149] atmosphere; the difference is insensible, even for the moon.* Hence it

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[8146e] $$d\theta = \frac{cdr}{rw^{\frac{1}{2}}} - \frac{\frac{1}{2}cdw}{w^{\frac{1}{2}}.(c^2+w)} = \frac{cdr}{(c^2+w).w^{\frac{1}{2}}}.\left\{\frac{(c^2+w)}{r} - \tfrac{1}{2}.\frac{dw}{dr}\right\} = \frac{cdr}{(c^2+w).w^{\frac{1}{2}}}.\left\{\frac{(c^2+w)}{r} - \frac{(c^2+w)}{r} + \varphi r^2\right\}$$

[8146f] $$= \frac{cr^2.\varphi\, dr}{(c^2+w).w^{\frac{1}{2}}}.$$

[8146g] This last expression of $d\theta$ is easily reduced to the form [8147], by re-substituting w [8146a]. Now from the definitions of θ, Θ [8140, 8140'], it is evident that $\theta - \Theta$ represents the refraction R of the ray in proceeding from O to A; so that we have
[8146h] $R = \theta - \Theta$, whose differential gives $dR = d\theta$; therefore $d\theta$ represents the differential of the refraction.

[8148a] * (3838) If we suppose the point A', in figure 106, page 444, to be the limit of the atmosphere, and EAG to be a right line tangent to the point A'; the *true* zenith distance of the body G, when observed at the point O, is equal to the angle
[8148b] $HOG = HEG - EGO = \theta - EGO$. To estimate the value of the small angle EGO, we shall observe that it is to the angle EAO, nearly in the ratio of the line OA to OG; so
[8148c] that we shall have nearly $EGO = EAO \times \frac{OA}{OG} = EAO \times \frac{OA}{OC} \times \frac{OC}{OG}$. Now whatever be the nature of the curve OA, provided its curvature is regular, the angle EAO will be of the same order as the refraction R of the body; and by putting it equal to R, and also
[8148d] observing that $\frac{OC}{OG}$ is of the order of the parallax P of the body, we shall have EGO of the order $P \times R \times \frac{OA}{OC}$; and as each term of this expression is small, the angle EGO must be very small. Thus for the moon, which has the greatest parallax, we have $P = \frac{1}{60}$
[8148e] of the radius nearly at its maximum; and the greatest, or horizontal refraction, is nearly 60′ [8281]; hence the greatest value of EGO is of the order $1' \times \frac{OA}{OC}$, and in general
[8148f] this angle is much less. Now the height of the atmosphere is not more than $\frac{1}{100}$ part of the radius; therefore the fraction $\frac{OA}{OC}$, when OA is vertical, is $\frac{1}{100}$; and when OA is
[8148g] horizontal, it is nearly $\sqrt{\frac{2}{100}}$, or $\frac{1}{7}$; hence we see in general that EGO is insensible, and that we may put $HOG = \theta$, as in [8149].

follows that *the integral of the expression of $d\delta$, taken from the origin of the curve to its other extremity, is the refraction of the body.* But to obtain this integral, we must determine the values of the constant quantities c and q, and the function φ. [8150]

The constant quantity c is easily determined by observing that, if we use [8151]
the radius a [8138'], *and commence the integral $f\varphi dr$ at the origin of the* [8152]
curve, putting Θ for the value of δ at that point; or, in other words, for [8153]
the apparent distance of the body from the zenith at that point; we shall have, by what precedes,*

Value of Θ at the surface of the earth.

$$\text{tang.}\Theta = \frac{\frac{c}{a}}{\sqrt{q^2 - \frac{c^2}{a^2}}}; \qquad [8154]$$

whence we deduce,

$$\frac{c}{a} = q.\sin.\Theta. \qquad [8155]$$

2. The value of q depends upon the integral $f\varphi dr$, consequently upon the nature of φ. To determine this function, we shall consider a ray of light which penetrates into a transparent body, terminated by plane surfaces.
The particle, before its entrance, is drawn *perpendicularly* towards the plane [8156]
surface, by which it enters the body. For, the action of bodies upon light being sensible only at very small distances, the parts of the body, which are at a sensible distance from the particle of light, will not have any sensible
action upon it; and we may, *in the calculation of the action of the body,* [8157]
consider it as a solid of infinite dimensions, terminated by a plane surface, [8157']
indefinite in every direction. In this hypothesis, it is evident that the action [8157'']
of the body upon the particle of light, is perpendicular to its surface.†

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* (3839) At the point O of the curve CA, we have $f\varphi dr = 0$, $r = a$, $v' = \Theta$ [8154a]
[8152, &c.]. Substituting these in [8146'], we get the expression [8154]; whose square
being multiplied by $q^2 - \frac{c^2}{a^2}$, gives $q^2.\text{tang.}^2\Theta = \frac{c^2}{a^2}.(1+\text{tang.}^2\Theta) = \frac{c^2}{a^2}\cdot\frac{1}{\cos.^2\Theta}$; then [8154b]
multiplying by $\cos.^2\Theta$, extracting the square root, and putting $\cos.\Theta.\text{tang.}\Theta = \sin.\Theta$, it becomes as in [8155].

† (3840) For the purpose of illustration, we have inserted the figure 107, page 448, [8158a]
where $ID'DCL$ represents the path of the ray, which enters the body $ABDE$, at the

[8157‴] *We shall first consider the particle before it enters the body,* using the following symbols;

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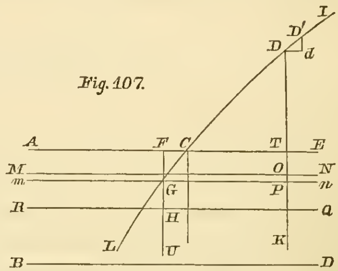
point C of its surface; this surface being considered as a plane, passing through the line AE, perpendicular to the plane of the figure. The lines MN, mn, RQ, BD, Dd, are
[8158b] parallel to the surface AE; and the lines FU, DTK, $D'd$, perpendicular to AE. Then taking C for the origin of the co-ordinates, and D for the situation of the particle
[8158c] of light at any time, we shall have, according to the above notation, $CT=x$, $TD=s'$.
[8158d] Moreover if we put $DO=s$, $OP=ds$, we shall have $MNnm$ for the section of the stratum treated of in [8158], which acts upon
[8158e] the particle of light at D, with a force $\rho.ds.\Pi(s)$ [8161], in the direction DT. The whole action of the body upon the particle will be obtained, by taking the integral of the expression $\rho.ds.\Pi(s)$, from $s=DT$ to
[8158f] $s=DK$, taking for K any point which is so distant from D as to be without the sphere of the sensible action upon the particle at D;

Fig. 107.



[8158g] and it is evident that we may suppose DK to be infinite, because the action of the points situated beyond K are supposed to be insensible on the particle at D; therefore the whole action of the body, upon the particle at D, will be represented by $\int_{s'}^{\infty}\rho.ds.\Pi(s)$ in the
[8158h] direction DT; and if we suppose the density ρ to be constant, this force will become $\rho.\int_{s'}^{\infty}ds.\Pi(s)=\rho.\Pi_1(s')$ [8161′].

[8158i] Now if we put $z=0$, and $y=s'$, in the equations of the motion of a particle [38], they will become,

[8158k] $$\frac{ddx}{dt^2}=P; \qquad \frac{dds'}{dt^2}=Q;$$

P, Q [34^{vi}], being the forces acting upon the particle in the directions parallel to the
[8158l] rectangular co-ordinates x, s', respectively, and tending to increase the co-ordinates [34^{vii}]. In the present case, the only force acting upon the particle is that in [8158h], which is represented by $\rho.\Pi_1(s')$, in the direction of the ordinate DT, and tending to *decrease* it;
[8158m] this force will therefore be *negative*, and we shall have $Q=-\rho.\Pi_1(s')$, and $P=0$.
[8158n] Substituting these in the equations [8158k], we get [8164, 8165]. From these equations we easily deduce [8167], by the process of integration explained in [8166]. The integral
[8158o] of [8164] gives $\frac{dx}{dt}=$ constant; therefore the velocity of the ray, in a direction parallel to the surface, remains constant, and is unaltered by the action of the refractive surface.
[8158p] *The same result is found in* [8177i], *after entering the surface of the body.*

$s =$ the distance of the particle of light from an infinitely thin stratum of the body, which is drawn parallel to its surface ; [8158]

$s' =$ the value of s corresponding to the external surface ; or the distance of the particle of light from the external surface ; [8159]

$\rho =$ the density of the stratum, whose thickness is ds ; [8160]

$\rho.ds.\Pi(s) =$ the action which the stratum, whose thickness is ds, exerts upon the particle ; and to have the whole action of the body we must integrate this expression from $s = s'$ to $s = \infty$, [8158f] ; [8161]

$\Pi_1(s') = \int_{s'}^{\infty} ds.\Pi(s)$; consequently $\Pi_1(s) = \int_{s}^{\infty} ds.\Pi(s)$, as in [8242] ; [8161']

$K = \int_0^{\infty} ds'.\Pi_1(s')$; or $K = \int_0^{s'} ds'.\Pi_1(s')$, when s' is of any sensible magnitude ; [8162]

$n =$ the velocity of the ray of light when at a sensible distance from the attracting body, and before entering it ; [8162']

$\mathrm{n} =$ the expression of the variable velocity of the ray of light, after it is acted upon by the attracted body. [8162'']

Now if we put x and s' for the rectangular co-ordinates of the particle of light; x being parallel to the surface of the body, and in the plane formed by the vertical line at that surface, and the direction of the luminous ray; we shall have, as in [8158k, m], [8163]

Equations before the ray enters the first surface.

$$\frac{ddx}{dt^2} = 0 ; \qquad [8164]$$

$$\frac{dds'}{dt^2} = -\rho.\Pi_1(s') ; \qquad [8165]$$

dt being the element of the time, supposing it to be constant. Therefore we shall have, by multiplying the first of these equations by $2dx$, the second by $2ds'$, and integrating their sum, [8166]

$$\frac{dx^2 + ds'^2}{dt^2} = \text{constant} - 2\int_0^{s'} \rho.ds'.\Pi_1(s') = (\text{velocity of the ray})^2. \qquad [8167]$$

To determine the constant quantity, we shall use the abridged symbols K, n [8162, 8162'] ; observing that when the particle of light is at a sensible distance from the attracting body, we shall evidently have $\int_0^{s'} ds'.\Pi_1(s') = K$, because the action of the body upon light is sensible only at very small distances ; hence we get,* [8168] [8169] [8170]

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* (3841) To avoid the consideration of negative signs, in the limits of the integrals, we shall adopt the method proposed in [8137v] ; and instead of considering the path of the particle, in its motion from I [fig. 107, page 448] towards the origin of the co-ordinates [8164a]

Velocity before entering the body.

[8171] $$n^2 = \text{constant} - 2K\rho;$$

consequently,

[8172] $$\text{constant} = n^2 + 2K\rho.$$

Hence we deduce,

[8173] $$\frac{dx^2 + ds'^2}{dt^2} = n^2 + 2K\rho - 2\rho . \int_0^{s'} ds' . \Pi_1(s') = (\text{velocity of the light})^2;$$

Velocity at the surface of the body.

[8174] *so that when the light enters into the body, where* $s' = 0$ *and the integral commences, we shall have, for the square of the velocity of the ray, the expression* $n^2 + 2K\rho$.

To obtain the value of the square of the velocity, when the light has penetrated into the body, by the quantity s', we shall observe that s' being

[8175] the distance of the particle from the surface of the body, it is attracted towards that surface by a stratum of the thickness s'; but this attraction is

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C, at the surface of the body, we shall consider it in a backward course, from C towards I.

[8164b] Then we have $CT = x$, $TD = s'$, $Dd = dx$, $D'd = ds'$; hence $DD' = \sqrt{dx^2 + ds'^2}$. Dividing this by the time dt, we get the general expression of the velocity of the particle

[8164c] of light, which we shall represent, as in [8162″], by the Roman letter n, when at the point D, or after it has entered within the sphere of activity of the attracting body; the symbol n being its value before it is acted upon by the body [8162′]. Hence we have

[8164d] generally $\mathrm{n} = \frac{DD'}{dt} = \frac{\sqrt{dx^2 + ds'^2}}{dt}$; substituting this in [8167], we get,

[8164e] $$\mathrm{n}^2 = \frac{dx^2 + ds'^2}{dt^2} = \text{constant} - 2\int_0^{s'} \rho . ds' . \Pi_1(s')$$

[8164f] $$= \text{constant} - 2\int_0^{\infty} \rho . ds' . \Pi_1(s') + 2\int_{s'}^{\infty} \rho . ds' . \Pi_1(s');$$

the last of these expressions being easily derived from that which precedes it, by merely changing the limits of the integrals. If we suppose ρ to be constant, the expression [8164f] becomes successively, by using K [8162],

[8164g] $$\mathrm{n}^2 = \frac{dx^2 + ds'^2}{dt^2} = \text{constant} - 2\rho . \int_0^{\infty} ds' . \Pi_1(s') + 2\rho . \int_{s'}^{\infty} ds' . \Pi_1(s')$$

[8164h] $$= \text{constant} - 2K\rho + 2\rho . \int_{s'}^{\infty} ds' . \Pi_1(s').$$

When the distance s' is so great that the particle is without the sphere of activity of the attracting body, the integral part of the expression [8164h] vanishes, and n changes into

[8164i] n; then the expression [8164h] becomes as in [8171]; hence we easily derive the value of the constant quantity, as in [8172]. Substituting this in [8164e], we get, as in [8173],

[8164k] $$\mathrm{n}^2 = \frac{dx^2 + ds'^2}{dt^2} = n^2 + 2K\rho - 2\int_0^{s'} \rho . ds' . \Pi_1(s').$$

At the point C of the surface of the body, where $s' = 0$, the integral part of the second member of [8164k] vanishes, and this expression of the square of the velocity becomes

[8164l] $n^2 + 2K\rho$, as in [8174].

destroyed by that of an inferior stratum of the same thickness; so that the particle moves in the same manner as if it were acted upon only by the strata which are situated below these; *it is therefore affected in the same* [8175′]
manner as when it is without the body, at the distance s' from its surface; therefore the attraction, which the body exerts upon it, is equal to $\rho.\Pi_1(s')$, [8176]
[8158*h*]. But in this case the attraction tends to increase s'; therefore, by putting x and s' for the co-ordinates of the particle, we shall have,*

After entering the first surface at an insensible distance.

$$\frac{ddx}{dt^2}=0;$$ [8177]

$$\frac{dds'}{dt^2}=\rho.\Pi_1(s').$$ [8178]

hence we deduce,

$$\frac{dx^2+ds'^2}{dt^2}=\text{constant}+2\rho.\textstyle\int_0^{s'}ds'.\Pi_1(s').$$ [8179]

The constant quantity is evidently the square of the velocity of the particle,

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* (3842) In computing the path of the particle CGL, fig. 107, page 448, after it has entered the body at C, we shall put $CF=x$, $FG=s'$, for the rectangular co-ordinates [8177*a*] of any point G of this part of the path. If we continue the vertical ordinate FG to H, making $FG=GH$, and draw the plane RHQ parallel to the surface AE, it is evident that the attraction of the different parts of $AEQR$ upon G will mutually destroy each [8177*b*] other; and the particle will be acted upon only by the part $RQDB$, which falls below the plane RQ; so that the particle of light will be affected in exactly the same manner as if it were at the distance $GH=s'$, without the surface, and acted upon by the whole body. [8177*c*] Then the force acting upon the particle, *perpendicular to the surface*, will be, as in [8158*h*], $\rho.\Pi_1(s')$, *and nothing parallel to the surface*, ρ being constant [8158*h*]. These forces give [8177*d*] the expressions [8177, 8178], which are similar to [8164, 8165], the sign of $\Pi_1(s')$ being [8177*e*] taken positive in [8178], because this force tends to *increase* FG or s'. Now multiplying [8177, 8178] by $2dx$, $2ds'$, respectively, and integrating the sum of these products, we [8177*f*] get [8179]; substituting the value of the constant quantity [8180], it becomes,

$$\mathrm{n}^2=\frac{dx^2+ds'^2}{dt^2}=n^2+2K\rho+2\rho.\textstyle\int_0^{s'}ds'.\Pi_1(s').$$ [8177*g*]

When the particle has penetrated into the body a sensible distance, s', we shall have $\int_0^{s'}ds'.\Pi_1(s')=K$ [8162]; substituting this in [8177*g*], it becomes,

$$\mathrm{n}^2=\frac{dx^2+ds'^2}{dt^2}=n^2+4K\rho,$$ [8177*h*]

as in [8182]. Finally we may remark, as in [8158*o*], that the integral of the equation [8177] gives the value of $\frac{dx}{dt}=$ constant; so that the velocity, parallel to the surface, is [8177*i*] constant.

[8180] *at the point where it enters into the body, and we have just seen* [8174] *that this square is equal to* $n^2+2K\rho$. To determine the value of the integral $\int ds'.\Pi_1(s')$, when the particle has penetrated sensibly within the body, we

[8181] may observe that this is very nearly equal to that value taken from $s'=0$ to $s'=\infty$, therefore it is equal to K [8162]; hence we have, when the particle has sensibly penetrated into the body,

After entering the first surface at a sensible distance.

[8182]
$$\frac{dx^2+ds'^2}{dt^2}=n^2+4K\rho. \quad [8177h]$$

[8183] *We shall put* θ *for the angle of incidence, which the ray of light makes* with the perpendicular to the surface, before its entrance into the body, and when at a sensible distance from it; and we shall have,*

[8184]
$$\sin.\theta=\frac{dx}{ndt};$$

[8185] *also* θ' *for the angle of refraction*, which the ray of light makes with the perpendicular to the surface, when it has sensibly penetrated into the body; and we shall have, as in [8184*d*],

[8186]
$$\sin.\theta'=\frac{dx}{\sqrt{dx^2+ds'^2}}=\frac{dx}{ndt.\sqrt{1+\frac{4K}{n^2}\cdot\rho}}.$$

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* (3843) In the differential triangle $DD'd$, fig. 107, page 448, we have

[8184*a*] $\sin.DD'd=\frac{Dd}{DD'}$; and by substituting $Dd=dx$, $DD'=\mathrm{n}dt$ [8164*b*, *d*], it becomes $\sin.DD'd=\frac{dx}{\mathrm{n}dt}$. Now before the particle enters the sphere of activity of the body, we have $\mathrm{n}=n$ [8164*c*, *d*], and $DD'd=\theta$ [8183]; then the equation [8184*a*] becomes

[8184*b*] $\sin.\theta=\frac{dx}{ndt}$, as in [8184]. After the ray has penetrated into the surface of the body, n

[8184*c*] changes into $\sqrt{n^2+4K\rho}$ [8182], and $DD'd$ changes into θ' [8185]; then [8184*a*]

[8184*d*] becomes $\sin.\theta'=\frac{dx}{dt.\sqrt{n^2+4K\rho}}$, which is easily reduced to the form [8186]. Now the differential expression [8164 or 8177] is the same in all parts of the path, namely

[8184*e*] $\frac{ddx}{dt^2}=0$; and its integral is $\frac{dx}{dt}=$ constant, or $\frac{dx}{ndt}=$ constant; so that this quantity is

[8184*f*] of the same value in [8186] as in [8184]. Finally, dividing [8186] by [8184], and then multiplying the result by $\sin.\theta$, we get [8187]; and from this we obtain the following proportion,

[8184*g*] $\sin.\theta : \sin.\theta' :: \sqrt{n^2+4K\rho} : n ::$ the velocity after refraction : the velocity before refraction, as in [8188].

The values of $\frac{dx}{dt}$ are the same in both cases [8184e], since we have [8186′]
always $\frac{ddx}{dt^2}=0$; therefore we shall have, as in [8184f, g],

Ratio of the sine of incidence to the sine of refraction.

$$\sin.\theta' = \frac{\sin.\theta}{\sqrt{1+\frac{4K}{n^2}\cdot\rho}}; \qquad \text{or,} \qquad [8187]$$

$$\frac{\text{sine of the angle of incidence } \theta}{\text{sine of the angle of refraction } \theta'} = \sqrt{1+\frac{4K}{n^2}\cdot\rho} = i, \quad [8191']; \qquad [8187']$$

that is, the sine of incidence is to the sine of refraction in a constant ratio, and this ratio [*in the Newtonian theory of light*] *is that of the velocity of light after having sensibly penetrated into the body, to its velocity before that time, and when at a sensible distance from the body.* [8188]

The quantity $4K\rho$ [*in the Newtonian theory*] *is the increment of the square of the velocity of light, when it has experienced the whole action of the transparent body. This quantity is not the same in different diaphanous bodies; and it is not in the ratio of their densities.* Possibly the function of the distance, which expresses the action upon light, may differ for every different body; or it may be the same, varying only in the different bodies by the product of the density multiplied by a constant coefficient, depending on the nature of the bodies. In both these suppositions, the whole action of the body upon light will be the same; and since, in the calculation, we require only the whole result of the action, we may use the second hypothesis as the most simple. *The constant coefficient just mentioned, will represent the respective intensity of the action of the bodies upon light, or their refrangible power. This coefficient is proportional to* $\frac{4K}{n^2}$; *so that we may represent the refrangible power of the body by that quantity. If we put* i *for the ratio of the sine of incidence to the sine of refraction, we shall have, by what has been said,** [8189] [8190] [8191] [8191′]

Increment of the square of the velocity.

Ratio i.

* (3844) We have, as in [8191′, 8187′], $i=\sqrt{1+\frac{4K}{n^2}\cdot\rho}$. Squaring this equation and reducing, we obtain [8192]. If we put for brevity, [8192a]

$$k=\frac{4K}{n^2}, \qquad [8192b]$$

the equation [8187] may be put under the form,

$$\sin.\theta=\sqrt{1+k\rho}.\sin.\theta' = i.\sin.\theta'. \qquad [8192c]$$

[8192] $$\frac{4K}{n^2}=\frac{i^2-1}{\rho}=\text{the refrangible power};$$

Refrangible power $\frac{4K}{n^2}$.

therefore we shall have, by this formula, the ratios of the refrangible powers of different substances.

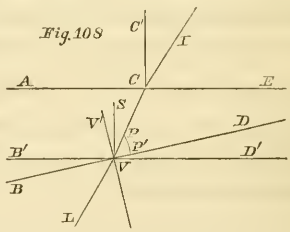
[8193] *As the rays of different colors have different refrangibility, it follows that their velocities are not the same, or that the intensity of the action of a body upon each of them is different.* The difference of velocity alone will not account for all the phenomena of the refrangibility of the rays; for in that case the difference of the refractions of the extreme rays, or, in other words,
[8193′] the dispersion of light, would be the same for all the bodies which refract equally the mean rays, which is contrary to observation.

We shall now consider the particle of light while moving within the body, and just at the point of quitting it, at a plane surface, which is inclined by
[8194] *the angle* ε *to the surface at which the particle entered into the body.* We shall put s' for the distance of this particle from the second surface, and x for its absciss parallel to that surface; and we shall then have,*

When approaching towards the second surface.

[8195] $$\frac{ddx}{dt^2}=0;$$

[8196] $$\frac{dds'}{dt^2}=\rho.\Pi_1(s').$$



* (3845) We shall suppose, in the annexed figure 108, that ACE is the first surface, and
[8195a] BVD the second surface of the body; $ICVL$ the path of the particle of light; $B'VD'$ parallel to ACE; CC' perpendicular to AE; VS perpendicular to $B'VD'$; VV' perpendicular to BVD. Then we shall have,

[8195b] angle $DVD'=SVV'=\varepsilon$, [8194];
angle $SVC=\theta'$; angle $V'VC=\theta'+\varepsilon$.

Now when the particle is approaching towards the
[8195c] point V, and is at any point P, its rectangular co-ordinates are $VP'=x$, and $PP'=s'$; this last being perpendicular to VP', or parallel to VV'; and it is very evident that the
[8195d] particle will then be acted upon, by the body, in like manner as in [8176]; and we shall get, for the differential equations of its motions, the expressions [8195, 8196], which are similar to those in [8177, 8178], using the same method as in [8164a]. Multiplying
[8195e] [8196] by $2ds'$, and integrating the product, we get [8197], corresponding to any point within the sphere of activity of the surface BVD.

Hence we deduce,

$$\frac{ds'^2}{dt^2} = \text{constant} + 2\rho.\textstyle\int_0^{s'} ds'.\Pi_1(s')\,;$$ [8197]

the integral being taken from $s'=0$. When s' has a sensible value, this integral is equal to $2K\rho$ [8169], and we shall have,* [8198]

$$\frac{dx}{dt} = n.\sqrt{1+\frac{4K}{n^2}.\rho}.\sin.(\theta'+\varepsilon)\,;$$ [8199]

$$\frac{ds'}{dt} = n.\sqrt{1+\frac{4K}{n^2}.\rho}.\cos.(\theta'+\varepsilon)\,;$$ [8200]

therefore,

$$\text{constant} = n^2.\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2(\theta'+\varepsilon) - 2K\rho\,;$$ [8201]

which gives,

When approaching towards the second surface.

$$\frac{ds'^2}{dt^2} = n^2.\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2(\theta'+\varepsilon) - 2K\rho + 2\rho.\textstyle\int_0^{s'} ds'.\Pi_1(s').$$ [8202]

This value of $\frac{ds'^2}{dt^2}$ *will vanish, before the ray arrives at the second surface, whenever* $\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2(\theta'+\varepsilon)$ *is less than*† $\frac{2K}{n^2}.\rho$. In this case it is [8203]

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* (3846) When the particle is within the body, at P, at a sensible distance from the surface BVD, its velocity in its path is $\sqrt{n^2+4K\rho}$, or $n.\sqrt{1+\frac{4K}{n^2}.\rho}$ [8182]. This [8199a] is resolved in the directions parallel to the co-ordinates x, s', or $P'V$, PP', by multiplying it by $\sin.VPP'$, $\cos.VPP'$, respectively; and as we have the [8199b] angle $VPP'=$ the angle $V'VC = \theta'+\varepsilon$ [8195b], they become as in the second members of [8199, 8200], respectively. The first members of these expressions being the well known values of the velocities in the directions of the co-ordinates. Substituting the values [8199c] [8200, 8162] in [8197], we get, by transposition, the value of the constant quantity [8201]; and by introducing it into the general expression of $\frac{ds'^2}{dt^2}$ [8197], when within the [8199d] sphere of activity of the second surface BVD, it becomes as in [8202]. This represents the value of $\frac{ds'^2}{dt^2}$, in the space included between the second surface BVD, and the point within the body where the particle is first sensibly affected by the attraction here treated of. [8199e]

† (3847) When $\frac{ds'}{dt}=0$, the expression [8202] gives, by dividing by n^2, and [8203a] transposing the two last terms, using also for brevity $k=\frac{4K}{n^2}$ [8192b],

[8203′] evident that the value of $\frac{dx}{dt}$ will remain always the same, and the particle will begin to recede from the surface, describing a curved branch, precisely similar to that which it passed over in approaching towards the surface;*

[8204] the vertex of the whole curve being at the point where $\frac{ds'}{dt}$ vanishes. The action of a body upon light being sensible only at very small distances, *the part of the trajectory, which is sensibly curved, may be considered as a point,*

[8203b] $$(1+k.\rho).\cos.^2(\theta'+\varepsilon)=\tfrac{1}{2}k.\rho-\frac{2\rho}{n^2}.\int_0^{s'}ds'.\Pi_1(s');$$

and as the integral $\int_0^{s'}ds'.\Pi_1(s')$ commences when $s'=0$, the second member of [8203b] must be less than $\frac{1}{2}k\rho$ when s' exceeds zero, as in [8203]. Dividing [8203b] by $1+k\rho$, and substituting the result in $\sin.^2(\theta'+\varepsilon)=1-\cos.^2(\theta'+\varepsilon)$, we get,

[8203c] $$\sin.^2(\theta'+\varepsilon)=\frac{1+\frac{1}{2}k\rho}{1+k\rho}+\frac{2\rho}{n^2.(1+k\rho)}.\int_0^{s'}ds'.\Pi_1(s');$$

hence it is evident that this limit of $\sin.^2(\theta'+\varepsilon)$ generally exceeds $\frac{1+\frac{1}{2}k\rho}{1+k\rho}$, because

[8203d] the integral $\int_0^{s'}ds'.\Pi_1(s')$ is either zero or positive, while s' varies from 0 to s'; this agrees with [8208].

[8204a] * (3848) The force acting on the particle being in the direction PP', fig. 108, page 454, *perpendicular* to the surface BD, the velocity in the direction $P'V$, parallel to that surface, will always remain the same; and $\frac{ds'}{dt}$ [8202] being a function of s', its value in

[8204b] the *second branch* of the curve, in receding from the surface BD, must be the same as in the *first branch* in its approach towards BD, for equal values of s'. Therefore the two branches must be similar; and the vertex of the curve is at the point where the motion is

[8204c] parallel to the surface BD, which evidently corresponds to $\frac{ds'}{dt}=0$, as in [8204].

The path of the ray in this case resembles the figure of a hyperbolic curve, the point of

[8204d] reflexion being in its principal vertex. The incident and reflected ray may then be considered as *the asymptotes of the curve*, since its curvature is confined to the insensible part of the path, which falls within the sphere of activity of the second surface. Similar

[8204e] remarks may be made relative to the form of the path of a refracted ray, which may, for the same reasons, be considered as of a hyperbolic nature. It is also evident, in this case, that

[8204f] the action of the body on the ray must be excessively great, in comparison with the force of gravity upon the surface of the earth, to produce such an essential change in the motion

[8204g] of a particle, in the small part of a second of time, during which the ray is within the sphere of activity of the surface.

and the two branches of the curve as two right lines, meeting in that point, [8205]
[8204*d*, &c.]; *so that the ray will appear to be reflected from the second* Angle of reflexion equal to the angle of incidence.
*surface, at an angle of reflexion which is equal to the angle of incidence.**
The limit where this reflexion takes place, is when the sine of the angle of
incidence $\theta'+\varepsilon$, upon the second surface, is represented by the following [8206]
expression [8207*a*, &c.]; Limit where the reflexion takes place.

$$\left\{\frac{1+\frac{2K}{n^2}\cdot\rho}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}=\sin.(\theta'+\varepsilon). \qquad [8207]$$

The reflexion always takes place when the sine of the angle of incidence exceeds this limit [8204*c*]; *but when it is less, the ray is transmitted;* and [8208]
it is evident that after the ray has passed the second surface, and is at a Motion just after passing the second surface.
sensible distance from it, we shall have,†

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* (3849) If the velocity of the ray $\frac{ds'}{dt}$ vanish, at the moment the ray arrives at the [8207*a*]
second surface, where $s'=0$, we shall have $\int_0^{s'}ds'.\Pi_1(s')=0$; and then the expression
[8203*c*] will become $\sin.^2(\theta'+\varepsilon)=\frac{1+\frac{1}{2}k\rho}{1+k\rho}$; whence we get the value of $\sin.(\theta'+\varepsilon)$, [8207*b*]
[8207]; and when this sine exceeds that value, the ray will be reflected at the second surface.

† (3850) The expression [8209] may be computed in the same manner as in finding [8209*a*]
[8202]; or we may derive [8209] from [8202], by the following considerations. While the particle is approaching towards the second surface BD, fig. 108, page 454, the
ordinate $PP'=s'$, and the integral $\int_0^{s'}ds'.\Pi_1(s')$, decrease in value; so that when the [8209*b*]
particle arrives at the surface at V, they will both *vanish together.* During this time
the expression of the square of the velocity $\frac{ds'^2}{dt^2}$ continually decreases with $\int_0^{s'}ds'.\Pi_1(s')$, [8209*c*]
until this integral vanishes at the point V. At this point the particle passes without the second surface BVD, and the integral $\rho.\int_0^{s'}ds'.\Pi_1(s')$ changes its sign, and becomes
$-\rho.\int_0^{s'}ds'.\Pi_1(s')$; observing that the force $\rho.\Pi_1(s')$ is equal to $\rho.\Pi_1(-s')$ [8176, &c.]; [8209*d*]
and that its direction always continues to be parallel to VV', towards the inner part of the [8209*e*]
body. Making this change of the sign in [8202], it becomes as in [8209], which will
therefore represent the value of the continually decreasing quantity $\frac{ds'^2}{dt^2}$, when the particle [8209*f*]
has passed the surface BD, and is without the body; considering s' as positive, in the
integral expression [8209]. When the particle is at a sensible distance without the body, [8209*g*]
we shall have $\int_0^{s'}ds'.\Pi_1(s')=K$ [8162]; and then [8209] will become as in [8211].

[8209] $$\frac{ds'^2}{dt^2} = n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\theta'+\varepsilon) - 2K\rho - 2\rho.\textstyle\int_0^{s'} ds'.\Pi_1(s');$$

[8210] the integral being taken from $s'=0$. At a sensible distance s', we have $\int_0^{s'} ds'.\Pi_1(s') = K$, as in [8162]; and then we have,

Motion when at a sensible distance without the second surface.

[8211] $$\frac{ds'^2}{dt^2} = n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\theta'+\varepsilon) - 4K\rho;$$

$\frac{ds'^2}{dt^2}$ will therefore vanish, whenever we have,*

[8212] $$\sin.(\theta'+\varepsilon) > \left\{\frac{1}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}.$$

In this case the particle will appear to be reflected from the surface, making the angle of reflexion equal to the angle of incidence. *Therefore, from*

[8213] $$\sin.(\theta'+\varepsilon) = \left\{\frac{1}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}} \text{ [8212], to } \sin.(\theta'+\varepsilon) = \left\{\frac{1+\frac{2K}{n^2}\cdot\rho}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}} \text{ [8207],}$$

Limits of the angle of reflexion.

the particle will appear to be reflected from the second surface, after it has

[8214] *passed through the diaphanous body; and from* $\sin.(\theta'+\varepsilon) = \left\{\frac{1+\frac{2K}{n^2}\cdot\rho}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}$ [8207], *to* $\sin.(\theta'+\varepsilon) = 1$, *the particle will appear to be reflected from the second surface, without ever arriving at that surface. When* $\sin.(\theta'+\varepsilon)$

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* (3851) The greatest limit of the value of $\int_0^{s'} ds'.\Pi(s')$, is K [8162]; substituting this in the second member of [8209], it becomes $n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\theta+\varepsilon')-4K\rho$;

[8212a] hence it is evident that if $n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\theta'+\varepsilon) < 4K\rho$, the second member of

[8212b] [8209] will become negative for certain values of s'; and with these values $\frac{ds'^2}{dt^2}$ will become negative, so that its square root, which represents the velocity of the particle,

[8212c] resolved in a direction perpendicular to the second surface, will be impossible. Substituting $\cos.^2(\theta'+\varepsilon) = 1-\sin.^2(\theta'+\varepsilon)$ in the preceding expression [8212a], we get,

[8212d] $$n^2+4K\rho-n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\sin.^2(\theta'+\varepsilon) < 4K\rho, \quad\text{or}\quad n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\sin.^2(\theta'+\varepsilon) > n^2.$$

Dividing this by $n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right)$, and extracting the square root, it becomes as in [8212].

is less than $\left\{\frac{1}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}$ [8212], *the particle will be transmitted through* [8215]
*the body without reflexion.** Then we shall have, at a sensible distance from the body,

After passing through the first body a sensible quantity.

$$\frac{dx^2}{dt^2}=n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\sin.^2(\theta'+\varepsilon) \quad [8199]; \qquad [8216]$$

$$\frac{ds'^2}{dt^2}=n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\theta'+\varepsilon)-4K\rho \quad [8211]; \qquad [8217]$$

which gives the square of the velocity of light equal to n^2 [8215c]; consequently it is the same as it was before the particle entered into the body. *Putting* θ'' *equal to the angle which the direction of the ray makes* [8218]
with the perpendicular to the second surface, after it has quitted the body, and is at a sensible distance from it, we shall have, as in [8186],

$$\sin.\theta''=\frac{dx}{\sqrt{dx^2+ds'^2}}; \qquad [8219]$$

therefore,

$$\sin.\theta''=\sqrt{1+\frac{4K}{n^2}\cdot\rho}\,.\sin.(\theta'+\varepsilon). \qquad [8220]$$

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* (3852) Putting $\cos.^2(\theta'+\varepsilon)=1-\sin.^2(\theta'+\varepsilon)$ in [8211], we get, by using for brevity $k=\frac{4K}{n^2}$ [8192b],

$$\frac{ds'^2}{dt^2}=n^2-n^2.(1+k\rho).\sin.^2(\theta'+\varepsilon), \qquad [8215a]$$

which always has a real value when $\sin.(\theta'+\varepsilon)<\left\{\frac{1}{1+k\rho}\right\}^{\frac{1}{2}}$; so that the ray will then be transmitted. The velocity $\frac{dx}{dt}$ [8199] being squared, gives [8216]; and that of $\frac{ds'^2}{dt^2}$ [8215b]
[8211], is the same as in [8217]. The sum of the two expressions [8216, 8217] gives, by making a slight reduction, and using k [8192b],

$$\frac{dx^2+ds'^2}{dt^2}=n^2.(1+k\rho)-4K\rho=n^2; \qquad [8215c]$$

and as the first member of this expression, $\frac{dx^2+ds'^2}{dt^2}$, represents the square of the velocity [8164d, c], after quitting the body, this velocity will be represented by n; being the same [8215d]
as in [8162'], before entering the body. Hence $\sqrt{dx^2+ds'^2}=ndt$; substituting this and $\frac{dx}{dt}$ [8216], in [8219], which is similar to the first formula in [8186], it becomes as [8215e]
in [8220].

If we suppose that the second surface is contiguous to the surface of a second
[8221] opake or diaphanous body, whose action upon light, at the distance s', is represented by $\rho'.\Psi_1(s')$, ρ' being its density, we shall have, while the ray remains within the first body,*

Before entering the second body.

[8222]
$$\frac{dds'}{dt^2}=\rho.\Pi_1(s')-\rho'.\Psi_1(s');$$

[8222'] which gives, by putting $K'=\int_0^\infty ds'.\Psi_1(s')$,

[8223]
$$\frac{ds'^2}{dt^2}=n^2.\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2(\theta'+\varepsilon)-2K\rho+2K'\rho'+2\rho.\int_0^{s'}ds'.\Pi_1(s')-2\rho'.\int_0^{s'}ds'.\Psi_1(s').$$

[8223'] In this case the ray will appear to be reflected from the common surface of the two bodies, without penetrating into the second body, whenever the sine
[8224] of incidence, or $\sin.(\theta'+\varepsilon)$, *is equal to or greater than the following quantity*;†

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* (3853) The force $\rho'.\Psi_1(s')$ tends to decrease the force $\rho.\Pi_1(s')$ [8196], and must
[8222a] therefore be subtracted from it, giving for the result the quantity $\rho.\Pi_1(s')-\rho'.\Psi_1(s')$, as in the second member of the equation [8222], which is similar to [8196, &c.]. Multiplying [8222] by $2ds'$, and integrating, we get,

[8222b]
$$\frac{ds'^2}{dt^2}=\text{constant}+2\rho.\int_0^{s'}ds'.\Pi_1(s')-2\rho'.\int_0^{s'}ds'.\Psi_1(s').$$

When the particle is within the first body, and at a sensible distance from its surface, $\frac{ds'}{dt}$ will be as in [8200]; and the preceding expression, by using K [8162], and K' [8222'], will become,

[8222c]
$$n^2.\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2(\theta'+\varepsilon)=\text{constant}+2K\rho-2K'\rho'.$$

Finding the value of the constant quantity [8222c], and substituting it in [8222b], we get [8223].

† (3854) At the surface which separates the two bodies we have $\int_0^{s'}ds'.\Pi_1(s')=0$,
[8225a] $\int_0^{s'}ds'.\Psi_1(s')=0$; and if the particle of light just touches that surface, without passing through it, we shall have $\frac{ds'}{dt}=0$; then [8223] becomes,

[8225b]
$$0=n^2.\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2(\theta'+\varepsilon)-2K\rho+2K'\rho';$$

and by substituting $\cos.^2(\theta'+\varepsilon)=1-\sin.^2(\theta'+\varepsilon)$, we get,

[8225c]
$$n^2.\left(1+\frac{4K}{n^2}.\rho\right).\sin.^2(\theta'+\varepsilon)=n^2+2K\rho+2K'\rho';$$

whence we easily deduce the value of $\sin.(\theta'+\varepsilon)$ [8225].

[8225d] If the velocity $\frac{ds'}{dt}$ be supposed to vanish before the particle attain the surface which

$$\left\{\frac{1+\frac{2K}{n^2}\,\rho+\frac{2K'}{n^2}\cdot\rho'}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}. \qquad \left[\text{Limit of } \sin.(\theta+\varepsilon')\right]. \qquad [8225]$$

If the particle passes from the first body into the second, it is evident, from [8225h, &c.], that at the distance s' from the surface, we shall have,

Motion within the second body, at a small distance from the surface.

$$\frac{ds'^2}{dt^2}=n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\theta'+\varepsilon)-2K\rho+2K'\rho'-2\rho.\textstyle\int ds'.\Pi_1(s')+2\rho'.\int ds'.\Psi_1(s'). \qquad [8226]$$

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separates the two bodies, the integrals [8225a] will not vanish, and we may assume for the value of $\sin.^2(\theta'+\varepsilon)$ the following expression, which is similar in form to that in [8225], with the addition of a term depending on x;

$$\sin.^2(\theta'+\varepsilon)=\frac{1+\frac{2K}{n^2}\cdot\rho+\frac{2K'}{n^2}\cdot\rho'+\frac{2\text{x}}{n^2}}{1+\frac{4K}{n^2}\cdot\rho}, \quad \text{or} \quad \cos.^2(\theta'+\varepsilon)=\frac{\frac{2K}{n^2}\cdot\rho-\frac{2K'}{n^2}\cdot\rho'-\frac{2\text{x}}{n^2}}{1+\frac{4K}{n^2}\cdot\rho}. \qquad [8225e]$$

Substituting this value of $\cos.^2(\theta'+\varepsilon)$ in [8223], putting $\frac{ds'}{dt}=0$, and neglecting the terms which mutually destroy each other, then dividing by 2, and transposing —x, we get,

$$\text{x}=\rho.\textstyle\int_0^{s'}ds'.\Pi_1(s')-\rho'.\int_0^{s'}ds'.\Psi_1(s'). \qquad [8225f]$$

Substituting this value of x in the expression of $\sin.^2(\theta'+\varepsilon)$ [8225e], we find that whenever x is negative, the expression of $\sin.(\theta'+\varepsilon)$ *will be less than that in* [8225], [8225g]
instead of being always greater or equal to it, as is asserted in [8224]. If the particle of light pass into the second body, the value ds' becomes negative, and for equal values of s' we have, as in [8209d], $\Pi_1(-s')=\Pi_1(s')$; so that the integrals $\int_0^{s'}ds'.\Pi_1(s')$, [8225h]
$\int_0^{s'}ds'.\Psi_1(s')$, change their signs without altering their numerical values, as in [8209d]; by this means the expression [8223] becomes as in [8226]. At a sensible distance from the surface, within the second body, we have $\int_0^{s'}ds'.\Pi_1(s')=K$ [8162], $\int_0^{s}ds'.\Psi_1(s')=K'$, [8222']; substituting these in [8226], it becomes as in [8227]; and by putting [8225i]
$\cos.^2(\theta'+\varepsilon)=1-\sin.^2(\theta'+\varepsilon)$, it changes [8227] into,

$$\frac{ds'^2}{dt^2}=n^2.\left(1+\frac{4K'}{n^2}\cdot\rho'\right)-n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\sin.^2(\theta'+\varepsilon). \qquad [8225k]$$

The value of $\frac{dx^2}{dt^2}$ remains always the same as in [8199 or 8216]; and by adding it to the expression [8216], we get $\frac{dx^2+ds'^2}{dt^2}=n^2.\left(1+\frac{4K'}{n^2}\cdot\rho'\right)=n^2+4K'\rho'$, which represents [8225$l$]
the square of the whole velocity of the ray of light, after it has penetrated a sensible distance within the second body, being the same as if the ray had passed directly into the second body, without passing through the first, as is evident from [8182], changing ρ into [8225m]
ρ', and K into K', so as to make it correspond to the second body. The same holds good whatever be the number of intervening bodies, as is observed by the author in [8235].

At a sensible distance within the surface of the second body, we shall have,

Sensibly within the second body.

as in [8225i],

[8227] $$\frac{ds'^2}{dt^2}=n^2.\left(1+\frac{4K}{n^2}\cdot\rho\right).\cos.^2(\vartheta'+\varepsilon)-4K\rho+4K'\rho'.$$

The particle will therefore be reflected whenever $\sin.(\vartheta'+\varepsilon)$ *is equal to or*

[8228] *greater than** $\left\{\dfrac{1+\frac{4K'}{n^2}\cdot\rho'}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}$; *which requires* $K'\rho'$ *to be less than* $K\rho$.

[8228'] *When* $\sin.(\vartheta'+\varepsilon)$ *is included between that limit and the quantity*

[8229] $\left\{\dfrac{1+\frac{2K}{n^2}\cdot\rho+\frac{2K'}{n^2}\cdot\rho'}{1+\frac{4K}{n^2}\cdot\rho}\right\}^{\frac{1}{2}}$ [8225], *the ray will still be reflected after penetrating the second body. When* $\sin.(\vartheta'+\varepsilon)$ *exceeds this last limit, the*

[8230] *ray will yet be reflected, but without penetrating the second body. If this last body naturally absorbs the light, the ray can be reflected only in the second manner; and then the observation of the limit in which it ceases to be reflected, will determine the value of* $K'\rho'$, *consequently that of the refractive power of the second body. Thus we may, by experiment, determine the refractive power even of opake bodies.*

Refractive power of opake bodies.

When a ray of light passes through several mediums, terminated by parallel

[8230'] *plane surfaces, it is evident, from the preceding anlysis,*

First. That the square of the velocity, perpendicular to the surface in the

[8231] *first medium, is increased by the quantity* Q *depending upon the action of this medium upon light.*

Second. That after passing from the first medium, and penetrating to a

[8232] *sensible distance into the second, the square of this velocity is increased by*

Refraction of light in

the difference $Q^{(1)}-Q$ *between the action of the second and the first*

[8232'] *medium; and so on for others.*

passing through several mediums.

Hence it follows that for a number of mediums, terminated by parallel

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* (3855) Putting $\frac{ds'}{dt}=0$ in [8225k], we get the value of $\sin.(\vartheta'+\varepsilon)$ [8228]. For

[8228a] this to represent a real angle, it must be equal to, or less than unity, which requires that the numerator should be equal to, or less than the denominator, or $K'\rho'<K\rho$, as in [8228'].

[8228b] The limit mentioned in [8229] is the same as in [8225], which may be a little modified by the term x, as in [8225e].

surfaces, and represented by $i+1$, the increment of this square is $Q^{(i)}$; therefore it is the same as if the light penetrate instantly into the last [8233]
medium; and as the square of the horizontal velocity, or that which is in*

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* (3856) Before the ray of light enters into the first body, the square of its velocity [8233a]
is represented by $\frac{dx^2+ds'^2}{dt^2}=n^2$ [8162', 8164d]. The direction of the ray forms the angle θ [8183], with the perpendicular to the surface; hence it is evident that its velocity, parallel to the surface, is $\frac{dx}{dt}=n.\sin.\theta$; and its velocity perpendicular to the surface [8233b]
is $\frac{ds'}{dt}=n.\cos.\theta$. The sum of the squares of these expressions, $\frac{dx^2+ds'^2}{dt^2}=n^2$, is [8233c]
changed into $\frac{dx^2+ds'^2}{dt^2}=n^2+4K\rho$ [8182], after entering the first surface of the first body. [8233d]
Now as $\frac{dx}{dt}$ is constant [8158o, 8177i, &c.], it is evident that the quantity $4K\rho$, which we shall represent by $Q=4K\rho$, expresses the increment of $\frac{ds'^2}{dt^2}$, arising from the action of [8233e]
the first surface upon the ray, while passing completely through it, till it gets at a sensible distance from its surface. Moreover it is plain that if the first body were taken away, and [8233f]
the second placed in its stead, the quantities K, ρ [8162, 8160], will change into K', ρ' [8222', 8221], respectively; and the same formula [8182] will give $\frac{dx^2+ds'^2}{dt^2}=n^2+4K'\rho'$; so that the increment of $\frac{ds'^2}{dt^2}$ in passing *directly into the second* [8233g]
body, without passing through the first, will be $4K'\rho'$, which we shall represent by $4K'\rho'=Q^{(1)}$. On the other hand, *if the ray pass through the first body, and then enter* [8233h]
into the second a sensible distance, the increment of $\frac{ds'^2}{dt^2}$ will be had, by subtracting the [8233i]
value of $\frac{ds'^2}{dt^2}=n^2.\cos.^2\theta$ [8233c], before entering the first body, from the expression [8233k]
[8227], after entering the second body a sensible distance, putting $\varepsilon'=0$, because all the surfaces are parallel [8230']. *Hence the increment of velocity, acquired in passing through the first body, and then entering into the second body a sensible distance, will be expressed by,*

$$n^2.\left(1+\frac{4K}{n^2}.\rho\right).\cos.^2\theta'-4K\rho+4K'\rho'-n^2.\cos.^2\theta=\text{increment of }\frac{ds'^2}{dt^2}. \qquad [8233l]$$

Substituting $\cos.^2\theta'=1-\sin.^2\theta'$ in the first member of this equation, and rejecting the terms which destroy each other, it becomes, by a different arrangement of the terms,

$$n^2.(1-\cos.^2\theta)-n^2.\left(1+\frac{4K}{n^2}.\rho\right).\sin.^2\theta'+4K'\rho'; \qquad [8233m]$$

and by substituting the value of $\sin.\theta'$ [8187], it changes into,

$$n^2.(1-\cos.^2\theta)-n^2.\sin.^2\theta+4K'\rho', \qquad [8233n]$$

the direction of the parallel surfaces, remains always the same, we see that,
[8234] *in these different mediums, the velocity of light is the same as if it enter immediately into each of them; and its direction is parallel to what it would be in this last case.*

In general, whatever be the number of mediums through which the light
[8235] *passes to enter a body, and whatever be the inclinations of the surfaces to each other, the velocity of the light, in this last body, will always be the same as if it pass directly into it from the first medium* [8225*m*].

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or simply $4K'\rho'$, by rejecting the terms which mutually destroy each other; being the same as is found in [8233*h*], *when the ray enters directly into the second body, without*
[8233*o*] *passing through the first body.* Similar results will be obtained with several successive planes, which is conformable to the statement of the author in [8232—8235]. Hence it
[8233*p*] follows, that if the refrangible power $\frac{4K}{n^2}$ be constant, as is the case with strata of air of different densities [8264*m*], the refraction of *the ray, after passing through several*
[8233*q*] *successive parallel plane strata of different densities, will be the same as if the incident ray pass directly into the last stratum, without going through the intermediate ones.* Now we have seen, in [8187, &c.], that if a ray of light, with an angle of incidence θ, pass
[8233*r*] into a stratum of the density ρ, with the angle of refraction θ', we shall have, by using the
[8233*s*] abridged symbol $k=\frac{4K}{n^2}$ [8192*b*], $\sin.\theta'=\frac{\sin.\theta}{\sqrt{1+k\rho}}$ [8187]. If the density of the stratum be ρ', and the angle of refraction $\theta_{,}$, with the same angle of incidence θ, the
[8233*t*] preceding expression will change into $\sin.\theta_{,}=\frac{\sin.\theta}{\sqrt{1+k\rho'}}$. Dividing this by the expression
[8233*u*] of $\sin.\theta'$, we get, by a slight reduction, $\sin.\theta_{,}=\frac{\sqrt{1+k\rho}}{\sqrt{1+k\rho'}}.\sin.\theta'$; which expresses the relation of θ', $\theta_{,}$; supposing θ' to be the angle of incidence of a ray moving in a medium
[8233*v*] of the density ρ, and refracted into the angle $\theta_{,}$, upon passing into a similar medium of the density ρ'. If we now suppose the density ρ' to differ from ρ by an infinitely small quantity, we shall have $\rho'=\rho+d\rho$, and $\theta_{,}=\theta'+d\theta'$; or,

[8233*w*] $$\sin.\theta_{,}=\sin.\theta'+d\theta'.\cos.\theta'=(1+d\theta'.\text{cotang}.\theta').\sin.\theta', \quad [60]\text{ Int.}$$

Substituting these values in [8233*u*], and dividing by $\sin.\theta'$, we get,

[8233*x*] $$1+d\theta'.\text{cotang}.\theta'=\frac{\sqrt{1+k\rho}}{\sqrt{1+k\rho+kd\rho}}=1-\frac{\frac{1}{2}kd\rho}{1+k\rho}; \quad \text{or,} \quad d\theta'.\text{cotang}.\theta'=-\frac{\frac{1}{2}kd\rho}{1+k\rho}.$$

Multiplying this last expression by $\text{tang}.\theta'$, we get,

[8233*y*] $$d\theta'=-\frac{\frac{1}{2}kd\rho}{1+k\rho}.\text{tang}.\theta';$$

which gives the refraction $d\theta'$, *in passing from a stratum of the density* ρ, *with an*
[8233*z*] *angle of incidence* θ', *into another similar stratum of the density* $\rho+d\rho$; *this refraction being proportional to the variation of density* $d\rho$, *as in* [8264*m*].

3. *We shall now represent by* ρ, *the density of a stratum of the atmosphere whose radius is* r. In the calculation of the action of this stratum upon the light, we may consider it as a plane, on account of the small extent of this action in comparison with the radius of the earth. The density of an inferior stratum at the distance s, is,* [8236]

$$\rho - s.\frac{d\rho}{dr} + \frac{s^2}{1.2}.\frac{d^2\rho}{dr^2} - \&c. = \text{density at the distance } (r-s). \qquad [8237]$$

The action of this last stratum upon a particle placed at the distance r from the earth's centre, is,

$$\Pi(s).\left\{\rho - s.\frac{d\rho}{dr} + \frac{s^2}{1.2}.\frac{d^2\rho}{dr^2} - \&c.\right\} = \text{action of the lower stratum.} \qquad [8238]$$

The action of a superior stratum at the distance s from the same particle, is,

$$\Pi(s).\left\{\rho + s.\frac{d\rho}{dr} + \frac{s^2}{1.2}.\frac{d^2\rho}{dr^2} - \&c.\right\} = \text{action of the upper stratum.} \qquad [8239]$$

The difference of these two forces [8238, 8239], is,

$$-2\Pi(s).\left\{s.\frac{d\rho}{dr} + \frac{s^3}{1.2.3}.\frac{d^3\rho}{dr^3} + \&c.\right\} = \text{whole action of the two strata.} \qquad [8240]$$

We must multiply this difference by ds, and integrate it from $s=0$ to $s=\infty$, to obtain the whole force with which the atmosphere attracts the luminous body towards the *earth's centre* [8237d], or the value of φ [8139′]†. Now we have, as in [8161′], [8241]

$$\Pi_1(s) = \int_s^{\infty} ds.\Pi(s)\,; \qquad [8242]$$

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* (3857) It is assumed, in the hypothesis in [8137^{v}], that the density ρ is a function of r; and when r changes into $r-s$, the density ρ becomes as in [8237], as is evident from Taylor's theorem [617]. Multiplying this by $\Pi(s)$, as in [8161], we obtain the action of this lower stratum, drawing downwards, as in [8238]; and by changing in it $-s$ into s, and putting, as in [8209d], $\Pi(-s)=\Pi(s)$, we obtain the action of the upper stratum [8239] drawing upwards; subtracting this from the former [8238], we get the whole action [8240], *towards the centre of the earth*, of the two strata, situated at the distance s from the particle of light, the one above, the other below. From this we obtain the whole atmospherical action by the process of integration mentioned in [8241]. [8237a] [8237b] [8237c] [8237d]

† (3858) Multiplying [8240] by ds, and integrating from $s=0$ to $s=\infty$, we have the following expression of the attractive force φ [8139′, 8241];

$$\varphi = -2.\frac{d\rho}{dr}.\int_0^{\infty} sds.\Pi(s) - \tfrac{2}{6}.\frac{d^3\rho}{dr^3}.\int_0^{\infty} s^3ds.\Pi(s) - \tfrac{2}{120}.\frac{d^5\rho}{dr^5}.\int_0^{\infty} s^5ds.\Pi(s) - \&c.\,; \qquad [8241a]$$

and we shall see, in [8246—8251], that all the terms may be neglected except the first, on account of their smallness. [8241b]

[8243] the integral being taken from $s=s$ to $s=\infty$. Therefore by taking the integral from $s=0$ to $s=s$, we shall have,*

[8244] $$\int_0^s ds.\Pi(s) = \text{constant}-\Pi_1(s).$$

Hence we easily deduce,

[8245] $$\int_0^s sds.\Pi.(s) = -s.\Pi_1(s)+\int_0^s ds.\Pi_1(s).$$

[8246] The function $s.\Pi_1(s)$ vanishes when† $s=0$; it also vanishes when s is infinite; for the function $\Pi_1(s)$ is then infinitely small, and infinitely less than $\frac{1}{s}$, since the action of bodies on light is insensible at very small distances. Therefore we have, by taking the integral from $s=0$ to $s=\infty$,

[8247] $$\int_0^\infty sds.\Pi(s) = \int_0^\infty ds.\Pi_1(s) = K;$$

[8248] K being in this case the same as in [8162]. The terms $\frac{1}{6}.\int_0^\infty s^3ds.\Pi(s)$, $\frac{1}{120}.\int_0^\infty s^5ds.\Pi(s)$, &c., may be neglected in comparison with $\int_0^\infty sds.\Pi(s)$, by reason of the small extent of the action of the bodies upon light. For an

[8249] example, we shall suppose that this action is represented by $\Pi(s)=Q.c^{-is}$;

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* (3859) We shall put $C=\int_0^\infty ds.\Pi(s)$, C being a constant quantity. Subtracting

[8245a] from it the expression [8242], we get $C-\Pi_1(s)=\int_0^s ds.\Pi(s)$; the limits of this last integral being 0 and s, as in [8244]. The differential of this being multiplied by s, changing the order of the terms, gives $sds.\Pi(s)=-s.d.\Pi_1(s)$. Integrating this last

[8245b] expression, it becomes as in [8245], as is easily proved by taking its differential, and neglecting the terms which mutually destroy each other.

† (3860) It is supposed that $\Pi_1(s)$ does not become infinite when $s=0$. This

[8246a] condition can be satisfied in various manners, as, for example, by the form of $\Pi(s)=Q.c^{-is}$, assumed in [8249], Q being constant. Multiplying this by ds, and

[8246b] integrating, we get $\int_0^s ds.\Pi(s)=\frac{Q}{i}-\frac{Q}{i}.c^{-is}$, as is evident by taking the differential of its second member. This vanishes when $s=0$, or $c^{-is}=1$; and when $s=\infty$ it becomes $\int_0^\infty ds.\Pi(s)=\frac{Q}{i}$; subtracting the first of these integrals from the second, we

[8246c] obtain $\int_s^\infty ds.\Pi(s)=\frac{Q}{i}.c^{-is}$; comparing this with [8161′], we obtain $\Pi_1(s)=\frac{Q}{i}.c^{-is}$.

[8246d] Hence $s.\Pi_1(s)=\frac{Q}{i}.\frac{s}{c^{is}}=\frac{Q}{i}.\frac{s}{1+is+\frac{1}{2}i^2s^2+\&c.}$, as is evident by developing c^{is}, in a series, by [55] Int. Now when $s=0$, we evidently have $\frac{s}{1+is+\frac{1}{2}i^2s^2+\&c.}=0$; and when $s=\infty$, the denominator is infinitely greater than the numerator, and in this case it also becomes 0, as in [8246]. Therefore if we take the limits of the integrals in [8245],

[8246e] from $s=0$ to $s=\infty$, we may neglect the term $-s.\Pi_1(s)$, and it will become as in [8247], $\int_0^\infty ds.\Pi_1(s)$; this last expression is equal to K [8162].

Q being a constant quantity, c the number whose hyperbolic logarithm is 1, and i *a very great number*, which renders c^{-is} insensible at a very small distance. The integrals $\int_0^\infty s ds.c^{-is}$, $\frac{1}{6}.\int_0^\infty s^3 ds.c^{-is}$, &c., become [8250]
$\frac{1}{i^2}$, $\frac{1}{i^4}$, &c.;* hence we see that $\frac{1}{6}.\int_0^\infty s^3 ds.\Pi(s)$, $\frac{1}{120}.\int_0^\infty s^5 ds.\Pi(s)$, are [8251]
insensible in comparison with $\int_0^\infty s ds.\Pi(s)$; and it is evident that the same takes place for every other function, which renders the action upon the light insensible at very small distances. Whence it follows that,†

$$\varphi = -2K.\frac{d\rho}{dr}. \qquad [8252]$$

Hence we have, as in [8252b],

$$\int_a^r \varphi dr = 2K.\{(\rho)-\rho\}; \qquad [8253]$$

(ρ) *being the density of the stratum of the atmosphere whose radius is* a. [8254]

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* (3861) We easily find, by taking the differential, that,

$$\frac{1}{n}.\int_0^s s^n ds.c^{-is} = -\frac{1}{ni}.\frac{s^n}{c^{is}} + \frac{1}{i}.\int_0^s s^{n-1} ds.c^{-is}. \qquad [8250a]$$

Now we find, by developing in like manner as in the last note [8246d], that $\frac{s^n}{c^{is}}$ vanishes when $s=0$, or when $s=\infty$; hence we have, between these limits,

$$\frac{1}{n}.\int_0^\infty s^n ds.c^{-is} = \frac{1}{i}.\int_0^\infty s^{n-1} ds.c^{-is}. \qquad [8250b]$$

In the case of $n=0$, we have $\int_0^s ds.c^{-is} = \frac{1}{i} - \frac{c^{-is}}{i}$, as appears by putting $Q=1$, in [8246b]; and when $s=\infty$, it becomes simply $\int_0^\infty ds.c^{-is} = \frac{1}{i}$. Now putting [8250c]
$n=1$ in [8250b], and using the expression [8250c], we obtain,

$$\int_0^\infty s ds.c^{-is} = \frac{1}{i}.\int_0^\infty ds.c^{-is} = \frac{1}{i^2}; \qquad [8250d]$$

putting also $n=3$, $n=2$, in [8250b], we get, by dividing by 2,

$$\tfrac{1}{6}.\int_0^\infty s^3 ds.c^{-is} = \frac{1}{2i}.\int_0^\infty s^2 ds.c^{-is} = \frac{1}{i^2}.\int_0^\infty s ds.c^{-is} = \frac{1}{i^4}. \qquad [8250e]$$

Again, the values $n=5$, $n=4$, give, in [8250b], by dividing by 24,

$$\tfrac{1}{120}.\int_0^\infty s^5 ds.c^{-is} = \frac{1}{24i}.\int_0^s s^4 ds.c^{-is} = \frac{1}{6i^2}.\int_0^s s^3 ds.c^{-is} = \frac{1}{i^6}, \text{ \&c.} \qquad [8250f]$$

This agrees with [8250, &c.].

† (3862) Substituting in the first term of φ [8241a] the value of the integral K [8247], and neglecting the other terms as in [8241b], we get the expression of φ [8252]. [8252a]
Multiplying it by dr, and integrating, we obtain $\int\varphi dr = -2K\int d\rho = 2K.\{\text{constant}-\rho\}$. If we take the constant so that the integral may vanish when $r=a$, and $\rho=(\rho)$, as in [8152], it becomes as in [8253]. [8252b]

[8255] When r is infinite, ρ is nothing, and the equation [8141'] gives,*

[8256] $$r^2dv=\frac{cdr}{\sqrt{q^2-4K(\rho)}}.$$

[8257] We have $r^2dv=cdt$ [8143], and when r is infinite we also have† $dr=ndt$; hence we obtain,

[8258] $$\frac{n}{\sqrt{q^2-4K(\rho)}}=1;$$

consequently,

[8259] $$q=n.\sqrt{1+\frac{4K}{n^2}.(\rho)}.$$

[8260] Substituting this value of q in [8155], we get,

[8261] $$\frac{c}{a}=n.\sqrt{1+\frac{4K}{n^2}.(\rho)}.\sin.\Theta;$$

hence the expression of $d\delta$ [8147] becomes,‡

[8256a] * (3863) Multiplying [8141'] by r^2, then putting $\frac{c^2}{r^2}=0$, $\rho=0$ [8255], $\int_a^r\varphi dr=2K(\rho)$ [8253], corresponding to $r=\infty$, it becomes as in [8256].

[8257a] † (3864) When r is infinite, the line $CA=r$, fig. 106, page 444, becomes parallel to the extreme tangent EAG, so that dr will represent the space passed over by the
[8257b] particle of light in the time dt, and as the velocity is n [8162'], this space must be represented by ndt, as in [8257]. Substituting this and $r^2dv=cdt$ [8257], in [8256],
[8257c] and then dividing by cdt, we get [8258]; whence we deduce the value of q [8259]; and by substituting it in [8155], we get [8261].

‡ (3865) Subtracting $2\int\varphi dr$ [8253] from q^2 [8259], we get,

[8262a] $$q^2-2\int\varphi dr=n^2.\left(1+\frac{4K}{n^2}.\rho\right).$$

[8262b] Multiplying [8261] by $\frac{a}{r}$ we obtain $\frac{c}{r}=n.\sqrt{1+\frac{4K}{n^2}.(\rho)}.\frac{a}{r}.\sin.\Theta$; lastly we have, in [8252], $\varphi dr=-2K.d\rho$. Substituting these in [8147], it becomes as in [8262]; the numerator and denominator being divided by n^3; and this is the fundamental equation for finding the refraction of the heavenly bodies.

[8262c] The expression of the refraction $d\delta$ [8262], although it is investigated upon the principles of the Newtonian theory of emission, may be considered as founded upon observation alone,
[8262d] without any necessary connection with that theory, or with the theory of undulation. For we shall show, in [8262t], that this formula can be derived from that in [8187 or 8192c]; and that this last formula is nothing more than the analytical expression of *the well known*

$$d\delta = -\frac{\frac{2K}{n^2}\cdot d\rho . \sqrt{1+\frac{4K}{n^2}\cdot(\rho)}\cdot\frac{a}{r}.\sin.\Theta}{\left(1+\frac{4K}{n^2}\cdot\rho\right)\cdot\sqrt{1+\frac{4K}{n^2}\cdot\rho-\left(1+\frac{4K}{n^2}\cdot(\rho)\right)\cdot\frac{a^2}{r^2}.\sin.^2\Theta}}. \quad (3)$$ Fundamental differential equation of the refraction of light. [8262]

results of observation, namely; first, that the ratio of the sine of the angle of incidence to the sine of the angle of refraction, in passing from one homogeneous medium into another of a different but given density, is constant; [8262e] *second, that the refractive forces of parallel strata of air, of different densities, are very nearly proportional to the densities* [8261m]. [8262f]

We shall therefore proceed to the investigation of the differential of the refraction $d\delta$ [8262], by means of the formula $\sin.\delta=\sqrt{1+k\rho}.\sin.\delta'$ [8192c], considering it as a result of observation [8262e]. [8262g] For this purpose we shall suppose the ray to return, as in [8137v], from the eye of the observer towards the heavenly body, in the direction OAA', fig. 106, page 444; and that at the point A it quits a stratum, whose density is ρ, to enter into a second stratum $LAFK$, whose density is $\rho+d\rho$. [8262h] The angle of incidence at the point A, which is situated in the surface LA, separating the first from the second stratum, is represented by v' [8140″]; and this angle is increased, upon entering into the second stratum, by the increment of the refraction $d\delta$ [8146h]; so that the angle of refraction becomes $FAA'=v'+d\delta$. The differential of $v+v'=\delta$ [8144] gives [8262i] $dv'=d\delta-dv$. Multiplying this by $\text{cotang.}v'$, and putting, in the last term of the second member of the product, $\text{cotang.}v'=\frac{dr}{rdv}$ [8145], we get, [8262k]

$$dv'.\text{cotang.}v'=d\delta.\text{cotang.}v'-\frac{dr}{r}. \qquad [8262l]$$

We have deduced the equation [8233y] from the assumed formula [8262g, &c.], and if we change the angle of incidence δ' [8233y] into v', and the refraction $d\delta'$ into $d\delta$, to conform to the present notation, it becomes,

$$d\delta=-\frac{\frac{1}{2}k\,d\rho}{1+k\rho}.\text{tang.}v'; \qquad [8262m]$$

and by substituting it in [8262l], we get,

$$dv'.\text{cotang.}v'=-\frac{\frac{1}{2}k\,d\rho}{1+k\rho}-\frac{dr}{r}. \qquad [8262n]$$

Integrating this expression by [59] Int., and adding the constant quantity $\log.C$, we obtain,

$$\log.\sin.v'=\log.C-\log.\sqrt{1+k\rho}-\log.r; \qquad \text{or,} \qquad \sin.v'=\frac{C}{r.\sqrt{1+k\rho}}; \qquad [8262o]$$

The value of C is easily found, by observing that at the point O, fig. 106, page 444, on the earth's surface, we have $r=a$, $\rho=(\rho)$, $v'=\Theta$ [8138—8140″]; substituting [8262p] these values in the second of the equations [8262o], we get, by a slight reduction, $C=\sqrt{1+k(\rho)}.a.\sin.\Theta$; hence the expression of $\sin.v'$ [8262o], becomes as in [8262r]; [8262q] whence we easily deduce the value of $\text{tang.}v'$ [8262s];

[8263] *In this equation it is supposed that the refractive forces of the strata of the atmosphere are proportional to the densities of its strata*, which follows from

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[8262r] $$\sin.v' = \frac{\sqrt{1+k(\rho)}.a.\sin.\Theta}{r.\sqrt{1+k\rho}};$$

[8262s] $$\text{tang}.v' = \frac{\sin.v'}{\sqrt{1-\sin.^2 v'}} = \frac{\sqrt{1+k(\rho)}.a.\sin.\Theta}{\sqrt{r^2.(1+k\rho)-\{1+k(\rho)\}.a^2.\sin.^2\Theta}} = \frac{\sqrt{1+k(\rho)}.\frac{a}{r}.\sin.\Theta}{\sqrt{1+k\rho-\{1+k(\rho)\}.\frac{a^2}{r^2}.\sin.^2\Theta}}.$$

Substituting this last value of $\text{tang}.v'$ in the expression of $d\delta$ [8262m], we obtain,

[8262t] $$d\delta = \frac{-\tfrac{1}{2}kd\rho.\sqrt{1+k(\rho)}.\frac{a}{r}.\sin.\Theta}{(1+k\rho).\sqrt{1+k\rho-\{1+k(\rho)\}.\frac{a^2}{r^2}.\sin.^2\Theta}};$$

and by replacing the expression of $k=\frac{4K}{n^2}$ [8192b], it becomes as in [8262].

This may be reduced to a more simple form by putting,

[8262u] $$\text{u} = \left\{\frac{1+k(\rho)}{1+k\rho}\right\}^{\frac{1}{2}}, \qquad \text{or} \quad \text{u}^2 = \frac{1+k(\rho)}{1+k\rho};$$

for its differential, considering u, ρ, as the variable quantities, is,

[8262v] $$d\text{u} = \frac{-\tfrac{1}{2}k\,d\rho}{(1+k\rho)}.\left\{\frac{1+k(\rho)}{1+k\rho}\right\}^{\frac{1}{2}};$$

and by substituting this in [8262t], after dividing its numerator and denominator under the radical by $\sqrt{1+k\rho}$, we get, without any reduction, the formula [8262w]; and from this we easily deduce the formula [8262x],

[8262w] $$d\delta = \frac{d\text{u}.\frac{a}{r}.\sin.\Theta}{\sqrt{1-\frac{a^2}{r^2}.\text{u}^2.\sin.^2\Theta}}$$

[8262x] $$= \frac{a.\sin.\Theta.d\text{u}}{\sqrt{r^2-a^2.\text{u}^2.\sin.^2\Theta}};$$

which is similar to the form proposed by La Grange, and used by Plana and others. With a slight modification we may deduce from it the formula of Kramp. This formula becomes integrable, with circular arcs, by putting $\frac{a}{r}=\text{u}^{-2m}$, as La Grange observed, and as we

[8262y] shall see in [8372—8376]. If we suppose $\frac{a}{r} = A\text{u}^{-m}+(1-A).\text{u}^{-2m-1}$, A being a constant quantity, we can obtain the integral of [8262w] by means of elliptical functions,

[8262z] as Plana has shown in a memoir, given in vol. 27 of the *Memorie della reale Accademia delle Scienze di Torino.*

[8262z′] The same value of u [8262v] being substituted in [8262r], gives $\sin.v' = \frac{a}{r}.\text{u}.\sin.\Theta$;

the experiments of Hauksbee. However it is possible that this may not be rigorously exact, and it will be useful to make a greater number of [8263′]

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hence,

$$\text{tang.}v' = \frac{\sin.v'}{\sqrt{1-\sin.^2 v'}} = \frac{\frac{a}{r}.\text{u}.\sin.\Theta}{\sqrt{1-\frac{a^2}{r^2}.\text{u}^2.\sin.^2\Theta}}; \qquad [8263a]$$

substituting this in $dv = \frac{dr}{r}.\text{tang.}v'$ [8145], we get, [8263b]

$$dv = \frac{\frac{a.dr}{r^2}.\text{u}.\sin.\Theta}{\sqrt{1-\frac{a^2}{r^2}.\text{u}^2.\sin.^2\Theta}} \qquad [8263c]$$

$$= \frac{adr.\text{u}.\sin.\Theta}{r.\sqrt{r^2-a^2.\text{u}^2.\sin.^2\Theta}}. \qquad [8263d]$$

Dividing [8262x] by [8263d], and multiplying the result by dv, we get,

$$d\theta = \frac{d\text{u}}{dr}.\frac{r}{\text{u}}.dv. \qquad [8263e]$$

These formulas will be of use hereafter.

Before closing this note we may observe, that the equation [8233u], upon which the preceding demonstration is founded, cannot, *in the utmost strictness, be considered as a direct result of observation, unless we neglect terms depending upon the square and higher powers of $k\rho$; because the terms depending on these powers have not in fact been ascertained by experiment.* For the results of different sets of observations differ from each other $\frac{1}{103}$ part in [8264f], and $\frac{1}{546}$ part in [8264i]. Now $\frac{1}{2}k\rho$ is less than 0,0003 [8277], and its square is less than $0{,}0003.\frac{1}{2}k\rho$, or less than $\frac{1}{3333}$ part of $\frac{1}{2}k\rho$; and as $\frac{1}{3333}$ is much less than $\frac{1}{103}$ or $\frac{1}{546}$, it follows that the terms depending on the square or higher powers of $\frac{1}{2}k\rho$, are much less than the errors of observation. Hence it follows that instead of the fundamental equation $\sin.\theta_{,} = \left(\frac{1+k\rho}{1+k\rho'}\right)^{\frac{1}{2}}.\sin.\theta'$ [8233u], we may assume many other forms, in which the terms depending on the first power of ρ, ρ', are the same; as, for example, $\sin.\theta_{,} = \left(\frac{1+\frac{1}{2}k\rho}{1+\frac{1}{2}k\rho'}\right).\sin.\theta'$, $\sin.\theta_{,} = \left(\frac{1+\frac{1}{4}k\rho}{1+\frac{1}{4}k\rho'}\right)^2$, &c.; and yet satisfy fully the degree of accuracy which has been obtained in these observations of the refraction. We may moreover remark that terms of the order $(\frac{1}{2}k\rho)^2$ are omitted by La Place in several parts of this chapter, as, for example, in [8287, 8368, &c.]; and particularly in [8494], where the corrections for the barometer and thermometer are supposed to be proportional to $\alpha = \frac{\frac{1}{2}k(\rho)}{1+k(\rho)} = \frac{1}{2}k(\rho).\{1-k(\rho)+\&\text{c.}\}$ [8277′]; now this quantity is not accurately proportional to the density (ρ), unless we reject the quantities of the order k^2, k^3, &c.; and quantities of exactly the same kind are such as have been neglected in the preceding demonstration.

[8263f] [8263g] [8263h] [8263i] [8263k] [8263l] [8263m] [8263n]

[8264] experiments upon this point.* But, whatever be the result, we may always use the preceding equation, supposing that ρ represents the refractive force

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[8264a] *(3866) Since the publication of this volume, a very important and accurate set of experiments has been instituted, by Biot and Arago, to determine the refractive power of atmospherical air and of several of the gases, for different temperatures and densities, by [8264b] direct observations of the changes which take place in passing at a very oblique angle, through a receiver containing the air or gas, of any proposed density, upon which the experiment is to be made. The results of these experiments have been published by them [8264c] in the *Mémoires de l'Institut*, 1806, tome vii. page 315, &c.; where they have given the value of $\frac{2K}{n^2}\cdot(\rho)$, corresponding to each set of observations, reduced to the temperature of [8264d] melting ice, and to the height of the barometer $0^{\text{metre}},76$ [8274]. The extreme results of nine sets of observations, made on different days, each set containing from ten to thirty observations, are,

Results of observations on the refraction, by Biot and Arago;

[8264e]
$$\frac{2K}{n^2}\cdot(\rho)=0,000293904,\qquad\text{and}\qquad \frac{2K}{n^2}\cdot(\rho)=0,000296777;$$

[8264f] the difference between these extreme values is 0,000002873, being nearly $\frac{1}{103}$ part of the whole refraction. By neglecting two somewhat doubtful sets, they obtain, for the mean of the remaining seven sets,

[8264g]
$$\frac{2K}{n^2}\cdot(\rho)=0,0002945856=\tfrac{1}{2}k(\rho)\quad[8192b].$$

by Delambre.

The value of the same quantity, determined by Delambre from the mean of more than [8264h] 500 observations of the altitudes of circumpolar stars, is $\frac{2K}{n^2}\cdot(\rho)=0,0002940470$, as in [8277]; and from this we can deduce the expression of α [8277', 8285, 8505]. Moreover the difference between the results of the astronomical observations of Delambre, [8264i] [8264h], and the direct observations of Biot and Arago [8264g], is 0,0000005386, being about $\frac{1}{546}$ part of the whole value.

[8264k] In page 322 of the same volume, is given a table of the results of the observations of Biot and Arago, on the refraction of air of different densities, as indicated by the heights of the barometer in the receiver. These heights in nine sets of observations, each [8264l] set being formed of from ten to twenty single observations, were respectively 50, 210, 1200, 2425, 2830, 4055, 5260, 6130, 8007, ten thousandth parts of a metre; *and* [8264m] *for every one of these heights they observed the refraction of air to be proportional to the density;* the greatest refraction in these observations being $5^{m}\ 58^{s}$, and the greatest [8264n] variation from the calculation of the refraction, supposing it to be in proportion to the density, being only $2^{s},1$. They have also obtained similar results with several of the gases, as oxygen, hydrogen, carbonic acid, &c., as in page 320 of the same volume; and they finally conclude, from all their observations, *that in varying the density from the most perfect vacuum, which they could obtain, to that which corresponds to the common* [8264o] *pressure of the atmosphere, the refraction produced by any particular gas is always*

Refraction in any gas is strictly proportional to the density.

of the stratum of the atmosphere, whose radius is r. *We shall hereafter suppose this force to be proportional to the density of the stratum*, which cannot vary but very little from the truth. [8264′]

INTEGRATION OF THE DIFFERENTIAL EQUATION OF THE MOTION OF LIGHT.

4. *To integrate the equation* [8262] *we must determine the expression of* ρ *in a function of* r; *that is to say, the law of the decrease of the density of the strata, corresponding to the elevation above the level of the sea. The two limits of this law are; first, a constant density; second, a density decreasing in geometrical progression, while the altitude increases in arithmetical progression; and we shall see, in* [8283, 8299, &c.], *that this last limit supposes a uniform temperature throughout the whole atmosphere.* We shall therefore consider the refraction in these two extreme cases. [8265]

Cassini's or first hypothesis of a uniform density of the air.

The hypothesis of a constant density amounts to the same thing as to suppose ρ *to vary only when infinitely near the external surface of the atmosphere.* We shall suppose at the external surface, $r = a + l$, putting, [8265′]

$$t^2 = \frac{1 + \frac{4K}{n^2} \cdot \rho}{\left\{ 1 + \frac{4K}{n^2} \cdot (\rho) \right\} \cdot \frac{a^2}{(a+l)^2} \cdot \sin.^2 \Theta} - 1 ; \qquad [8266]$$

and then the equation [8262] becomes,*

$$d\theta = -\frac{dt}{1+t^2} ; \qquad [8267]$$

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rigorously proportional to the density, without requiring the least modification whatever; observing, however, that different gases of the same density have different refractive powers. [8264p]

* (3867) The hypothesis of a uniform density is equivalent to that proposed by Cassini, who supposes the surface of the earth and the summit of the atmosphere to be concentric spherical surfaces, and that the whole refraction takes place at the upper surface, or upon the entering of the ray into the atmosphere. In this case the density changes from 0 to (ρ), while the ray passes through an insensible space near the summit of the atmosphere, and during this small moment of time r may be considered as constant, and equal to $a + l$; then if, for brevity, we put $\left\{ 1 + \frac{4K}{n^2} \cdot (\rho) \right\} \cdot \frac{a^2}{(a+l)^2} \cdot \sin.^2 \Theta = A^2$, [8267a]
also $\frac{4K}{n^2} = k$ [8192b], the expression [8262] will become $d\theta = -\frac{\frac{1}{2} k d\rho . A}{(1+k\rho) . \sqrt{1+k\rho - A^2}}$, [8267b]

which gives by integration,

[8268] $$\delta\theta = \text{ang.tang.}T - \text{ang.tang.}T';$$

consequently,

[8269] $$\text{tang.}\delta\theta = \frac{T-T'}{1+TT'};$$

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and $t^2 = \frac{1+k\rho}{A^2} - 1$ [8266]. This last equation gives $1+k\rho = A^2.(1+tt)$, and its

[8267c] differential, divided by -2, is $-\frac{1}{2}kd\rho = -A^2.t\,dt$. The same equation also gives $\sqrt{1+k\rho-A^2} = At$. Substituting these in $d\theta$ [8267b], it becomes,

[8267d] $$d\theta = -\frac{A^3 t\,dt}{A^3.(1+tt).t} = -\frac{dt}{1+tt},$$

as in [8267]. Its integral is $\theta = \text{constant} - \text{ang.tang.}t$ [51] Int.; and if we take the limits from $t = T$ to $t = T'$, as in [8271], it becomes as in [8268]. If we put

[8267e] $T = \text{tang.}a$, $T' = \text{tang.}b$, we get $\text{tang.}(a-b) = \frac{T-T'}{1+TT'}$ [30] Int.; substituting this in the tangent of the expression [8268], it becomes as in [8269]. The value of T^2 [8272], is

[8267f] deduced from t^2 [8266], by putting $\rho = (\rho)$; and the value of T'^2 [8272'], is deduced from the same value of t^2, by putting $\rho = 0$, as is observed in [8271].

The expression of $\delta\theta$ [8269] may be reduced to terms of Θ, α, l [8277', 8275], in the following manner. Putting for brevity,

[8267g] $$\lambda = \frac{l}{a}, \qquad e = \frac{(a+l)^2}{a^2.\sin.^2\Theta} = \frac{1+2\lambda+\lambda^2}{\sin.^2\Theta};$$

[8267h] and using also $\dfrac{1}{1+\frac{4K}{n^2}.(\rho)} = 1-2\alpha$ [8286a], we find that the values of T^2, T'^2, [8272, 8272'], become,

[8267i] $$T^2 = e-1; \qquad T'^2 = e.(1-2\alpha)-1 = (e-1)-2\alpha e = T^2-2\alpha e.$$

Extracting the square root of the last expression of T'^2, we get,

[8267k] $$T' = T - \frac{\alpha e}{T} - \frac{\alpha^2 e^2}{2T^3} - \&c.$$

Hence we obtain, by successive reductions, and observing that $1+T^2 = e$ [8267i],

[8267l] $$T-T' = \frac{\alpha e}{T}.\left\{1+\frac{\alpha e}{2T^2}+\&c.\right\};$$

[8267m] $$1+TT' = 1+T^2-\alpha e-\&c. = e-\alpha e-\&c. = e.(1-\alpha-\&c.).$$

Substituting these values in [8269], and writing $\delta\theta$ for its tangent, we obtain, by neglecting terms of the order α^3,

[8267n] $$\delta\theta = \frac{\alpha}{T}.\left\{1+\alpha+\frac{\alpha e}{2T^2}\right\}.$$

As this expression of $\delta\theta$ is of the order α, we may neglect terms of the order λ^2, in finding the value of T; and then substituting e [8267g], in T^2 [8267i], we obtain, by successive reductions,

observing that $\delta\theta$ *denotes the refraction, or the quantity which is to be added to* Θ, *to obtain the zenith distance of the body cleared from refraction,* [8270]
[8150]. We must also observe that the integral is taken from $\rho=(\rho)$ to $\rho=0$. *T is the value of* t, *at the origin of the curve, where* $\rho=(\rho)$, and [8271]
T' is its value at the other end, or at the summit of the atmosphere, where $\rho=0$; hence we have,

$$T^2=\frac{(a+l)^2}{a^2.\sin.^2\Theta}-1\,; \qquad [8272]$$

$$T'^2=\frac{(a+l)^2}{\left\{1+\frac{4K}{n^2}.(\rho)\right\}.a^2.\sin.^2\Theta}-1. \qquad [8272']$$

To deduce from these formulas the horizontal refraction, we must put [8273]
$\sin.\Theta=1$, and ascertain the values of l and $\frac{K}{n^2}.(\rho)$. At the level of the [8274]
sea and at the temperature of melting ice, the height of the barometer being

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$$T^2=\frac{1+2\lambda}{\sin.^2\Theta}-1=\frac{1-\sin.^2\Theta}{\sin.^2\Theta}+\frac{2\lambda}{\sin.^2\Theta}=\text{cotang.}^2\Theta+\frac{2\lambda}{\sin.^2\Theta} \qquad [8267o]$$

$$=\text{cotang.}^2\Theta.\left\{1+\frac{2\lambda}{\cos.^2\Theta}\right\}; \qquad [8267p]$$

$$T=\text{cotang.}\,\Theta.\left\{1+\frac{\lambda}{\cos.^2\Theta}\right\}; \qquad \frac{1}{T}=\text{tang.}\Theta.\left\{1-\frac{\lambda}{\cos.^2\Theta}\right\}. \qquad [8267q]$$

This expression of $\frac{1}{T}$ is to be substituted in [8267*n*]; and as the term of [8267*n*], depending on e, is of the order α^2, we may, in finding the value of this term, neglect λ in e [8267*g*], and put simply $e=\frac{1}{\sin.^2\Theta}$, also $\frac{1}{T^2}=\text{tang.}^2\Theta$ [8267*q*]; whence [8267r]
$\frac{e}{T^2}=\frac{\text{tang.}^2\Theta}{\sin.^2\Theta}=\frac{1}{\cos.^2\Theta}$; by this means the expression [8267*n*] becomes,

$$\delta\theta=\alpha.\text{tang.}\Theta.\left\{1-\frac{\lambda}{\cos.^2\Theta}\right\}.\left\{1+\alpha+\frac{\frac{1}{2}\alpha}{\cos.^2\Theta}\right\} \qquad [8267s]$$

$$=\alpha.\text{tang.}\Theta.\left\{1+\frac{\frac{1}{2}\alpha.(1+2.\cos.^2\Theta)-\lambda}{\cos.^2\Theta}\right\}. \qquad [8267t]$$

Re-substituting the value of λ [8267*g*], it becomes of exactly the same form as the expression [8474], which La Place has deduced from his assumed formula for the density [8267u]
of the atmosphere [8412]; so that this important and accurate formula, for computing the refraction for altitudes exceeding ten degrees, can be deduced from the simple process of [8267v]
Cassini, as well as from the more complicated hypothesis of La Place, as has been remarked by Mr. Ivory.

[8274′] $0^m,76$, we shall have, for the height l of a homogeneous atmosphere,*

[8275] $$l = 7974 \text{ metres};$$ [Height of a homogeneous atmosphere.]

[8276] which is the value deduced from a great number of observations of the height of mountains, determined by the barometer, and compared with their heights measured trigonometrically. By a great number of observations of the refraction at the same temperature, and at the same height of the barometer [8274′], it has been found that,

[8277] $$\frac{2K}{n^2}\cdot(\rho) = 0{,}000294047, \ [8264h]; \qquad \log.\frac{2K}{n^2}\cdot(\rho) = 6{,}4684168;$$

[8277′] $$\frac{\frac{2K}{n^2}\cdot(\rho)}{1+\frac{4K}{n^2}\cdot(\rho)} = 0{,}000293876 = \alpha, \ [8285, 8505]; \qquad \log.\alpha = 6{,}4681641.$$

Then we have, *for the radius of the earth*,†

[8278] $$a = 6366198 \text{ metres}; \qquad \log.a = 6{,}8038801.$$ [Radius of the earth.]

Horizontal refraction, the density being uniform.

By means of these values we find for the horizontal refraction, supposing the atmosphere to be of a uniform density,

[8279] $$\delta\theta = 3979'',5, \quad [21^m 29^s].$$ [Horizontal refraction in Cassini's hypothesis of a uniform density.]

[8280] *Although astronomers do not wholly agree with each other as to the value of the horizontal refraction, yet they all find it much greater under this pressure and temperature. The mean between their results is,*

[8277a] * (3868) In the state of the atmosphere mentioned in [8274], the mercury is about 10467 times as heavy as air, according to the observations of Biot and Arago, [8277b] [8264a, &c.]. Multiplying 10467 by $0^{\text{metre}},76$ [8274′], we obtain the height of a homogeneous atmosphere 7955 metres nearly, being about 19 metres less than the estimate of La Place [8275].

[8278a] † (3869) The whole arc of the meridian is put equal to 40000000 metres; and if we divide this by 6,28318.., we shall get the radius of the earth a [8278], supposing the [8278b] earth to be spherical. Substituting this value of a and those of l, $\frac{2K}{n^2}\cdot(\rho)$ [8275, 8277], in [8272, 8272′], we shall obtain the values of T, T'; and then from [8269] we get $\delta\theta$, as in [8279]. This expression differs very much from the result of observation [8278c] 6500″, as given by La Place in [8281]; or from that by Bradley's rule, 6418″ [8383m]. The values of l, a [8275, 8278], give,

[8278d] $$\frac{l}{a} = 0{,}00125256; \qquad \frac{a}{l} = 798{,}37; \qquad \log.\frac{l}{a} = 7{,}09779;$$

which are used hereafter.

$$\delta\theta = 6500'', \quad [=35^m 6^s] \quad [8278c]. \quad \text{[Observed horizontal refraction.]}$$ [8281]

Hence it appears that the supposition of a uniform density differs too much from observation of the refraction to be admitted as a true hypothesis. [8282]

5. *We shall now consider the hypothesis of a uniform temperature.* If we put, [8283]

Second hypothesis, or that of a uniform temperature of the air.

$$\frac{a}{r} = 1-s; \qquad \text{or,} \qquad r-a = rs;$$ [8284]

$$\alpha = \frac{\frac{2K}{n^2}\cdot(\rho)}{1+\frac{4K}{n^2}\cdot(\rho)} = 0{,}000293876, \quad [8277']; \quad \text{or} \quad \frac{K}{n^2} = \frac{1}{2(\rho)}\cdot\frac{\alpha}{1-2\alpha};$$ [8285]

the equation [8262] will become,*

$$d\delta = -\frac{\alpha.\frac{d\rho}{(\rho)}.(1-s).\sin.\theta}{\left\{1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)\right\}.\sqrt{\cos.^2\theta-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)+(2s-s^2).\sin.^2\theta}}; \quad (4)$$ [8286]

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* (3870) From the value of $\frac{K}{n^2}$ [8285], we get,

$$1+\frac{4K}{n^2}.\rho = 1+\frac{2\alpha}{1-2\alpha}\cdot\frac{\rho}{(\rho)}; \qquad 1+\frac{4K}{n^2}.(\rho) = \frac{1}{1-2\alpha}.$$ [8286a]

Substituting these values and that of $\frac{a}{r}$ [8284], in [8262], it becomes as in [8286b]; and by multiplying the numerator and denominator by $(1-2\alpha)^{\frac{3}{2}}$, it is reduced to the form [8286c];

$$d\delta = \frac{\frac{-\alpha}{1-2\alpha}\cdot\frac{d\rho}{(\rho)}\cdot\sqrt{\frac{1}{1-2\alpha}}.(1-s).\sin.\theta}{\left(1+\frac{2\alpha}{1-2\alpha}\cdot\frac{\rho}{(\rho)}\right).\sqrt{1+\frac{2\alpha}{1-2\alpha}\cdot\frac{\rho}{(\rho)}-\frac{1}{1-2\alpha}.(1-s)^2.\sin.^2\theta}}$$ [8286b]

$$= \frac{-\alpha.\frac{d\rho}{(\rho)}.(1-s).\sin.\theta}{\left\{1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)\right\}.\sqrt{1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)-(1-s)^2.\sin.^2\theta}}.$$ [8286c]

Now we have, by development,

$$1-(1-s)^2.\sin.^2\theta = (1-\sin.^2\theta)+(2s-s^2).\sin.^2\theta = \cos.^2\theta+(2s-s^2).\sin.^2\theta;$$ [8286d]

substituting this in the radical of the expression [8286c], it becomes as in [8286]. We may observe that the symbol s [8284], is wholly different from that in [8158], which is used in § 1, 2, 3. [8286e]

[8286'] α being very small [8277'], we may suppose, without any sensible error, that $1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)$ is equal to the mean quantity between its two

[8287] extreme values 1 and $1-2\alpha$; therefore we shall suppose it equal to $1-\alpha$.

Symbols. The temperature of the atmosphere being supposed uniform, if we put,

[8288] $p=$ the pressure or the elastic force of the air corresponding to the density ρ;

[8288'] $(p)=$ the pressure corresponding to the density (ρ);

[8289] $g=$ the gravity corresponding to the radius r;

[8289'] $(g)=$ the gravity corresponding to the radius a;

[8290] $l=$ the height of a column of air of the density (ρ), which being acted upon by the gravity (g), will be in equilibrium with the pressure (p); in other words, l is the height of a homogeneous atmosphere;

[8290'] $c=2{,}718281828..$, the number whose hyperbolic logarithm is equal to unity.

Then we shall have very nearly, as the result of observation,

[8291]
$$p=(p).\frac{\rho}{(\rho)};$$

[8292]
$$g=(g).\frac{a^2}{r^2}.$$

Density of the air proportional to the pressure.

The *decrement* of the pressure p, produced by the *elevation* dr, is evidently equal to the small column of air ρdr, multiplied by the gravity g;

[8293] therefore we shall have,

[8294]
$$dp=-(g).\frac{a^2}{r^2}.\rho dr$$

[8294']
$$=(g).a.\rho.d\frac{a}{r};$$

consequently,*

[8295]
$$(p).\frac{d\rho}{(\rho)}=(g).a.\rho.d.\frac{a}{r}.$$

[8295a] * (3871) We have, in [8293], $dp=-g\rho dr$; and by substituting the value of g [8292], it becomes as in [8294]. Now the differential of [8291] is $dp=(p).\frac{d\rho}{(\rho)}$; and

[8295b] by substitution in [8294], we obtain, as in [8295],

[8295c]
$$(p).\frac{d\rho}{(\rho)}=-(g).\frac{a^2}{r^2}.\rho dr=(g).a\rho.d.\frac{a}{r}.$$

Dividing [8295] by $\frac{(p).\rho}{(\rho)}$, we get $\frac{d\rho}{\rho}=\frac{(g).(\rho).a}{(p)}.d.\frac{a}{r}$; and by integration,

Hence we deduce by integration,

$$\rho=(\rho).c^{\left(\frac{a}{r}-1\right).a.\frac{(g).(\rho)}{(p)}};$$ [8296]

c being as in [8290′]; and from the definition of l [8290] we have,* [8297]

$$(p)=(g).(\rho).l;$$ [8298]

hence we get, *in the case of a uniform temperature of the atmosphere* [8283], Density if the temperature is uniform.

$$\rho=(\rho).c^{-\frac{as}{l}}; \qquad \text{or} \qquad \frac{\rho}{(\rho)}=c^{-\frac{as}{l}};$$ [8299]

therefore the equation [8286] becomes, by reducing the radical to a series, ascending according to the powers of s,†

$$\log.\rho=\frac{(g).(\rho).a}{(p)}.\left(\frac{a}{r}+\text{constant}\right).$$ [8295d]

When $r=a$, and $\rho=(\rho)$, this becomes,

$$\log.(\rho)=\frac{(g).(\rho).a}{(p)}.\left(1+\text{constant}\right);$$ [8295e]

subtracting this from the preceding expression of $\log.\rho$, we get,

$$\log.\frac{\rho}{(\rho)}=\frac{(g).(\rho).a}{(p)}.\left(\frac{a}{r}-1\right).$$ [8295f]

Multiplying the second member of this expression by $\log.c=1$ [8290′], and then reducing to natural numbers, it becomes as in [8296].

* (3872) The weight (p) of the homogeneous column of air l, is found by multiplying its height l by the gravity (g), and by the density (ρ); hence we get [8297a] $(p)=(g).(\rho).l$, as in [8298]. From this we get $\frac{(g).(\rho)}{(p)}=\frac{1}{l}$; and by multiplying it by $a.\left(\frac{a}{r}-1\right)=-as$ [8284], we obtain $\left(\frac{a}{r}-1\right).a.\frac{(g).(\rho)}{(p)}=-\frac{as}{l}$. Substituting [8297b] this last expression in [8296], we get [8299].

† (3873) In order to develop the radical in the denominator of [8286], we shall put for brevity $D=\cos.^2\theta-2\alpha.\left(1-c^{-\frac{as}{l}}\right)+2s.\sin.^2\theta$; and we shall get by development, [8300a] using the value of $\frac{\rho}{(\rho)}$ [8299],

$$\frac{1-s}{\sqrt{\cos.^2\theta-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)+(2s-s^2).\sin.^2\theta}}=\frac{1-s}{\sqrt{D-s^2.\sin.^2\theta}}=\frac{1-s}{D^{\frac{1}{2}}}.\left\{1+\tfrac{1}{2}.\frac{s^2.\sin.^2\theta}{D}+\&c.\right\}$$ [8300b]

$$=\frac{1}{D^{\frac{1}{2}}}-\frac{s.(D-\frac{1}{2}s.\sin.^2\theta)}{D^{\frac{3}{2}}}+\&c.=\frac{1}{D^{\frac{1}{2}}}-\frac{s.\left\{\cos.^2\theta+\frac{3}{2}s.\sin.^2\theta-2\alpha.\left(1-c^{-\frac{as}{l}}\right)\right\}}{D^{\frac{3}{2}}}+\&c.$$ [8300c]

General value of $d\theta$ when the temperature is uniform.

[8300]
$$d\theta = \frac{\alpha.\frac{a}{l}.ds.c^{-\frac{as}{l}}.\sin.\Theta}{(1-\alpha).\left\{\cos.^2\Theta-2\alpha.\left(1-c^{-\frac{as}{l}}\right)+2s.\sin.^2\Theta\right\}^{\frac{1}{2}}} \quad 1$$
$$-\frac{\alpha.\frac{as}{l}.ds.c^{-\frac{as}{l}}.\sin.\Theta.\left\{\cos.^2\Theta+\tfrac{3}{2}s.\sin.^2\Theta-2\alpha.\left(1-c^{-\frac{as}{l}}\right)\right\}}{(1-\alpha).\left\{\cos.^2\Theta-2\alpha.\left(1-c^{-\frac{as}{l}}\right)+2s.\sin.^2\Theta\right\}^{\frac{3}{2}}}-\&c. \quad 2 \qquad ;\ (5)$$

[8300′] *The first term of this differential expression is much greater than the others, which are nearly insensible* [8368] *; we shall first integrate this term.* For this purpose we shall put,

[8301]
$$s=s'+\alpha.\frac{\left(1-c^{-\frac{as}{l}}\right)}{\sin.^2\Theta};$$

and we shall have, by means of the formula [629, 629a],*

Multiplying this expression by $-\dfrac{\alpha.\frac{d\rho}{(\rho)}.\sin.\Theta}{\left\{1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)\right\}}$; and then substituting

[8300d] $\frac{d\rho}{(\rho)}=-\frac{ads}{l}.c^{-\frac{as}{l}}$ [8299], also $1-\alpha$ for $1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)$ [8287], we obtain the expression of $d\theta$ [8286], under the form which is given in [8300].

[8301a] * (3874) Comparing the equation $x=\varphi.(t+\alpha z)$ [629a], with [8301], we get $x=s$, $\varphi=1$, $t=s'$, $z=\frac{1-c^{-\frac{as}{l}}}{\sin.^2\Theta}$; and if we suppose $u=\psi(x)=c^{-\frac{as}{l}}$ [629a], we shall have $u=c^{-\frac{as}{l}}$; then [629] becomes,

[8301b]
$$u=\mathrm{u}+\alpha Z.\frac{d\mathrm{u}}{ds'}+\frac{1}{1.2}.\alpha^2.\frac{d.\left(Z^2.\frac{d\mathrm{u}}{ds'}\right)}{ds'}+\frac{1}{1.2.3}.\alpha^3.\frac{d^2.\left(Z^3.\frac{d\mathrm{u}}{ds'}\right)}{ds'^2}+\&c.;$$

u and Z being the values of u, z, respectively, when $\alpha=0$; so that we have,

[8301c]
$$\mathrm{u}=c^{-\frac{as'}{l}};\qquad Z=\frac{1-c^{-\frac{as'}{l}}}{\sin.^2\Theta};\qquad \frac{d\mathrm{u}}{ds'}=-\frac{a}{l}.c^{-\frac{as'}{l}}.$$

Substituting these values in the preceding expression of u or $c^{-\frac{as}{l}}$, it becomes as in [8302].

$$c^{-\frac{as}{l}} = c^{-\frac{as'}{l}} - \frac{\alpha . a}{l.\sin.^2\theta}.\left(1-c^{-\frac{as'}{l}}\right).c^{-\frac{as'}{l}} \qquad 1$$

$$-\frac{\alpha^2.a}{1.2.l.\sin.^4\theta}.\frac{d.\left\{\left(1-c^{-\frac{as'}{l}}\right)^2.c^{-\frac{as'}{l}}\right\}}{ds'} \qquad 2 \qquad [8302]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\frac{\alpha^i.a}{1.2.3...i.l.\sin.^{2i}\theta}.\frac{d^{i-1}.\left\{\left(1-c^{-\frac{as'}{l}}\right)^i.c^{-\frac{as'}{l}}\right\}}{ds'^{i-1}} \qquad i$$

$$-\&c.$$

Therefore the first term [8300′], which is now under consideration, becomes,* by observing that $\frac{a}{l}.ds.c^{-\frac{as}{l}} = -d.c^{-\frac{as}{l}}$, [8303]

First term of $d\vartheta$.

$$\frac{\alpha.\frac{a}{l}.\sin.\theta.ds'}{(1-\alpha).\{\cos.^2\theta+2s'.\sin.^2\theta\}^{\frac{1}{2}}}.\left\{\begin{array}{ll} c^{-\frac{as'}{l}} & 1 \\ -\frac{\alpha}{\sin.^2\theta}.\frac{d.\left\{\left(c^{-\frac{as'}{l}}-1\right).c^{-\frac{as'}{l}}\right\}}{ds'} & 2 \\ +\frac{\alpha^2}{1.2.\sin.^4\theta}.\frac{d^2.\left\{\left(c^{-\frac{as'}{l}}-1\right)^2.c^{-\frac{as'}{l}}\right\}}{ds'^2} & 3 \\ \cdots\cdots\cdots\cdots & \\ \pm\frac{\alpha^i}{1.2.3...i.\sin.^{2i}\theta}.\frac{d^i.\left\{\left(c^{-\frac{as'}{l}}-1\right)^i.c^{-\frac{as'}{l}}\right\}}{ds'^i} & i+1 \\ \mp\&c. & \end{array}\right\}; \qquad [8304]$$

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* (3875) Multiplying [8301] by $2.\sin.^2\theta$, and adding $\cos.^2\theta$ to the product, we get, by transposing the terms containing α,

$$\cos.^2\theta - 2\alpha.\left(1-c^{-\frac{as}{l}}\right)+2s.\sin.^2\theta = \cos.^2\theta+2s'.\sin.^2\theta. \qquad [8304a]$$

Substituting this in the denominator of [8300 line 1], and retaining only this term of $d\vartheta$, as in [8300′], it becomes, by using [8303],

$$d\vartheta = -\frac{\alpha.\sin.\theta.d.c^{-\frac{as}{l}}}{(1-\alpha).\sqrt{\cos.^2\theta+2s'.\sin.^2\theta}}. \qquad [8304b]$$

Now substituting the value of $d.c^{-\frac{as}{l}}$, deduced from [8302], considering s, s', as the

[8305] *the upper sign being used when i is even, and the lower when i is odd.* In general we have,*

Development of the differential expression.

[8306]
$$\pm\frac{\alpha^i}{1.2.3...i.\sin.^{2i}\Theta}\cdot\frac{d^i.\left\{\left(c^{-\frac{as'}{l}}-1\right)^i.c^{-\frac{as'}{l}}\right\}}{ds'^{\,i}}=\frac{\left(\alpha.\frac{a}{l}\right)^i}{1.2.3...i.\sin.^{2i}\Theta}\cdot\left\{\begin{array}{ll}(i+1)^i.c^{-(i+1).\frac{a}{l}.s'} & 1\\ -i.i^i.c^{-i.\frac{a}{l}.s'} & 2\\ +\frac{i.(i-1)}{1.2}.(i-1)^i.c^{-(i-1).\frac{a}{l}.s'} & 3\\ -\&c. & \end{array}\right\}.$$

We must multiply each term of this development by the following expression, being the factor without the braces of the formula [8304];

Factor in the value of $d\theta$.

[8307]
$$\frac{\alpha.\frac{a}{l}.\sin.\Theta.ds'}{(1-\alpha).\sqrt{\cos.^2\Theta+2s'.\sin.^2\Theta}};$$

[8304c] variable quantities, we get [8304]; observing that in this last formula $1-c^{-\frac{as'}{l}}$ is changed into $-\left(c^{-\frac{as'}{l}}-1\right)$; and by this means the sign $\mp$ is introduced.

* (3876) Developing the first member of [8305a], it becomes as in its second member. Its differential, divided by ds', is as in [8305b]; its second differential, divided by ds'^2, as in [8305c]; and so on for others.

[8305a]
$$\left(c^{-\frac{as'}{l}}-1\right)^i.c^{-\frac{as'}{l}}=c^{-(i+1).\frac{as'}{l}}-i.c^{-i.\frac{as'}{l}}+\frac{i.(i-1)}{1.2}.c^{-(i-1).\frac{as'}{l}}-\&c.;$$

[8305b]
$$\frac{d.\left\{\left(c^{-\frac{as'}{l}}-1\right).c^{-\frac{as'}{l}}\right\}}{ds'}=-\frac{a}{l}.\left\{\begin{array}{l}(i+1).c^{-(i+1).\frac{as'}{l}}-i.i.c^{-i.\frac{as'}{l}}\\ +\frac{i.(i-1)}{1.2}.(i-1).c^{-(i-1).\frac{as'}{l}}-\&c.\end{array}\right\};$$

[8305c]
$$\frac{d^2.\left\{\left(c^{-\frac{as'}{l}}-1\right).c^{-\frac{as'}{l}}\right\}}{ds'^2}=\left(\frac{a}{l}\right)^2.\left\{\begin{array}{l}(i+1)^2.c^{-(i+1).\frac{as'}{l}}-i.i^2.c^{-i.\frac{as'}{l}}\\ +\frac{i.(i-1)}{1.2}.(i-1)^2.c^{-(i-1).\frac{as'}{l}}-\&c.\end{array}\right\}.$$

And by continuing to take the differentials in this manner, we easily obtain the general form of the expression [8306]. Each term of this development is to be multiplied by the general factor [8307], which occurs in [8304].

and then take the integrals from $s'=0$ to $s'=1-\frac{\alpha.\left(1-c^{-\frac{a}{l}}\right)}{\sin.^2\Theta}$.* [8308]

But since at this last limit $c^{-\frac{as'}{l}}$ is extremely small, because c exceeds 2, [8290′], and $\frac{a}{l}$ is a very great number, being nearly equal to 800 [8278*d*]; *we see that the integrals may, without fear of any appreciable error, be taken from* $s'=0$ *to* $s'=\infty$. This being premised we shall consider the differential,† [8309]

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* (3877) In finding the value θ, or rather $\delta\theta$, we must evidently take the integral of the expression of $d\theta$ [8300, &c.], from $r=a$ to $r=\infty$ [8138, 8138′]; *or from* $\frac{a}{r}=1$ *to* $\frac{a}{r}=0$; *and this is the same as taking it from* $s=0$ *to* $s=1$ [8284], respectively. Now from [8301] we have $s'=s-\alpha.\frac{\left(1-c^{-\frac{as}{l}}\right)}{\sin.^2\Theta}$; and if $s=0$, it gives $s'=0$; but if $s=1$, it becomes $s'=1-\alpha.\frac{\left(1-c^{-\frac{a}{l}}\right)}{\sin.^2\Theta}$; agreeing with the limits of s' in [8308]. Now we have nearly $\frac{a}{l}=798$ [8278*d*], and $c=2{,}718..$ [8290′]; hence $c^{-\frac{a}{l}}=\left(\frac{1}{2{,}718}\right)^{798}=10^{-347}$ nearly, which may be put $=0$, on account of its excessive smallness; and then the last limit of s' [8307*b*], becomes $s'=1-\frac{\alpha}{\sin.^2\Theta}$. Near the horizon, where the refraction is greatest, $\sin.^2\Theta$ is nearly equal to unity, and $\alpha=0{,}00029..$ [8277′]; therefore this last limit of s' is nearly equal to unity. While s' varies from $s'=1$ to $s'=\infty$, the factor $c^{-\frac{as'}{l}}$ does not exceed the extremely small quantity 10^{-347} [8307*c*]; so that this factor and its powers, which occur in [8302–8306], may be considered as vanishing, while s' varies from 1 to ∞, or from $1-\frac{\alpha}{\sin.^2\Theta}$ to ∞; we may therefore extend the limits of s' [8307*d*], and take the integrals from $s'=0$ to $s'=\infty$, as in [8309]; always considering that the radical, in the denominator of the factor [8307], does not vanish, or the factor itself become infinite. We may remark that the limit $s'=\infty$, being substituted in [8301], gives $s=\infty$; therefore the limits of s [8307*a*] may be extended, in these integrals, from $s=0$ to $s=\infty$. These limits are used in [8416*a*, &c.] [8307*a*] [8307*b*] [8307*c*] [8307*d*] [8307*e*] [8307*f*] [8307*g*] [8307*h*] [8307*i*]

† (3878) When we substitute, in [8304], the developed expressions [8306], it will [8310*a*]

General form of the term of $d\theta$.

[8310]
$$\frac{\frac{a}{l}\cdot ds'.c^{-\frac{\mathrm{r}a}{l}.s'}.\sin.\Theta}{\sqrt{\cos.^2\Theta+2s'.\sin.^2\Theta}},$$

and shall put,

[8311]
$$\tfrac{1}{2}.\frac{\cos.^2\Theta}{\sin.^2\Theta}+s'=\frac{l}{\mathrm{r}a}.t^2;\qquad \text{or}\quad t=\sqrt{\frac{\mathrm{r}a}{2l}.\frac{\cos.^2\Theta}{\sin.^2\Theta}+\frac{\mathrm{r}as'}{l}};$$

then the preceding differential becomes,

[8312]
$$\sqrt{\frac{2a}{\mathrm{r}l}}.dt.c^{\frac{\mathrm{r}a}{2l}.\frac{\cos.^2\Theta}{\sin.^2\Theta}-t^2}.$$

[8313] The integral being taken from $t=\sqrt{\frac{\mathrm{r}a}{2l}}.\frac{\cos.\Theta}{\sin.\Theta}=\mathrm{t}$ [8313*b*], to $t=\infty$.*
Now putting between those limits,

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evidently produce terms of the form [8310], multiplied by powers of $\frac{\alpha.\frac{a}{l}}{\sin.^2\Theta}$, and by the numerical coefficients depending on α, i; these quantities, as well as the new numerical
[8310*b*] coefficient r, being constant in the integrations relative to s'. Now the first of the expressions [8311], being multiplied by $2.\sin.^2\Theta$, gives, by extracting the square root,

[8310*c*]
$$\sqrt{\cos.^2\Theta+2s'.\sin.^2\Theta}=\sqrt{\frac{2l}{\mathrm{r}a}}.\sin.\Theta.t.$$

[8310*d*] Moreover we get, from [8311], $s'=\frac{l}{\mathrm{r}a}.t^2-\tfrac{1}{2}.\frac{\cos.^2\Theta}{\sin.^2\Theta}$, whose differential is $ds'=\frac{2l}{\mathrm{r}a}.tdt$. Substituting these in [8310], it becomes as in [8312].

[8313*a*] * (3879) The limits of s' [8309] are $s'=0$, $s'=\infty$; and by substituting them in t [8311], we get the limits of t, as in [8313]. Now if we put for brevity
[8313*b*] $\mathrm{t}=\sqrt{\frac{\mathrm{r}a}{2l}}.\frac{\cos.\Theta}{\sin.\Theta}$, these limits will be represented by $t=\mathrm{t}$, and $t=\infty$. The integral of the expression [8312], may evidently be put under the form,

[8313*c*]
$$\sqrt{\frac{2a}{\mathrm{r}l}}.c^{\frac{\mathrm{r}a}{2l}.\frac{\cos.^2\Theta}{\sin.^2\Theta}}.\textstyle\int_{\mathrm{t}}^{\infty}dt.c^{-tt};$$

[8313*d*] so that if we substitute the assumed value of the integral [8314], and then multiply the result by $\frac{\alpha}{1-\alpha}$, we shall get, within the proposed limits,

[8313*e*]
$$\int_0^\infty\frac{\alpha.\frac{a}{l}.ds'.c^{-\frac{\mathrm{r}a}{l}.s'}.\sin.\Theta}{(1-\alpha).\sqrt{\cos.^2\Theta+2s'.\sin.^2\Theta}}=\frac{\alpha}{1-\alpha}.\sqrt{\frac{2a}{l}}.\left\{\frac{1}{\sqrt{\mathrm{r}}}.\Psi(\mathrm{r})\right\}.$$

$$\int_{\iota}^{\infty} dt.c^{-tt} = c^{-\frac{\mathrm{r}a}{2l}\cdot\frac{\cos.^2\Theta}{\sin.^2\Theta}}.\Psi(\mathrm{r});$$ [8314]

Symbol $\Psi(\mathrm{r})$.

$$\Psi(\mathrm{r}) = c^{\frac{\mathrm{r}a}{2l}\cdot\frac{\cos.^2\Theta}{\sin.^2\Theta}}.\int_{\iota}^{\infty} dt.c^{-tt}.$$ [8314']

We shall have, by noticing only the first term of $d\delta$ [8304],*

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* (3880) We shall represent by $Ac^{-\mathrm{r}.\frac{as'}{l}}$ any term of the expression of $d\delta$ [8304], between the braces, developed as in [8306]; and when this is connected with the factor without the braces in [8304], which is given separately in [8307], it will become equal to the function [8310], multiplied by $A.\frac{\alpha}{1-\alpha}$; therefore its integral will be equal to the expression [8313e] multiplied by A; so that it will become, [8314a] [8314b]

$$\frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{2a}{l}}\cdot\left\{\frac{A}{\sqrt{\mathrm{r}}}\cdot\Psi(\mathrm{r})\right\};$$ [8314c]

the factor $\frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{2a}{l}}$, without the braces, being the same as in [8315]; therefore the part $\frac{A}{\sqrt{\mathrm{r}}}\cdot\Psi(\mathrm{r})$ must represent the part between the braces, in [8315], arising from the term $Ac^{-\mathrm{r}.\frac{as'}{l}}$ [8314a] in [8306]. Hence it appears, that to deduce from the part of $d\delta$ [8304] the corresponding part of $\delta\delta$, we must reduce the terms between the braces, in [8304], to the form [8306], and then change $c^{-\mathrm{r}.\frac{as'}{l}}$ into $\frac{\Psi(\mathrm{r})}{\sqrt{\mathrm{r}}}$; and by this process we can easily derive the terms of [8315] from those in [8304]. Thus the term in [8304 line 1] is $c^{-\frac{as'}{l}}$, corresponding to $\mathrm{r}=1$; hence $\frac{\Psi(\mathrm{r})}{\mathrm{r}}=\Psi(1)$, as in [8315 line 1]. The term in [8304 line 2], developed as in [8306], corresponds to $i=1$, in [8306], and becomes $\frac{\alpha.\frac{a}{l}}{\sin.^2\Theta}\cdot\left\{2c^{-2.\frac{as'}{l}}-c^{-\frac{as'}{l}}\right\}$, which produces $\frac{\alpha.\frac{a}{l}}{\sin.^2\Theta}\cdot\left\{\frac{2\Psi(2)}{\sqrt{2}}-\Psi(1)\right\}$, as in [8315 line 2]. The term in [8304 line 3], developed as in [8306], putting $i=2$, gives, [8314d] [8314e] [8314f] [8314g] [8314h]

$$\frac{\alpha^2.\frac{a^2}{l^2}}{1.2.\sin.^4\Theta}\cdot\left\{\frac{3^2.\Psi(3)}{\sqrt{3}}-2.\frac{2^2.\Psi(2)}{\sqrt{2}}+\Psi(1)\right\},$$ [8314i]

as in [8315 line 3]. The next term of [8304], corresponding to $i=3$ in [8306], produces, in like manner, the term in [8315 line 4]; and so on for other terms.

[8315]
$$\delta\theta=\frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{2a}{l}}\cdot\left\{\begin{array}{ll}\Psi(1) & 1\\ +\dfrac{\alpha.\frac{a}{l}}{\sin.^2\Theta}.\{2^{\frac{1}{2}}.\Psi(2)-\Psi(1)\} & 2\\ +\dfrac{\alpha^2.\frac{a^2}{l^2}}{1.2.\sin.^4\Theta}.\{3^{\frac{3}{2}}.\Psi(3)-2.2^{\frac{3}{2}}.\Psi(2)+\Psi(1)\} & 3\\ +\dfrac{\alpha^3.\frac{a^3}{l^3}}{1.2.3.\sin.^6\Theta}.\{4^{\frac{5}{2}}.\Psi(4)-3.3^{\frac{5}{2}}.\Psi(3)+3.2^{\frac{5}{2}}.\Psi(2)-\Psi(1)\} & 4\\ +\dfrac{\alpha^4.\frac{a^4}{l^4}}{1.2.3.4.\sin.^8\Theta}.\{5^{\frac{7}{2}}.\Psi(5)-4.4^{\frac{7}{2}}.\Psi(4)+6.3^{\frac{7}{2}}.\Psi(3)-4.2^{\frac{7}{2}}.\Psi(2)+\Psi(1)\} & 5\\ +\&c. & \end{array}\right\}.$$

This expression may be put under the following form;*

Part of $\delta\theta$ deduced from the first term of $d\theta$.

[8316]
$$\delta\theta=\frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{2a}{l}}\cdot\left\{\begin{array}{ll}c^{-\frac{\alpha a}{l.\sin.^2\Theta}}.\Psi(1) & 1\\ +\dfrac{\alpha a}{l.\sin.^2\Theta}.2^{\frac{1}{2}}.c^{-\frac{2\alpha a}{l.\sin.^2\Theta}}.\Psi(2) & 2\\ +\dfrac{\alpha^2.a^2}{1.2.l^2.\sin.^4\Theta}.3^{\frac{3}{2}}.c^{-\frac{3\alpha a}{l.\sin.^2\Theta}}.\Psi(3) & 3\\ +\&c. & \end{array}\right\}.$$

[8316a] * (3881) If we put for a moment for brevity $\dfrac{\alpha.\frac{a}{l}}{\sin.^2\Theta}=x$, and then connect together the terms of [8315] depending on $\Psi(1)$, also those depending on $\Psi(2)$, &c., it will become,

[8316b]
$$\delta\theta=\frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{2a}{l}}\cdot\left\{\begin{array}{l}\left\{1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\&c.\right\}.\Psi(1)\\ +2^{\frac{1}{2}}.x.\left\{1-2x+\frac{(2x)^2}{1.2}-\frac{(2x)^3}{1.2.3}+\&c.\right\}.\Psi(2)\\ +3^{\frac{3}{2}}.\frac{x^2}{1.2}.\left\{1-3x+\frac{(3x)^2}{1.2}-\frac{(3x)^3}{1.2.3}+\&c.\right\}.\Psi(3)\\ +\&c.\end{array}\right\}.$$

The only remaining difficulty is to ascertain the form of $\Psi(\mathrm{r})$ [8314′], or to take the integral $\int_{\mathrm{t}}^{\infty} dt.c^{-tt}$, from $t=\sqrt{\frac{\mathrm{r}a}{2l}}\cdot\frac{\cos.\theta}{\sin.\theta}=\mathrm{t}$ to $t=\infty$, [8313]. [8317]
In the case of the horizontal refraction $\cos.\theta=0$, $\sin.\theta=1$; [8318]
and then $\Psi(\mathrm{r})$ *is independent of* r, *and is represented by* $\Psi(\mathrm{r})=\int_0^{\infty} dt.c^{-tt}$, [8314′]. [8319]
To determine this integral, we shall consider the double integral $\int\int ds.dx.c^{-s.(1+xx)}$, taking the integrals from $s=0$ to $s=\infty$, and from $x=0$ to $x=\infty$. Integrating in the first place relative to s, we obtain,* [8320] [8321]

$$\int_0^{\infty}\int_0^{\infty} ds.dx.c^{-s.(1+xx)}=\int_0^{\infty}\frac{dx}{1+xx}\quad [1534e']. \qquad [8322]$$

The integral $\int_0^x\frac{dx}{1+xx}=\text{angle}(\text{tang}.x)$ [51] Int.; and this integral, taken [8323]
from $x=0$ to $x=\infty$, is equal to a right angle, or $\int_0^{\infty}\frac{dx}{1+xx}=\frac{1}{2}\pi$; π [8323′]
being the semi-circumference whose radius is 1; hence we have, Definite integrals.

$$\int_0^{\infty}\int_0^{\infty} ds.dx.c^{-s.(1+xx)}=\tfrac{1}{2}\pi,\quad [1534f]. \qquad [8324]$$

We shall now investigate this integral in another manner, supposing $sx^2=t^2$, which gives $dx=dt.s^{-\frac{1}{2}}$; s being supposed constant in the differentration. Then the double integral [8324] will become, [8325]

$$\int_0^{\infty}\int_0^{\infty} ds.s^{-\frac{1}{2}}.dt.c^{-s-tt}=\int_0^{\infty} ds.s^{-\frac{1}{2}}.c^{-s}.\int_0^{\infty} dt.c^{-tt}. \qquad [8326]$$

Now from [56] Int. we have,

$$c^{-x}=1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\&c.;\qquad c^{-2x}=1-2x+\frac{(2x)^2}{1.2}-\frac{(2x)^3}{1.2.3}+\&c.; \qquad [8316c]$$

$$c^{-3x}=1-3x+\frac{(3x)^2}{1.2}-\frac{(3x)^3}{1.2.3}+\&c., \&c. \qquad [8316d]$$

Substituting these in [8316*b*], it becomes,

$$\delta\theta=\frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{2a}{l}}\cdot\left\{c^{-x}.\Psi(1)+2^{\frac{1}{2}}.x.c^{-2x}.\Psi(2)+\tfrac{1}{2}.3^{\frac{3}{2}}.x^2.c^{-3x}.\Psi(3)+\&c.\right\}; \qquad [8316e]$$

and by re-substituting the value of x [8316*a*], it becomes as in [8316].

* (3882) The computation in [8323—8331] is exactly the same as that in [1534*e*–*l*], changing the symbol v into s. Thus the integral in [8322] relative to s, is the same as that in [1534*e*′] relative to v. That in [8324] is the same as [1534*f*]. The assumed value of t^2 [8325] is the same as in [1534*g*]. The integral [8326] is the same as [1534*h*]. The symbol t' [8328] is equivalent to τ [1534*i*]. The integral [8329] is the same as [1534*i*], and [8330] is the same as [1534*k*]. Finally, the theorem [8331] is the same as in [1534*l*]. [8325a] [8325b] [8325c]

[8327] We shall put $\int_0^\infty dt.c^{-tt} = K$, and then the preceding double integral will
[8328] become $K\int_0^\infty ds.s^{-\frac{1}{2}}.c^{-s}$. If we now substitute $s = t'^2$ in this last integral, it becomes,*

[8329] $$\int_0^\infty ds.s^{-\frac{1}{2}}.c^{-s} = 2\int_0^\infty dt'.c^{-t't'} = 2K \quad [8327];$$

therefore we shall have,

[8330] $$\int_0^\infty ds.s^{-\frac{1}{2}}.c^{-s}.\int_0^\infty dt.c^{-tt} = 2K^2 = \int_0^\infty\int_0^\infty ds.dx.c^{-s.(1+xx)} = \tfrac{1}{2}\pi \quad [8324].$$

Hence we have $K = \frac{1}{2}\sqrt{\pi}$; and by re-substituting the values of K and
Definite integral. $\psi(\mathrm{r})$ [8327, 8319], we finally obtain,

[8331] $$\int_0^\infty dt.c^{-tt} = \tfrac{1}{2}\sqrt{\pi} = \psi(\mathrm{r}) \quad [1534l,\ 8319];$$

therefore the expression of the horizontal refraction is,†

[8332] $$\delta\theta = \frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{a\pi}{2l}}\cdot\left\{\begin{array}{ll} 1 + \frac{\alpha.a}{l}.\{2^{\frac{1}{2}}-1\} & \qquad 1 \\ + \frac{\alpha^2.a^2}{1.2.l^2}.\{3^{\frac{3}{2}}-2.2^{\frac{3}{2}}+1\} & \qquad 2 \\ + \frac{\alpha^3.a^3}{1.2.3.l^3}.\{4^{\frac{5}{2}}-3.3^{\frac{5}{2}}+3.2^{\frac{5}{2}}-1\} & \qquad 3 \\ + \&c. & \end{array}\right\}.$$

Part of the horizontal refraction $\delta\theta$ deduced from the first term of $d\theta$.

This expression may be put under the following form [8331b];

[8333] $$\delta\theta = \frac{\alpha}{1-\alpha}\cdot\sqrt{\frac{a\pi}{2l}}\cdot\left\{c^{-\frac{\alpha a}{l}} + \frac{\alpha.a}{l}.2^{\frac{1}{2}}.c^{-\frac{2\alpha a}{l}} + \frac{\alpha^2.a^2}{1.2.l^2}.3^{\frac{3}{2}}.c^{-\frac{3\alpha a}{l}} + \&c.\right\}.$$

To obtain the value of $\psi(\mathrm{r})$, when the body is a little elevated above the horizon, we shall put,

[8334] $$T^2 = \frac{\mathrm{r}a}{2l}\cdot\frac{\cos.^2\theta}{\sin.^2\theta};$$

Symbol T.

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* (3883) The limits of s, x [8321], evidently give those of t, t', which are inserted in the formulas [8322—8331]. For the first limit of s is $s=0$, and this gives in
[8328a] [8325] $t=0$, and in [8328] $t'=0$. The second limit of $s=\infty$ [8307i], gives in [8325], for any value of x, $t=\infty$, and in [8328] $t'=\infty$; which are the limits used in this article.

† (3884) From [8331] it follows that $\psi(1)$, $\psi(2)$, $\psi(3)$, &c., are each equal to
[8331a] $\frac{1}{2}\sqrt{\pi}$, and $\sin.\theta = 1$ [8318]; substituting these in [8315] it becomes as in [8332]. The
[8331b] same substitutions being made in [8316], it becomes as in [8333].

and we shall have, by developing and taking the integral from $t=0$ to $t=T$,*

$$\int_0^T dt.c^{-tt} = T - \tfrac{1}{3}.T^3 + \frac{1}{1.2}\cdot\frac{T^5}{5} - \frac{1}{1.2.3}\cdot\frac{T^7}{7} + \&c.\,; \qquad [8335]$$

and we shall also have, by a different development,† Definite integrals.

$$\int_0^T dt.c^{-tt} = c^{-TT}.T.\left\{1 + \frac{2T^2}{3} + \frac{(2T^2)^2}{1.3.5} + \frac{(2T^2)^3}{1.3.5.7} + \&c.\right\}. \qquad [8336]$$

These two series cease to be converging, for every value of T. The first series is alternately greater and less than the integral, according as the series is terminated by a positive or negative term; so that if we add to any number of its terms the half of the following term, the error will be less than that half.‡ This furnishes a simple method of estimating the degree of approximation. Then by subtracting the value of the series from $\int_0^\infty dt.c^{-tt} = \frac{1}{2}\sqrt{\pi}$ [8331], we obtain the value of the integral $\int_T^\infty dt.c^{-tt}$. [8337] [8338]

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* (3885) Developing c^{-tt} by [56] Int., we get,

$$\int dt.c^{-tt} = \int dt.\left\{1 - t^2 + \frac{1}{1.2}\cdot t^4 - \frac{1}{1.2.3}\cdot t^6 + \&c.\right\} = t - \tfrac{1}{3}.t^3 + \frac{1}{1.2}\cdot\frac{t^5}{5} - \&c.\,; \qquad [8335a]$$

which vanishes when $t=0$; and when $t=T$, it becomes as in [8335].

† (3886) It is easy to prove, by taking the differential, that,

$$\int t^n dt.c^{-tt} = \frac{1}{n+1}\cdot t^{n+1}.c^{-tt} + \frac{2}{n+1}\cdot\int t^{n+2}dt.c^{-tt}. \qquad [8336a]$$

Now putting successively $n=0$, $n=2$, $n=4$, $n=6$, &c., we get, by repeated substitutions,

$$\int dt.c^{-tt} = t c^{-tt} + 2\int t^2 dt.c^{-tt} = t c^{-tt} + \frac{2}{3}.t^3 c^{-tt} + \frac{2^2}{3}\cdot\int t^4 dt.c^{-tt} \qquad [8336b]$$

$$= t c^{-tt} + \frac{2}{3}.t^3 c^{-tt} + \frac{2^2}{3.5}\cdot t^5 c^{-tt} + \frac{2^3}{3.5}\cdot\int t^6 dt.c^{-tt},\ \&c.\,; \qquad [8336c]$$

and so on *ad infinitum*. This series vanishes when $t=0$; and when $t=T$, it becomes, with some slight reductions, as in [8336].

‡ (3887) Suppose the sum of n terms of the series [8335] to be represented by $\int_0^T dt.c^{-tt} - m$, and $n+1$ terms of the same series by $\int_0^T dt.c^{-tt} + p$, their difference $p+m$ [8337a] represents the value of the term $n+1$ of the series; and by adding its half to the first of [8337b] these expressions, the sum becomes $\int_0^T dt.c^{-tt} + \frac{1}{2}.(p-m)$. This differs from the real value of the integral $\int_0^T dt.c^{-tt}$, by the quantity $\frac{1}{2}.(p-m)$, which is less than half the last term [8337c] $p+m$; because m and p have the same signs, by hypothesis.

[8338'] When T is equal to, or exceeds 3, we shall have the value of the integral, by means of the series,*

[8339] $$\int_T^{\infty} dt.c^{-tt} = \frac{c^{-TT}}{2T} \cdot \left\{ 1 - \frac{1}{2T^2} + \frac{1.3}{2^2.T^4} - \frac{1.3.5}{2^3.T^6} + \&c. \right\}.$$

This series has also the advantage of being alternately greater and less than the integral.

To reduce a series to the form of a continued fraction.

We may give to this series the form of a continued fraction, by the following method, which may be used in other cases; and by this means the series may be put under an infinite number of forms.

If we put,

[8340] $$u = \frac{1}{1-t} \cdot \left\{ 1 - \frac{q}{(1-t)^2} + \frac{1.3.q^2}{(1-t)^4} - \frac{1.3.5.q^3}{(1-t)^6} + \&c. \right\};$$

it will be easy to prove, by differentration, that we shall have,†

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* (3888) We easily prove, by taking the differential and reducing, that,

[8339a] $$\int t^{-n} dt.c^{-tt} = -\tfrac{1}{2}.t^{-n-1}.c^{-tt} - \tfrac{1}{2}.(n+1).\int t^{-n-2} dt.c^{-tt}.$$

Now putting successively $n=0$, $n=2$, $n=4$, &c., and making several substitutions, we get,

[8339b] $$\int dt.c^{-tt} = -\tfrac{1}{2}.t^{-1}.c^{-tt} - \tfrac{1}{2}.\int t^{-2} dt.c^{-tt} = -\tfrac{1}{2}.t^{-1}.c^{-tt} + \frac{1}{2^2}.t^{-3}.c^{-tt} + \frac{1.3}{2^2}.\int t^{-4} dt.c^{-tt}$$

[8339c] $$= -\tfrac{1}{2}.t^{-1}.c^{-tt} + \frac{1}{2^2}.t^{-3}.c^{-tt} - \frac{1.3}{2^3}.t^{-5}.c^{-tt} - \frac{1.3.5}{2^3}.\int t^{-6} dt.c^{-tt};$$

and so on for higher values of n. Hence we shall have generally,

[8339d] $$\int dt.c^{-tt} = \text{constant} - \frac{c^{-tt}}{2t} \cdot \left\{ 1 - \frac{1}{2t^2} + \frac{1.3}{2^2.t^4} - \frac{1.3.5}{2^3.t^6} + \&c. \right\}.$$

The constant quantity is found by making the integral vanish when $t=T$; hence we have,

[8339e] $$\int_T^t dt.c^{-tt} = \frac{c^{-TT}}{2T} \cdot \left\{ 1 - \frac{1}{2T^2} + \frac{1.3}{2^2.T^4} - \&c. \right\} - \frac{c^{-tt}}{2t} \cdot \left\{ 1 - \frac{1}{2t^2} + \frac{1.3}{2^2.t^4} - \frac{1.3.5}{2^3.t^6} + \&c. \right\}.$$

Now putting $t=\infty$, the whole integral becomes as in [8339], representing the value of $\int_T^{\infty} dt.c^{-tt}$.

† (3889) The expression of u [8340] is easily put under the form [8340a], and its differential, considering u, t, as the variable quantities, is as in [8340b];

[8340a] $$u = \frac{1}{1-t} - \frac{q}{(1-t)^3} + \frac{1.3.q^2}{(1-t)^5} - \frac{1.3.5.q^3}{(1-t)^7} + \&c.;$$

[8340b] $$\frac{du}{dt} = \quad + \frac{1}{(1-t)^2} - \frac{1.3.q}{(1-t)^4} + \frac{1.3.5.q^2}{(1-t)^6} - \&c.$$

Multiplying [8340a] by $1-t$, and [8340b] by q, then adding the products, and

$$q.\frac{du}{dt}+(1-t).u=1.$$ [8341]

We shall now consider u as the generating function of the series,

$$u=y_1+y_2.t+y_3.t^2+y_4.t^3+\&c.;$$ [8342]

the differential equation [8341] gives, by noticing only the coefficients of the term t^r [8340c],

$$(r+1).q.y_{r+2}+y_{r+1}-y_r=0;$$ [8343]

and when $r=0$, we shall have, as in [8340d],

$$q.y_2+y_1=1;$$ [8344]

which is the same as to put $y_0=1$, in [8344]. *We may here observe that* [8345]
the generating function u *of* y_r, *in every linear equation of finite differences, in which the coefficients are rational and integral functions of* r, *may be determined in the foregoing manner, by means of a differential* [8346]
equation, relative to infinitely small quantities, of the same order as the highest power of* r *in its coefficients.*

Now every linear equation of the second order of finite differentials, may easily be reduced to a continued fraction, by the method we have used in [8347]
[2289—2292]. *We shall now consider generally the equation,*

General equation.

$$y_r=a_r.y_{r+1}+b_r.y_{r+2}.$$ [8348]

Dividing by y_{r+1}, we obtain, for all values of r,

$$\frac{y_r}{y_{r+1}}=a_r+b_r.\frac{y_{r+2}}{y_{r+1}};$$ [8349]

dividing unity by both sides of this equation, we obtain,

$$\frac{y_{r+1}}{y_r}=\frac{1}{a_r+b_r.\frac{y_{r+2}}{y_{r+1}}};$$ [8350]

and by changing r into r+1, we get,

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neglecting the terms of the second member which mutually destroy each other, we get [8341]. Substituting in this the assumed value of u, we find that the coefficient of t^r is [8340c]
equal to the first member of [8343]; and by putting it equal to nothing, so as to satisfy the equation [8341], we get [8343]. Finally the terms of the resulting equation, which are [8340d]
independent of t, are as in [8344].

* (3890) Thus, for example, the *coefficient* r+1, in the first term of [8343], contains the *first* power of r, corresponding to the differential equation [8341] of the *first* [8346a]
order.

[8351]
$$\frac{y_{r+2}}{y_{r+1}} = \frac{1}{a_{r+1} + b_{r+1}.\frac{y_{r+3}}{y_{r+2}}}.$$

If we continue this process we shall finally obtain,*

General equation reduced to a series.

[8352]
$$\frac{y_{r+1}}{y_r} = \cfrac{1}{a_r + \cfrac{b_r}{a_{r+1} + \cfrac{b_{r+1}}{a_{r+2} + \&c.}}}$$

[8352′] Putting $a_r = 1$, $b_r = (r+1).q$, in [8348], it becomes as in the equation [8343], namely,†

[8353]
$$y_r = y_{r+1} + (r+1).q.y_{r+2};$$

and then from [8352] we get,

[8354]
$$\frac{y_{r+1}}{y_r} = \cfrac{1}{1 + \cfrac{(r+1).q}{1 + \cfrac{(r+2).q}{1 + \cfrac{(r+3).q}{1 + \&c.}}}}$$

If we put $r = 0$, we shall have, by observing that $y_0 = 1$ [8345],

[8355]
$$y_1 = \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1 + \cfrac{4q}{1 + \&c.}}}}}$$

y_1 is the coefficient independent of t, in the development of the value of u [8340], in a series according to the powers of t, as in [8342], namely;

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* (3891) Substituting the value of $\frac{y_{r+2}}{y_{r+1}}$ [8351], in the second member of [8350], we get,

[8352a]
$$\frac{y_{r+1}}{y_r} = \cfrac{1}{a_r + \cfrac{b_r}{a_{r+1} + b_{r+1}.\frac{y_{r+3}}{y_{r+2}}}}.$$

[8352b] Now increasing r by unity in [8351], we get the expression of $\frac{y_{r+3}}{y_{r+2}}$, to be substituted in the second member of [8352a]; and by proceeding in this manner, we finally obtain [8352].

† (3892) Substituting the values of a_r, b_r [8352′], in [8348], it becomes as in
[8354a] [8353 or 8343]; and by the same substitutions, [8352] becomes as in [8354]. When $r = 0$, we have $y_0 = 1$ [8345], and then [8354] changes into [8355].

$$u=\frac{1}{1-t}\cdot\left\{1-\frac{q}{(1-t)^2}+\frac{1.3.q^2}{(1-t)^4}-\&c.\right\}=y_1+y_2.t+y_3.t^2+\&c.; \qquad [8356]$$

hence we have,

$$y_1=1-q+1.3.q^2-1.3.5.q^3+\&c. \qquad [8357]$$

Now if we put,

$$q=\frac{1}{2T^2}; \qquad [8358]$$

we shall have,*

$$\int_T^\infty dt.c^{-tt}=\frac{c^{-TT}}{2T}\cdot\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+\cfrac{4q}{1+\&c.}}}}} \qquad [8359]$$

Under this form we may employ it for all values of T; but for facility in calculation it is best to use it only when q does not exceed $\frac{1}{4}$; in other cases the two first series will more easily give the integral. To use the preceding continued fraction, we must reduce it to common fractions, alternately greater or less than the integral. The two first fractions of this series are $\frac{1}{1}$ and $\frac{1}{1+q}$, as in [8362]. The successive fractions of this series have such relation to each other, that the numerator of the i^{th} fraction is equal to the numerator of the $(i-1)$ fraction, *plus* the numerator of the $(i-2)$ fraction, multiplied by $(i-1).q$. The denominators are found in the same manner. These successive fractions are as follows;† [8360] [8361]

$$\frac{1}{1};\quad \frac{1}{1+q};\quad \frac{1+2q}{1+3q};\quad \frac{1+5q}{1+6q+3q^2};\quad \frac{1+9q+8q^2}{1+10q+15q^2};\ \&c. \qquad [8362]$$

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* (3893) Putting $q=\frac{1}{T^2}$ in [8357], we get $y_1=1-\frac{1}{2T^2}+\frac{1.3}{(2T^2)^2}-\frac{1.3.5}{(2T^2)^3}+\&c.$ [8359a]

Substituting this in [8339], we get $\int_T^\infty dt.c^{-tt}=\frac{c^{-TT}}{2T}\cdot y_1$; and by using the value of y_1, [8355], it becomes as in [8359]. [8359b]

† (3894) The continued fraction in [8359] may be put under the form,

$$0+\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+\&c.}}}} \qquad [8362a]$$

[8362′] *We shall now consider the second term of* $d\theta$, in the formula [8300], and shall investigate its influence on the refraction. This term is at its greatest value in the horizontal refraction, and in this case this second term becomes,*

Second term of $\delta\theta$.

[8363]
$$-\frac{\alpha.\frac{as}{l}.ds.c^{-\frac{as}{l}}.\left\{\tfrac{3}{2}.s-2\alpha.\left(1-c^{-\frac{as}{l}}\right)\right\}}{(1-\alpha).\left\{2s-2\alpha.\left(1-c^{-\frac{as}{l}}\right)\right\}^{\frac{3}{2}}}. \quad \text{[Second term of } d\theta\text{]}$$

The most sensible part of this integral corresponds to a very small value of s, because then the denominator is very small.† Therefore we may,

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Comparing this with [2290g', n, o, p], we obtain the following series of fractions, approximating towards the true value of the series;

Value of i,			1	2	3	4	5	&c.
Upper index,	0	1	1	1	1	1	1	&c.
[8362b] Fractions,	$\frac{1}{0}$;	$\frac{0}{1}$;	$\frac{1}{1}$;	$\frac{1}{1+q}$;	$\frac{1+2q}{1+3q}$;	$\frac{1+5q}{1+6q+3q^2}$;	$\frac{1+9q+8q^2}{1+10q+15q^2}$;	&c.
Lower index,	1	q	$2q$	$3q$	$4q$	$5q$	$6q$	&c.

The series of fractions [8362b] agrees with [8362]; and the rule for finding the successive terms, given in [8361—8362], is the same as in [2290q].

[8363a] * (3895) The second term of $d\theta$ [8300], which is neglected in [8300′, &c.], becomes for the horizontal refraction, where $\cos.\theta=0$, $\sin.\theta=1$ [8318], the same as in [8363]. Moreover it is evident that this assumed value of $\theta=100^{\circ}$, must give the greatest value to this second term, because the denominator becomes small, α being only 0,0003 [8277′],
[8363b] and $\frac{a}{l}=800$, nearly [8278d].

† (3896) To estimate roughly the magnitude of the quantity [8363], we shall substitute in it the approximate values of α, $\frac{a}{l}$ [8363b], and it becomes nearly equal to the following expression;

[8365a]
$$-\tfrac{1}{4}.\frac{sds.c^{-800s}.\{\frac{3}{2}.s-0{,}0006.(1-c^{-800s})\}}{\{2s-0{,}0006.(1-c^{-800s})\}^{\frac{3}{2}}}.$$

If we put successively $s=\frac{1}{8}$, $s=\frac{1}{80}$, $s=\frac{1}{800}$, $s=\frac{1}{8000}$, the corresponding values of
[8365b] c^{-800s} will be represented by c^{-100}, c^{-10}, c^{-1}, $c^{-\frac{1}{10}}$; and by using the value of c [8290′], they become nearly equal to $(\frac{1}{10})^{43}$; 0,00004; 0,4; 0,91, respectively. Now all the terms of [8365a] are multiplied by this factor c^{-800s}, which occurs in the

in this denominator and in the factor $\frac{3}{2}s-2\alpha.\left(1-c^{-\frac{as}{l}}\right)$, develop $c^{-\frac{as}{l}}$ in a series, and notice only its first terms. If we retain only the two first terms, and neglect the rest, which may be done without any sensible error, we shall have, [8364]

$$-\frac{\alpha.\frac{a}{l}.s^{\frac{1}{2}}ds.c^{-\frac{as}{l}}.\left\{3-4\alpha.\frac{a}{l}\right\}}{2^{\frac{5}{2}}.(1-\alpha).\left(1-\alpha.\frac{a}{l}\right)^{\frac{3}{2}}}. \qquad \text{[Second term of } d\theta\text{]} \qquad [8365]$$

If we take the integral from $s=0$ to $s=\infty$ [8307*i*], we shall have,* [8366]

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numerator; hence it is evident that the factor of ds, in [8365*a*], must be insensible, except *s be extremely small;* and in this last case the term $0{,}0006.(1-c^{-800s})$, corresponding to the preceding values of s [8635*b*], will be respectively $0{,}0006$; $0{,}0006$; $0{,}0003$; $0{,}00006$; and it vanishes when $s=0$; so that for very small values of s, it can be developed in a converging series, ascending according to the powers of s; and we shall have, from [56] Int., [8365*c*]

$$1-c^{-\frac{as}{l}}=1-\left\{1-\frac{as}{l}+\tfrac{1}{2}.\frac{a^2.s^2}{l^2}-\&c.\right\}=\frac{as}{l}-\&c. \qquad [8365d]$$

Hence we shall have, by neglecting the square and higher powers of s,

$$\tfrac{3}{2}s-2\alpha.\left(1-c^{-\frac{as}{l}}\right)=\tfrac{1}{2}s.\left\{3-4\alpha.\frac{a}{l}\right\}; \qquad [8365e]$$

$$\left\{2s-2\alpha.\left(1-c^{-\frac{as}{l}}\right)\right\}^{\frac{3}{2}}=\left\{2s-2s.\alpha.\frac{a}{l}\right\}^{\frac{3}{2}}=2^{\frac{3}{2}}s^{\frac{3}{2}}.\left\{1-\alpha.\frac{a}{l}\right\}^{\frac{3}{2}}. \qquad [8365f]$$

Substituting the developments [8365*e*, *f*] in [8363], it becomes as in [8365].

* (3897) The integral of the expression [8365] is easily reduced to the form,

$$-\frac{\alpha.\left(3-4\alpha.\frac{a}{l}\right)}{8.(1-\alpha).\left(1-\alpha.\frac{a}{l}\right)^{\frac{3}{2}}}.\sqrt{2.\frac{a}{l}}.\textstyle\int_0^\infty ds.s^{\frac{1}{2}}.c^{-\frac{as}{l}}; \qquad [8367a]$$

and we have $\frac{a}{l}.\int ds.s^{\frac{1}{2}}.c^{-\frac{as}{l}}=-s^{\frac{1}{2}}.c^{-\frac{as}{l}}+\frac{1}{2}\int s^{-\frac{1}{2}}ds.c^{-\frac{as}{l}}$, as is easily proved by taking the differentials of both members and reducing. Moreover we have by development, as in [55] Int., [8367*b*]

$$s^{\frac{1}{2}}.c^{-\frac{as}{l}}=\frac{s^{\frac{1}{2}}}{c^{\frac{as}{l}}}=\frac{s^{\frac{1}{2}}}{1+\frac{as}{l}+\frac{1}{2}.\frac{a^2.s^2}{l^2}+\&c.}; \qquad [8367c]$$

and this last expression evidently vanishes at the two limits $s=0$, $s=\infty$; so that it may be neglected in [8367*b*], if we take the integral between these values of s; and we

[8367]
$$-\frac{\alpha.\left(3-4\alpha.\frac{a}{l}\right).\sqrt{\frac{\pi l}{2a}}}{8.(1-\alpha).\left(1-\alpha.\frac{a}{l}\right)^{\frac{3}{2}}}\,; \qquad \left[\text{Second term of } \delta\theta\right]$$

Second term of $\delta\theta$ is insensible.

which never exceeds three or four seconds, and is therefore insensible near the horizon, where the refraction suffers great variations. We may

[8368] therefore, in all cases, neglect the second term of $d\theta$, which is given in [8300 line 2], and retain only the first term, contained in [8300 line 1].

[8369] If we use the values of $\frac{2K}{n^2}.(\rho)$, l, and a [8277, 8275, 8278], we shall find that in this hypothesis of a uniform temperature of the atmosphere, corresponding to the *zero* of the centigrade thermometer, and to a height of

[8370] the barometer of $0^{\text{metre}},76$, the horizontal refraction will be $7390'',71$.* *This exceeds the observed value by* $900''$, *nearly* [8281], *which shows the error of the hypothesis of a uniform temperature.* Indeed we know that the

[8370′] temperature decreases as the elevation increases; now as the air is condensed by the cold, it follows that the difference of density between a stratum of the atmosphere and that immediately above it, is by this means decreased. The limit of this decrease is when the difference is nothing, or the density

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[8367d] shall have $\frac{a}{l}.\int_0^\infty ds.s^{\frac{1}{2}}.c^{-\frac{as}{l}}=\frac{1}{2}\int_0^\infty s^{-\frac{1}{2}}ds.c^{-\frac{as}{l}}$; and by putting $s=\frac{ls'}{a}$ in this second expression, it becomes,

[8367e]
$$\frac{a}{l}.\int_0^\infty ds.s^{\frac{1}{2}}.c^{-\frac{as}{l}}=\tfrac{1}{2}.\sqrt{\frac{l}{a}}.\int_0^\infty ds'.s'^{-\frac{1}{2}}.c^{-s'};$$

[8367f] the limits of s' [8367e] being evidently the same as those of s. Substituting in the last expression of [8367e] the integral $\int_0^\infty ds'.s'^{-\frac{1}{2}}.c^{-s'}=2K=\sqrt{\pi}$ [8329, 8331], we get,

[8367g]
$$\frac{a}{l}.\int_0^\infty ds.s^{\frac{1}{2}}.c^{-\frac{as}{l}}=\tfrac{1}{2}.\sqrt{\frac{l}{a}}.\sqrt{\pi}=\tfrac{1}{2}.\sqrt{\frac{\pi l}{a}}.$$

[8367h] Substituting this in [8367a], it becomes as in [8367]. Now substituting in [8367] the values of a, l, α [8278, 8275, 8277′], and multiplying by the radius in seconds $636620''$, it becomes as in [8368].

[8370a] * (3898) This value is computed from the formula [8333], using the values of α, l, &c. mentioned in [8367h], and reducing it to seconds, by multiplying by the radius in seconds. By a rough calculation it was found that the result given in [8370] is nearly correct.

is constant; and we have seen, in [8279], that in this case the horizontal refraction is too small. *The constitution of the atmosphere and the refractions are therefore between the two limits given by the hypothesis we have just considered; but we may obtain more approximate limits in the following manner.* [8371]

Third hypothesis, from which we can deduce the formulas of Bouguer, Simpson and Bradley.

6. *The differential expression of* $d\theta$ [8262] *can be rigorously integrated by supposing,*

$$\frac{a}{r}=\left\{\frac{1+\frac{4K}{n^2}\cdot\rho}{1+\frac{4K}{n^2}\cdot(\rho)}\right\}^{m}. \qquad [8372]$$

If we then put,

$$\left\{\frac{1+\frac{4K}{n^2}\cdot\rho}{1+\frac{4K}{n^2}\cdot(\rho)}\right\}^{m-\frac{1}{2}}.\sin.\Theta=z; \qquad [8373]$$

it becomes,*

$$d\theta=\frac{-dz}{(2m-1).\sqrt{1-zz}}. \qquad [8374]$$

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* (3899) If we substitute the value of u [8262*u*] in [8372, 8373], and use k [8192*b*], they become respectively,

$$\frac{a}{r}=\left\{\frac{1+k\rho}{1+k(\rho)}\right\}^{m}=\mathrm{u}^{-2m}; \qquad z=\mathrm{u}^{1-2m}.\sin.\Theta. \qquad [8374a]$$

Substituting this second value of $\frac{a}{r}$ in [8262*w*], and then z, with its differential $dz=-(2m-1).\mathrm{u}^{-2m}.d\mathrm{u}.\sin.\Theta$, we get successively, [8374*b*]

$$d\theta=\frac{d\mathrm{u}.\mathrm{u}^{-2m}.\sin.\Theta}{\sqrt{1-\mathrm{u}^{2-4m}.\sin.^2\Theta}}=-\frac{dz}{(2m-1).\sqrt{1-zz}}. \qquad [8374c]$$

This last expression is of the same form as in [8374], and its integral is,

$$\delta\theta=\text{constant}-\frac{\text{ang.}(\sin.z)}{2m-1}\ \text{[49] Int.} \qquad [8374d]$$

This integral is to be taken from its first limit at the surface of the earth, where ρ becomes (ρ) [8254], to its second limit at the upper surface of the atmosphere, where $\rho=0$. At the first limit the value of z [8373] becomes $z=\sin.\Theta$, and at the second limit it becomes $z=\left\{1+\frac{4K}{n^2}.(\rho)\right\}^{-m+\frac{1}{2}}.\sin.\Theta$, as in [8375]. Therefore the expression of $\delta\theta$ [8374*d*], must vanish when $z=\sin.\Theta$; hence we get $0=\text{constant}-\frac{\Theta}{2m-1}$; [8374*e*] [8374*f*]

[8375] Integrating this expression, from $z = \sin.\Theta$ to $z = \frac{\sin.\Theta}{\left\{1+\frac{4K}{n^2}\cdot(\rho)\right\}^{m-\frac{1}{2}}}$, we obtain, as in [8374g, &c.],

Formula similar to Bouguer's or Simpson's.

[8376] $$\delta\theta = \frac{1}{2m-1}\cdot\left\{\Theta - \text{ang.sin.}\left(\frac{\sin.\Theta}{\left[1+\frac{4K}{n^2}\cdot(\rho)\right]^{m-\frac{1}{2}}}\right)\right\}.$$

This expression may be put under the form,*

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subtracting this from [8374d], we obtain,

[8374g] $$\delta\theta = \frac{1}{2m-1}\cdot\{\Theta - \text{ang.sin.}z\};$$

now substituting the second value or limit of z [8374f], it becomes as in [8376].

[8374h] If we put $m = 3,902\ldots$, and $\left\{1+\frac{4K}{n^2}\cdot(\rho)\right\}^{-m+\frac{1}{2}} = 0,9978668785$, in [8376], we shall get,

Bouguer's formula for the refraction.

[8374i] $$\delta\theta = \frac{2}{13.61}\cdot\{\Theta - \text{ang.}(\sin. = 0,9978668785.\sin.\Theta)\}.$$

being the same as the rule given by Bouguer, in the memoir which gained the prize of the Academy of Arts and Sciences of Paris, in 1729. If we put, $m = 3\frac{1}{4}$, and

[8374k] $\left\{1+\frac{4K}{n^2}\cdot(\rho)\right\}^{-m+\frac{1}{2}} = 0,9986$, we shall get,

[8374l] $$\delta\theta = \frac{2}{11}\cdot\{\Theta - \text{ang.}(\sin.0,9986.\sin.\Theta)\};$$

Simpson's formula for the refraction.

which is Simpson's formula, published in 1743; about fourteen years after Bouguer's

[8374m] method had been made known. These methods may be considered as identical, since the only difference between them is in the data, deduced from observations. The name of

[8374n] Simpson has been generally annexed to this process of computing the refraction, though by right it appertains to Bouguer; remarking, however, that their methods of investigation are different.

* (3900) We shall put for brevity $A = \left\{1+\frac{4K}{n^2}\cdot(\rho)\right\}^{m-\frac{1}{2}}$, $2x = (2m-1).\delta\theta$.

[8377a] Substituting these in [8376], after multiplying it by $2m-1$, we get,

[8377b] $$2x = \Theta - \text{ang.sin.}\frac{\sin.\Theta}{A}, \qquad \text{or} \qquad \Theta - 2x = \text{ang.sin.}\frac{\sin.\Theta}{A};$$

and by taking its sine, we obtain $\sin.(\Theta - 2x) = \frac{\sin.\Theta}{A}$. The first member of this expression

[8377c] being developed, is $\sin.\Theta.\cos.2x - \cos.\Theta.\sin.2x$ [22] Int. Substituting this, and multiplying by $\frac{A}{\cos.\Theta}$, we get $A.\{\text{tang.}\Theta.\cos.2x - \sin.2x\} = \text{tang.}\Theta$; whence $\text{tang.}\Theta = \frac{A.\sin.2x}{A.\cos.2x - 1}$.

[8377d] Now from [30] Int., we have $\text{tang.}(\Theta - x) = \frac{\text{tang.}\Theta - \text{tang.}x}{1+\text{tang.}\Theta.\text{tang.}x}$; and if we substitute the

Formula similar to Bradley's.

$$\text{tang.}\left[\frac{2m-1}{2}\right].\delta\theta=\left\{\frac{\left[1+\frac{4K}{n^2}.(\rho)\right]^{m-\frac{1}{2}}-1}{\left[1+\frac{4K}{n^2}.(\rho)\right]^{m-\frac{1}{2}}+1}\right\}.\text{tang.}\left[\Theta-\left(\frac{2m-1}{2}\right).\delta\theta\right].$$ [8377]

For the horizontal refraction, we have $\Theta=100^\circ$; consequently,* [8378]

$$\text{tang.}\left\{\Theta-\left[\frac{2m-1}{2}\right].\delta\theta\right\}=\frac{1}{\text{tang.}\left[\frac{2m-1}{2}\right].\delta\theta}.$$ [8379]

Now $\frac{K}{n^2}.(\rho)$ is a very small fraction; and if we neglect its square, we shall have by development, as in [8379*b*], the following approximate equation;

$$\frac{\left\{1+\frac{4K}{n^2}.(\rho)\right\}^{m-\frac{1}{2}}-1}{\left\{1+\frac{4K}{n^2}.(\rho)\right\}^{m-\frac{1}{2}}+1}=\left[\frac{2m-1}{2}\right].\frac{2K}{n^2}.(\rho);$$ [8380]

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preceding value of tang.Θ, then multiply the numerator and denominator by $A.\cos.2x-1$, we shall get [8377*f*]; from this we deduce [8377*g*], by multiplying the numerator and denominator by $\cos.x$; then making successive reductions, using [22, 24] Int., we obtain [8377*h*]; [8377*e*]

$$\text{tang.}(\Theta-x)=\frac{A.(\sin.2x-\cos.2x.\text{tang}.x)+\text{tang}.x}{A.(\cos.2x+\sin.2x.\text{tang}.x)-1}$$ [8377*f*]

$$=\frac{A.(\sin.2x.\cos.x-\cos.2x.\sin.x)+\sin.x}{A.(\cos.2x.\cos.x+\sin.2x.\sin.x)-\cos.x}$$ [8377*g*]

$$=\frac{A.\sin.(2x-x)+\sin.x}{A.\cos.(2x-x)-\cos.x}=\frac{A.\sin.x+\sin.x}{A.\cos.x-\cos.x}=\frac{A+1}{A-1}.\frac{\sin.x}{\cos.x}=\left(\frac{A+1}{A-1}\right).\text{tang}.x.$$ [8377*h*]

Multiplying this last expression by $\frac{A-1}{A+1}$, we get $\text{tang}.x=\left(\frac{A-1}{A+1}\right).\text{tang}.(\Theta-x)$; and by re-substituting the assumed values of A, x [8377*a*], it becomes as in [8377]. [8377*i*]

* (3901) Putting $\Theta=100^\circ$, we have,

$$\text{tang.}\left\{\Theta-\left[\frac{2m-1}{2}\right].\delta\theta\right\}=\text{cotang.}\left[\frac{2m-1}{2}\right].\delta\theta=\frac{1}{\text{tang.}\left[\frac{2m-1}{2}\right].\delta\theta},$$ [8379*a*]

as in [8379]. Moreover if we develop $\left\{1+\frac{4K}{n^2}.(\rho)\right\}^{m-\frac{1}{2}}$ according to the powers of K, neglecting the square and higher powers, it becomes $1+(m-\frac{1}{2}).\frac{4K}{n^2}.(\rho)$, or $1+(2m-1).\frac{2K}{n^2}.(\rho)$. Substituting this in the first member of [8380], it becomes as in its second member, always neglecting terms of the order K^2. [8379*b*] [8379*c*]

therefore we shall have very nearly,*

[8381] $$\mathrm{tang.}\left[\frac{2m-1}{2}\right].\delta\theta = \sqrt{\left[\frac{2m-1}{2}\right].\frac{2K}{n^2}.(\rho)};$$

and by taking the arc itself for its tangent, which may be done in this case without any sensible error, we shall have,

[8382] $$\delta\theta = \left\{\frac{\frac{4K}{n^2}.(\rho)}{2m-1}\right\}^{\frac{1}{2}}.$$

Horizontal refraction in this third hypothesis.

[8383] *If we have no other object than to consider the refraction, we may determine* m, *so that the second member of the equation* [8382] *may represent the observed refraction*, which we have supposed to be $6500''$ [8281]. Then the general expression of $\mathrm{tang.}\left[\frac{2m-1}{2}\right].\delta\theta$ [8377], will give for all altitudes the value of the refraction $\delta\theta$. *This method has been followed by several astronomers*,†

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* (3902) Substituting the values [8379, 8380] in the second member of [8377], we get,

[8381a] $$\mathrm{tang.}\left[\frac{2m-1}{2}\right].\delta\theta = \left[\frac{2m-1}{2}\right].\frac{2K}{n^2}.(\rho).\frac{1}{\mathrm{tang.}\left[\frac{2m-1}{2}\right].\delta\theta};$$

[8381b] Multiplying by $\mathrm{tang.}\left[\frac{2m-1}{2}\right].\delta\theta$, and extracting the square root, it becomes as in [8381]. Writing the arc $\left[\frac{2m-1}{2}\right].\delta\theta$ for its tangent, and dividing by $\frac{2m-1}{2}$, it becomes as in [8382].

† (3903) Bradley is one of the astronomers who used this method; for his table of

[8383a] refraction, which is given by Dr. Maskelyne, in the "Requisite Tables," to be used with the Nautical Almanac, is founded on the formula [8377], supposing $\frac{2m-1}{2}=3$, or $m=\frac{7}{2}$;

Bradley's rule for the refraction.

[8383b] and taking the constant coefficient in the second member, so that the horizontal refraction may be 33^m, which he supposed to be its mean value, corresponding to 50° of Fahrenheit's thermometer, and 29,6 inches of the barometer. If we change, in the first member of [8377], the tangent into its arc on account of its smallness, and then divide by

[8383c] $\frac{2m-1}{2}=3$, we shall obtain the formula as it is given by Bradley, $\delta\theta = 57^s.\mathrm{tang.}(\circleddash - 3\delta\theta)$, which makes the horizontal refraction nearly equal to 33^m. The table of refractions founded on this rule, and on the horizontal refraction 33^m, has been much used by the English astronomers; it may be expressed in a logarithmic form in the following manner;

[8383d] log.tang.(app. zenith dist. — 3.refraction $\delta\theta$) — 8,2438534 = log. of refract. $\delta\theta$, in sexages. sec.

[8383d′] If the elevation of the barometer in English inches be H, and the height of Fahrenheit's

in constructing tables of refraction, and these tables agree very well with observation. But in order to correspond exactly with the real state of [8384]
nature, *it is necessary that the constitution of the atmosphere, here adopted, should not only agree with the observed refraction, but also with the height of the barometer, and with the observed diminution of the heat as the altitude of the observer increases.* We shall therefore examine both of these phenomena in the preceding hypothesis [8372].

We shall resume the equation [8294'],

$$dp = (g).a\rho.d.\frac{a}{r}. \qquad [8385]$$

If we substitute $\left(\frac{1+\frac{4K}{n^2}.\rho}{1+\frac{4K}{n^2}.(\rho)}\right)^m$ for $\frac{a}{r}$ [8372], we shall have after integration,

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thermometer T, we shall have for the actual refraction, by Bradley's rule, as it is given by Dr. Maskelyne, [8383e]

$$\text{actual refraction} = \text{mean refraction} \times \frac{400^\circ}{350^\circ + T} \times \frac{H}{29{,}6}. \qquad [8383f]$$

The mean refraction being supposed to correspond to $T = 50^\circ$, and $H = 29{,}6$ inches. [8383g]
From this rule it appears that if Fahrenheit's thermometer vary 1° from its mean value $T = 50^\circ$, the refraction $\delta\theta$ will vary by the quantity $\frac{\delta\theta}{400} = 0{,}00250.\delta\theta$; and by [8383h]
multiplying it by $\frac{180^\circ}{100^\circ}$, we get $0{,}00450.\delta\theta$, for the variation corresponding to one degree of the centigrade thermometer, instead of $0{,}00375.\delta\theta$, given by Gay-Lussac, [8488]; so that Bradley's correction is too great by one fifth part. To correct Bradley's factor $\frac{400^\circ}{350^\circ+T}$ or $\frac{400^\circ}{400^\circ+(T-50^\circ)}$, we must increase the term 400°, in this last [8383i]
expression, by one fifth part, making it $\frac{480^\circ}{480^\circ+(T-50^\circ)}$; and the corrected expression [8383f] will be,

$$\text{actual refraction} = \text{mean refraction} \times \frac{480^\circ}{430^\circ+T} \times \frac{H}{29{,}6}. \qquad [8383k]$$

According to Bradley the mean horizontal refraction, when $T = 50^\circ$ and $H = 29{,}6$ inches, is $1980^s = 6111'',1$. Now at the temperature of melting ice, we have $T = 32^\circ$; [8383l]
and if we suppose the height of the barometer to be $0^{\text{metre}},76 = 29^{\text{inches}},922$, we shall get for the refraction corresponding, the following expression,

$$6111'',1 \times \frac{480}{462} \times \frac{29{,}922}{29{,}6} = 6418''; \qquad [8383m]$$

which is $82''$ less than that which is assumed by La Place, in [8281], as being more in conformity with the best observations. [8383n]

[8385′] by observing that p, ρ, become nothing at the same point,*

[8386] $$p=(g).a.(\rho).\frac{\left\{\frac{\rho}{(\rho)}\cdot\left[1+\frac{4K}{n^2}\cdot\rho\right]^m+\frac{1}{(m+1).\frac{4K}{n^2}\cdot(\rho)}\cdot\left\{1-\left[1+\frac{4K}{n^2}\cdot\rho\right]^{m+1}\right\}\right\}}{\left[1+\frac{4K}{n^2}\cdot(\rho)\right]^m}.$$

[8386a] * (3904) Putting for brevity $b=1+\frac{4K}{n^2}\cdot(\rho)$, $x=1+\frac{4K}{n^2}\cdot\rho$, we get, from [8372],

[8386b] $$\frac{a}{r}=\frac{x^m}{b^m};\qquad d.\frac{a}{r}=\frac{mx^{m-1}\,dx}{b^m};\qquad \rho=\frac{n^2.(x-1)}{4K}.$$

Substituting these values in [8385] we obtain,

[8386c] $$dp=(g).a.\frac{n^2.(x-1)}{4K}\cdot\frac{mx^{m-1}\,dx}{b^m}=\frac{(g).a.n^2}{4K.b^m}\cdot\{mx^m\,dx-mx^{m-1}\,dx\};$$

and by integration,

[8386d] $$p=\frac{(g).a.n^2}{4K.b^m}\cdot\left\{\frac{mx^{m+1}}{m+1}-x^m+\text{constant}\right\}.$$

[8386e] Now when $\rho=0$, we have $p=0$ [8385′]; and in this case the value of x [8386a] becomes $x=1$. Substituting these expressions in [8386d], we get $0=\frac{m}{m+1}-1+\text{constant}$, or $\text{constant}=\frac{1}{m+1}$; hence [8386d] becomes as in [8386g].

[8386f] This value of p is easily reduced to the form [8386h], and by introducing the factor $\frac{n^2}{4K}$, within the braces, it becomes as in [8386i]. Finally, putting ρ for its value [8386b], it is reduced to the last form in [8386k], which is the same as [8386].

[8386g] $$p=\frac{(g).a.n^2}{4K.b^m}\cdot\left\{\frac{mx^{m+1}}{m+1}-x^m+\frac{1}{m+1}\right\}$$

[8386h] $$=\frac{(g).a.n^2}{4K.b^m}\cdot\left\{x^m.(x-1)+\frac{1}{m+1}\cdot(1-x^{m+1})\right\}$$

[8386i] $$=\frac{(g).a}{b^m}\cdot\left\{x^m.\frac{n^2.(x-1)}{4K}+\frac{(1-x^{m+1})}{(m+1).\frac{4K}{n^2}}\right\}$$

[8386k] $$=\frac{(g).a}{b^m}\cdot\left\{x^m\rho+\frac{(1-x^{m+1})}{(m+1).\frac{4K}{n^2}}\right\}=\frac{(g).a.(\rho)}{b^m}\cdot\left\{\frac{\rho}{(\rho)}\cdot x^m+\frac{(1-x^{m+1})}{(m+1).\frac{4K}{n^2}\cdot(\rho)}\right\}.$$

This expression gives very nearly,*

$$p=(g).a.(\rho).m.\frac{2K}{n^2}.(\rho).\frac{\rho^2}{(\rho)^2}.$$ [8387]

(p) [8288', 8288] being the pressure at the surface of the sea, where $\rho=(\rho)$, and $\frac{a}{r}=1$. Now we have, in [8298], [8388]

$$(p)=(g).(\rho).l;$$ [8389]

hence we obtain,

$$\frac{p}{(p)}=m.\frac{a}{l}.\frac{2K}{n^2}.(\rho).\frac{\rho^2}{(\rho)^2}.$$ [8390]

This equation gives, at the surface of the earth,†

$$m=\frac{\frac{l}{a}}{\frac{2K}{n^2}.(\rho)};$$ [8391]

therefore the horizontal refraction $\delta\theta$ becomes,

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* (3905) Using for brevity the symbol $k=\frac{4K}{n^2}$ [8192b], we get, by development, [8387a]

$(1+k\rho)^m=1+mk\rho+\&c.;\qquad 1-[1+k\rho]^{m+1}=-(m+1).k\rho-\frac{(m+1).m}{2}.k^2\rho^2-\&c.;$ [8387b]

$$\{1+k(\rho)\}^m=1+\&c.$$ [8387c]

Substituting these in [8386], it becomes, by neglecting terms of the order k^2,

$$p=(g).a.(\rho).\left\{\frac{\rho}{(\rho)}.(1+mk\rho)-\frac{\rho}{(\rho)}.(1+\tfrac{1}{2}mk\rho)\right\}=(g).a.(\rho).m.\tfrac{1}{2}k(\rho).\frac{\rho^2}{(\rho)^2};$$ [8387d]

being the same as in [8387]. Dividing this by (p) [8389], we get [8390].

† (3906) At the surface of the earth we have $\rho=(\rho)$, $p=(p)$ [8388]; and then [8390] becomes $1=m.\frac{a}{l}.\frac{2K}{n^2}.(\rho)$; hence we get m [8391]. Substituting this in [8382], it becomes, by using $\frac{4K}{n^2}=k$ [8387a], [8391a] [8391b]

$$\delta\theta=\left\{\frac{\frac{k(\rho)}{4l}}{\frac{4l}{ak(\rho)}-1}\right\}^{\frac{1}{2}}=\left\{\frac{\frac{1}{4}\{k(\rho)\}^2}{\frac{l}{a}-\frac{1}{4}k(\rho)}\right\}^{\frac{1}{2}}=\frac{\frac{1}{2}k(\rho)}{\sqrt{\frac{l}{a}-\frac{1}{4}k(\rho)}}.$$ [8391c]

Re-substituting the value of k [8391b], it becomes as in [8392]; and by using the values mentioned in [8392'], and multiplying by the radius in seconds, it becomes as in [8393].

[8392]
$$\delta\theta = \frac{\frac{2K}{n^2}\cdot(\rho)}{\sqrt{\frac{l}{a}-\frac{1}{2}\cdot\frac{2K}{n^2}\cdot(\rho)}}.$$

[8392′] If we substitute for a, l, and $\frac{2K}{n^2}\cdot(\rho)$, their values [8278, 8275, 8277], we shall have,

[8393]
$$\delta\theta = 5630''.$$

[8394] *This is less than the observed refraction* [8281], *but greater than that which results from a constant density.* Hence the actual constitution of the atmosphere is an intermediate one between the supposition of a uniform temperature, and that which results from the assumed relation between r and ρ [8372].

[8395] In this last hypothesis *the densities of the strata of the atmosphere decrease in arithmetical progression, when those altitudes increase in a similar*
[8395′] *progression.** For if we put $r = a.(1+s)$, we shall have very nearly,

[8396]
$$as = 4m.\frac{K}{n^2}.(\rho).a.\left\{1-\frac{\rho}{(\rho)}\right\} = 2l.\left\{1-\frac{\rho}{(\rho)}\right\};$$

[8397] as being the height of the stratum of the atmosphere. The limit of the atmosphere is at the point where $\rho = 0$, and then [8396] becomes
[8397′] $as = 2l$; *the height of the atmosphere is in this case double that of its height in the hypothesis of a constant density* [8290].

Height of the atmosphere in this third hypothesis.

The expression of p [8390] gives,†

[8396a] * (3907) Developing the first expression of $\frac{a}{r}$ [8374a], according to the powers of k, and neglecting k^2, we get $\frac{a}{r} = 1-mk.\{(\rho)-\rho\}$. Putting this equal to $\frac{a}{r} = \frac{1}{1+s} = 1-s$,
[8396b] nearly [8395′], we get $s = mk.\{(\rho)-\rho\} = mk.(\rho).\left\{1-\frac{\rho}{(\rho)}\right\}$. Multiplying this by a, and re-substituting the value of k [8387a], it becomes as in the first expression of as,
[8396c] [8396]. Substituting $4m.\frac{K}{n^2}.(\rho).a = 2l$ [8391], it becomes as in the last form of [8396]. Whence it evidently follows that when the density ρ varies in arithmetical progression, the altitude as varies in a similar progression, as in [8395]. Finally when $\rho = 0$, we get, from [8396], $as = 2l$, as in [8397′].

[8398a] † (3908) Substituting $m.\frac{a}{l}.\frac{2K}{n^2}.(\rho) = 1$ [8391a], in [8390], we get $\frac{p}{(p)} = \frac{\rho^2}{(\rho)^2}$;

$$\frac{p.(\rho)}{(p).\rho} = \frac{\rho}{(\rho)} = 1 - \frac{as}{2l}. \qquad \text{[Expression of the temperature.]}$$ [8398]

The function $\frac{p.(\rho)}{(p).\rho}$ *is important to be considered, because it expresses the law of the heat of the strata of the atmosphere.* For, at the same temperature, it has been found by experiment that the pressure of the air is proportional to its density; but this ratio increases with the temperature, and may be supposed to represent it;* since the particles of air appear to be subjected only to the repulsive force of heat, and it is natural to suppose that this force increases in the same ratio as the heat. *It follows from the preceding expression of* $\frac{p.(\rho)}{(p).\rho}$ [8398], *that the heat of the strata of the atmosphere decreases as their density in an arithmetical progression* [8400e]. A diminution of $\frac{1}{250}$ in the value of $\frac{p.(\rho)}{(p).\rho}$, is the same as to suppose a diminution of $\frac{1}{250}$ in the value of $1-\frac{as}{2l}$; thus in setting out from the surface of the earth, we must ascend to the height of $\frac{2l}{250}$ or $63^{\text{metres}},8$, [8402]

[8399]

Temperature of the atmosphere.

[8400]

[8401]

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multiplying it by $\frac{(\rho)}{\rho}$, we obtain, as in [8398], $\frac{p.(\rho)}{(p).\rho} = \frac{\rho}{(\rho)}$; but from [8396] we have $\frac{\rho}{(\rho)} = 1 - \frac{as}{2l}$; hence by substitution we have $\frac{p.(\rho)}{(p).\rho} = 1 - \frac{as}{2l}$, as in [8398]. [8398b]

* (3909) If we suppose the pressure p to be proportional to the density ρ, multiplied by the heat h, we shall have $p = Ch.\rho$; C being a constant quantity; and if we suppose [8400a]
h to be equal to unity at the surface of the earth, the preceding equation will become at that surface $(p) = C.(\rho)$. Dividing the expression of p, by that of (p), we get [8400b]
$\frac{p}{(p)} = \frac{\rho}{(\rho)}.h$, or $h = \frac{p.(\rho)}{(p).\rho}$, as in [8399, &c.]; hence we have, by using [8398b], [8400c]
$h = 1 - \frac{as}{2l} = \frac{\rho}{(\rho)}$; and it follows from this equation that if the height as increases in [8400d]
arithmetical progression, the heat h and the density ρ will decrease in arithmetical [8400e]
progression nearly, as in [8400]. For the sake of symmetry, if we put the heat at the surface of the earth equal to (h), or $(h) = 1$, the preceding equation [8400d] may be [8400f]
put under the form $\frac{(h)-h}{(h)} = \frac{as}{2l}$. The first member of this equation evidently represents [8400g]
the decrement of heat; and if we suppose this to be $\frac{1}{250}$, as in [8401], we shall have $as = \frac{2l}{250} = 0{,}008.l = 63^{\text{metres}},8$ [8275], as in [8402]. [8400h]

[8400h], to experience a decrease of $\frac{1}{250}$ in the elastic force of the air,
[8402′] with the same degree of density. *This corresponds very nearly to a fall of one degree* in the centigrade thermometer. All the observations concur in showing that this elevation is too small, and that the decrease of the heat is less rapid; therefore this third hypothesis, now under consideration, does*
[8403] *not represent either the observed refraction, or the observed law of the diminution of heat.*

In the hypothesis of a constant density, we have,†

[8404] $$\frac{p\cdot(\rho)}{(p)\cdot\rho} = 1 - \frac{as}{l};$$

therefore we need only to ascend to half the height required by the preceding hypothesis to obtain a diminution of one degree in the thermometer;
[8405] *consequently this hypothesis is still farther than the other from satisfying*

[8402a] * (3910) This appears from the formula [8488], by putting $x = -1$, which changes the value 1, at the surface, into $1-0{,}00375$, or $1-\frac{1}{266}$ nearly. Now it appears from
[8402b] the experiment of Gay Lussac [8448], that an elevation of 6980 metres produces a depression of $40°\frac{1}{4}$ in the centigrade thermometer, which corresponds to 173 metres for
[8402c] one degree, instead of $63^{\text{metres}},8$ [8402]. Hence it appears that the diminution of heat is much less rapid, by observation, than by the third hypothesis here used, as is remarked in [8403].

[8404a] † (3911) In the hypothesis of a constant density, the height of the atmosphere is l, [8275]; and if the height of the observer be as [8395′], the height of the superincumbent atmosphere, at that elevation, will be $l-as$. Now the pressure being in proportion to
[8404b] the height of the atmosphere, we have $(p):p::l:l-as$; hence $\frac{p}{(p)} = 1-\frac{as}{l}$; and
[8404c] since by hypothesis $\frac{(\rho)}{\rho}=1$, we have $\frac{p\cdot(\rho)}{(p)\cdot\rho} = 1-\frac{as}{l}$, as in [8404]. The first member of this equation is represented by h [8400c]; moreover $(h)=1$ [8400f];
[8404c′] hence we have in this hypothesis $\frac{h}{(h)} = 1-\frac{as}{l}$, or $as = l.\frac{(h)-h}{(h)}$; whereas the
[8404d] hypothesis of a density decreasing with the height, in arithmetical progression, gives, as in [8400g], $as = 2l.\frac{(h)-h}{(h)}$; therefore the preceding value of as is only half of this last expression, as is observed in [8405]. If we put as in [8400g, h] $\frac{(h)-h}{(h)} = \frac{1}{250}$, and use l,
[8404e] [8275], we shall have, from [8404c'], $as = \frac{l}{250} = 32^{\text{metres}}$, nearly, which differs from
[8404f] observation, 173^{metres} [8402b], much more than that deduced from the hypothesis used in [8400, &c.], namely $63^{\text{metres}},8$ [8402].

the observations upon the refraction and heat. At the same time we see that the nearer we approach to the observed refraction, the nearer it will agree relative to the heat.

7. *The constitution of the atmosphere being comprised between the two limits of a density decreasing in arithmetical progression, and a density decreasing in geometrical progression, an hypothesis which partakes of both these progressions, ought, it would seem, to represent both the refraction and the observed decrement of the heat of the atmospherical strata. The following hypothesis has both these advantages, and also furnishes a very simple method of computation.* [8406]

We shall resume the equation [8262], substituting $\frac{a}{r}=1-s$ [8284], and we shall have very nearly, from [8286],* [8407]

$$d\delta=\frac{-\alpha.d.\frac{\rho}{(\rho)}.\sin.\theta}{(1-\alpha).\sqrt{\cos.^2\theta-2\alpha.\left[1-\frac{\rho}{(\rho)}\right]+2s-2s.\cos.^2\theta}}; \qquad [8408]$$

s being a very small fraction, we may neglect the term $-2s.\cos.^2\theta$, in comparison with $\cos.^2\theta$; thus we shall have, [8409]

$$d\delta=\frac{-\alpha.\frac{d\rho}{(\rho)}.\sin.\theta}{(1-\alpha).\sqrt{\cos.^2\theta-2\alpha.\left[1-\frac{\rho}{(\rho)}\right]+2s}}. \qquad [8410]$$

We shall now put,

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* (3912) Substituting $\sin.^2\theta=1-\cos.^2\theta$, in the denominator of $d\delta$ [8286], and using the mean value of the factor $1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)=1-\alpha$ [8286′, 8287], it becomes, by neglecting s^2,

$$d\delta=-\frac{\alpha.\frac{d\rho}{(\rho)}.(1-s).\sin.\theta}{(1-\alpha).\sqrt{\cos.^2\theta-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)+2s-2s.\cos.^2\theta}}. \qquad [8408a]$$

We may neglect the term containing s, in the numerator, because this quantity is of the same order, relative to s, as the term in [8300 line 2], which is shown to be insensible in [8368]; hence the expression of $d\delta$ [8408a] becomes as in [8408]; and by neglecting the small term $-2s.\cos.^2\theta$, in the radical, it becomes as in [8410]. [8408b]

[8411] $$u = s - \alpha . \left\{ 1 - \frac{\rho}{(\rho)} \right\}; \quad \text{or,}$$

Fourth hypothesis, proposed by La Place.

[8411'] $$s = u + \alpha . \left\{ 1 - \frac{\rho}{(\rho)} \right\};$$

[8412] $$\rho = (\rho) . \left\{ 1 + \frac{fu}{l'} \right\} . c^{-\frac{u}{l'}}; \quad \text{or,}$$

[8412'] $$\frac{\rho}{(\rho)} = \left\{ 1 + \frac{fu}{l'} \right\} . c^{-\frac{u}{l'}}.$$

f and l' being two indeterminate constant quantities. *This value of* ρ *partakes both of the arithmetical and geometrical progressions.** We must
[8413] determine f and l' so as to represent the height of the barometer and the horizontal refraction; and if this value satisfies also the observed decrease of the heat of the strata of the atmosphere, we may consider it as representing the true constitution of the atmosphere, to be used in the construction of a table of refraction. Substituting the values of s, ρ, [8411', 8412], in [8410], we obtain,

[8414] $$d\delta = \frac{\alpha . \frac{du}{l'} . \left[1 - f + \frac{fu}{l'} \right] . c^{-\frac{u}{l'}} . \sin . \theta}{(1-\alpha) . \sqrt{\cos .^2 \theta + 2u}}.$$

We shall now put,

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* (3913) For extremely small values of s, it is evident, by inspection of [8411, 8412], that u, s, ρ, vary nearly in arithmetical progression; observing that,

[8413a] $$c^{-\frac{u}{l'}} = 1 - \frac{u}{l'} + \tfrac{1}{2} . \frac{u^2}{l'^2} - \&c. \quad [56] \text{ Int.},$$

[8413b] and that $\frac{u}{l'}$ is very small [8436, 8438]. When s and u become larger, the geometrical function $c^{-\frac{u}{l'}}$ begins more sensibly to operate in [8412]; and the expression of ρ [8412] partakes sensibly of the nature of both kinds of progressions, as in [8413]. Now from

[8413c] [8412'] we get $\frac{\rho}{(\rho)} = c^{-\frac{u}{l'}} + \frac{fu}{l'} . c^{-\frac{u}{l'}}$; whose differential is,

[8413d] $$\frac{d\rho}{(\rho)} = - \frac{du}{l'} . \left\{ 1 - f + \frac{fu}{l'} \right\} . c^{-\frac{u}{l'}}.$$

Substituting in [8410] the values [8411', 8413d], it becomes as in [8414]. Lastly, substituting $u = l' . \mathrm{t}^2 - \frac{1}{2} . \cos .^2 \theta$ [8415], in [8414], we get [8416].

$$\cos.^2\Theta + 2u = 2l'.\mathrm{t}^2; \quad \text{or} \quad \mathrm{t} = \sqrt{\frac{\cos.^2\Theta + 2u}{2l'}}; \qquad [8415]$$

and we shall have,

$$d\delta = \frac{2\alpha dt.\sin.\Theta}{(1-\alpha).\sqrt{2l'}}.\left\{1 - f - \frac{f.\cos.^2\Theta}{2l'} + f\mathrm{t}^2\right\}.c^{\frac{\cos.^2\Theta}{2l'} - \mathrm{t}^2}. \qquad [8416]$$

The integral of this expression is to be taken from* $\mathrm{t} = \frac{\cos.\Theta}{\sqrt{2l'}}$ to $\mathrm{t} = \infty$; [8417]
therefore if we put,

$$T = \frac{\cos.\Theta}{\sqrt{2l'}}; \qquad [8418]$$

$$\textstyle\int_T^\infty d\mathrm{t}.c^{-\mathrm{tt}} = c^{-TT}.\Psi(T); \qquad [8419]$$

$$\Psi(T) = c^{TT}.\textstyle\int_T^\infty d\mathrm{t}.c^{-\mathrm{tt}}; \qquad [8419']$$

we shall have,†

General expression of the refraction [8420] in La Place's hypothesis.

$$\delta\delta = \frac{2\alpha.\sin.\Theta}{(1-\alpha).\sqrt{2l'}}.(1 - \tfrac{1}{2}f - fT^2).\Psi(T) \qquad 1$$

$$+ \frac{\alpha f}{2.(1-\alpha).l'}.\sin.\Theta.\cos.\Theta. \qquad 2$$

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* (3914) The integral is to be taken, as in [8307*i*], from $s=0$ to $s=\infty$; and therefore as in [8271], from $\rho = (\rho)$ to $\rho = 0$. At the first of these limits, namely $s=0$, $\rho=(\rho)$, we have $u=0$ [8411]; whence $\mathrm{t} = \frac{\cos.\Theta}{\sqrt{2l'}}$ [8415]; at the second limit, where $s=\infty$, $\rho=0$, we have $v=\infty$ [8411]; whence $\mathrm{t}=\infty$ [8415]. These limits of t are the same as are given in [8417]. [8416*a*] [8416*b*] [8416*c*]

† (3915) Substituting T instead of its value [8418], we can separate $d\delta$ [8116] into two parts, depending on different functions of t; and its integral may therefore be put under the following form; [8420*a*]

$$\delta\delta = \frac{2\alpha.\sin.\Theta}{(1-\alpha).\sqrt{2l'}}.\left\{(1-f-fT^2).c^{TT}.\textstyle\int_T^\infty d\mathrm{t}.c^{-\mathrm{tt}} + fc^{TT}.\int_T^\infty \mathrm{t}^2.c^{-\mathrm{tt}}d\mathrm{t}\right\}. \qquad [8420b]$$

Now we have generally $\int \mathrm{t}^2 d\mathrm{t}.c^{-\mathrm{tt}} = \text{constant} - \frac{1}{2}\mathrm{t}.c^{-\mathrm{tt}} + \frac{1}{2}\int d\mathrm{t}.c^{-\mathrm{tt}}$, as is easily proved [8420*c*] by taking the differential and reducing. If we suppose these integrals to commence with $\mathrm{t} = T$, we must put the constant quantity equal to $\frac{1}{2}T.c^{-TT}$, so as to make the terms without the sign $\int$, vanish when $\mathrm{t} = T$. Then at the other limit, where $\mathrm{t} = \infty$, the [8420*d*] integral [8420*c*] becomes $\int_T^\infty \mathrm{t}^2.c^{-\mathrm{tt}}d\mathrm{t} = \frac{1}{2}T.c^{-TT} + \frac{1}{2}.\int_T^\infty d\mathrm{t}.c^{-\mathrm{tt}}$; substituting this [8420*e*] in [8420*b*], and making a slight reduction, we get,

$$\delta\delta = \frac{2\ .\sin.\Theta}{(1-\alpha).\sqrt{2l'}}.\left\{(1 - \tfrac{1}{2}f - fT^2).c^{TT}.\textstyle\int_T^\infty d\mathrm{t}.c^{-\mathrm{tt}} + \frac{1}{2}fT\right\}. \qquad [8420f]$$

Substituting $\Psi(T)$ [8119′] instead of its value, and in the last term putting for T its value [8118], we get the expression of $\delta\delta$ [8420].

[8421] T [8418] vanishes at the horizon when $\cos.\Theta = 0$ [8318]; then we have,
Horizontal refraction in La Place's hypothesis.
from [8419′, 8331],* $\Psi(T) = \tfrac{1}{2}.\sqrt{\pi}$; therefore the horizontal refraction is,

[8422]
$$\delta\theta = \frac{\alpha.\sqrt{\pi}}{(1-\alpha).\sqrt{2l'}}.(1-\tfrac{1}{2}f);$$

[8423] *so that the refraction would be nothing if f were equal to 2; it would be negative if $f>2$.*

We shall now determine the pressure p of the atmosphere, in this hypothesis. We have very nearly, by [8294, &c.],†

[8424]
$$dp = -(g).a.\rho ds;$$

[8425] therefore by substituting for s its value [8411′], we get,

[8426]
$$dp = -(g).a.\rho du + \alpha.(g).a.\frac{\rho d\rho}{(\rho)}.$$

Substituting for ρ its value [8412], namely,

[8427]
$$\rho = (\rho).\left(1+\frac{fu}{l'}\right).c^{-\frac{u}{l'}};$$

[8428] then integrating, and observing that $(g) = \dfrac{(p)}{(\rho).l}$ [8298], we obtain,‡

Pressure p.

[8429]
$$\frac{p}{(p)} = \frac{al'}{l}.\left(1+\frac{fu}{l'}\right).c^{-\frac{u}{l'}} + f.\frac{al'}{l}.c^{-\frac{u}{l'}} + \tfrac{1}{2}.\alpha.\frac{a}{l}.\frac{\rho^2}{(\rho)^2}.$$

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* (3916) When $\cos.\Theta = 0$, we have $T=0$ [8418]; and the expression of $\Psi(T)$
[8422a] [8419′] becomes $\Psi(T) = \int_0^\infty dt.c^{-tt} = \tfrac{1}{2}.\sqrt{\pi}$ [8331]. Substituting these values of $\cos.\Theta$, T, $\Psi(T)$, in [8420], it becomes as in [8422].

† (3917) The differential of [8284] is $d.\dfrac{a}{r} = -ds$; hence [8294′] becomes as in
[8424a] [8424]. The differential of s [8411′] is $ds = du - \alpha.\dfrac{d\rho}{(\rho)}$; substituting this in [8424], it becomes as in [8426].

‡ (3918) Substituting the value of ρ [8427] in the first term of the second member of [8426], we get,

[8428a]
$$dp = -(g).a.(\rho).\left(1+\frac{fu}{l'}\right).c^{-\frac{u}{l'}}du + \alpha.(g).a.\frac{\rho d\rho}{(\rho)}.$$

Dividing this by (p), and substituting the value of (g) [8428], we obtain,

[8428b]
$$\frac{dp}{(p)} = -\frac{a}{l}.\left(1+\frac{fu}{l'}\right).c^{-\frac{u}{l'}}du + \alpha.\frac{a}{l}.\frac{\rho d\rho}{(\rho)^2}.$$

The integral of this expression is the same as in [8429], as is easily proved by taking its differential, and neglecting the terms which destroy each other.

At the surface of the earth we have $\rho=(\rho)$ [8254], $p=(p)$ [8288, 8288′], [8430]
$u=0$ [8416b]; hence we get,*

$$l'.(1+f)=\frac{l}{a}-\tfrac{1}{2}\alpha. \qquad [8431]$$

If we suppose the horizontal refraction to be 6500″ [8281], or in parts of the radius 0,01021018, we shall have,† [8432]

$$0{,}01021018=\frac{\alpha.\sqrt{\pi}}{(1-\alpha).\sqrt{2l'}}.(1-\tfrac{1}{2}f). \qquad [8433]$$

These two equations give, Equations to find f, l'.

$$f=\frac{l}{al'}-\frac{\alpha}{2l'}-1 \quad [8431]; \qquad [8434]$$

$$(0{,}01021018)^2.(1-\alpha)^2.8l'^3=\alpha^2.\pi.\left\{3l'-\frac{l}{a}+\frac{\alpha}{2}\right\}^2. \qquad [8435]$$

Substituting the preceding values of α, a, l, we obtain,‡

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* (3919) Substituting the values [8430] in [8429], it becomes,

$$1=\frac{al'}{l}+f.\frac{al'}{l}+\tfrac{1}{2}\alpha.\frac{a}{l}; \qquad [8431a]$$

multiplying it by $\frac{l}{a}$, we get,

$$\frac{l}{a}=l'+fl'+\tfrac{1}{2}\alpha=l'.(1+f)+\tfrac{1}{2}\alpha; \qquad [8431b]$$

whence by transposition we obtain [8431].

† (3920) The horizontal refraction 6500″ [8432] being divided by the radius in seconds 636620″, gives the quantity 0,01021018, which is to be put equal to $\delta\theta$ [8422]; [8433a]
hence we get [8433]. The value of f [8434] is the same as that in [8431]; and from this we get $1-\frac{1}{2}f=\frac{1}{2l'}.\left(3l'-\frac{l}{a}+\frac{\alpha}{2}\right)$; substituting this in [8433], we find, [8433b]

$$0{,}01021018=\frac{\alpha.\sqrt{\pi}}{(1-\alpha).(2l')^{\frac{3}{2}}}.\left\{3l'-\frac{l}{a}+\frac{\alpha}{2}\right\}. \qquad [8433c]$$

Squaring this and multiplying by $(1-\alpha)^2.8l'^3$, we obtain [8435].

‡ (3921) Dividing [8435] by $\alpha^2\pi$, then substituting the values of α, a, l, [8277′, 8278, 8275], it becomes very nearly $3072.l'^3=(3l'-0{,}00110563)^2$. From this [8435a]
cubic equation we deduce the value of l' [8436]; substituting this and the values of α, a, l, above-mentioned, we get, from [8434], $f=\frac{0{,}00110563..}{l'}-1=0{,}49042$, as in [8437]. [8435b]

We may observe that the cubic equation in l' [8435a], has two other roots, which are however of no use in the problem for finding the refraction, because they lead to results [8435c]

[8436] $l' = 0{,}000741816;$ $\log.l' = 6{,}87030;$

[8437] $f = 0{,}49042.$ $\log.f = 9{,}69057.$

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which are inconsistent with the actual state and limits of the atmosphere. One of these
[8435d] values is $l' = 0{,}00028...$, which, by the substitution in the first expression of f [8435b], gives $f = 2{,}93...$; and with this value of f, we find that the expression of the horizontal refraction [8422] becomes *negative*; therefore this value of l' must be rejected. The
[8435e] other value of l' is $l' = 0{,}0019...$, which gives $f = -0{,}42...$ [8435b], and by
[8435f] substituting the values of l', f, in [8412], we get $\rho = (\rho).\{1-220.u\}.c^{-524.u}$ nearly. At the upper limit of the atmosphere, where $\rho = 0$, this expression gives $1-220.u = 0$,
[8435g] or $u = 0{,}00455...$ Substituting these values of u, f, ρ, also $\alpha = 0{,}00029..$ [8277′],
[8435h] in the expression of s [8411], we get $s = 0{,}00455 + 0{,}00029 = 0{,}00484$; and as a is less than 4000 miles, we shall have, for the corresponding height as of the atmosphere,
[8435i] an altitude which is less than 20 miles, being much smaller than it is proved to be, by the appearance of the twilight, and by observation of the heights of meteors, that sometimes
[8435k] appear in the atmosphere; which make it between 40 and 50 miles in height. The
[8435l] value of l', which is used by La Place in [8437, &c.], *gives a very accurate formula for the refraction at all altitudes* [8440]; *and a much more simple formula in* [8501], *for*
[8435m] *altitudes which exceed* 12°; so that if nothing more were required than to find the refraction, we might be content with the formulas of La Place [8438—8440, 8474, &c.],
[8435n] which are derived from this value of l'. But we shall see, in [8445o], *that these formulas give a too rapid decrease of the pressure of the atmosphere at high elevations, and that the decrement of the heat is also too rapid, particularly in low altitudes* [8444p, q, z].
[8435o] To obviate these defects, and yet preserve the same forms of calculation as those which are used by La Place, it is proposed by Plana, in vol. 27, page 206, &c. of the *Memorie della reale Accademia delle Scienze de Torino,* to use La Place's value of l' [8436], in
[8435p] computing the refraction; but in finding the temperature and pressure of the atmosphere,
[8435q] to use the value of l' resulting from the observations of Gay Lussac [8445g]. This value of l' is easily deduced from [8441, 8446], using the observations given in [8445g]. For we shall see in [8491p] that the expression of the heat h [8400c], may be reduced to the form,

[8435r] $$h = \frac{p.(\rho)}{(p).\rho} = \frac{1+0{,}00375.x}{1+0{,}00375.(x)};$$

x, (x), being the heights of the thermometer in the upper and lower stations respectively. This formula is equivalent to that in [8446], when $(x) = 0°$. Now substituting, in
[8435s] [8435r], the values $(x) = 30°{,}75$, $x = -9°{,}75$, observed by Gay Lussac [8445g], we obtain,

[8435t] $$h = \frac{p.(\rho)}{(p).\rho} = \frac{1-0{,}00375\times 9°.5}{1+0{,}00375\times 30°{,}75} = 0{,}864668;$$

[8435u] hence $\frac{\rho}{(\rho)} = \frac{p}{0{,}864668.(p)}$; and by using the values of (p) [8445g], p [8445k],

We shall therefore have, in this constitution of the atmosphere,*

La Place's formulas for the refraction and density.

$$u = s - 0{,}000293876.\left(1-\frac{\rho}{(\rho)}\right); \qquad [8438]$$

$$s = u + 0{,}000293876.\left(1-\frac{\rho}{(\rho)}\right); \qquad [8438']$$

$$\rho = (\rho).\{1+u.661{,}107\}.c^{-u.1348{,}04}; \qquad [8439]$$

$$\delta\theta = 8611'',6.\{0{,}75479-0{,}49042.T^2\}.\sin.\theta.\frac{2\Psi(T)}{\sqrt{\pi}} + 30930'',3.\sin.2\theta. \qquad [8440]$$

We shall now determine the corresponding law of the diminution of heat, or, in other words, the expression of $\frac{p.(\rho)}{(p).\rho}$ [8399]. Substituting for p and ρ their preceding values [8429, 8412], we shall have, for the expression of the temperature,†

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s [8444t], we obtain $\frac{\rho}{(\rho)}$, also u from [8411]. Finally, substituting $h=0{,}864668$ [8435t] for the first member of [8441], and putting u, $\frac{\rho}{(\rho)}$, $\frac{l}{a}$, [8435u, 8278d], in its second member, we get a quadratic equation in l'; whence we may determine l', also the value of f [8434], and the general expression of ρ [8412]. But we have not thought it to be necessary to go into any farther detail on this method, as the author uses a different process, in [8546, &c.], for determining the decrement of the heat, in measuring altitudes by the barometer. [8435v] [8435w] [8435x]

* (3922) Substituting in [8411] the value of α [8277'], we get [8438, 8438']. Again, by using the values of l', f [8436, 8437], we obtain [8439] from [8412]; observing that the preceding values give $\frac{f}{l'}=661{,}107$, and $\frac{1}{l'} = 1348{,}04$, nearly. Lastly, substituting the same values of α, a, l, l', in $\delta\theta$ [8420], then multiplying by the radius in seconds 636620″, we get [8440]; remarking that in order to reduce it to the form [8440], we must multiply the term [8420 line 1] by $\frac{\sqrt{\pi}}{\sqrt{\pi}}$, and then connect the term of the denominator $\sqrt{\pi}$, with $2\Psi(T)$ in [8440], and the term of the numerator $\sqrt{\pi}$, is included in the coefficient 8611″,6 [8440]. In [8420 line 2] the term $\sin.\theta.\cos.\theta$ is changed into $\frac{1}{2}.\sin.2\theta$ [31] Int. [8438a] [8438b] [8438c] [8438d] [8438e]

† (3923) The value of ρ [8412] gives $\frac{(\rho)}{\rho}=\left(1+\frac{fu}{l'}\right)^{-1}.c^{\frac{u}{l'}}$. Now multiplying the expression of $\frac{p}{(p)}$ [8429], by $\frac{\rho}{(\rho)}$, and using the preceding value of $\frac{(\rho)}{\rho}$ in the first and second terms of the second member of the product, we get [8441]. This is reduced to the form [8442], by substituting the values of $\frac{a}{l}$, α, f, l' [8278d, 8277', 8437, 8436]. [8441a] [8441b]

[8441] $$\frac{p.(\rho)}{(p).\rho} = \frac{al'}{l} + \frac{f.\frac{al'}{l}}{1+\frac{fu}{l'}} + \tfrac{1}{2}\alpha.\frac{a}{l}.\frac{\rho}{(\rho)};$$ [Expression of the heat h]

La Place's formulas for the temperature of the atmosphere.

consequently,

[8442] $$\frac{p.(\rho)}{(p).\rho} = 0,592243 + \frac{0,290448}{1+u.661,107} + 0,117311.\frac{\rho}{(\rho)}.$$ [Expression of the heat h]

To compare this with observation, we shall suppose,

[8443] $$u = 0,00092727;$$

and we shall then have,*

[8444] $$\frac{\rho}{(\rho)} = 0,46214; \qquad as = 6909^{\text{metres}},44.$$

[8444a] * (3924) Substituting the assumed value of u [8443] in [8439], we get $\frac{\rho}{(\rho)}$ [8444]; and the same values being substituted in [8438'], gives s; then multiplying it by a,
[8444b] [8278], we obtain as [8444]. Using these values, we reduce [8442] to the form [8445].
[8444c] Putting the expression [8445] equal to its assumed value [8446], we get the value of x, [8447]. This is the computed depression of the centigrade thermometer, corresponding to the elevation $as = 6909^{\text{metres}},44$ [8444], above the surface of the earth. Now the
[8444d] height to which Gay Lussac ascended, was nearly equal to this quantity, namely,
[8444e] 6980^{metres} [8448], and the observed depression of the thermometer $40^{\circ},25$ [8449], differs but very little from this computation. Mr. Plana, in his remarks on this article, in
[8444f] vol. 27, page 205 of the *Memorie della reale Accademia delle Scienze de Torino*, uses a value of u which varies a little from that in [8443], and gives a more correct result in the value of as [8444]. He assumes $u = 0,00093708$, and then from [8439] he gets
[8444g] $\frac{\rho}{(\rho)} = 0,457903$; from [8438'] he obtains $s = 0,00109639$; whence $as = 6979^{\text{metres}},83$,
[8444h] being very nearly the same as by observation [8448]. Substituting these results in [8442],
[8444i] he obtains $\frac{p.(\rho)}{(p).\rho} = 0,8253$, instead of the value $0,8266$ [8445]; and then from [8446] gets $0,8253 = 1+0,00375.x$; consequently $x = -46^{\circ},6$, being nearly as in [8447].
[8444k] This result of La Place's theory agrees nearly with the observation in [8449], but makes the decrement of heat somewhat too rapid. This rapidity is also found to be much too
[8444l] great in the lower regions of the atmosphere. For if we suppose $u = 0,0000139$, we
[8444m] shall have, in [8439], $\frac{\rho}{(\rho)} = 0,9904553$, and then from [8438, 8278], $s = 0,0000167$, $as = 106^{\text{metres}}$. Substituting these values in the expression of the heat h [8442], it becomes $h = 0,9962377 = 1-0,00376$. Comparing this with the general expression
[8444n] $h = 1+0,00375.x$ [8446], we get very nearly $x = -1^{\circ}$. Hence it appears, that from La Place's formula for the decrement of heat [8442], it is only necessary to ascend to the

Hence we deduce,

$$\frac{p.(\rho)}{(p).\rho} = 0{,}8266. \qquad [8445]$$

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height of 106 metres [8444m], to decrease the temperature of the atmosphere 1° of the centigrade thermometer. Now this is much greater than it is found to be by observation; [8444o]
since the result of Ramond's observations is 164$^{\text{metres}}$,7; that of Humboldt 161$^{\text{metres}}$, and [8444p]
that of Gay Lussac 173$^{\text{metres}}$ [8402b]. [8444q]

At the upper limit of the atmosphere we have $\rho = 0$, and then La Place's formula [8438] becomes $u = s - 0{,}000293876$. Now from the observations of the twilight, &c. [8444r]
[8435k], it is found that the atmosphere is about 40 or 50 miles in height, or about $\frac{1}{90}$ part [8444s]
of the radius; making $s = 0{,}011..$ [8284]; or for facility of calculation $s = 0{,}011293876$. [8444t]
Substituting this in u [8444r], we get $u = 0{,}011$; and then the expression of the heat h, in the second member of [8442], becomes $h = 0{,}6274$. Putting this equal to its [8444u]
value, in the second member of [8446], we obtain $0{,}6274 = 1 + 0{,}00375.x$, which gives [8444v]
$x = -99°{,}36$; adding this to the height of the thermometer at the lower station 30°,75, [8449], we get $-68°{,}61$, for the height of the centigrade thermometer, at the summit of the atmosphere, according to the formulas of La Place [8442], and by reducing it to the [8444w]
scale of Fahrenheit, it becomes $-68°{,}61 \times \frac{180°}{100°} + 32° = -91°{,}5$. This depression of [8444x]
the thermometer is considerably greater than the estimate of Fourier, in the *Mémoires de l' Institut*, tome vii, page 598; since from various considerations he assumes, as a probable result, that the temperature of the space, in which the earth moves, is not far from $-40°$ [8444y]
of Réaumur's thermometer, or $-58°$ of Fahrenheit's scale. *From what has been said* [8444y′]
it is evident that La Place's formulas [8438—8440, 8442], *give a too rapid decrement of the heat; and that this defect is most striking in the lower regions of the atmosphere.* [8444z]

In like manner *the pressure of the atmosphere p, deduced from La Place's formula* [8445a]
[8429], *decreases with greater rapidity than is indicated by the observations of Gay* [8445b]
Lussac. To prove this we shall observe, that the formula [8429] is easily reduced to the form [8445d], and by substituting the values $\frac{f}{l'} = 661{,}107$, $\frac{1}{l'} = 1348{,}04$, [8438b]; [8445c]
$f = 0{,}49042$ [8437]; it becomes as in [8445e]. Finally, substituting the values $u = 0{,}00093708$, $\frac{\rho}{(\rho)} = 0{,}457903$, [8444g]; also $\frac{a}{l}$ [8278d], it becomes as in [8445f].

$$\frac{p}{(p)} = \frac{a}{l}.\left\{ l'.\left(1 + f + \frac{f}{l'}.u\right).c^{-\frac{u}{l'}} + \tfrac{1}{2}\alpha.\frac{\rho^2}{(\rho)^2} \right\} \qquad [8445d]$$

$$= \frac{a}{l}.\left\{ l'.(1{,}49042 + u.661{,}107).c^{-u.1348{,}04} + \tfrac{1}{2}\alpha.\frac{\rho^2}{(\rho)^2} \right\} \qquad [8445e]$$

$$= \frac{a}{l}.0{,}000173349 = 0{,}37791. \qquad [8445f]$$

[8445'] We shall see, in [8488], that x *being the number of degrees of the centigrade thermometer, we shall have, for the expression of the temperature,*

[8446] $$\frac{p.(\rho)}{(p).\rho} = 1+0{,}00375.x.$$ [Expression of the heat h]

Putting this equal to 0,8266 [8445], we shall find,

[8447] $$x = -46°{,}24.$$

[8448] The most decisive experiment of this kind is that of Gay Lussac, who having ascended at Paris in a balloon to the height of 6980^{metres} above the
[8448'] level of the Seine, observed the thermometer at that height to be $-9°{,}5$,
[8449] when it was $30°{,}75$ at the observatory. The difference $-40°{,}25$ corresponds, as near as can be expected, with the preceding result, particularly if we take into consideration the changes which the peculiar state of the atmosphere may produce in these results. Thus we can determine by means of observations of the mean horizontal refraction in

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The observations of Gay Lussac, at the surface of the earth, and at his greatest elevation, were,

[8445g] $$(p) = 0^{\text{met.}}{,}76568\,; \quad p = 0^{\text{met.}}{,}3288\,; \quad (x) = 30°{,}75\,; \quad x = -9°{,}5\,;$$

[8445g'] (x) being the height of the centigrade thermometer at the surface of the earth, and x its height at his greatest elevation. Substituting this value of (p) in [8445*f*], we get
[8445h] $p = 0{,}37791 \times 0^{\text{met.}}{,}76568 = 0^{\text{met.}}{,}28936$. In this calculation we have used the value of
[8445i] $l = 7974^{\text{met.}}$ [8275], instead of $l = 7974^{\text{met.}}.\{1+0{,}00375.(x)\} = 7974^{\text{met.}} \times 1{,}115312$, [8500, 8445*g*]; therefore the factor $\frac{a}{l}$ [8445*f*], and the value of p [8445*f*, *h*], must be decreased in the ratio of 1,115312 to 1, so that we shall have,

[8445k] $$p = \frac{0^{\text{met.}}{,}28936}{1{,}115312} = 0^{\text{met.}}{,}2594.$$

The actual value of p, by observation, was $p = 0^{\text{met.}}{,}3288$ [8445*g*]; the temperature
[8445l] of the column of mercury being $-9°{,}5$. To reduce this to the temperature of the lower station $30°{,}75$ [8445*g*], which differs $40°{,}25$ from that of the upper, we must, as in [8490], multiply it by $1 + \frac{40{,}25}{5412} = \frac{5452{,}25}{5412{,}00}$; and by this means the corrected observation
[8445m] becomes $p = 0^{\text{met.}}{,}3288 \times \frac{5452{,}25}{5412{,}00} = 0^{\text{met.}}{,}3312$. *The value* $p = 0^{\text{met.}}{,}2594$ [8445*k*],
[8445n] *deduced from La Place's formula, being less than by observation, by* $0^{\text{met.}}{,}0718$. *Reducing these to English inches, we find the value of* p, *by observation, to be*
[8455o] *nearly* $13^{\text{inches}}{,}0$; *and by La Place's formula* $10^{\text{inches}}{,}2$; *their difference being* $2^{\text{inches}}{,}8$.

any climate, the mean diminution of the heat depending on the elevation above the surface of the earth; or reciprocally, the refraction may be determined from the diminution of the heat.* [8449′]

If we wish, by means of the preceding law, to determine the refraction upon a mountain, *at the elevation* h *above the level of the sea*, we must first determine the values of u and ρ corresponding to that height. We shall denote these values by U and (ρ') respectively, and we shall have the following equations; † [8450] [8451]

Formulas for the refraction observed upon a great elevation.

$$(\rho')=(\rho).\left(1+\frac{fU}{l'}\right).c^{-\frac{U}{l'}}; \qquad [8452]$$

$$\frac{\mathrm{h}}{a}=U+\alpha.\left(1-\frac{(\rho')}{(\rho)}\right). \qquad [8453]$$

Thus by putting $u=U+u'$, we shall have, ‡ [8454]

$$\rho=(\rho').\left(1+\frac{f'u'}{l'}\right).c^{-\frac{u'}{l'}}; \qquad [8455]$$

supposing that,

$$f'=\frac{fl'}{l'+fU}, \qquad \text{or} \qquad 1+\frac{fU}{l'}=\frac{f}{f'}. \qquad [8456]$$

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* (3925) These remarks are grounded upon the supposition that La Place's theory of the diminution of heat [8441] is correct; but we have seen in [8444k, o, &c.], that the decrement, as we ascend in the atmosphere, is too rapid. [8449a]

† (3926) Changing ρ into (ρ'), and u into U, in [8412], it becomes as in [8452]. The elevation h [8450] above the horizon, is evidently equal to $r-a=rs$ [8284], or as nearly; hence $s=\frac{\mathrm{h}}{a}$; substituting this in [8411′], after changing ρ into (ρ'), and u into U, we get [8153]. [8452a]

‡ (3927) Substituting $u=U+u'$ in [8412], it becomes,

$$\rho=(\rho).\left[1+\frac{fU+fu'}{l'}\right].c^{-\frac{U}{l'}-\frac{u'}{l'}}. \qquad [8455a]$$

Dividing this by the expression [8452], we get $\frac{\rho}{(\rho')}=\left(1+\frac{fu'}{l'+fU}\right).c^{-\frac{u'}{l'}}$; and by substituting $\frac{f}{l'+fU}=\frac{f'}{l'}$ [8456], it becomes as in [8455]. [8455b]

[8456′] It is sufficient therefore to change,* in the preceding formulas, (ρ) into (ρ'), and f into f'.

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* (3928) All the preceding calculations may be repeated, supposing the observer to be [8456a] elevated above the surface of the earth by the quantity h [8450]; or, in other words, we may suppose that a becomes $a'=a+\mathrm{h}$, s becomes s', α becomes α', and (ρ) [8456b] becomes (ρ'). Then, instead of the equations [8284, 8285], we shall have,

[8456c] $$\frac{a'}{r}=1-s'; \qquad \alpha'=\frac{\frac{2K}{n^2}\cdot(\rho')}{1+\frac{4K}{n^2}\cdot(\rho')}.$$

Taking into consideration the smallness of $\frac{K}{n^2}\cdot(\rho')$ [8277, 8451], we may change $1+\frac{4K}{n^2}\cdot(\rho')$ into $1+\frac{4K}{n^2}\cdot(\rho)$, in the denominator of the expression of α' [8456c],

[8456d] and then it becomes $\alpha'=\frac{\frac{2K}{n^2}\cdot(\rho')}{1+\frac{4K}{n^2}\cdot(\rho)}$, which is equal to the expression [8285],

[8456e] multiplied by $\frac{(\rho')}{(\rho)}$; hence we have very nearly $\alpha'=\alpha.\frac{(\rho')}{(\rho)}$. Substituting in the first of the equations [8456c], the value of $a'=a+\mathrm{h}$ [8456b], it becomes $\frac{a+\mathrm{h}}{r}=1-s'$; and

[8456f] by subtracting $\frac{a}{r}=1-s$ [8284], we get $\frac{\mathrm{h}}{r}=s-s'$. Now if we neglect terms of the order $\frac{\mathrm{h}s}{a}$, we may change the denominator r into a, and we shall have $s'=s-\frac{\mathrm{h}}{a}$.

[8456g] Substituting the values of s, $\frac{\mathrm{h}}{a}$ [8411′, 8453], it becomes as in [8456h], which is easily reduced to the form [8456i];

[8456h] $$s'=\left\{u+\alpha.\left(1-\frac{\rho}{(\rho)}\right)\right\}-\left\{U+\alpha.\left(1-\frac{(\rho')}{(\rho)}\right)\right\}$$

[8456i] $$=u-U+\frac{\alpha.(\rho')}{(\rho)}.\left\{1-\frac{\rho}{(\rho')}\right\}.$$

Substituting the value of u [8454], and using α' [8456e], it becomes $s'=u'+\alpha'.\left\{1-\frac{\rho}{(\rho')}\right\}$, which is of the same form as that of s [8411′], making the

[8456k] accents on the symbols s, u, α, (ρ), as in [8456b, &c.]; moreover the same changes in the accents being made in ρ [8412], gives ρ [8455]; therefore the same changes must be

[8456l] made in $d\vartheta$ [8414], or $\delta\vartheta$ [8420, &c.], to obtain the value of the refraction corresponding to the proposed elevation h of the observer.

Hence it follows that the horizontal refractions, at the level of the sea and at the height h, *are to one another as** $(1-\frac{1}{2}f).f'$ *is to* $(1-\frac{1}{2}f').f.c^{-\frac{U}{l'}}$ [8457]

To obtain the refraction below the horizon, we shall observe that a luminous ray, which proceeds from a body below the horizon, describes the first branch of a curve which is concave towards the earth, and approaches towards it till the moment it becomes horizontal, and then it recedes from it, describing a second branch which is similar to the first. Hence it is evident that the refraction of this body being added to the refraction of a second body, which is seen as much elevated above the horizon as the first appears below it, is equal to double the horizontal refraction, at the point where the direction of the ray is horizontal, which takes place when† $2u' = -\cos.^2\theta$. [8458] [8459]

Refraction of a body below the horizon.

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* (3929) Accenting the symbols as in [8456*b*], we find that the expression of the horizontal refraction $\delta\theta$, at the surface of the earth [8422], gives the horizontal refraction $\delta\theta'$, at the elevation h, as in the first form of [8457*b*]. From this we easily deduce the second form [8457*b*], by substituting the value of $\delta\theta$ [8422]. Now $\alpha'-\alpha$ being very small [8277', 8456*e*], we may put $\frac{1-\alpha}{1-\alpha'}=1$, and we shall obtain the first form of [8457*c*]; and by substituting the value of α' [8456*e*], we get the second form of [8457*c*]. [8457*a*]

$$\delta\theta' = \frac{\alpha'.\sqrt{\pi}}{(1-\alpha').\sqrt{2l'}}.(1-\tfrac{1}{2}f') = \frac{\alpha'}{\alpha}.\frac{1-\alpha}{1-\alpha'}.\frac{(1-\frac{1}{2}f')}{(1-\frac{1}{2}f)}.\delta\theta \qquad [8457b]$$

$$= \frac{\alpha'}{\alpha}.\frac{(1-\frac{1}{2}f')}{(1-\frac{1}{2}f)}.\delta\theta = \frac{(\rho')}{(\rho)}.\frac{(1-\frac{1}{2}f')}{(1-\frac{1}{2}f)}.\delta\theta. \qquad [8457c]$$

Now at the elevation h, where $u=U$, and $\rho=(\rho')$ [8451], the expression [8412] becomes, as in [8452],

$$(\rho') = (\rho).\left(1+\frac{fU}{l'}\right).c^{-\frac{U}{l'}} = (\rho).\frac{f}{f'}.c^{-\frac{U}{l'}} \quad [8456]. \qquad [8457d]$$

Substituting this last value of (ρ') in the last expression of $\delta\theta'$ [8457*c*], we get,

$$\delta\theta' = \frac{f}{f'}.\frac{(1-\frac{1}{2}f')}{(1-\frac{1}{2}f)}.c^{-\frac{U}{l}}.\delta\theta. \qquad [8457e]$$

Hence we easily deduce the relation between $\delta\theta'$, $\delta\theta$, which is given in [8457].

† (3930) In the annexed figure 109, we shall suppose AQ to be the surface of the earth; B the place of the observer; $PEDIH$ the path of the ray of light, which enters the atmosphere at E, in the direction of the right line $PEGK$, and is bent into the curved path $EDBH$, passing by the observer at B, and continuing its course until it [8459*a*] [8459*b*]

[8459′] We shall therefore easily obtain, by this method, the refraction of a body which is seen below the horizon.

The preceding formulas contain the three indeterminate quantities l', f,
[8460] and α, which we have determined in [8436, 8437, 8277′], by means of the horizontal refraction, and the observed heights of the barometer and
[8461] thermometer. We may, instead of the horizontal refraction, use the observations upon the diminution of the temperature. To construct a table of refraction, we must know either the law of this diminution, or the
[8461′] horizontal refractions corresponding to those heights, which will require a long series of observations; but we can by this means obtain a much more accurate table than those which are now used. It will however be liable to some degree of uncertainty, because the law of nature, relative to the

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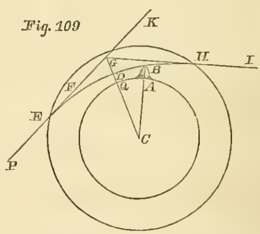
quits the atmosphere at H, in the direction HI. The right lines $PEGK$, IHG, are tangents to the path of the ray at the points E, H, and they form, by their intersection
[8459c] at G, the external angle HGK, which is equal to the whole refraction of the ray in passing from E to H; and this is evidently equal to the double of the horizontal refraction, corresponding to the point D, which divides EDH into two equal and similar
[8459d] parts or branches ED, DH. The same double refraction is divided into two unequal parts, when observed at any other point B; thus the refraction, in proceeding from E to B, represents the refraction of a depressed body situated at P; and the refraction from
[8459e] B to H is equal to the refraction of an elevated body placed at I, whose apparent altitude is the same as the apparent depression of P; so
[8459f] that the sum of the refractions of the bodies P, I, is equal to twice the horizontal refraction at the middle
[8459g] point D. At the point D the ray is perpendicular to the radius CD; therefore we shall have $dr=0$, or $ds=0$ [8407]; and as ρ is a function of r or s, [8137v, &c.], we shall also have $d\rho=0$; substituting these values in the differential of u [8411], we obtain,
[8459h] at the same point D, $du=0$. The differential $d\theta$ does not however vanish at the point D, because it is proportional to dv, as we shall see in [8548 or 8556]; but if we substitute the preceding
[8459i] value $du=0$, in the expression of $d\theta$ [8414], after accenting the letters α, u, &c. as in [8456b], we shall find that its numerator will vanish, and we shall have $d\theta=0$, unless
[8459k] the denominator vanish also; hence it is evident that we must put this denominator equal to nothing, and as one of its factors $1-\alpha'$ is nearly equal to unity [8277′, 8456e], we
[8459l] must put the other factor $\sqrt{\cos.^2\theta+2u'}=0$; whence we get $2u'=-\cos.^2\theta$, as in [8459].

Fig. 109



density of the strata of the atmosphere, is not exactly that which is assumed in [8439], but varies from a thousand unknown causes. *For this reason astronomers place confidence only in such observations as are made at an altitude of at least eleven or twelve degrees.* [8462] *Fortunately, at such altitudes the refraction becomes independent of these causes, and we can ascertain its value with great accuracy by observations of the barometer and thermometer, at the place of observation, as we shall now proceed to show.* [8462']

Uncertainty of observations at altitudes below 10°.

THEORY OF ASTRONOMICAL REFRACTIONS, CORRESPONDING TO APPARENT ALTITUDES EXCEEDING TWELVE DEGREES.

8. We shall now resume the differential equation [8286], and by reducing the radical to a series, it becomes,*

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* (3931) Dividing the numerator and denominator of the factor

$$\frac{\sin.\theta}{\sqrt{\cos.^2\theta - 2\alpha.\left(1-\frac{\rho}{(\rho)}\right)+(2s-s^2).\sin.^2\theta}}, \qquad [8463a]$$

which occurs in [8286], by $\cos.\theta$, we find that it may be put under the form,

$$\left\{1-\frac{2\alpha}{\cos.^2\theta}.\left(1-\frac{\rho}{(\rho)}\right)+(2s-s^2).\text{tang.}^2\theta\right\}^{-\frac{1}{2}}.\text{tang.}\theta. \qquad [8463b]$$

Developing this by the binomial theorem, and substituting the result in [8286], it becomes as in [8463]. Now if we neglect terms of the third order in α, s, we may reject those of the second order between the braces, in [8463], by this means it becomes,

$$\left\{1+\frac{\alpha.\left(1-\frac{(\rho)}{\rho}\right)}{\cos.^2\theta}-s.\text{tang.}^2\theta\right\}. \qquad [8463c]$$

The factor of [8463] without the braces is,

$$-\alpha.\frac{d\rho}{(\rho)}.\frac{(1-s)}{1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)}.\text{tang.}\bar{\theta}=-\alpha.\frac{d\rho}{(\rho)}.\left\{1-s+2\alpha.\left(1-\frac{\rho}{(\rho)}\right)\right\}.\text{tang.}\theta,\text{ nearly}, \qquad [8463d]$$

neglecting s^2, &c. Multiplying this last expression by that in [8463c], and neglecting α^2, s^2, &c., we get, from [8463],

$$d\delta=-\alpha.\frac{d\rho}{(\rho)}.\left\{1+\frac{\alpha.\left(1-\frac{\rho}{(\rho)}\right)}{\cos.^2\theta}-s.\text{tang.}^2\theta-s+2\alpha.\left(1-\frac{\rho}{(\rho)}\right)\right\}.\text{tang.}\theta; \qquad [8463e]$$

in which the coefficient of $-s$, between the braces, is $1+\text{tang.}^2\theta=\frac{1}{\cos.^2\theta}$, and that of $\alpha.\left(1-\frac{\rho}{(\rho)}\right)$ is $\frac{1}{\cos.^2\theta}+2=\frac{1+2.\cos.^2\theta}{\cos.^2\theta}$, as in [8464]. This expression may be put under the form, [8463f]

[8463] $$d\delta = \frac{-\alpha.\frac{d\rho}{(\rho)}.(1-s).\text{tang.}\Theta}{1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)}.\left\{\begin{array}{l} 1+\frac{\alpha.\left(1-\frac{\rho}{(\rho)}\right)}{\cos.^2\Theta}-(s-\tfrac{1}{2}s^2).\text{tang.}^2\Theta \\ +\tfrac{3}{2}.\left\{\frac{\alpha.\left(1-\frac{\rho}{(\rho)}\right)}{\cos.^2\Theta}-(s-\tfrac{1}{2}s^2).\text{tang.}^2\Theta\right\}^2+\&c. \end{array}\right\}.$$

[8463′] *If we neglect the products of three dimensions in* α, s, we shall have,

[8464] $$d\delta = -\alpha.\frac{d\rho}{(\rho)}.\left\{1-\frac{s}{\cos.^2\Theta}+\alpha.\left(1-\frac{\rho}{(\rho)}\right).\frac{(1+2.\cos.^2\Theta)}{\cos.^2\Theta}\right\}.\text{tang.}\Theta.$$

[8465] Integrating from $\rho=(\rho)$ to $\rho=0$, we shall have, as in [8463i],

[8466] $$\delta\delta = \alpha.\left\{1+\frac{\frac{1}{2}\alpha.(1+2.\cos.^2\Theta)}{\cos.^2\Theta}+\frac{1}{\cos.^2\Theta}.\int_{(\rho)}^{0}\frac{s\,d\rho}{(\rho)}\right\}.\text{tang.}\Theta.$$

[8467] The integral $\int\frac{s\,d\rho}{(\rho)}$ is equal to $\frac{s\rho}{(\rho)}-\int\frac{\rho\,ds}{(\rho)}$, as is easily proved by taking the differentials and reducing. If we take the limits of the integrals from $\rho=(\rho)$ to $\rho=0$, observing that at the first of these limits we have

[8467′] $s=0$, and at the second limit $s=1$, or $r=\infty$ [8284, 8254, 8255], we shall obtain,

[8468] $$\int_{(\rho)}^{0}\frac{s\,d\rho}{(\rho)} = -\int_{0}^{1}\frac{\rho\,ds}{(\rho)}.$$

To find this last integral, we shall observe that, as p denotes the pressure of the air, we shall have,*

[8469] $$dp = -g\rho.dr = -g.\frac{r^2}{a}.\rho\,ds;$$

[8463g] $$d\delta = \alpha.\left\{-\frac{d\rho}{(\rho)}+\frac{\frac{1}{2}\alpha.(1+2.\cos.^2\Theta)}{\cos.^2\Theta}.\left(-\frac{2d\rho}{(\rho)}+\frac{2\rho.d\rho}{(\rho)^2}\right)+\frac{1}{\cos.^2\Theta}.\frac{s\,d\rho}{(\rho)}\right\}.\text{tang.}\Theta.$$

Its integral is,

[8463h] $$\delta\delta = \alpha.\left\{-\frac{\rho}{(\rho)}+\frac{\frac{1}{2}\alpha.(1+2.\cos.^2\Theta)}{\cos.^2\Theta}.\left(-\frac{2\rho}{(\rho)}+\frac{\rho^2}{(\rho)^2}\right)+\frac{1}{\cos.^2\Theta}.\int\frac{s\,d\rho}{(\rho)}+\text{constant}\right\}.\text{tang.}\Theta;$$

[8463i] the constant quantity being taken so as to make the terms without the sign $\int$ vanish when $\rho=(\rho)$; and then putting $\rho=0$, it will become as in [8466].

* (3932) We have, in [8295a], $dp=-g\rho.dr$, as in [8469]. Now the differential

[8469a] of $1-s$ [8284], gives $-\frac{adr}{r^2}=-ds$, or $dr=\frac{r^2.ds}{a}$; substituting this in the preceding

[8469b] value of dp, we get the second expression in [8469]; and by substituting in it the value

[8469c] of g [8470], we get $dp=-(g).a.\rho ds$; which can also be much more readily obtained

now (g) being the gravity at the surface of the earth [8289′], we shall have $g=(g).\frac{a^2}{r^2}$ [8292]; therefore, [8470]

$$p = \text{constant} - (g).a.\int \rho\, ds. \qquad [8471]$$

Thus the integral $\int \rho\, ds$ is equal to the whole pressure (p) at the earth's surface, divided by $(g).a$ [8469*f*]; and as this pressure is equal to $(g).(\rho).l$ [8298], we shall have, [8472]

Oriani's formula.

$$\int_0^1 \frac{\rho\, ds}{(\rho)} = \frac{l}{a}; \qquad [8473]$$

hence the expression of $\delta\theta$ [8466] becomes,

La Place's formula for the refraction, for altitudes exceeding 12°.

$$\delta\theta = \alpha.\left\{1+\frac{\frac{1}{2}\alpha.(1+2.\cos.^2\Theta)-\frac{l}{a}}{\cos.^2\Theta}\right\}.\text{tang.}\Theta; \quad (A) \qquad [8474]$$

This expression has the advantage of being independent of any hypothesis*

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by substituting $d.\frac{a}{r}=-ds$ [8284], in [8294′]. The integral of [8469*c*] gives p [8471]. At the commencement of the integral we have $p=(p)$; hence [8471] becomes $p=(p)-(g).a.\int_0^s \rho ds$. At the other limit, where $\rho=0$, $p=0$, $s=1$ [8467′, 8284], this equation becomes $0=(p)-(g).a.\int_0^1\rho ds$; whence $\int_0^1 \rho ds=\frac{(p)}{(g).a}$, as in [8472]; substituting (p) [8298], we get $\int_0^1\rho ds=\frac{(\rho).l}{a}$, which is easily reduced to the form [8473]. Substituting [8473] in [8468], we get $\int_{(\rho)}^0 \frac{s d\rho}{(\rho)}=-\frac{l}{a}$; hence [8466] becomes as in [8474]. We may here observe that the integral [8473] for the whole height of the atmosphere, *and for all laws of the decrease of the density*, was first discovered by Oriani, in nearly the same form as it is here given. [8469*d*] [8469*e*] [8469*f*] [8469*g*] [8469*h*]

* (3933) If we substitute $\frac{1}{\cos.^2\Theta}=1+\text{tang.}^2\Theta$ [34‴] Int. in [8474], it becomes as in [8474*b*]. Reducing and putting for brevity,

$$A=\alpha.\left(1+\tfrac{3}{2}\alpha-\frac{l}{a}\right); \qquad nA^2=\alpha.\left(\frac{l}{a}-\tfrac{1}{2}\alpha\right); \qquad p=1-2nA; \qquad [8474a]$$

it becomes of the form [8474*d*].

La Place's second formula reduced.

$$\delta\theta=\alpha.\left\{1+\tfrac{1}{2}\alpha.(1+\text{tang.}^2\Theta)+\alpha-\frac{l}{a}.(1+\text{tang.}^2\Theta)\right\}.\text{tang.}\Theta \qquad [8474b]$$

$$=\alpha.\left\{1+\tfrac{3}{2}\alpha-\frac{l}{a}\right\}.\text{tang.}\Theta-\alpha.\left\{\frac{l}{a}-\tfrac{1}{2}\alpha\right\}.\text{tang.}^3\Theta \qquad [8474c]$$

$$=A.\text{tang.}\Theta-nA^2.\text{tang.}^3\Theta. \qquad [8474d]$$

[8474'] *on the constitution of the atmosphere; since it depends solely upon the density and temperature of the air at the place of observation.* For the values of

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The numerical values of A, n, p, are easily deduced from those of α, $\frac{l}{a}$, [8277', 8278d], which give,

[8474e] $A=0{,}0002936375$; $nA^2=0{,}0000003249..$; $n=3{,}768..$; $p=0{,}9977871...$

[8474f] From La Place's expression of the refraction [8474], reduced to the form [8474d], we may easily deduce the methods of computation proposed by Simpson and Bradley,
[8474g] [8374l, 8383c, d], neglecting terms of the order $\text{tang.}^5\Theta$, A^3, $\delta\theta^3$. For the expression
[8474h] [8474d] may be put under the form [8474l], as is evident by multiplying together the factors of the second member, and neglecting terms of the order just mentioned. Moreover
[8474i] the expression [8474l] may be reduced to the form [8474m], by substituting $A.\text{tang.}\Theta=\delta\theta$
[8474k] [8474d] in the two small terms between the braces, containing A. Finally, by using [34''', 54] Int., we obtain successively the formulas [8474n, o];

[8474l] $$\delta\theta=A.(1+nA).\{\text{tang.}\Theta-nA.\text{tang.}\Theta-nA.\text{tang.}^3\Theta\}$$
[8474m] $$=A.(1+nA).\{\text{tang.}\Theta-n\delta\theta-n\delta\theta.\text{tang.}^2\Theta\}$$
[8474n] Bradley's formula $$=A.(1+nA).\left\{\text{tang.}\Theta-\frac{n\delta\theta}{\cos.^2\Theta}\right\}$$
[8474o] deduced from that of La Place. $$=A.(1+nA).\text{tang.}(\Theta-n\delta\theta).$$

This last expression is the same as Bradley's formula [8383c], putting $n=3$, instead of
[8474p] $n=3{,}768..$ [8474e]; also $A.(1+nA)=57^s$, instead of $A.(1+nA)=60^s{,}634$, which is deduced from the values [8474e]; observing that the quantity 57^s, used by Bradley, will be increased to rather more than 60^s, by reducing it to the temperature and height of
[8474q] the barometer [8274], corresponding to the quantities used by La Place in estimating the values of n, A, p [8474e].

[8474r] Again, the expression [8474d] $\delta\theta=A.\text{tang.}\Theta$—&c., being multiplied by $-nA.\text{tang.}^2\Theta$, gives $-n\delta\theta.A.\text{tang.}^2\Theta=-nA^2.\text{tang.}^3\Theta$—&c. Substituting this in the last term of [8474d], and neglecting terms of the order A^3, as in [8474g], we get,

[8474s] $$\delta\theta=A.\{1-n\delta\theta.\text{tang.}\Theta\}.\text{tang.}\Theta.$$

Dividing this expression of $\delta\theta$, by the coefficient of A, we obtain,

[8474t] $$A=\delta\theta.\{1+n\delta\theta.\text{tang.}\Theta\}.\text{cotang.}\Theta=\delta\theta.\{\text{cotang.}\Theta+n\delta\theta\}.$$

Multiplying this value of A by $2n.\sin.\Theta$, and substituting $2nA=1-p$ [8474a], we get,

[8474u] $$(1-p).\sin.\Theta=2n\delta\theta.\{\cos.\Theta+n\delta\theta.\sin.\Theta\}.$$

Simpson's formula, deduced from that of La Place. From this we deduce the first of the following values of $p.\sin.\Theta$, and by making successive reductions, using [44, 43, 22] Int., we obtain the final result in [8474x];

[8474v] $$p.\sin.\Theta=(1-2n^2.\delta\theta^2).\sin.\Theta-2n\delta\theta.\cos.\Theta$$
[8474w] $$=\cos.2n\delta\theta.\sin.\Theta-\sin.2n\delta\theta.\cos.\Theta$$
[8474x] $$=\sin.(\Theta-2n\delta\theta).$$

This last expression is the same as Simpson's formula [8374l], putting $p=0{,}9986$,

(ρ) and l [8290, 8298], are given by the observations made at that place, [8475]
by the barometer and thermometer. It is therefore important to know to what apparent altitude of any heavenly body we can use this formula.

If we examine the preceding development of $d\theta$ in a series [8463], it will be evident that the most considerable term which we have neglected is the following;*

$$-\tfrac{3}{2}.\alpha.\frac{d\rho}{(\rho)}.s^2.\text{tang.}^5\theta. \qquad \text{[Neglected term.]}$$ [8476]

Estimate of the value [8477] of the neglected terms.

It may become sensible in low altitudes, where tang.θ is large. This term is decreased by those of the same order in the formula,† so that in those

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$2n=\frac{1}{2}^1$, instead of the values of p, $2n$, in [8474e]. Hence we see that the formula [8474] is equivalent either to Bradley's formula [8383c], or to that of Simpson [8374l], [8474y]
neglecting terms of the order $\delta\theta^3$; and we have seen in [8267t, u], that La Place's [8474z]
formula [8474] is identically the same as that which is deduced from Cassini's hypothesis. [8475a]
Hence it appears that in altitudes exceeding eleven or twelve degrees, which includes by far the most important and useful observations, the refraction can be obtained by either of these [8475b]
methods, with almost the same degree of accuracy. At lower altitudes the formula of La Place [8440] has decidedly the advantage over the methods of Cassini, Bouguer, Simpson, [8475c]
or Bradley; but is not, on some accounts, so satisfactory as the method proposed by Mr. Ivory, of which we shall hereafter give some account. [8475d]

* (3934) The value of $\alpha=0{,}00029...$ [8277′], is much smaller than the general value of s; therefore the terms between the braces, in [8463], depending on α^2, αs, [8476a]
must generally be much smaller than those which depend on s^2; moreover when θ is [8476b]
nearly equal to a right angle, and tang.θ very large, the terms multiplied by the powers of tang.θ must be much increased. Now the highest power of tang.θ, connected with s^2, [8476c]
between the braces, in [8463], and free from α, is tang.$^4\theta$. If we neglect the terms connected with the lower powers of tang.θ, we shall obtain for the greatest of these terms of $d\theta$ [8463], the expression $\frac{3}{2}s^2.\text{tang.}^4\theta$ within the braces, multiplied by the [8476d]
factor without the braces, or $-\alpha.\frac{d\rho}{(\rho)}.\text{tang.}\theta$, nearly; being the same as in [8476].

† (3935) The term between the braces in [8463], which produces the quantity [8476], appears under the form $\frac{3}{2}.\left\{\frac{\alpha.\left(1-\frac{\rho}{(\rho)}\right)}{\cos.^2\theta}-s.\text{tang.}^2\theta\right\}^2$, neglecting smaller terms. [8477a]
Now the term of this expression depending on α, tends to decrease that which arises from s; and this is what the author alludes to in [8477], neglecting the consideration of some [8477b]
small terms, which serve to increase that in [8476]; but not in any important degree, in [8477c]
comparison with those which are retained, as is evident from the last note.

altitudes of the heavenly bodies in which its integral [8476] is insensible, we may, without fear of any important error, use the formula [8474]. The
[8478] integral of this term [8476], taken from $\rho=(\rho)$ to $\rho=0$, becomes,*

[8479] $$3\alpha.\mathrm{tang.}^5\theta.\int_0^1\frac{\rho}{(\rho)}.sds.$$

If we suppose the temperature to be equal throughout the whole atmosphere, we shall have, as in [8299],

[8480] $$\rho=(\rho).c^{-\frac{as}{l}};$$

therefore the expression [8479] becomes,†

[8481] $$3\alpha.\mathrm{tang.}^5\theta.\int_0^1\frac{\rho}{(\rho)}.sds=3\alpha.\mathrm{tang.}^5\theta.\frac{l^2}{a^2}.$$

The value of this integral is greater in the hypothesis of a uniform temperature, than when the temperature is supposed to decrease as the elevation increases. For if we imagine the temperature at first to be
[8482] uniform, and then to decrease as the elevation increases, it is evident that the particle of the atmosphere, represented by ρds, will descend, and the

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[8478a] * (3936) The integral of [8476] may be put under the form $-\frac{3\alpha.\mathrm{tang.}^5\theta}{2.(\rho)}.\int s^2d\rho$; but we have $\int s^2d\rho=s^2\rho-2\int\rho sds$, as is easily proved by differentration. At the first
[8478b] limit $\rho=(\rho)$ and $s=0$ [8307a]; therefore the term $s^2\rho$ vanishes; it also vanishes at the second limit, where $\rho=0$ and $s=1$ [8307a]; hence the preceding integral
[8478c] becomes $\int_{(\rho)}^0 s^2d\rho=-2\int_0^1\rho sds$. Substituting this in the term [8478a], it becomes $\frac{3\alpha.\mathrm{tang.}^5\theta}{(\rho)}.\int_0^1\rho sds$, as in [8479].

† (3937) If we substitute the value of ρ [8480], in [8479], it becomes,

[8481a] $$3\alpha.\mathrm{tang.}^5\theta.\int_0^1 sds.c^{-\frac{as}{l}}.$$

Now we have generally,

[8481b] $$\int_0^s sds.c^{-\frac{as}{l}}=\frac{l^2}{a^2}.\left(1-c^{-\frac{as}{l}}\right)-\frac{ls}{a}.c^{-\frac{as}{l}},$$

as is easily proved by taking the differential and reducing. This vanishes when $s=0$, so that it is not necessary to add any constant quantity; and when $s=1$, the term $c^{-\frac{as}{l}}$ becomes excessively small, and may be put equal to nothing [8307c]; then the
[8481c] integral [8481b] becomes $\int_0^1 sds.c^{-\frac{as}{l}}=\frac{l^2}{a^2}$. Substituting this in [8481a], it becomes as in [8481].

product $\rho s ds$, which corresponds to it, will become less; consequently the integral $\int \frac{\rho s ds}{(\rho)}$ will become less. Therefore the formula [8474] will be exact for all altitudes in which $3\alpha.\text{tang.}^5\theta.\frac{l^2}{a^2}$ is insensible. Now if we [8483]
use the values of α, l, a [8277′, 8275, 8278], and suppose $\theta = 88^{\circ}$, we shall find this quantity equal to* $3'',486$, which is hardly sensible. *At* [8483′]
greater apparent altitudes, the error of the formula [8474] *becomes wholly insensible*; it is therefore important to determine accurately the elements of this formula.

9. Its chief elements are, *First. The variations of the density of the air, arising from the variations in the pressure and temperature of the* [8484]
atmosphere. Second. The refraction of atmospheric air at a given temperature and pressure. The change in the density of the air, arising from the variations of the pressure it suffers, is well known by means of [8485]
the law, which makes the density proportional to that pressure. The accuracy of this law has been confirmed by a great number of experiments, made within the limits of the variations of the barometer, from the level of the sea to the greatest heights to which we can ascend. The dilatation of air by means of heat, has been the object of research of several observers, who differ sensibly from each other in their results. I requested Gay Lussac to repeat these experiments with all possible care, graduating accurately the air thermometer and the mercurial thermometer, and paying the greatest [8485′]
attention to dry the air and the tubes which he used; for it appears to me that the difference in the results of various observers depends chiefly on the humidity. He found by the mean of twenty-five experiments, noticing the dilation of the glass, and the corrections for the variations of the barometer during the time of each experiment, that a mass of air, expressed by unity, at the temperature of *zero* of the centigrade thermometer, became 1,375 at the heat of boiling water, under a pressure equivalent to that of a [8486]
column of mercury of $0^{\text{metre}},76$ height. He also observed that when the air thermometer was at 50°, the mercurial thermometer likewise stood at 50°; the difference given by the mean result of the twenty-five experiments we have just mentioned, being insensible. Thus the range of both

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* (3938) Substituting the values of α, l, a, θ [8483], in the second member of [8481], and then multiplying by the radius in seconds, it becomes as in [8483′]. [8483a]

thermometers appears to be the same in the interval from 0° to 100°.
[8487] *Therefore if we put x for the number of degrees of a mercurial centigrade thermometer*, we shall find that a mass of air, which is represented by unity at the temperature *zero*, becomes, at the temperature of x degrees, $1+0{,}00375.x$; so that we have,

[8488] $$1+0{,}00375.x = \frac{\text{volume of air at temperature } x \text{ degrees}}{\text{volume of air at temperature } zero.}.$$

[8489] The density of the air is proportional to its pressure. *We shall take for unity its density at the temperature zero, and at the height of the barometer* $0^{\text{metre}},76$. *We shall then express its height, corrected for the effect of the*
[8489'] *dilatation of mercury, and reduced to the zero of temperature, by* $0^{\text{met.}},76.(1+y)$. This correction is easily made by observing that, for each degree of the
[8490] thermometer, the mercury expands* $\frac{1}{5412}$. The density of the air at the temperature of (x) degrees, will be,†

[8490a] * (3939) Plana, in his remarks on the refraction, vol. 27, page 166, of the *Memorie della reale Accademia delle Scienze di Torino*, instead of $\frac{1}{5412}$, uses $\frac{1}{5550}$, in conformity with the observations of Dulong and Petit, detailed in their memoir, which
[8490b] gained the prize of the Academy of Arts and Sciences of Paris, in 1818. This memoir is printed in *Cahier* 18, of the *Journal de l'Ecole Royale Polytechnique*, in 1820.

[8491a] † (3940) If the temperature remain at *zero*, but the pressure of the atmosphere, as it is indicated by the height of the barometer, increase from $0^{\text{metre}},76$ to $0^{\text{metre}},76.(1+y)$,
[8491b] the density of the air will increase from 1 to $1+y$ [8489]. Now if the temperature vary from *zero* to (x) degrees above zero, the density $1+y$ will vary inversely as 1 to $1+0{,}00375.(x)$, as appears from the experiments of Gay Lussac [8488]; and the density
[8491c] will therefore become $\frac{1+y}{1+0{,}00375.(x)}$, as in [8191]; observing that in finding $1+y$, or
[8491d] rather $0^{\text{metre}},76.(1+y)$, we must correct the observed height of the barometer for the
[8491e] expansion of the mercury in the tube, as in [8490]. Multiplying the expression [8491] by the value of α [8277'], we get its corrected value [8494]. For perspicuity and
[8491e'] symmetry, we have changed La Place's symbol x into (x), in the formulas [8491–8535]; and we shall now suppose, as in [8288, 8487, &c.], that p, ρ, x, represent respectively
[8491f] the pressure, the density, and the temperature of a stratum of air, at the distance r from
[8491g] the earth's centre; also (p), (r), (x), the similar quantities for a stratum of air at the surface of the earth, or at the distance a from the centre. The values p, (p), will be proportional to the observed heights of the barometer, corrected, as in [8491d], for the
[8491h] expansion of the mercury by the heat, so as to reduce them to the same temperature. Now it is evident, from what is said in [8491b, &c.], that the expressions

$$\frac{1+y}{1+0{,}00375.(x)} = \text{density of air.} \quad \begin{bmatrix}\text{Height of the thermometer } (x) \text{ degrees.} \\ \text{Height of the barometer } 0^{\text{met.}},76.(1+y).\end{bmatrix}$$ [8491]

We shall suppose that α [8285] corresponds to the temperature zero, and to the height of the barometer $0^{\text{metre}},76$. It appears natural to suppose that the refractive force of air is proportional to its density, and this is, in fact, confirmed by the experiments of Hauksbee, [Biot and Arago, 8264m]. The value of α, corresponding to the temperature of (x) degrees, and to the height of the barometer $(1+y).0^{\text{metre}},76$, will therefore be, [8492] [8493]

$$\frac{\alpha.(1+y)}{1+0{,}00375.(x)} = \text{corrected value of } \alpha. \quad \begin{bmatrix}\text{Height of the thermometer } (x) \text{ degrees.} \\ \text{Height of the barometer } 0^{\text{met.}},76.(1+y).\end{bmatrix}$$ [8494]

Moreover at the temperature of *zero*, and at the height of the barometer $0^{\text{metre}},76$, we have by what precedes [8275], [8495]

$$l = 7974 \text{ metres.*} \quad [\text{Height of a homogeneous atmosphere when } (x)=0.]$$ [8496]

The value of l does not vary with the height of the barometer, while the temperature remains unchanged; for the equation, [8497]

$$(p) = l.(g).(\rho) \quad [8298],$$ [8498]

proves, that as in this case (p) is proportional to (ρ), the value of l will

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$\frac{p}{1+0{,}00375.x}$, $\frac{(p)}{1+0{,}00375.(x)}$, will represent the values of p, (p), respectively, supposing [8491i] the densities to remain the same as before, but the temperatures to be reduced to *zero* of [8491k] the centigrade thermometer; and then the pressures, reduced as above, will be proportional to the densities ρ, (ρ), by the law of Mariotte; hence we have, [8491l]

$$\frac{p}{1+0{,}00375.x} : \frac{(p)}{1+0{,}00375.(x)} :: \rho : (\rho).$$ [8491m]

From this we deduce,

$$\frac{p.(\rho)}{(p).\rho} = \frac{1+0{,}00375.x}{1+0{,}00375.(x)}.$$ [8491n]

The first member of this expression represents the expression of the decrement of heat [8400c], in proceeding from the lower stratum of the temperature (x), to the upper stratum of the temperature x; hence we have, [8491o]

$$h = \frac{p.(\rho)}{(p).\rho} = \frac{1+0{,}00375.x}{1+0{,}00375.(x)};$$ [8491p]

which is used in [8435r, &c.]. We have here followed the method and notation of the author, but it would be more simple to use the process in [8400a, &c.].

* (3941) The observations of Biot and Arago [8277b], give more correctly $l = 7955$ metres. [8496a]

The height l of the homogeneous atmosphere does not vary except with a change of temperature.

[8499] remain always the same.* *But if the temperature varies, then l varies in the inverse ratio of* (ρ), *and we shall have,*

[8500] $$l = 7974^{\text{metres}}.\{1+0,00375.(x)\}.$$

This being premised the formula [8474] becomes, by observing that $a = 6366198$ metres [8278],†

General formula for the refraction, in altitudes above 12°.

[8501] $$\delta\theta = \left.\begin{array}{l} \dfrac{\alpha.(1+y).\text{tang.}\Theta}{1+0,00375.(x)} \\ + \dfrac{\frac{1}{2}\alpha^2.(1+y)^2}{(1+0,00375.(x))^2}\cdot\dfrac{(1+2.\cos.^2\Theta).\text{tang.}\Theta}{\cos.^2\Theta} \\ -\alpha.(1+y).0,00125254.\dfrac{\text{tang.}\Theta}{\cos.^2\Theta} \end{array}\right\}. \quad (B)$$

[8502] The only indeterminate quantity which now remains is α, and one of the best methods of finding its value is by means of observations of the heights of circumpolar stars, in their greatest and least altitudes. Delambre, by the comparison of a great number of astronomical observations, found the
[8503] refraction to be equal to $186'',728$, at $50°$ of apparent altitude, the temperature being zero, and the height of the barometer $0^{\text{metre}},76$.

[8499a] * (3942) Supposing the temperature to remain unchanged, while the pressure (p), the density (ρ), and the altitude l vary, so as to become $p_{,}$, $\rho_{,}$, $l_{,}$, respectively, we shall have, from [8498], $p_{,}=l_{,}.(g).\rho_{,}$. Dividing this value of $p_{,}$ by that of (p) [8498],
[8499b] we get $\frac{p_{,}}{(p)}=\frac{l_{,}}{l}\cdot\frac{\rho_{,}}{(\rho)}$, or $l_{,}=l.\frac{p_{,}}{(p)}\cdot\frac{(\rho)}{\rho_{,}}$. But the density in this case being
[8499c] proportional to the pressure [8485], we have $(p):(\rho)::p_{,}:\rho_{,}$; whence $\frac{p_{,}.(\rho)}{(p).\rho_{,}}=1$. Substituting this in the preceding expression of $l_{,}$, we get $l_{,}=l$, as in [8497]. Now if
[8499d] we suppose $p_{,}=(p)$, the expression of $l_{,}$ [8499b], becomes $l_{,}=l.\frac{(\rho)}{\rho_{,}}$; moreover if the height of the barometer be $0^{\text{metre}},76$, we shall have $y=0$ [8489']; and if the temperature
[8499e] be (x), the density of the air [8491] becomes $\rho_{,}=\frac{(\rho)}{1+0,00375.(x)}$. Substituting this in $l_{,}$ [8499d], we get $l_{,}=l.\{1+0,00375.(x)\}$, as in [8500, 8496].

† (3943) Substituting in $\delta\theta$ [8474] for α its corrected value [8494], for l its value [8500], and for a its value [8278], it becomes as in [8501]; observing that
[8501a] $\frac{7974}{6366198}=0,00125254$, nearly, and that these corrected values of α, l, change αl into $7974^{\text{met.}}.\alpha.(1+y)$.

Hence we find,*

$$\alpha = 187'',087 \,;$$ [8504]

Value of α, by astronomical observations.

or in parts of the radius,

$$\alpha = 0,000293876.$$ [8505]

10. *The humidity of the air has not hitherto been noticed in computing the refraction, and it is a question whether it has any sensible effect.* We shall now take this subject into consideration; and for this purpose we shall recall to mind several results of observation relative to the evaporation of [8506] different fluids. It has been found by experiment that a mass of any kind of gas whatever, *when fully saturated with water, contains the same quantity of vapor as would ascend in the same void space, at the same temperature,* [8507] supposing that there is a sufficient supply of water for the whole evaporation.

Expansion of gas and vapor.

It has also been observed *that while the pressure remains the same, every sort of gas expands the same quantity by the addition of a given degree of heat, and every vapor expands the same quantity as a gas. It has also been found, at the same temperature, that the density of any gas, or vapor, is* [8508] *proportional to its pressure, or to its elastic force.*

If we place, in a vacuum, a vessel filled with water, the elastic force of the vapor, which ascends from it, increases with the temperature, according to a law which has been ascertained by experiment. It has been found that [8509] this elastic force increases very nearly in a geometrical progression, while the temperature increases in an arithmetical progression, so that the logarithms of the force increase very nearly in an arithmetical progression. This is not however strictly accurate. If we suppose, at the moment of [8510] ebullition, when the height of the barometer is $0^{\text{metre}},76$, that this height expresses the elastic force of the aqueous vapor, we shall find that we can very nearly satisfy the experiments of Dalton, upon such forces, by supposing [8511] the elastic force of this vapor, at any temperature whatever, to be represented by,

Expression of the elastic force of gas or vapor.

$$\text{elastic force of vapor} = 0^{\text{metre}},76.(10)^{i.0,0154547-i^2.0,0000625826} \,;$$ [8512]

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* (3944) Putting, in [8501], $y=0$, $(x)=0$, $\Theta=50^{\circ}$, $\delta\theta=186'',728$, we obtain, [8504a]

$$186'',728=\alpha+2\alpha^2-2\alpha\times 0,00125254=0,99749492\,\alpha+2\alpha^2.$$ [8504b]

Dividing this by the coefficient of α, we get $\alpha=187'',197-2,005.\alpha^2$; so that α is nearly $187'',197$, and $2,005.\alpha^2=2,005\times(187'',197)^2.\sin.1''=0'',110$, nearly; hence [8504c] $\alpha=187'',197-0'',110=187'',087$, as in [8504]. Dividing this by the radius $636620''$, it becomes as in [8505].

$i.$
[8513] *i being the number of degrees of the centigrade mercurial thermometer above $100°$, this number being supposed negative for lower temperatures.* Therefore we shall have the tabular logarithm of this elastic force expressed in decimals of a metre, by adding to the logarithm of $0^{\text{metre}},76$, the exponent of 10, in the expression [8512], so that we shall have,

Elastic force of vapor.

[8514] $$\text{tab. log. of elastic force} = 9,8808136 + i.0,0154547 - i^2.0,0000625826;$$

[8514′] $$i = -100° + \text{height of the centigrade thermometer in degrees.}$$

[8515] *The formula [8514] may be used from $i = -\infty$ to $i = 50°$, or $60°$, and it can be applied to all fluids; observing to count the quantity i for each one of them, from the term of their ebullition. For this remarkable*
[8516] *principle has invariably been found to hold good, namely, that if we count from the point of ebullition, or in general from any other point, where the elastic force is the same, any increment of temperature will produce an equal increment in the elastic force.*

Remarkable property of the elasticity of gases.

In whatever manner the vapor exists in the atmosphere, it is evident that the action of the moist air upon light is composed of the action of the air and of that of the vapor. We shall suppose that with an equal elastic force, and with the same temperature, the actions of vapor and air, upon the light,
$p, q.$ are in the ratio of p to q; so that we shall have,

[8517] $$\frac{p}{q} = \frac{\text{the action of vapor upon the light}}{\text{the action of air upon the light}};$$

[8517′] *q being the action of air upon the light at the temperature zero, and under a pressure corresponding to the height of the barometer $0^{\text{metre}},76$. We shall*
[8518] *also put $z.0^{\text{metre}},76$ for the elastic force which the aqueous vapor*, suspended in a given mass of air, at the same temperature zero and under the same pressure, will have if this vapor be placed alone in the same space, considered as void. Then it is evident that, at this temperature, the
[8519] humidity of the air will add, to its action on the light, the quantity* $z.(p-q)$;

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* (3945) If we suppose the whole pressure or elastic force of the air and vapor to be
[8519a] represented by $0^{\text{metre}},76$, as in [8518], and that the part $z.0^{\text{metre}},76$ arises from the vapor,
[8519b] we shall have $0^{\text{metre}},76.(1-z)$, for the remaining part depending on the air. Now the
[8519c] action of the vapor, when the pressure is $0^{\text{metre}},76$, is represented by p [8517, &c.];
[8519d] therefore the part corresponding to the pressure $z.0^{\text{metre}},76$ [8519a], is pz. In like manner the action of the air, under the pressure $0^{\text{metre}},76$, being represented by q,
[8519e] [8517′], the part corresponding to the pressure $0^{\text{metre}},76.(1-z)$ [8519b], is $q.(1-z)$.
[8519f] Adding this to the part pz [8519d], we get $q + z.(p-q)$, for the combined action of the

and if the temperature be x degrees, the density of the vapor being decreased about 0,00375 [8488] for each degree, the correction of the refractive force of the air, arising from its humidity, will be, [8520]

$$\frac{z.(p-q)}{1+x.0,00375}.$$ [8521]

In order to ascertain the value of $p-q$, we shall suppose the quantity $\frac{K}{n^2}$ [8192, &c.] to be the same in its liquid state as in its state of vapor. This is indeed the most natural hypothesis which can be admitted; it is analogous to that which is used in the theory of refraction, by supposing that the density of the air does not vary the value of $\frac{K}{n^2}$. In passing from a vacuum into water, the ratio of the sine of incidence to the sine of refraction is, according to Newton, $\frac{529}{396}$. This gives, by means of the formula [8192], [8522] [8523]

$$\frac{4K}{n^2}.\rho=\left(\frac{529}{396}\right)^2-1;$$ [8524]

ρ *being the density of water.* It follows, from the experiments of Dalton, Saussure and Watts, that, with equal elastic forces and temperatures, we have, [8525]

the density of aqueous vapor $=\frac{10}{14}\times$ the density of air; [8526]

and according to Lavosier, at the temperature 12°,5, and at the pressure $0^{\text{metre}},76$, the density of air is $\frac{\rho}{842}$; therefore we have, by putting ρ' *for the density of the aqueous vapor at that temperature*,* [8527]

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air and vapor; and as the action of the air alone, when free from moisture and under the same pressure $0^{\text{metre}},76$, is q [8517′], the increment from the action of the vapor must be $z.(p-q)$, as in [8519]. [8519g]

* (3946) Multiplying [8524] by $\frac{z\rho'}{\rho}$, we get $\frac{4K}{n^2}.z\rho'=\frac{z\rho'}{\rho}.\left\{\left(\frac{529}{326}\right)^2-1\right\}$; now from [8526, 8527], we have $\rho'=\frac{10}{14}.\frac{\rho}{842}$; substituting this in the preceding equation, it becomes as in [8528]. Dividing this last expression by 2, and then multiplying its second member by the radius in seconds 636620″, it becomes very nearly as in [8529]. This corresponds to the temperature 12°,5 [8527], and to reduce it to the temperature 0, we must multiply it by $1+0,00375\times12,5$ [8488]; by this means it becomes as in [8530], which represents the value of pz. [8528a] [8528b] [8528c]

[8528] $$\frac{4K}{n^2}.z\rho' = \frac{10.z}{14\times842}.\left\{\left(\frac{529}{396}\right)^2 - 1\right\}.$$

Reducing the numerical coefficient of this expression to seconds, we get,

[8529] $$\frac{2K}{n^2}.\rho'.z = 211'',84.z.$$

Multiplying this quantity by $1+12,5\times0,00375$, we shall have, at the temperature *zero*,

[8530] $$\frac{2K}{n^2}.\rho'.z = 221'',77.z.$$

At the temperature zero, (ρ) being the density of the air, under a pressure equal to that of a height of $0^{\text{metre}},76$ in the barometer, we have by observation,*

[8531] $$\frac{2K}{n^2}.(\rho).z = 187'',09.z\,;$$

therefore the humidity of the air adds, to the refractive force, the quantity,

[8532] $$\frac{34'',68.z}{1+x.0,00375}.$$

Multiplying this quantity by the tangent of the apparent zenith distance θ,
[8533] we shall obtain, as is evident from what has been said, the increment of the refraction depending on the humidity of the air. Therefore this increment is,†

[8534] $$\frac{34'',68.z}{1+x.0,00375}.\text{tang.}\theta.$$ [Increment of refraction.]

If we suppose the air to be saturated with water, the tabular logarithm of z

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[8531a] * (3947) Multiplying $\frac{2K}{n^2}.(\rho)$ [8277], by z, and by the radius in seconds 636620″, it becomes as in [8531] nearly. This represents the value of qz, which is used above.
[8531b] Subtracting it from pz [8530], we get $z.(p-q)=34'',68.z$; and by substituting it in [8521], we get [8532].

† (3948) If we suppose the height of the barometer to be $0^{\text{metre}},76$, we shall have $y=0$ [8489′]; and by retaining only the chief term of $\delta\theta$ [8501], we have
[8533a] $\delta\theta = \frac{\alpha.\text{tang.}\theta}{1+0,00375.x}$. The quantity $\alpha = \frac{2K}{n^2}.(\rho)$ nearly [8277′], is increased, by the vapor in the air, as in [8532], by the quantity $34'',68.z$; and this produces in $\delta\theta$ [8533a] the
[8533b] term $\frac{34'',68.z.\text{tang.}\theta}{1+0,00375.x}$, as in [8534].

will be, by what precedes,*

$$\text{tab. log. } z = -(100-x).0{,}0154547-(100-x)^2.0{,}0000625826. \qquad [8535]$$

Hence we have deduced the following values of the increment of refraction, depending on the humidity of the air, from fifteen to forty degrees of temperature of the centigrade thermometer;

Increment of refraction when the air is saturated with vapor.

Degrees of the centigrade thermometer.	Increment of refraction for extreme humidity.	
$x = 15°$	$0'',563 \,.\, \text{tang.}\,\theta$;	
$x = 20°$	$0'',744 \,.\, \text{tang.}\,\theta$;	
$x = 25°$	$0'',977 \,.\, \text{tang.}\,\theta$;	[8536]
$x = 30°$	$1'',274 \,.\, \text{tang.}\,\theta$;	
$x = 35°$	$1'',651 \,.\, \text{tang.}\,\theta$;	
$x = 40°$	$2'',122 \,.\, \text{tang.}\,\theta$.	

It follows, from this table, that the effect of the humidity of the air upon the refraction is very small; *the excess of the refractive power of the aqueous vapor, above that of the air, being in great measure compensated by its* [8537]
decrease of density. We may however notice it, by means of the preceding table, in case the humidity is very great. For this purpose we must determine, by hygrometrical observations, the ratio of the quantity of vapor contained in a given mass of air, to the quantity which would be found in [8538]
this mass, in the case of extreme humidity. We must then multiply by this ratio the increment of the refraction [8536], depending on this extreme humidity.*

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* (3949) The elastic force of the vapor is represented, in [8518], by $z.0^{\text{metre}},76$; [8535a]
putting this equal to its value, given in [8512], then dividing by $0^{\text{metre}},76$, and taking the logarithms of the results, we get, in the case of extreme humidity,

$$\text{tab. log. } z = i.0{,}0154547-i^2.0{,}0000625826. \qquad [8535b]$$

Now the values of x, i [8187, 8513], give $i=x-100° = -(100°-x)$; substituting this in the preceding expression of $\log.z$, we get [8535]. Substituting this in [8534], [8535c]
and then putting successively x equal to 15°, 20°, 25°, 30°, 35°, 40°, we get the increments of refraction as in the table [8536], nearly.

† (3950) Before closing this chapter we may remark, that besides the methods of computing the refraction which we have already explained, others have been proposed by [8538a]
several mathematicians and astronomers; some by means of new and more accurate observations, for determining the constant coefficients of former methods; others by [8538b]
combining these observations with the various hypotheses which are assumed, in a somewhat

[8538′] If we wish to notice the figure of the earth in the theory of refraction, we must observe that we can always imagine an osculatory circle to be drawn,

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arbitrary manner, to express the decrement of temperature, in ascending in the atmosphere; and we may particularly mention, in addition to the persons already spoken of, Brook
[8538c] Taylor, Euler, Mayer, Maskelyne, Lambert, Bernoulli, Young, Brinkley, Groombridge, Oriani, Bessel, Ivory, Plana, &c. Previously to the publications of the Mécanique
Kramp. Céléste, the theory of refraction had been treated of, in a very elaborate and satisfactory
[8538d] manner, by Kramp, in his *Analyse des Réfractions Astronomiques et Terrestres*, published at Strasbourg in 1798, using functions similar to those which are employed by La Place,
La Grange. [8538e] in [8316, &c.]. The refraction had also been treated of by La Grange, in the Berlin Memoirs for 1772, by the method which is explained in [8262x, 8374a, &c.]. Since the publication of La Place's method [8411, &c.], a table of refraction has been given by
Bessel. [8538f] Bessel, founded on Bradley's observations, and published in the *Fundamenta Astronomiæ*, in 1818. In calculating this table, Bessel follows, in a great measure, the methods given in this chapter; assuming for the expression of the temperature h, of the strata of the
[8538g] atmosphere, the formula $h=\frac{p.(\rho)}{(p).\rho}=e^{-ias}$; i being a small constant and positive quantity, determined so as to satisfy, very nearly, the observations of Bradley, in altitudes exceeding
[8538h] four degrees; but this formula makes the horizontal refraction too great. In the
Young. Transactions of the Royal Society of London, for 1819 and 1821, are given two memoirs
[8538i] of Dr. Young on the refraction. The first of them is founded on Leslie's hypothesis of the density,

[8538k] Leslie's hypothesis.

$$\rho=\frac{p.(\rho)}{(p)}.\left\{1+n.\left(\frac{(\rho)}{\rho}-\frac{\rho}{(\rho)}\right)\right\}.$$

n being a small constant quantity, to be determined by observation. In the second of these
[8538l] memoirs, for the purpose of obtaining an easy integration, he assumes p to be proportional
[8538m] to $\frac{3}{2}\rho^{\frac{3}{2}}-\frac{1}{2}\rho^2$. The result of the observations and calculations of Brinkley, are given in
Brinkley. the same Transactions for 1810, using Bradley's method [8383c], but changing, by means
[8538n] of new observations, his coefficients 57^s and 3, into $56^s,9$ and 3,2 respectively; also
Groombridge. the coefficient 400 [8383i] into 500. Groombridge has given two valuable papers on
[8538o] the refraction, in the same Transactions for 1810 and 1814, using also Bradley's method; but in the last of these papers he changes Bradley's coefficients 57^s and 3 [8383c], into $58^s,132967$ and 3,6342956 respectively, to conform to the results of his own observations.
Ivory. We might mention several others, but we shall restrict ourselves to that of Mr. Ivory, who
[8538p] has made an important improvement in the calculation of the refraction, in his paper, published in the same Transactions for 1823, pages 409 to 495. The limits of this note prevent our giving a full account of this interesting memoir, and we shall therefore only state the general principles of his method. He remarks that *the formulas of La Place*
[8538q] [8440, 8501] *give the mean refractions with greater accuracy than any other tables*
[8538r] *whatever, whether founded on theory or observation; and he finds that the table of*

touching the earth's surface at the point where the observer is situated, so [8538″]
that the plane of the circle may pass through the observed heavenly body;

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refractions, computed from these formulas, differ scarcely any from the results of his own calculations; but he objects to La Place's law of the decrement of temperature and pressure, which we have also shown to be defective, in [8444*o*, *z*, 8445*a*]; and he states, as the results of experience, that there is no ground whatever for attributing to the gradation of heat in the atmosphere, any other law than that of *an equable decrease as the* [8538*s*]
altitude increases; observing that this law prevails, very nearly, from the surface of the earth to the greatest height ascended by Gay Lussac; the decrement of heat, at any elevation in the atmosphere, being very nearly one degree of the centigrade thermometer [8538*t*]
for an ascent of 173 metres, or 95 English fathoms. This result differs considerably from La Place's hypothesis in low altitudes [8444*o*]. To obtain a formula in which the decrement of the temperature is nearly equable, Mr. Ivory finally assumes the two following equations [8538*v*, *w*], for the determination of the relation between the altitude and density of any stratum of the atmosphere, neglecting quantities of the order s^2, and changing the [8538*u*]
notation, so as to conform to that which is used by La Place, in this chapter. Formulas of Ivory's method.

$$\frac{\rho}{(\rho)}=c^{-u}; \qquad [8538v]$$

$$s=\frac{l}{a}\cdot\{(1-f).u+2f.(1-c^{-u})\}; \qquad [8538w]$$

$$l=7974^{\text{metres}}.\{1+0{,}00375.(x)\} \quad [8500]. \qquad [8538x]$$

This value of l being that at the surface of the earth, corrected for the temperature, as in [8500]. The differential of s [8538*w*], being multiplied by a, gives, [8538*y*]

$$ads=l.\{1-f+2f.c^{-u}\}.du. \qquad [8538z]$$

Substituting this in $dp=-(g).\rho.ads$ [8424], and then dividing by $(p)=(g).(\rho).l$, [8298], we get [8539*a*], which is easily reduced to the form [8539*b*], by using [8538*v*];

$$\frac{dp}{(p)}=-\frac{\rho}{(\rho)}.\{1-f+2f.c^{-u}\}.du \qquad [8539a]$$

$$=-(1-f).c^{-u}du-2f.c^{-2u}du. \qquad [8539b]$$

The integral of this last expression gives [8539*c*]; dividing it by [8538*v*], and substituting h [8435*r*], we get [8539*d*].

$$\frac{p}{(p)}=(1-f).c^{-u}+f.c^{-2u}; \qquad [8539c]$$

$$h=\frac{p.(\rho)}{(p).\rho}=1-f+f.c^{-u}=\frac{1+0{,}00375.x}{1+0{,}00375.(x)}. \qquad [8539d]$$

From various considerations, Mr. Ivory estimates f to be very nearly equal to $\frac{1}{4}$; and [8539*e*]
by substituting this value in [8538v, w, 8539c, d], he obtains the following system of equations, which he assumes as the best adapted for defining the mean state of the atmosphere;

[8539] then as the figure of the strata of the atmosphere is very nearly the same as that of the earth, it is evident that the circles which are concentric to this

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Ivory's theory.

[8539f] $$\frac{\rho}{(\rho)} = c^{-u};$$

[8539g] $$s = \frac{l}{a} \cdot \left\{ \tfrac{3}{4} u + \tfrac{1}{2} \cdot (1 - c^{-u}) \right\};$$

[8539h] $$\frac{p}{(p)} = \tfrac{3}{4} c^{-u} + \tfrac{1}{4} c^{-2u}$$

[8539i] $$= \tfrac{3}{4} \cdot \frac{\rho}{(\rho)} + \tfrac{1}{4} \cdot \left(\frac{\rho}{(\rho)} \right)^2;$$

[8539k] $$h = \frac{p \cdot (\rho)}{(p) \cdot \rho} = \tfrac{3}{4} + \tfrac{1}{4} \cdot c^{-u}$$

[8539l] $$= \tfrac{3}{4} + \tfrac{1}{4} \cdot \frac{\rho}{(\rho)}.$$

We shall now examine these formulas in the two extreme cases. First, where u is so small that we may neglect the square and higher powers of u. Second, where u is so
[8539m] *large as to correspond to the observation of Gay Lussac* [8445g]. In the first case, where u is very small, we have from [56] Int. $c^{-u} = 1 - u$, $c^{-2u} = 1 - 2u$; substituting these in [8539, f, g, h, k], and reducing, we get,

[8539n] $$\frac{\rho}{(\rho)} = 1 - u; \qquad s = \tfrac{5}{4} u \cdot \frac{l}{a}; \qquad \frac{p}{(p)} = 1 - \tfrac{5}{4} u; \qquad h = 1 - \tfrac{1}{4} u.$$

[8539o] If we put this value of h equal to that in [8446], we get $1 - \frac{1}{4} u = 1 + 0{,}00375 . x$; and by supposing $x = -1^{\circ}$, we obtain $u = 0{,}015$. Substituting this in s [8539n], we get
[8539p] $as = 0{,}01875 . l$; and by using the value $l = 8894^{\text{metres}}$ [8445i], corresponding to the temperature at the surface of the earth, in the experiment mentioned in [8445g], it
[8539q] becomes $as = 166^{\text{metres}}$; which represents, according to Ivory's theory, the elevation corresponding to a change of temperature of 1° of the centigrade thermometer, near the surface of the earth; agreeing very nearly with the observations in [8444p]. In ascending
[8539r] to the very small height as, we decrease the pressure of the atmosphere from l to $l - as$; hence $\frac{p}{(p)} = \frac{l - as}{l} = 1 - \frac{as}{l}$, as in [8404b]; and by substituting the value of $\frac{as}{l} = \frac{5}{4} u$ [8539n], it becomes $\frac{p}{(p)} = 1 - \frac{5}{4} u$, as in [8539n]; and this last expression,
[8539s] being divided by $h = 1 - \frac{1}{4} u$ [8539n], must evidently give, as in [8539n],
[8539t] $\frac{p}{(p) . h} = \frac{\rho}{(\rho)} = 1 - u$. We shall now compare the formulas [8539f—k] with the
[8539u] experiment of Gay Lussac [8445g]. Here we have $as = 6980^{\text{metres}}$ [8448]; substituting this, and l [8539p], in [8539g], after multiplying it by $\frac{2a}{l}$, we get,

[8539v] $$2 . \frac{as}{l} = \tfrac{3}{2} u + 1 - c^{-u} = 1{,}56960, \quad \text{or} \quad \tfrac{3}{2} u - c^{-u} = 0{,}56960.$$

osculatory circle, will also be nearly of the same curvature as the corresponding strata of the atmosphere; and we may determine the [8539']

From this last equation we obtain successively,

$$u = 0{,}70811\,; \qquad c^{u} = 2{,}03015\,; \qquad c^{-u} = 0{,}49257. \qquad [8539w]$$

Substituting these values in [8539f, h, k], we get,

$$\frac{\rho}{(\rho)} = 0{,}49257\,; \qquad \frac{p}{(p)} = 0{,}43008\,; \qquad h = 0{,}87314. \qquad [8539x]$$

Now from [8445g, n] we have, by observation, $(p) = 0^{\text{met.}},76568$, $p = 0^{\text{met.}},3312$; [8539y] consequently $\frac{p}{(p)} = 0{,}4326$, being nearly the same as by Ivory's theory [8539x]. Moreover the expression of h [8539x], by the theory, agrees very well with the result of observation in [8435t]; consequently the expression of $\frac{\rho}{(\rho)}$, obtained in [8539x], must [8539z] also be very nearly conformable to the theory. *Hence we see that this method of Ivory gives, with a great degree of accuracy, the temperature and density of the atmosphere in* [8540a] *these two extreme cases; and it gives also very nearly the same mean refractions as La Place's method, as we have observed in* [8538r].

At the summit of the atmosphere ρ vanishes, and we must then have $c^{-u} = 0$ [8539f], and $u = \infty$. Substituting this in [8539d], we get $1 - f = \frac{1+0{,}00375.x}{1+0{,}00375.(x)}$; hence we [8540b] deduce the following expression of x;

$$x = (x) - f.\{266°6 + (x)\}\,; \qquad [8540c]$$

and by putting, as in [8539e], $f = \frac{1}{4}$, it becomes,

$$x = \tfrac{3}{4}.(x) - 66°{,}7. \qquad [8540d]$$

If we assume, with Bradley, that the mean refraction is 50° of Fahrenheit's scale, corresponding to $(x) = 10°$ of the centigrade thermometer [8383g], we shall find that [8540e] this last value of x becomes $x = -59°{,}2$ of the centigrade thermometer, or $-74°$ of Fahrenheit's scale; differing a few degrees from the estimate of Fourier [8144y]. If [8540f] the temperature of the planetary space be constant, f must vary with the climate of the place, and we shall have, from [8540c], $f = \frac{(x)-x}{266°{,}6+(x)}$. We shall here close our [8540g] remarks on Ivory's method, and shall refer any one who wishes to pursue the subject to his memoir, and to two very interesting and important papers of Plana, given in vols. 27, 32, of the *Memorie della reale Accademia delle Scienze di Torino*, where the various methods [8540h] Plana. of computing the refraction, upon principles assumed by several astronomers and mathematicians, as Euler, Mayer, La Grange, Lambert, Kramp, Oriani, Young, Bessel, &c., [8540i] are minutely discussed; pointing out the peculiarities of their methods, and showing that the chief improvement made by La Place consists in introducing the fourth hypothesis, [8540k] [8411, 8412]; which is not so satisfactory as that of Ivory [8539f—l]. For greater

refraction of the body, by supposing the earth to be a sphere, with a radius equal to that of the proposed osculatory circle. Hence we see, *first*, *that*
[8540] *the refraction always takes place in the vertical plane; second, that the refraction is not the same in every direction relative to the horizon, since the osculatory circles are not the same in every direction;* but it is easy to prove
[8541] that this difference is insensible, when the object is somewhat elevated above the horizon. At the horizon there may be a difference of a few seconds.*

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accuracy, Plana reduces many of the integrals, which they use, to the form of elliptical
[8540l] functions. He particularly discusses, with much care, the hypothesis in [8538s], where the decrement of the heat of the atmosphere follows strictly the arithmetical progression of
[8540m] the altitudes. But the limits of this work will not allow us to go into any detail on this subject; we shall therefore close this note with the remark, that a very good historical
Mathieu. account of the writers on the refraction is given by Mathieu, at the end of the sixth volume
[8540n] of Delambre's *Histoire de l'Astronomie au dix-huitième siècle*, published at Paris, in 1827.

* (3951) To compute this, we must insert the value of the radius a, corresponding to
[8541a] the proposed osculatory circle, in the expressions [8434, 8435]. This will alter a little the values of f, l' [8437, 8436], as well as the expressions [8438, 8439, &c.]; and it is evident that the effect of this change in the value of a, which at its maximum alters it only $\frac{1}{300}$ part, must be very small in $\delta\theta$ [8440, 8474, &c.].

CHAPTER II.

ON TERRESTRIAL REFRACTIONS.

11. The *terrestrial refraction is nothing more than the part of the astronomical refraction, comprised between the origin of the curve of the ray of light, and the point where this curve meets the terrestrial object. This part being always small, in comparison with the whole refraction, we are enabled to use several simplifications, which we shall now explain.* [8542]

When the elevation of the object is very small, in respect to its distance, instead of giving the refraction in terms of the elevation of the object above the horizon, it is much more accurate and simple to express it in terms of the angle v [8139], which is found by the radii drawn from the earth's centre to the observer and to the object. We then have,* as in § 1, 3, [8543]

Differential formulas for the terrestrial refraction.

$$dv = \frac{\frac{a.dr}{r^2}.\sin.\theta.\sqrt{1+\frac{4K}{n^2}.(\rho)}}{\sqrt{1+\frac{4K}{n^2}.\rho-\left(1+\frac{4K}{n^2}.(\rho)\right).\frac{a^2}{r^2}.\sin.^2\theta}}; \qquad [8544]$$

$$d\delta = -\frac{\frac{2K}{n^2}.\frac{d\rho}{dr}.rdv}{1+\frac{4K}{n^2}.\rho}. \qquad [8545]$$

---

* (3952) Multiplying the numerator and denominator of the expression of dv [8263c] by $\sqrt{1+k\rho}$, then re-substituting the value of u [8262u], and that of $k=\frac{4K}{n^2}$ [8192b], [8544a] we get, by a slight reduction, the value of dv [8544]. Dividing the expression of $d\delta$ [8544b] [8262] by that of dv [8544], and then multiplying the result by dv, we get [8545].

The elevation of the object being supposed very small, we shall have very nearly,*

[8546] $$\rho = (\rho).\left(1 - \frac{ias}{l}\right);$$

[8547] *i being a constant coefficient, depending on the diminution of the heat of the strata of the atmosphere, corresponding to their elevations.* This value of ρ gives very nearly,†

[8548] $$\delta\theta = \frac{2K}{n^2}.(\rho).\frac{ia}{l}.v.$$

[8548'] $\delta\theta$ is the sum of the terrestrial refractions at the object and at the observer,

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* (3953) If we suppose the temperature to be uniform, we shall have, from [8299], after developing it by means of [56] Int.,

[8546a] $$\rho = (\rho).c^{-\frac{as}{l}} = (\rho).\left\{1 - \frac{as}{l} + \tfrac{1}{2}.\frac{a^2s^2}{l^2} - \&c.\right\};$$

[8546b] and if we neglect s^2, it becomes $\rho = (\rho).\left\{1 - \frac{as}{l}\right\}$, which is of the same form as [8546], supposing $i = 1$. The more complicated hypothesis relative to the temperature,

[8546c] in [8439], can also be reduced to the form [8546], as we shall see in [8555]; and this gives $i = 0,7159$, as in [8553d]. Ivory's hypothesis gives, near the surface of the

[8546d] earth, $u = \frac{4}{5}.\frac{as}{l}$, nearly [8539n]; substituting this in the value of ρ [8539n], we get $\rho = (\rho).\left\{1 - \frac{4}{5}.\frac{as}{l}\right\}$; whence $i = 0,8$ [8546] nearly, in this hypothesis.

† (3954) Substituting $ds = \frac{a.dr}{r^2}$ [8469a] in the differential of [8546], we get,

[8547a] $$d\rho = -\frac{ia}{l}.(\rho).ds = -\frac{ia}{l}.(\rho).\frac{a.dr}{r^2}; \qquad \text{or} \qquad \frac{d\rho}{dr} = -\frac{ia}{l}.(\rho).\frac{a}{r^2};$$

hence [8545] becomes, by neglecting terms of the order K^2,

[8547b] $$d\theta = \frac{2K}{n^2}.(\rho).\frac{ia}{l}.\frac{a}{r}.dv.$$

If we also neglect terms of the order sK, we may put $\frac{a}{r} = 1$ [8284], and then we have

[8547c] $d\theta = \frac{2K}{n^2}.(\rho).\frac{ia}{l}.dv$; whose integral is $\delta\theta = \frac{2K}{n^2}.(\rho).\frac{ia}{l}.v$, as in [8548]. Hence we see that by neglecting terms of the order s^2, we shall have *the whole refraction $\delta\theta$*

[8547d] *proportional to the intercepted arc v, whatever be the altitude of the observer and object;* and that a change in either of these altitudes produces no change in the refraction, always

[8547e] supposing the difference in the altitudes to be so small that we can neglect terms of the order s^2.

and this sum is the double of the refraction at either of these points, because the refraction is nearly the same at both points;* therefore the [8549] terrestrial refraction of an object, *which has but a small elevation, is very nearly represented by the following expression;*

$$\frac{K}{n^2}\cdot(\rho)\cdot\frac{ia}{l}\cdot v = \text{the terrestrial refraction.}$$ [8550]

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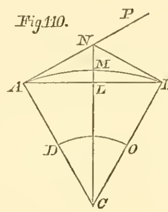
* (3955) In the annexed figure 110, C represents the centre of the earth; DO its surface; H the place of the observer; A the place of the observed object; AMH the [8549a] path of the ray; ALH the chord or line connecting the points A, H; ANP, HN, tangents to the path at the points A, H, intersecting each other in N; lastly, the line [8549b] CN cuts the path of the ray in M, and the chord AH in L. Then it is evident that the whole refraction of the ray, in proceeding from A to H, in its path AMH, is equal to the inclination of the two tangents AN, HN, which is measured by the external angle [8549c] HNP of the triangle ANH, or by the sum of the two internal angles NAH, NHA; so that we shall have $\delta\theta = NAH + NHA$. If $CH = CA$, the triangle CHN will be [8549d] similar and equal to CAN, and we shall have the angles NAH, NHA, equal to each other, whence $NHA = \frac{1}{2}\delta\theta$; and as NHA is evidently the terrestrial refraction at the point H, we shall have, for this refraction, the expression $\frac{1}{2}\delta\theta = \frac{K}{n^2}\cdot(\rho)\cdot\frac{ia}{l}\cdot v$ [8548], agreeing [8549e] with the formula [8550]. This result is accurate when the points A, H, are equally elevated above the surface, or $CA = CH$; and we have seen, in [8547e], that a small change in either of these altitudes does not materially affect the result; always supposing that we can neglect terms of the order s^2, in the value [8549f] of ρ [8546, &c.]. Finally, substituting in [8550] the values $\frac{K}{n^2}\cdot(\rho)$, a, l [8277, 8278, 8275], we get,

Fig. 110.



$$\frac{iv}{8{,}5194} = \text{terrestrial refraction.}$$ [8549g]

In the case of uniform temperature, we have $i = 1$ [8546b], and then the expression of the terrestrial refraction [8549g] becomes as in [8552]. In La Place's theory of the temperature, we have $i = 0{,}7159$ [8553d]; and by substituting this in [8549g], it [8549h] becomes $\frac{v}{11{,}9}$, as in [8556]. Lastly, if we use Ivory's value $i = 0{,}8$ [8546d], the refraction [8549g] becomes $\frac{v}{10{,}6}$ nearly. These quantities represent the mean values of [8549i] the terrestrial refraction in these different hypotheses; but it is found by observation that the terrestrial refraction varies extremely in different states of the atmosphere, being sometimes not more than $\frac{1}{24}v$, and at other times as much as $\frac{1}{2}v$. [8549k]

[8551] In case the temperature of the atmosphere is uniform, $i=1$ [8546b]; and then we have, at the temperature of melting ice, and at the height of the barometer $0^{\text{metre}},76$, the terrestrial refraction equal to,

Terrestrial refraction.

[8552]
$$\frac{v}{8,5194} = \text{terrestrial refraction.} \qquad \left[\begin{array}{c}\text{In the hypothesis of a uniform}\\ \text{temperature.}\end{array}\right]$$

If we adopt the law which is used in § 7, we shall have very nearly, at low elevations,*

[8553]
$$\rho = (\rho).\{1-u.686,93\};$$

[8554]
$$u = s - u.0,20137; \qquad \left[\begin{array}{c}\text{La Place's hypothesis of the}\\ \text{temperature [8142].}\end{array}\right]$$

which give,

[8555]
$$\rho = (\rho).\{1-s.571,551\}. \qquad [\text{La Place's value of } \rho.]$$

hence we deduce the terrestrial refraction, as in the following expression;

[8556]
$$\frac{v}{11,9003} = \text{terrestrial refraction.} \qquad \left[\begin{array}{c}\text{In La Place's hypothesis of the}\\ \text{temperature [8442.]}\end{array}\right]$$

It appears to me that this value ought to be adopted, at least until we can obtain, by direct observation, the factor i. *The value of i is very variable; it may even happen that, from some particular circumstances, the density*
[8556′] *of the atmospherical strata will increase with the elevation near the earth's surface, instead of decreasing; and then the refraction, instead of elevating*
[8557] *objects, will depress them.*† From such causes great differences have been observed in the terrestrial refractions [8549k].

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* (3956) If we neglect terms of the order s^2 or u^2 [8438], we shall have, by
[8553a] development, as in [56] Int., $c^{-u.1348,04} = 1-u.1348,04$; substituting this in [8439], we
[8553b] get $\rho = (\rho).\{1+u.[661,107-1348.04]\}$; which is easily reduced to the form [8553];
[8553c] whence we get $1-\frac{\rho}{(\rho)} = u.686.93$. Substituting this in the value of u [8438], we obtain [8554], or $u = s.0,832037$; hence the value of ρ [8553] becomes,

[8553c′]
$$\rho = (\rho).\{1-s.686,93\times0,832037\} = (\rho).\{1-s.571,551\},$$

as in [8555]. Comparing this with the assumed value $\rho = (\rho).\left(1-s.\frac{ia}{l}\right)$ [8546], we
[8553d] obtain $\frac{ia}{l} = 571,551$; and by using $\frac{a}{l} = 798,37$ [8278d], we get $i = 0,7159$. With this value of i, we find that the terrestrial refraction [8549g] becomes as in [8556].

Mirage.

† (3957) The phenomenon of *mirage*, which is sometimes observed on the surface of the sea, and on dry sandy plains, as in those of Egypt, arises from this source; as has been
[8556a] shown by Monge. This subject is fully discussed by Biot, in an excellent paper *sur les*

We have no occasion to ascertain these refractions, except for correcting the observed altitudes of the object; and we can determine these altitudes directly, [8558]
by integrating the expression of dv [8544]. For, if we use the value of ρ [8546], and then integrate [8544] from $s=0$, we shall obtain,*

$$\frac{v}{\sin.\Theta}=\frac{\sqrt{\cos.^2\Theta+2s.\sin.^2\Theta-\frac{4K}{n^2}.(\rho).\frac{ia}{l}.s}-\cos.\Theta}{\sin^2\Theta-\frac{2K}{n^2}.(\rho).\frac{ia}{l}}. \qquad [8559]$$

Height of the observed object.

Hence we deduce,†

$$as=\frac{av^2}{2}.\left\{1-\frac{2K}{n^2}.(\rho).\frac{ia}{l.\sin.^2\Theta}\right\}+av.\cot.\Theta; \qquad [8560]$$

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refractions extraordinaires qui s'observent très-près de l'horizon, which is published in the [8556b]
Mémoires de l'Institut de France, for the year 1809. This paper contains a full theory of [8556c]
these refractions, together with many observations, made by Biot near Dunkirk.

* (3958) Substituting $\frac{a.dr}{r^2}=ds$ [8469a], and $\frac{a^2}{r^2}=(1-s)^2$ [8284], in [8263c], [8559a]
then dividing the numerator and denominator by u, we get $dv=\frac{ds.\sin.\Theta}{\sqrt{\text{u}^{-2}-(1-s)^2.\sin.^2\Theta}}$. [8559b]
Now by neglecting terms of the order k^2, we have $\text{u}^{-2}=1+k\rho-k(\rho)$ [8262u]; and if we use the value of ρ [8546], it becomes $\text{u}^{-2}=1-k(\rho).\frac{ias}{l}$. Substituting this in the [8559c]
preceding value of dv, putting also $1-\sin.^2\Theta=\cos.^2\Theta$; $(1-s)^2=1-2s$, neglecting s^2, then dividing by $\sin.\Theta$, we get, [8559d]

$$\frac{dv}{\sin.\Theta}=\frac{ds}{\sqrt{\cos.^2\Theta+s.\left\{2.\sin.^2\Theta-k(\rho).\frac{ia}{l}\right\}}}. \qquad [8559e]$$

The integral of this expression is as in [8559], as is easily proved by taking its differential; v, s, being the variable quantities; observing that the term $-\cos.\Theta$ is [8559f]
introduced into the numerator of [8559] as the constant quantity, so as to make v vanish when $s=0$; and that k is the same as in [8192b].

† (3959) Putting for brevity,

$$A=1-\tfrac{1}{2}k(\rho).\frac{ia}{l.\sin.^2\Theta}, \qquad \text{or} \qquad A.\sin.^2\Theta=\sin.^2\Theta-\tfrac{1}{2}k(\rho).\frac{ia}{l}, \qquad [8560a]$$

we get, from [8559],

$$\frac{v}{\sin.\Theta}=\frac{\sqrt{\cos.^2\Theta+2As.\sin.^2\Theta}-\cos.\Theta}{A.\sin.^2\Theta}. \qquad [8560b]$$

Multiplying this by $A.\sin.^2\Theta$, transposing $\cos.\Theta$, and squaring, we get,

[8561] *as being very nearly the height of the observed object, above the level of the observer.* It is easy to show that this expression agrees with that which we obtain by correcting the height by means of the preceding expression of the refraction.*

To determine as, whatever be the apparent angle Θ, we must integrate
[8562] the expression of dv [8544]; and this integration requires that the law, according to which the density of the strata of the atmosphere decreases, should be known. If we use that which is assumed in [8412], we may easily integrate the expression of dv, by the analysis explained in that article; and from this integral we may deduce the value of s, in terms of
[8563] v. But when the apparent altitudes are rather large, we can obtain this value, independent of any hypothesis, relative to the constitution of the atmosphere; as we have seen, in [8501], that the astronomical refraction is independent of it.

[8560c] $$\{\cos.\Theta + Av.\sin.\Theta\}^2 = \cos.^2\Theta + 2As.\sin.^2\Theta.$$

Developing the first member, rejecting $\cos.^2\Theta$ from both sides, and then dividing by $A.\sin.^2\Theta$, we get $2v.\text{cotang}.\Theta + Av^2 = 2s$; multiplying this by $\frac{1}{2}a$, we get as [8560].
[8560d] Now the height of the object is $r - a = rs$ [8284]; and as r is nearly equal to a, this becomes very nearly equal to as, as in [8560].

[8561a] * (3960) If there were no refraction, the ray would proceed directly from the object at A, to the observer at O, in figure 106, page 444; and if we put the external angle
[8561b] AOH, of the triangle COA, equal to Θ, and the angle $OCA = v$, we shall have the angle $CAO = \Theta - v$; and since $\sin.COA : \sin.CAO :: CA : CO$, we have in symbols, by development,

[8561c] $$a.\sin.\Theta = r.\sin.(\Theta - v) = r.\sin.\Theta.\cos.v - r.\cos.\Theta.\sin v = r.\sin.\Theta.\{\cos.v - \sin.v.\text{cotang}.\Theta\}.$$

Neglecting v^3 on account of its smallness, we may put $\sin.v = v$, $\cos.v = 1 - \frac{1}{2}v^2$, [43, 44] Int.; substituting these in [8561c], then dividing by $\sin.\Theta$, and reducing, we get,

[8561d] $$r - a = r.\{\tfrac{1}{2}v^2 + v.\text{cotang}.\Theta\} = \tfrac{1}{2}av^2 + av.\text{cotang}.\Theta, \quad \text{nearly.}$$

This represents the altitude $r - a$ very nearly, when there is no refraction. If we now suppose the zenith distance Θ to be decreased by the refraction $\frac{1}{2}\delta\vartheta$ [8549e], the expression $\text{cotang}.\Theta$ will become $\text{cotang}.\Theta + \frac{\frac{1}{2}\delta\vartheta}{\sin.^2\Theta}$ [54] Int.; v being unaltered [8570];
[8561e] consequently the height $r - a$ [8561d], will be increased by the term $\frac{1}{2}.\frac{av.\delta\vartheta}{\sin.^2\Theta}$; and by using the value of $\frac{1}{2}\delta\vartheta$ [8549e], it becomes $\frac{K}{n^2}.(\rho).\frac{ia^2v^2}{l.\sin.^2\Theta}$. Hence the corrected

[8561f] value of the height is $r - a = \frac{1}{2}av^2.\left\{1 - \frac{2K}{n^2}.(\rho).\frac{ia}{l.\sin.^2\Theta}\right\} + av.\text{cotang}.\Theta$; being the same as in [8560].

If we suppose $\frac{a}{r}=1-s$ [8284], we shall have,* [8564]

$$dv=\frac{ds.\sin.\theta}{\sqrt{\cos.^2\theta+2s.(1-\frac{1}{2}s).\sin.^2\theta-2\alpha\left(1-\frac{\rho}{(\rho)}\right)}}. \qquad [8565]$$

By reducing it to a series, we shall have,

$$dv=ds.\left\{1-s.(1-\tfrac{1}{2}s).\text{tang.}^2\theta+\tfrac{3}{2}.s^2.\text{tang.}^4\theta+\frac{\alpha.\left(1-\frac{\rho}{(\rho)}\right)}{\cos.^2\theta}+\&c.\right\}.\text{tang.}\theta\,; \qquad [8566]$$

which gives by integration, commencing the integral from $s=0$, [8567]

$$v=s.\left\{1-\tfrac{1}{2}s.\text{tang.}^2\theta+\frac{s^2.(1+2.\sin.^2\theta)}{6.\cos.^2\theta}.\text{tang.}^2\theta\right\}.\text{tang.}\theta+\frac{\alpha.\text{tang}\,\theta}{\cos.^2\theta}.\left\{s-\frac{\int\rho ds}{(\rho)}\right\}. \qquad [8568]$$

Let as' be the height calculated without taking notice of the refraction, so that $s=s'-\delta s$. The refraction does not alter the value of v, because it elevates the objects in a vertical plane, and a point viewed from the two extremities of a base is seen in the common intersection of two vertical planes which pass through these extremities and the observed object; now this common intersection is a radius of the earth; *therefore the value of v remains the same as it is when we neglect the refraction.* Thus by substituting [8569] [8569'] [8570]

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* (3961) From [8262u] we get, by successive reductions,

$$u^{-2}=\frac{1+k\rho}{1+k(\rho)}=1-\frac{k(\rho)}{1+k(\rho)}.\left(1-\frac{\rho}{(\rho)}\right)=1-2\alpha.\left(1-\frac{\rho}{(\rho)}\right);\quad [8277'] \qquad [8565a]$$

substituting this and $1-\sin.^2\theta=\cos.^2\theta$ in dv [8559b], it becomes as in [8565]. If we divide the numerator and denominator of dv [8565], by $\cos.\theta$, we may put it under the following form;

$$dv=ds.\left\{1+2s.(1-\tfrac{1}{2}s).\text{tang.}^2\theta-\frac{2\alpha}{\cos.^2\theta}.\left(1-\frac{\rho}{(\rho)}\right)\right\}^{-\frac{1}{2}}.\text{tang.}\theta. \qquad [8565b]$$

Developing this expression, and neglecting terms of the order s^3, α^2, it becomes as in [8566]. The coefficient of s^2, in the terms between the braces in [8566], is,

$$\tfrac{1}{2}.\text{tang.}^2\theta+\tfrac{3}{2}.\text{tang.}^4\theta=\tfrac{1}{2}.\text{tang.}^2\theta.\left\{1+3.\frac{\sin.^2\theta}{\cos.^2\theta}\right\}=\frac{\text{tang.}^2\theta}{2.\cos.^2\theta}.\{\cos.^2\theta+3.\sin.^2\theta\}$$
$$=\frac{(1+2.\sin.^2\theta)}{2.\cos.^2\theta}.\text{tang.}^2\theta\,; \qquad [8565c]$$

and by substituting it in [8566], we obtain,

$$dv=\left\{ds-sds.\text{tang.}^2\theta+\frac{(1+2.\sin.^2\theta)}{2.\cos.^2\theta}.\text{tang.}^2\theta.s^2ds\right\}.\text{tang.}\theta+\frac{\alpha.\text{tang.}\theta}{\cos.^2\theta}.\left\{ds-\frac{\rho ds}{(\rho)}\right\}. \qquad [8565d]$$

Integrating this expression, we get v [8568].

$s'-\delta s$ for s, and neglecting the products $s\delta s$ and $\alpha\delta s$, we shall have,*

[8571] $$0=-\delta s.\,\text{tang.}\theta+\frac{\alpha.\text{tang.}\theta}{\cos.^2\theta}.\left(s'-\frac{\int\rho ds}{(\rho)}\right).$$

Hence we deduce,

[8572] $$a\delta s=\frac{\alpha}{\cos.^2\theta}.\left\{as'-\frac{a.\int_0^s\rho\,ds}{(\rho)}\right\}.$$

[8573] $(g).a.\int_0^s\rho ds$ [8469e], is the pressure of the atmosphere, at the place of the
[8573′] observer, *minus* its pressure at the observed object.† Let ε be the difference of the heights of the barometer at these points, the mercury being in both places reduced to the temperature zero; and if we suppose that (ρ)

[8571a] * (3962) If we suppose the coefficient of α, in [8568], to be represented by B, and the remaining terms of v by A, we shall have $v=A+\alpha B$; A and B being functions of s. Now if we put A and B for the values of A, B, respectively, when there is no refraction, or $\alpha=0$, we shall have generally, by Taylor's theorem, when s changes into
[8571b] $s'-\delta s$ [8569], $A=\mathrm{A}-\left(\frac{d\mathrm{A}}{ds}\right).\delta s+\&c.$; $B=\mathrm{B}-\left(\frac{d\mathrm{B}}{ds}\right).\delta s+\&c.$ Substituting these in the expression of v [8571a], we get,

[8571c] $$v=\mathrm{A}-\left(\frac{d\mathrm{A}}{ds}\right).\delta s+\&c.+\alpha.\left\{\mathrm{B}-\left(\frac{d\mathrm{B}}{ds}\right).\delta s+\&c.\right\}.$$

Now when there is no refraction we have $\alpha=0$, and then the expression of v [8571a] becomes $v=\mathrm{A}$; and as v does not alter by the refraction [8570], we may substitute this value of v in [8571c], and then by neglecting A, which occurs in both members, we shall
[8571d] have $0=-\left(\frac{d\mathrm{A}}{ds}\right).\delta s+\alpha.\mathrm{B}$, neglecting the other terms depending on δs^2, $\alpha\delta s$, on account of their smallness. If we retain only the chief term of A [8571a, 8568], we shall have $A=s.\text{tang.}\theta$; whence $\left(\frac{d\mathrm{A}}{ds}\right)=\text{tang.}\theta$; we also have, from [8571a, 8568],

[8571e] $$B=\frac{\text{tang.}\theta}{\cos.^2\theta}.\left\{s-\frac{\int\rho ds}{(\rho)}\right\},\qquad\text{or}\qquad \mathrm{B}=\frac{\text{tang.}\theta}{\cos.^2\theta}.\left\{s'-\frac{\int\rho ds}{(\rho)}\right\};$$

[8571f] substituting these values in [8571d], we get [8571]. Multiplying this by $\frac{a}{\text{tang.}\theta}$, and transposing $a\delta s$, we get [8572].

[8573a] † (3963) We have, in [8469e], $(g).a.\int_0^s\rho ds=(p)-p$, as in [8573]. Dividing this by $(g).(\rho).l=(p)$ [8298], we get $\frac{a.\int_0^s\rho\,ds}{(\rho).l}=\frac{(p)-p}{(p)}$; now $\frac{(p)-p}{(p)}$ is evidently
[8573b] equal to the ratio $\frac{\varepsilon}{0^{\text{met.}},76}$ [8573]; and by substituting it in the preceding equation, then multiplying by l, we get [8575].

corresponds to that temperature, and to the height of the barometer $0^{\text{metre}},76$, we shall have, [8574]

$$\frac{a.\int_0^s \rho ds}{(\rho)} = \frac{\varepsilon l}{0^{\text{metre}},76}; \quad \text{or} \qquad [8575]$$

$$-\frac{\alpha}{(\rho)}.a.\int_0^s \rho ds = -\frac{\alpha l}{0^{\text{metre}},76}.\varepsilon. \qquad [8575']$$

We must vary this value in the proportion of the supposed density (ρ) to the true density; but as the value of α varies in the inverse ratio,* it follows that, ε remaining the same, the value of $\frac{\alpha.a.\int \rho ds}{(\rho)}$ will always remain the same. By substituting for α its value [8277'], we get, from [8575'], [8576]

$$-\alpha.\frac{a.\int_0^s \rho ds}{(\rho)} = -3,08338.\varepsilon. \qquad [8577]$$

To obtain the inclination of the visible horizon to the true horizon, when we are at any elevation above the level of the sea, we must ascertain the values of $\frac{dr}{rdv}$ in the several parts of the trajectory of the luminous ray, which touches the surface of the sea. The preceding expression of dv gives, when $\theta = 100°$,† [8578]

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* (3964) We have very nearly, from [8277'], $\alpha = \frac{2K}{n^2}.(\rho)$; so that if (ρ) varies, the value of α will vary in the same ratio; and as α, (ρ), have an inverse operation upon each other in the first member of the formula [8575'], the value, in its second member, will remain unaltered by any variation of the density (ρ); as is observed in [8576]. Now substituting, in the second member of [8575'], the values of α, l [8277', 8275], we get [8577]. [8576a] [8576b]

† (3965) The symbol v' [8140''] represents the angle formed by the radius r, and the tangent or path of the ray, at the point A, in the figure 106, page 444; and its complement expresses the angle of *depression* of this tangent below the *true horizon*, which will therefore be represented by $100°-v'$; so that if the path AO of the ray touch the earth at the point O, making there the angle $\theta = 100°$, we shall have $100°-v' = D$, for the depression of the visible horizon below the true horizon. Hence we obtain, from [8145], $\text{tang}.D = \text{cotang}.v' = \frac{dr}{rdv}$; or, on account of the smallness of the depression, $D = \frac{dr}{rdv}$; now we have, by putting $\sin.\theta = 1$ in [8263d], and multiplying by r, $rdv = \frac{au.dr}{\sqrt{r^2-a^2u^2}}$; substituting this in the preceding value of D, we obtain, [8579a] [8579b] [8579c] [8579d]

[8579] $$\frac{dr}{rdv}=\frac{r}{a}\cdot\sqrt{1-\frac{a^2}{r^2}-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)}$$

[8579′] $$=\sqrt{\frac{r^2}{a^2}-1-2\alpha.\frac{r^2}{a^2}.\left(1-\frac{\rho}{(\rho)}\right)};$$

or very nearly,

[8580] $$\frac{dr}{rdv}=\sqrt{2s-2\alpha.\left(1-\frac{\rho}{(\rho)}\right)}.$$ [Depression of the horizon.]

We must observe that $\frac{dr}{rdv}$ is the tangent of the angle of depression of the
[8581] visible horizon, at the height as; and this tangent may be taken for the angle itself [8579c].

If the height is rather small, we shall have, for the angle of depression,*

[8582] $$\sqrt{2s.\left(1-\alpha.\frac{ia}{l}\right)}.$$ [Depression of the horizon when s is small.]

[8579e] $$D=\frac{\sqrt{r^2-a^2u^2}}{au}=\sqrt{\frac{r^2}{a^2}.u^{-2}-1};$$

and by using the value of u^{-2} [8565a], it is reduced to the form [8579′]. Finally, by
[8579f] neglecting terms of the order s^2, we have $\frac{r^2}{a^2}=1+2s$ [8284]; substituting this in [8579′], and neglecting αs, we get [8580].

[8582a] * (3966) When s is very small, we have $1-\frac{\rho}{(\rho)}=\frac{ias}{l}$ [8546]; substituting this in [8580], we get [8582].

CHAPTER III.

ON THE EXTINCTION OF THE LIGHT OF THE PLANETS IN THE ATMOSPHERE, AND ON THE SUN'S ATMOSPHERE.

12. The *extinction of the light of the heavenly bodies, in passing through the atmosphere, has so near a relation to the theory of refraction, that we shall here take notice of it; and shall put,* [8583]

$\varepsilon =$ the intensity of the light of a body, when it has arrived at a stratum of [8584]
the atmosphere whose radius is r, and density ρ; *its intensity, at the time of its entrance into the atmosphere, being taken for unity;* [8585]

we shall then have,*

$$d\varepsilon = -Q.\rho.\varepsilon.\sqrt{dr^2+r^2dv^2}; \qquad [8586]$$

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* (3967) In figure 106, page 444, we have $AF = dr$, $A'F = rdv$; whence we get $AA' = \sqrt{dr^2+r^2dv^2}$, representing the arc described by the ray in the time dt. [8586a]
Multiplying this by the intensity of the ray ε, and by the density ρ, we get $\rho.\varepsilon.\sqrt{dr^2+r^2dv^2}$; which is proportional to the decrement of density $d\varepsilon$ [8587]; so that by [8586b]
taking Q for the constant ratio of these quantities, we shall have, as in [8586],

$$d\varepsilon = -Q.\rho.\varepsilon.\sqrt{dr^2+r^2dv^2} = -Q.\rho.\varepsilon.dr.\sqrt{1+\frac{r^2dv^2}{dr^2}}. \qquad [8586c]$$

Now we have, in [8263d], $\frac{rdv}{dr} = \frac{a\,\mathrm{u}.\sin.\Theta}{\sqrt{r^2-a^2.\mathrm{u}^2.\sin.^2\Theta}}$; and by substituting its square in the first member of the following equation, we get,

$$\sqrt{1+\frac{r^2dv^2}{dr^2}} = \frac{r}{\sqrt{r^2-a^2.\mathrm{u}^2.\sin.^2\Theta}} = \frac{\mathrm{u}^{-1}}{\sqrt{\mathrm{u}^{-2}-\frac{a^2}{r^2}.\sin.^2\Theta}}; \qquad [8586d]$$

substituting this in $d\varepsilon$ [8586c], and then dividing by ε, we get,

$$\frac{d\varepsilon}{\varepsilon} = -Q.\rho.dr.\frac{\mathrm{u}^{-1}}{\sqrt{\mathrm{u}^{-2}-\frac{a^2}{r^2}.\sin.^2\Theta}}. \qquad [8586e]$$

Multiplying the numerator and denominator of the last factor of this expression by

[8587] *Q being a constant coefficient.* For it is evident that the differential $d\varepsilon$ of the intensity of light is proportional to the product of its intensity ε, by the density of the stratum ρ, and by the arc described by the ray of light. Substituting the value of rdv, given in the preceding chapter, we shall have,

[8588]
$$\frac{d\varepsilon}{\varepsilon}=\frac{-Q.\rho.dr.\sqrt{1+\frac{4K}{n^2}.(\rho)}}{\sqrt{1+\frac{4K}{n^2}.\rho-\left(1+\frac{4K}{n^2}.(\rho)\right).\frac{a^2}{r^2}.\sin.^2\Theta}};$$

[8589] In this expression of $\frac{d\varepsilon}{\varepsilon}$ we may suppose the factor $\sqrt{1+\frac{4K}{n^2}.\rho}$ to be equal to unity. If the body is sensibly elevated above the horizon, the denominator will be reduced to $\cos.\Theta$ nearly. Then by integration,
[8590] observing that* $\int\rho dr=(\rho).l$, we shall have, as in [8590d],

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$\sqrt{1+k(\rho)}$, and then substituting $u^{-1}.\sqrt{1+k(\rho)}=\sqrt{1+k\rho}$ [8262u], we obtain,

$$\frac{d\varepsilon}{\varepsilon}=-Q.\rho.dr.\frac{u^{-1}.\sqrt{1+k(\rho)}}{\sqrt{u^{-2}.[1+k(\rho)]-[1+k(\rho)].\frac{a^2}{r^2}.\sin.^2\Theta}}$$

[8586f]
$$=-Q.\rho.dr.\frac{\sqrt{1+k\rho}}{\sqrt{1+k\rho-[1+k(\rho)].\frac{a^2}{r^2}.\sin.^2\Theta}};$$

which is the same as in [8588], using $k=\frac{4K}{n^2}$ [8192b]. If we divide the numerator and denominator of this last expression by $\sqrt{1+k\rho}$, the new denominator will be,

[8586g]
$$\sqrt{\left\{1-\left(\frac{1+k(\rho)}{1+k\rho}\right).\frac{a^2}{r^2}.\sin.^2\Theta\right\}};$$

and as k is of the order 0,0006 [8277, 8192b], $\frac{a}{r}=1$ nearly, this denominator becomes very nearly equal to $\sqrt{1-\sin.^2\Theta}=\cos.\Theta$; therefore the expression [8586f] is very
[8586h] nearly represented by $\frac{d\varepsilon}{\varepsilon}=-Q.\rho.dr.\frac{1}{\cos.\Theta}=-\frac{Q}{\cos.\Theta}.\rho.dr$; which may be used when
[8586i] the object is somewhat elevated above the horizon, so that $\cos.\Theta$ may have a sensible value.

[8590a] * (3968) Putting $\frac{a}{r}=1$ nearly, in [8294], we get $dp=-(g).\rho.dr$; whose integral is $p=(p)-(g).\int\rho.dr$; (p) being the constant quantity added to the integral, so as to make p equal to (p) at the surface of the earth, where $\int\rho.dr=0$. At the
[8590b] entrance of the light into the atmosphere, p becomes 0, and we shall suppose that r then

$$\log.\varepsilon = -\frac{Q.(\rho).l}{\cos.\Theta}. \qquad [8591]$$

If we put E *for the value of* ε, *or the intensity of the light at the zenith,* [8592]
where $\cos.\Theta = 1$, we shall have,*

$$\log.E = -Q.(\rho).l; \qquad [8592']$$

$$\log.\varepsilon = \frac{\log.E}{\cos.\Theta}; \qquad [8593]$$

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becomes r; hence we have $0 = (p) - (g).\int_a^r \rho.dr$, or $(g).\int_a^r \rho.dr = (p) = (g).(\rho).l$, [8590c]
[8298]; dividing this by (g), we get $\int_a^r \rho.dr = (\rho).l$, as in [8590]. Substituting this in the integral of the equation [8586h], we get,

$$\log.\varepsilon = -\frac{Q}{\cos.\Theta}.\int_a^r \rho.dr = -\frac{Q}{\cos.\Theta}.(\rho).l, \qquad [8590d]$$

as in [8591].

* (3969) At the zenith, where $\cos.\Theta = 1$, and $\varepsilon = E$, the equation [8591] [8593a]
becomes $\log.E = -Q.(\rho).l$, as in [8592']; substituting this in the second member of [8591], we obtain [8593]. Moreover we have, in [8298], $(\rho).l = \frac{(p)}{(g)}$; and as (g) is given, $(\rho).l$ must be proportional to (p), or to the height of the barometer, as in [8594]. [8593b]
Now if we put, as in [8290'], c equal to the number whose hyperbolic logarithm is unity, and take the constant quantity f, so that $E = c^{-f}$, or $\text{hyp.log}.E = -f$, the equation [8593c]
[8593] will give,

$$\text{hyp.log}.\varepsilon = -\frac{f}{\cos.\Theta} = \text{hyp.log}.c^{-\frac{f}{\cos.\Theta}}; \quad \text{or,} \quad \varepsilon = c^{-\frac{f}{\cos.\Theta}}. \qquad [8593d]$$

Effect of the earth's atmosphere on the intensity of light.

Hence it appears that if a heavenly body be observed, whose zenith distance is Θ, *the* [8593e]
intensity of its light, which is represented by unity upon entering into the earth's atmosphere [8585], *will be reduced to* $c^{-\frac{f}{\cos.\Theta}}$, *at the eye of the observer.* The same process will answer in finding the decrement of the intensity of a ray of light, in passing [8593f]
from the surface of the earth to the summit of the atmosphere, by taking the integral of $\frac{d\varepsilon}{\varepsilon} = -\frac{Q\rho.dr}{\cos.\Theta}$ [8586h], in an inverse order; or from the surface of the earth to the [8593g]
summit of the atmosphere. In this case we shall have the same equations as in [8590—8593], supposing the intensity of the ray, at the surface of the earth, to be [8593h]
represented by unity; then its intensity, upon quitting the atmosphere at the zenith, will

Effect of the sun's atmosphere on the intensity of light.

be $E = c^{-f}$ [8593c]; or $\varepsilon = c^{-\frac{f}{\cos.\Theta}}$, if the zenith distance be Θ [8593f]. In like [8593i]
manner, if the *intensity of a ray of light, proceeding from a luminous place upon the sun's body, be represented by unity, it will be reduced to* $E = c^{-f}$, *upon quitting the sun's* [8593k]

[8594] $(\rho).l$ *being proportional to the observed height of the barometer* [8593b]; *it*
[8594′] *is plain that* $\log.E$, *and generally the logarithm of the intensity of the light of any heavenly body, is proportional to that height of the barometer.* It is also evident that the two preceding logarithms may be considered as tabular ones in the equation [8593], as in [8593n, o].

Intensity of the light of the heavenly bodies.

We may easily obtain the value of E by comparing the intensities of the light of the same heavenly body, as for example, that of the moon, at two different heights.* In this way, *Bouguer found that the light of a body, seen*
[8595] *in the zenith, is reduced, after having passed through the atmosphere, to*
[8596] 0,8123. The tabular logarithm of this number is $\log.E = -0{,}0902835$;

Formula for the intensity.

therefore, by dividing this logarithm by the sine of the apparent altitude of the heavenly body, we shall obtain the logarithm of the intensity of the light at that altitude.

Very near the horizon, the diminution of light depends, like the refraction, upon the constitution of the atmosphere. If we adopt the hypothesis which
[8597] we have given in [8411, 8412], we shall easily obtain, by the analysis of that article, the corresponding value of the intensity of the light. But we may, without fear of any sensible error, use the hypothesis of a uniform
[8598] temperature. In this hypothesis, we have† $\rho dr = -l d\rho$; therefore, by

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[8593l] *atmosphere at the zenith of that place, or to* $\varepsilon = c^{-\frac{f}{\cos.\Theta}}$, *if the distance from the*
[8593m] *zenith of that place of the sun's body, or the inclination to the vertical, be represented by* Θ. We may finally remark, that we may use either tabular or hyperbolic logarithms
[8593n] in [8593], because they are proportional to each other, and occur in both members of the equation; and the same may be done in the expression of $\log.E$ [8592′]; taking care,
[8593o] however, to adapt the constant quantity Q to the kind of logarithms which are used; being the tabular logarithms in [8596, &c.].

[8595a] * (3970) At the zenith, where $\cos.\Theta = 1$, the observed intensity of the light is E, [8585, 8592]; the general value, corresponding to any zenith distance Θ, being ε, [8585, 8584]. Now if the ratio of these intensities is observed, and found to be as 1 to
[8595b] b, we shall have $\varepsilon = bE$, or $\log.\varepsilon = \log.b + \log.E$; substituting this in [8593], we
[8595c] get $\log.b + \log.E = \frac{\log.E}{\cos.\Theta}$; whence $\log.E = \frac{\cos.\Theta}{1-\cos.\Theta}.\log.b$; whence we easily deduce the value of E, as in [8596].

[8598a] † (3971) The differential of the logarithm of [8299], is $\frac{d\rho}{\rho} = -\frac{ads}{l} = -\frac{dr}{l}$ nearly, [8559a]; multiplying this by $-l\rho$, we get $-ld\rho = \rho.dr$, as in [8598].

putting the element of the refraction equal to $d\delta$, we shall have very nearly,*

$$\frac{d\varepsilon}{\varepsilon} = -\frac{H.d\delta}{\sin.\Theta}.$$ [8599]

Expression of the intensity, [8600] in terms of the refraction.

H being a constant quantity. *Therefore the logarithm of the intensity of light, of any heavenly body, is proportional to its refraction, divided by the cosine of its apparent altitude* [8599d].

We have seen in [8503], that, at the apparent altitude of 50°, the refraction is 186″,728; and in the hypothesis of a uniform temperature, the refraction in the horizon is 7390″,71 [8370]; hence we easily find that [8601]
the light at the horizon is† $\frac{1}{3779,1}$. We may, by these formulas, determine, [8602]
in an eclipse of the moon, the quantity of light which falls upon the moon's

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* (3972) If we neglect $\frac{4K}{n^2}\cdot(\rho)$, $\frac{4K}{n^2}\cdot(\rho)$, in comparison with 1, and put $\frac{a}{r}=1$, [8599a]
as we have done in the preceding notes, we shall find that the radical, in the denominator of the value $d\delta$ [8262], becomes $\sqrt{1-\sin.^2\Theta}=\cos.\Theta$; consequently this expression of [8599b]
$d\delta$ is reduced to $d\delta=-\frac{2K}{n^2}.d\rho.\frac{\sin.\Theta}{\cos.\Theta}=\frac{2K}{n^2}.\frac{\rho dr}{l}.\frac{\sin.\Theta}{\cos.\Theta}$ [8598]; but from [8586h], we [8599c]
have $-\frac{\rho dr}{\cos.\Theta}=\frac{d\varepsilon}{\varepsilon}.\frac{1}{Q}$; hence by substitution $d\delta=-\frac{2K}{n^2}.\frac{1}{Ql}.\frac{d\varepsilon}{\varepsilon}.\sin.\Theta$; and by putting $\frac{2K}{n^2.Ql}=\frac{1}{H}$, it becomes $d\delta=-\frac{1}{H}.\frac{d\varepsilon}{\varepsilon}.\sin.\Theta$; multiplying this by $\frac{H}{\sin.\Theta}$, we get [8599]. Its integral is $\log.\varepsilon=-H.\frac{\delta\delta}{\sin.\Theta}$; $\delta\delta$ being the whole astronomical refraction. [8599d]
This result is the same as in [8600].

† (3973) If we put ε' for the value of ε, when the zenith distance is $\Theta=50°$, and the refraction $\delta\delta=186'',728$ [8601], we shall have $\log.\varepsilon'=-H.\frac{186'',728}{\sqrt{\frac{1}{2}}}$ [8599d]. At [8602a]
the horizon, where $\Theta=100°$, and the refraction $\delta\delta=7390'',71$ [8601], we shall suppose that ε becomes ε'', and we shall have $\log.\varepsilon''=-H.7390'',71$ [8599d]. Dividing the [8602b]
expression of $\log.\varepsilon''$, by that of $\log.\varepsilon'$, we get,

$$\frac{\log.\varepsilon''}{\log.\varepsilon'}=\frac{7390'',71.\sqrt{\frac{1}{2}}}{186,728}=39,5801.\sqrt{\tfrac{1}{2}};\quad\text{ or }\quad\log.\varepsilon''=39,5801.\sqrt{\tfrac{1}{2}}.\log.\varepsilon'.$$ [8602c]

Now when $\Theta=50°$, we have, from [8593], $\log.\varepsilon'=\frac{\log.E}{\sqrt{\frac{1}{2}}}$; hence,

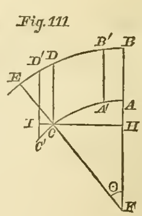
$$\log.\varepsilon''=39,5801\times\log.E=-3,57343,$$ [8602d]

using the value of $\log.E$ [8596]. The natural number corresponding to this logarithm is $\frac{1}{3745}$; hence we have $\varepsilon''=\frac{1}{3745}$, which differs a little from the result in [8602].

[8602′] disk, in consequence of the refraction of the sun's rays in passing through the earth's atmosphere, taking also into consideration the extinction of the light during that passage.

13. *According to the experiments of Bouguer, the light of the sun's disk*
[8603] *is less intense, near its limb, than at its centre.* At a distance from the limb
[8604] equal to a quarter of the semi-diameter, he found the intensity of light to be less than at its centre in the ratio of 35 to 48. Now any portion of the sun's disk, when it is transported by the sun's rotatory motion from the centre towards the limb of its disk, ought to appear with a more brilliant light, since it is viewed under a less angle; and it is natural to suppose that *each point of the sun's surface emits an equal quantity of light in every*
[8605] *direction. We shall put* Θ *for the arc of a great circle of the sun's surface, included between the luminous point and the centre of the sun's disk, the*
[8605′] *sun's radius being taken for unity;* a very small portion α of the surface,
[8606] being transported to the distance Θ from the centre of the disk, will appear to be reduced to the space* $\alpha.\cos.\Theta$; the intensity of its light must therefore
[8606′] be increased, in the ratio of unity to $\cos.\Theta$. But on the contrary it appears to be diminished. This difference is easily accounted for, by means of the atmosphere surrounding the sun. We have seen in the preceding article

Sun's light most intense near the centre of its disk.



* (3974) We shall suppose, in the annexed figure 111, that F is
[8606a] the sun's centre; FAB the line drawn towards the observer on the earth; AC' the sun's surface; BDE the surface of the sun's atmosphere; $AA' = CC'$ an infinitely small arc of the sun's surface;
[8606b] the line HCI is perpendicular to FAB; and the lines $A'B'$, CD, $C'ID'$, parallel to AB; lastly, the angle $CFA = \Theta$. Then if a part of the sun's disk, whose base is AA', height above the plane of the
[8606c] figure h, and area $h \times AA' = \alpha$, be transferred, by the sun's rotatory motion about its axis, which is supposed to be perpendicular to the
[8606d] plane of the present figure, until the arc AA' arrive to the situation CC', the base CC', when viewed from the earth, will appear under a less angle than when in the situation AA'
[8606e] in the ratio of $\cos.\Theta$ to 1; so that the apparent magnitude of the base CC' is reduced to $CI = AA'.\cos.\Theta$. Multiplying this by the height h, which is not altered by the rotatory
[8606f] motion, we get $h \times AA' \times \cos.\Theta = \alpha.\cos.\Theta$ [8606c], for the reduced value of the part α. Now the part CC' sends forth as many rays in a direction CD, parallel to AB, as the part AA' does in the direction parallel to AB. Therefore the intensity of the light at C,
[8606g] to that at A, must be as AA' to CI, or as CC' to CI; that is, as 1 to $\cos.\Theta$, as in [8606′].

*that the intensity of light which results from it is equal to** $c^{-\frac{f}{\cos.\Theta}}$, c being [8607]
the number whose hyperbolic logarithm is unity. Now the intensity of
light is c^{-f} at the centre of the disk; therefore at a point which is distant [8608]
from the limb by a quarter of the semi-diameter, it will be $\frac{1}{\cos.\Theta}.c^{-\frac{f}{\cos.\Theta}}$,
[8607f]; $\sin.\Theta$ being equal to $\frac{3}{4}$; therefore we shall have,† [8609]

$$\sqrt{\tfrac{16}{7}}.c^{-f.\sqrt{\frac{16}{7}}}=\tfrac{35}{48}.c^{-f}. \qquad [8610]$$

This equation gives the following value of f;

$$f=1{,}42459; \qquad [8611]$$

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* (3975) We have seen in [8593k, l], that a ray of light, *whose intensity is represented by unity at the surface of the sun's body,* will be decreased to c^{-f}, in passing vertically [8607a]
through the sun's atmosphere, in the direction AB or CE, in fig. 111 of the preceding [8607b]
note. Moreover, if the ray pass through the sun's atmosphere in the oblique direction CD,
forming the angle $ECD=\Theta$, with the vertical FCE, its intensity, upon quitting the [8607c]
sun's atmosphere at D, will be represented by $c^{-\frac{f}{\cos.\Theta}}$ [8593l]. This intensity at the [8607d]
point D is to be increased in the ratio of 1 to $\cos.\Theta$, as in [8606′], because it is
supposed that as many particles of light proceed from the surface CC', between the [8607e]
parallel lines CD, CD', as in a vertical direction, or parallel to CE. Hence the
intensity at the point D becomes $\frac{1}{\cos.\Theta}.c^{-\frac{f}{\cos.\Theta}}$, as in [8609]. [8607$f$]

† (3976) If the point C of the sun's disc, figure 111, page 556, be supposed to be
distant from the sun's limb by $\frac{1}{4}$ of the semi-diameter, we shall evidently have [8609a]
$\sin.CFA=\sin.\Theta=\frac{3}{4}$; whence $\cos.\Theta=\sqrt{\frac{7}{16}}$. Substituting these in the expression of [8609b]
the intensity at the point D [8607f], it becomes $\sqrt{\frac{16}{7}}.c^{-f.\sqrt{\frac{16}{7}}}$; while at the centre of [8609c]
the disk, at B, it is represented by c^{-f} [8593k]. Now, according to the observations of
Bouguer, these quantities are to each other as 35 to 48 [8604]; hence we easily obtain
the equation [8610]; or, as it may be written, [8609d]

$$c^{f.\left(1-\frac{4}{\sqrt{7}}\right)}=\frac{35}{48}\cdot\frac{\sqrt{7}}{4}; \quad \text{whence} \quad f=\frac{\text{hyp. log.}\ \frac{35}{48}\cdot\frac{\sqrt{7}}{4}}{1-\frac{4}{\sqrt{7}}}=1{,}42459..; \qquad [8609e]$$

as in [8611]. This value of f represents the hyperbolic logarithm of c^{f}; whence we
get c^{-f} [8612], nearly. [8609f]

whence we obtain,

[8612] $$c^{-f}=0{,}240686=E. \quad [8593c]$$

[8613] From this it follows that the *intensity, at the centre of the sun's disk, is reduced, by the extinction of the light in the sun's atmosphere, from* 1 *to* 0,240686. A column of air at the temperature *zero*, and under a pressure [8614] corresponding to the height of the barometer $0^{\text{metre}},76$, must have a height of* 54622^{metres}, to decrease the light in this manner. This will therefore be the height of the sun's atmosphere, reduced to the preceding density, if, [8614′] with the same density, it extinguishes the light as in our atmosphere.

The brilliancy of the sun is decreased by its atmosphere.

Hence we see that the sun would appear much more luminous, if the atmosphere which surrounds it were taken away. To determine how much its light is weakened, we shall observe that, by supposing *the sun's* [8615] *semi-diameter equal to unity, and putting* $\cos.\theta=x$, the whole light will [8616] be† $2\pi.\int_0^1 dx.c^{-\frac{f}{x}}$. It is true that the intensity of the light is sensibly

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* (3977) Substituting the value of $\log.E$ [8596], and that of l [8275], in [8592′], [8614a] after dividing it by $-l$, we get $Q.(\rho)=\frac{0{,}0902835}{7974^{\text{m}}}=\frac{1}{88322^{\text{m}}}$. Now upon the hypothesis [8614b] assumed in [8614′], $Q.(\rho)$ is the same for the sun as for the earth; therefore this value of $Q.(\rho)$ will correspond to the sun's atmosphere; and by substituting it in the equation [8593a], we get $l=-88322^{\text{metres}}.\log.E$; and since $\log.E=-0{,}6185492$, [8614c] [8612], we get $l=88322^{\text{metres}}\times 0{,}6185492$; being nearly the same as in [8614], which represents the height of the sun's homogeneous atmosphere, supposing it to be of a uniform density, as in [8614].

[8616a] † (3978) We shall suppose the arc CC', figure 111, page 556, to revolve about the line FAB, as an axis, so as to describe, by its revolution, an annulus, whose surface is [8616b] $2\pi.CH.CC'$; 2π being the circumference of the circle AC, whose radius is unity, [8616c] [8605′]. Now $AC=\theta$ [8605]; hence $CH=\sin.\theta$, $CC'=d\theta$; therefore the expression of the surface of this annulus is,

[8616d] $$2\pi.CH.CC'=2\pi.\sin.\theta.d\theta=-2\pi.d.\cos.\theta=-2\pi.dx. \quad [8615]$$

To obtain the intensity of the rays which proceed from this annulus, in the direction parallel to FAB, we must multiply the surface of the annulus $-2\pi.dx$ [8616d], by the [8616e] expression of the intensity [8593l], $c^{-\frac{f}{\cos.\theta}}$, or $c^{-\frac{f}{x}}$ [8615]; and it becomes $-2\pi.dx.c^{-\frac{f}{x}}$. Its integral between the limits $x=1$ and $x=0$, gives the whole [8616f] intensity of the sun, supposing it to be covered with an atmosphere; hence this intensity

proportional to $c^{-\frac{f}{x}}$, only between the limits $\Theta = 0$ and $\Theta = 88^\circ$; and beyond this last limit the intensity follows another law [8616n, o, p]. But [8617]
the sine of 88° differs so little from unity, that we may neglect the portion [8617′]
of the solar disk which corresponds to this difference, or else assume, as in the other parts of the disk, that the intensity of light is proportional to [8618]
$c^{-\frac{f}{x}}$. *Therefore if we suppose that the intensity of the sun's light is* [8619]
represented by unity, when its atmosphere is taken away, or f equal to
nothing [8616h], we shall have $\int_0^1 dx.c^{-\frac{f}{x}}$ for the expression of the [8619′]
intensity, in its decreased state, by the action of the sun's atmosphere.

To obtain this integral, we shall put $\frac{1}{f} = q$, and $z = \frac{1}{qx}$; and then it [8620]

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is expressed by $-2\pi.\int_1^0 dx.c^{-\frac{f}{x}}$; or, as it may be written, $2\pi.\int_0^1 dx.c^{-\frac{f}{x}}$, by merely [8616$g$]
changing the order of the limits of the integral. If we suppose $f = 0$, as in [8619′], the [8616h]
intensity of any ray $c^{-\frac{f}{\cos.\Theta}}$ [8616e], becomes equal to unity, being the same as when
the atmosphere is taken away [8593k]; and in this case the expression [8616g] is reduced [8616i]
to the form $2\pi.\int_0^1 dx = 2\pi$. Hence it appears, that if the sun's atmosphere be taken [8616k]
away, the whole intensity of the light, proceeding from the sun's body, will be represented
by 2π; but if it have an atmosphere, the whole intensity will be $2\pi.\int_0^1 dx.c^{-\frac{f}{x}}$ [8616g]; [8616l]
so that if the whole intensity of the sun's light, undiminished by the atmosphere, be represented by unity, its actual intensity, when diminished by the action of its atmosphere, [8616m]
will be represented by $\int_0^1 dx.c^{-\frac{f}{x}}$, as in [8619′]. These results require some modification, [8616n]
in consequence of the terms which are neglected in [8589, &c.]; since these terms impair the accuracy of the formula [8609], when Θ becomes large; [8616o]
in the same manner as we have seen, in [8483], that the formula for the refraction [8474] cannot be used when Θ exceeds 88°, on account of the neglect of similar quantities in computing that formula. [8616p]
The effect of these neglected terms, in computing the intensity of the sun's light, are not however of much importance, because $\sin.\Theta$ is nearly equal to unity, as is observed in [8617′]. [8616q]

[8621] becomes,* $-\int \frac{dz.c^{-z}}{qz^2}$; the limits of the integral being from $z=\infty$ to $z=\frac{1}{q}$;

[8622] $$-\int \frac{dz.c^{-z}}{qz^2} = \frac{c^{-z}}{qz^2}.\left\{1-\frac{1.2}{z}+\frac{1.2.3}{z^2}-\frac{1.2.3.4}{z^3}+\&c.\right\}+\text{constant}.$$

[8623] The integral must be taken from $z=\infty$ to $z=\frac{1}{q}=f$; so that the constant quantity is nothing; consequently the integral becomes,

[8624] $$-\int_{\infty}^{\frac{1}{q}} \frac{dz.c^{-z}}{qz^2} = q.c^{-f}.\{1-1.2.q+1.2.3.q^2-1.2.3.4.q^3+\&c.\} = \int_0^1 dx.c^{-\frac{f}{x}}. \quad [8619', \&c]$$

We can reduce this series to a continued fraction, by the method explained in [8340, &c.]. For this purpose we shall put,

[8625] $$u = 1-2q.(1-t)+1.2.3.q^2.(1-t)^2-\&c.;$$

and we shall have,†

[8620a] * (3979) If we change, as in [8620], the constant quantity f into $\frac{1}{q}$, and the variable quantity x into $\frac{1}{qz}$, we shall get $\frac{f}{x}=z$; substituting these in [8619'], it becomes as in [8621]. The limits $x=0$, $x=1$ [8619'], being substituted in z [8620a],

[8620b] give the corresponding limits of z, as in [8621]. Now we have generally,

[8620c] $$\int \frac{dz.c^{-z}}{qz^m} = -\frac{c^{-z}}{qz^m} - m.\int \frac{dz.c^{-z}}{qz^{m+1}};$$

[8620d] as is easily proved by taking the differential and reducing; therefore if we put successively $m=2$, $m=3$, $m=4$, &c., we shall get, by repeated substitutions,

[8620e] $$-\int \frac{dz.c^{-z}}{qz^2} = \frac{c^{-z}}{qz^2}+2.\int \frac{dz.c^{-z}}{qz^3} = \frac{c^{-z}}{qz^2}-2.\frac{c^{-z}}{qz^3}-2.3.\int \frac{dz.c^{-z}}{qz^4}$$
$$= \frac{c^{-z}}{qz^2}-2.\frac{c^{-z}}{qz^3}+2.3.\frac{c^{-z}}{qz^4}+2.3.4.\int \frac{dz.c^{-z}}{qz^5}; \ \&c.$$

[8620f] This last expression is easily reduced to the form [8622]; which vanishes at the first limit $z=\infty$ [8623]; and at the second limit $z=\frac{1}{q}$ [8623], it becomes as in [8624].

† (3980) Multiplying [8625] by $(1-t)^2$, we get,

[8626a] $$u.(1-t)^2 = (1-t)^2-2q.(1-t)^3+2.3.q^2.(1-t)^4-\&c.$$

Its differential, being multiplied by $\frac{q}{dt}$, considering u, t, as the variable quantities, gives,

$$q.\frac{du}{dt}.(1-t)^2-2qu.(1-t)-u+1=0.$$ [8626]

We shall consider u as the generating function of y_r, so that we shall have,

$$u=y_1+y_2.t+y_3.t^2 \ldots\ldots +y_{r+1}.t^r+\&c.$$ [8627]

Substituting this value of u in [8626], and then putting the coefficient of t^{r-1} equal to nothing, we get the following equation of finite differences;* [8628]

$$qr.y_{r+1}-(2qr+1).y_r+qr.y_{r-1}=0;$$ [8629]

in the case of $r=1$, this coefficient will give,

$$0=qy_2-(2q+1).y_1+1;$$ [8630]

which may be included in the preceding equation, by supposing $y_0=\frac{1}{q}$. [8631]

Now the equation of finite differences in y_r [8629], gives,†

$$\frac{y^{r-1}}{y^r}=\frac{2qr+1}{qr}-\frac{y_{r+1}}{y^r}.$$ [8632]

$$q.\frac{du}{dt}.(1-t)^2-2qu.(1-t)=-2q.(1-t)+2.3.q^2.(1-t)^2-2.3.4.q^3.(1-t)^3+\&c.$$ [8626b]

Adding 1 to both members of this equation, and then substituting for the second member its value u [8625], we get $q.\frac{du}{dt}.(1-t)^2-2qu.(1-t)+1=u$, as in [8626]. This is easily reduced to the form [8626d], by observing that $\frac{d.(ut)}{dt}=\frac{du}{dt}.t+u$, and $\frac{d.(ut^2)}{dt}=\frac{du}{dt}.t^2+2ut$; [8626c]

$$q.\frac{du}{dt}-2q.\frac{d.(ut)}{dt}+q.\frac{d.(ut^2)}{dt}-u+1=0.$$ [8626d]

* (3981) Substituting the value of u [8627] in the first *four* terms of the equation [8626d], and retaining only the quantities depending on t^{r-1}, we get the four following terms respectively;

$$\{qr.y_{r+1}-2qr.y_r+qr.y_{r-1}-y_r\}.t^{r-1}.$$ [8629a]

Now to satisfy the equation [8626] for all values of t, the coefficient of t^{r-1} must be put equal to nothing; hence we get [8629], by a slight transposition of the terms. The coefficient which is independent of t, being also put equal to nothing, gives [8630]. [8629b]

† (3982) Dividing [8629] by $qr.y_r$, and transposing the two first terms, it becomes as in [8632]. Now if we multiply the assumed equation [8633], by $(1+q.[r-1]+z_r).\frac{y_{r-1}}{y_r}$, it becomes $1+q.[r-1]+z_r=qr.\frac{y_{r-1}}{y_r}$; and by substituting the value of $\frac{y_{r-1}}{y_r}$ [8632], we get $1+q.(r-1)+z=2qr+1-qr.\frac{y_{r+1}}{y_r}$; or by reduction [8632a] [8632b]

We shall now suppose,

[8633] $$\frac{y_r}{y_{r-1}} = \frac{qr}{1+q.(r-1)+z_r};$$

and we shall have,

[8634] $$z_r = q.(r+1) - \frac{q^2.r.(r+1)}{1+qr+z_{r+1}};$$

[8635] $$z_r = \frac{q.(r+1)}{1+\frac{qr}{1+z_{r+1}}}.$$

Hence we deduce,

[8636] $$z_1 = \cfrac{2q}{1+\cfrac{q}{1+\cfrac{3q}{1+\cfrac{2q}{1+\cfrac{4q}{1+\cfrac{3q}{1+\&c.}}}}}}$$

therefore,*

[8637] $$\frac{y_1}{y_0} = \frac{q}{1+z_1} = \cfrac{q}{1+\cfrac{2q}{1+\cfrac{q}{1+\cfrac{3q}{1+\cfrac{2q}{1+\&c.}}}}}$$

[8632c] $z_r = q.(r+1) - qr.\frac{y_{r+1}}{y_r}$. If we substitute, in this, the value $\frac{y_{r+1}}{y_r} = \frac{q.(r+1)}{1+qr+z_{r+1}}$,
[8632d] deduced from [8633], by writing $r+1$ for r, it will become $z_r = q.(r+1) - \frac{q^2r.(r+1)}{1+qr+z_{r+1}}$, as in [8634]. This may also be put under the form,

[8632e] $$z_r = q.(r+1).\left\{1 - \frac{qr}{1+qr+z_{r+1}}\right\} = q.(r+1).\frac{1+z_{r+1}}{1+qr+z_{r+1}} = q.(r+1).\frac{1}{1+\frac{qr}{1+z_{r+1}}}.$$

as in [8635]. From this we get, by putting successively $r=1$, $r=2$, $r=3$, &c.,

[8632f] $$z_1 = \frac{2q}{1+\frac{q}{1+z_2}}; \qquad z_2 = \frac{3q}{1+\frac{2q}{1+z_3}}; \qquad z_3 = \frac{4q}{1+\frac{3q}{1+z_4}}, \text{ \&c.};$$

and by successive substitutions we obtain z_1, as in [8636].

[8637a] * (3983) Putting $r=1$ in [8633], we get $\frac{y_1}{y_0} = \frac{q}{1+z_1}$; and by substituting z_1, [8636], it becomes as in [8637].

Putting the two expressions of u [8625, 8627] equal to each other, we get,*

$$y_1 = 1 - 1.2.q + 1.2.3.q^2 - 1.2.3.4.q^3 + \&c.$$ [8638]

Moreover we have $y_0 = \frac{1}{q}$ [8631]; hence we obtain,†

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* (3984) The part of u [8625], which is independent of t, is evidently equal to $1-2q+1.2.3.q^2-\&c.$; and the corresponding part of u [8627], is y_1. Putting these two expressions equal to each other, we get [8638]. [8638a]

† (3985) Substituting $y_0 = \frac{1}{q}$ [8631], in [8637], we get $qy_1 = \cfrac{q}{1+\cfrac{2q}{1+\cfrac{q}{1+\&c.}}}$; [8639a]

multiplying this by c^{-f}, and then substituting the value of y_1 [8638], we obtain [8639]. The first member of this expression is the same as the value of $\int_0^1 dx.c^{-\frac{f}{x}}$ [8624]; and by using the symbols [8641], supposing also the fraction [8645] to be represented for brevity by F, we shall have,

$$\int_0^1 dx.c^{-\frac{f}{x}} = q.c^{-f}.F = \cfrac{q.c^{-f}}{1+\cfrac{\varepsilon^{(1)}}{1+\cfrac{\varepsilon^{(2)}}{1+\cfrac{\varepsilon^{(3)}}{1+\&c.}}}}$$ [8639b]

The continued fraction F, arranged as in the second member of [8645], is similar to that in [8362a]; changing the numerators q, $2q$, $3q$, &c. into $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\varepsilon^{( )}$, &c. respectively; also i into r. Hence we obtain, as in [8632b], the following series of fractions for determining the value of F; [8639c]

Value of $r+1$.	0	1	2	3	4	5	&c.	[8639d]
Upper index.	0	1	1	1	1	1	&c.	[8639e]
Fractions.	$\frac{1}{0}$;	$\frac{0}{1}$;	$\frac{1}{1}$;	$\frac{1}{1+\varepsilon^{(1)}}$;	$\frac{1+\varepsilon^{(2)}}{(1+\varepsilon^{(1)})+\varepsilon^{(2)}}$;	$\frac{(1+\varepsilon^{(2)})+\varepsilon^{(3)}}{1+\varepsilon^{(1)}+\varepsilon^{(2)}+\varepsilon^{(3)}.(1+\varepsilon^{(1)})}$;	&c.	[8639f]
Abridged forms of the fractions.	$\frac{1}{0}$;	$\frac{0}{1}$;	$\frac{N^{(1)}}{D^{(1)}}$;	$\frac{N^{(2)}}{D^{(2)}}$;	$\frac{N^{(3)}}{D^{(3)}}$;	$\frac{N^{(4)}}{D^{(4)}}$;	&c.	[8639g]
Lower index.	1	$\varepsilon^{(1)}$,	$\varepsilon^{(2)}$;	$\varepsilon^{(3)}$;	$\varepsilon^{(4)}$;	$\varepsilon^{(5)}$;	&c.	[8639h]

Hence $N^{(1)} = 1$; $N^{(2)} = 1$; $N^{(3)} = 1+\varepsilon^{(2)} = N^{(2)}+\varepsilon^{(2)}.N^{(1)}$;

$$N^{(4)} = (1+\varepsilon^{(2)})+\varepsilon^{(3)} = N^{(3)}+\varepsilon^{(3)}.N^{(2)}, \quad \&c.;$$ [8639i]

as in the general formula [8643]. In like manner,

$$D^{(1)} = 1; \quad D^{(2)} = 1+\varepsilon^{(1)}; \quad D^{(3)} = (1+\varepsilon^{(1)})+\varepsilon^{(2)} = D^{(2)}+\varepsilon^{(2)}.D^{(1)};$$

$$D^{(4)} = (1+\varepsilon^{(1)}+\varepsilon^{(2)})+\varepsilon^{(3)}.(1+\varepsilon^{(1)}) = D^{(3)}+\varepsilon^{(3)}.D^{(2)}, \quad \&c.;$$ [8639k]

as in [8644].

[8639] $$q.c^{-f}.(1-1.2.q+1.2.3.q^2-\&c.)=\cfrac{q.c^{-f}}{1+\cfrac{2q}{1+\cfrac{q}{1+\cfrac{3q}{1+\cfrac{2q}{1+\&c.}}}}}=\int_0^1 dx.c^{-\frac{f}{x}}.$$

[8640] We shall put,

[8641] $\varepsilon^{(1)}=2q;\quad \varepsilon^{(2)}=q;\quad \varepsilon^{(3)}=3q;\quad \varepsilon^{(4)}=2q;\quad \varepsilon^{(5)}=4q;\quad \varepsilon^{(6)}=3q;\quad \varepsilon^{(7)}=5q$, &c.;

and shall then form a series of fractions, beginning with $\frac{1}{1}$ and $\frac{1}{1+2q}$.

[8642] This series is to be continued, by putting $N^{(r)}$ for the numerator, and $D^{(r)}$ for the denominator, of the r^{th} fraction; and then computing their values, by means of the following formulas;

[8643] $$N^{(r)}=N^{(r-1)}+\varepsilon^{(r-1)}.N^{(r-2)};$$

[8644] $$D^{(r)}=D^{(r-1)}+\varepsilon^{(r-1)}.D^{(r-2)}.$$

Then the values of the following fraction,

[8645] $$\cfrac{1}{1+\cfrac{2q}{1+\cfrac{q}{1+\&c.}}}=0+\cfrac{1}{1+\cfrac{\varepsilon^{(1)}}{1+\cfrac{\varepsilon^{(2)}}{1+\&c.}}}$$

which occurs in [8639], will be included between the two fractions

[8646] $\frac{N^{(r)}}{D^{(r)}}$ and $\frac{N^{(r+1)}}{D^{(r+1)}}$. Hence we find that $\int_0^1 dx.c^{-\frac{f}{x}}$ is nearly equal to* $\frac{1}{12}$;

The sun would be twelve times as brilliant if its atmosphere were removed.

and it follows, from [8619], *that if the sun's atmosphere were taken away, it would appear twelve times as luminous.* This result depends however on

[8647] the experiment of Bouguer, which ought to be repeated several times with much care, upon several points of the sun's disk.

[8646a] * (3986) We have, in [8611, 8620], $q=\frac{1}{f}=\frac{1}{1,42459}=0,7$, nearly. Substituting this in [8641], we get $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\varepsilon^{(3)}$, &c.; hence we can form the series of fractions

[8646b] [8639f]. The fifth of these fractions is 0,50, and the sixth 0,48 nearly. Their mean gives nearly $F=0{,}49$. Multiplying it by $q=0{,}7$, and by $c^{-f}=0{,}24$ [8612], we obtain $F.q.c^{-f}=0{,}08$, or $\frac{1}{12}$ nearly, as in [8646].

CHAPTER IV.

ON THE MEASURE OF HEIGHTS BY A BAROMETER.

14. THE measure of heights by a barometer depends, like the theory of refraction, upon the law by which the density of the strata of the atmosphere decreases. We shall use the following symbols; Symbols.

$a =$ the distance from the centre of the earth to the lower station of the observer; [8648]

$a+r =$ the distance from the centre of the earth to the upper station of the observer; [8648']

$(g) =$ the force of gravity at the lower station; [8648'']

$g =$ the force of gravity at the place of the particle of air; [8648''']

$(p) =$ the pressure of the atmosphere at the lower station; [8649]

$p =$ the pressure of the atmosphere at the place of the particle of air; [8649']

$\rho =$ the density of the air at the distance $a+r$ from the earth's centre; [8649'']

$z =$ the heat of a particle of air at the distance $a+r$ from the earth's centre; and when $r=0$, z becomes equal to q [8657, 8661]. [8649''']

Then we shall have, as in [8469, 8653*d*],

$$dp = -g\rho . dr. \qquad [8650]$$

The pressure p is proportional to the product of the density ρ of the particle [8649''], by its heat z [8649''']; therefore we shall have [8653*c*], [8651]

$$p = K\rho . z; \qquad [8652]$$

K being a constant coefficient. Hence we obtain,* K.

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* (3987) Dividing [8650] by [8652], we get [8653]; or $\frac{gdr}{z} = -K.\frac{dp}{p}$; whose integral is $\int \frac{gdr}{z} = \text{constant} - K.\log.p$; and as this integral begins when $p=(p)$ [8662*a*], we get, constant $= K.\log.(p)$; consequently $\int \frac{gdr}{z} = K.\log.\frac{(p)}{p}$, as in [8654]. We [8653*a*] [8653*b*]

[8653] $$\frac{dp}{p} = -\frac{gdr}{Kz};$$

which gives by integration, and using (p) [8649],

[8654] $$\int \frac{gdr}{z} = K.\log.\frac{(p)}{p}.$$

[8655] The relation between g and (g), gives very nearly,*

[8656] $$g = (g).\frac{a^2}{(a+r)^2} = (g).\left(1-\frac{2r}{a}\right);$$

r'.

[8657] therefore, by putting $r' = r.\left(1-\frac{r}{a}\right)$, we shall have,

[8658] $$\int \frac{gdr}{z} = (g).\int \frac{dr'}{z} = K.\log.\frac{(p)}{p} \quad [8654].$$

[8659] To integrate these functions, it is necessary to find z in terms of r'; but as the integral extends only through a small interval, in comparison with the whole height of the atmosphere, it is evident that every function which represents the temperatures of the upper and the lower stations, and makes the temperature decrease nearly in an arithmetical progression, from the one to the other, is admissible; and we may select that form which most simplifies the calculations. Therefore we shall suppose,

[8660] $$z = \sqrt{q^2 - ir'}.$$ [Expressioo of the heat z]

[8661] q = the temperature of the air at the lower station;

[8661′] i = an indeterminate constant quantity, which is to be taken so that the expression of z [8660] may represent the temperature of the air at the upper station. We shall then have, as in [8662e],

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[8653c] may remark that the equation [8652] is like that in [8400a], changing C into K, and h into z. Moreover, the symbols used in this chapter are similar to those in [8653d] [8288–8289′, &c.], *the radius* r [8138] *being changed into* $a+r$ [8648′]; *but the differential of the radius* dr *is unaltered*, as in [8650, &c.].

[8655a] * (3988) Changing, in [8292], r into $a+r$, to conform to the present notation, [8653d], we get the first expression in [8656]. The second form is deduced from the first, by observing that $\frac{a^2}{(a+r)^2} = \left(1+\frac{r}{a}\right)^{-2} = 1-\frac{2r}{a}$, nearly. Substituting this in the [8655b] first member of [8658], it becomes $(g).\int \frac{dr}{z}.\left(1-\frac{2r}{a}\right)$, which is easily reduced to the form in the second member of [8658], by observing that the differential of r' [8657], is [8655c] $dr' = dr.\left(1-\frac{2r}{a}\right)$.

$$\int \frac{dr'}{z} = \frac{2r'}{q+z};$$ [8662]

hence we find, as in [8662g],*

$$r' = \frac{(q+z)}{2} \cdot \frac{K}{(g)} \cdot \log. \frac{(p)}{p};$$ [8663]

we shall use, in this equation, the tabular logarithms instead of hyperbolic logarithms, which only affects the constant quantity K. *We shall put l for the temperature corresponding to that of melting ice*; and shall suppose,† [8664] l.

$$q = l + t; \qquad z = l + t';$$ [8665]

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* (3989) Substituting the value of z [8660] in the first member of [8662c], and then integrating, we obtain the third expression in [8662c], as is easily proved by differentration. As the integral commences when $r = 0$, it is evident that the constant quantity must be [8662a] $\frac{2q}{i}$; hence we obtain the first expression in [8662d], which is easily reduced to its second form by substituting z instead of its value [8660]. Multiplying the numerator and denominator of this last expression by $q+z$, and substituting $q^2 - z^2 = ir'$ [8660], we get, by successive operations, the final expression in [8662e], being the same as in [8662]; [8662b]

$$\int \frac{dr'}{z} = \int \frac{dr'}{\sqrt{q^2 - ir'}} = \text{constant} - \frac{2}{i} \cdot \sqrt{q^2 - ir'}$$ [8662c]

$$= \frac{2}{i} \cdot \{q - \sqrt{q^2 - ir'}\} = \frac{2.(q-z)}{i}$$ [8662d]

$$= \frac{2.(q^2 - z^2)}{i.(q+z)} = \frac{2\,ir'}{i.(q+z)} = \frac{2r'}{q+z}.$$ [8662e]

Substituting this last value in [8658], we get,

$$\int \frac{gdr}{z} = \frac{2.(g).r'}{q+z} = K.\log. \frac{(p)}{p}.$$ [8662f]

Dividing the two last expressions in [8662f] by the coefficient of r', we obtain r' [8663]. [8662g] It is proper to state that the hypothesis for the temperature z, assumed by La Place in [8660], is substantially the same as the ancient hypothesis of De Luc, modified by La Grange; as has been shown by Plana, in vol. 27, page 194, of the work mentioned in [8540h]. [8662h]

† (3990) The heat is supposed to be expressed in degrees of the centigrade thermometer, supposing l to be its measure when the temperature is equal to that of melting ice. Then the temperature at the lower station being t [8668], the corresponding heat will be expressed by $q = l + t$ [8661, 8665]; and at the upper station, where the temperature is t' [8668], the heat becomes $z = l + t'$, as in [8649''', 8665]. Substituting the values of q, z [8665], in [8663], we get [8666]. [8665a] [8665b]

then we shall have,

[8666] $$r' = \frac{Kl}{(g)} \cdot \left\{ 1 + \frac{(t+t')}{2l} \right\} . \log . \frac{(p)}{p} .$$

The comparison of a great number of measures of the heights of mountains, by the barometer, with their trigonometrical measures, has been made by Ramond; who found, in the parallel of 50°, that the coefficient,

[8667] $$\frac{Kl}{(g)} = 18336^{\text{metres}}.$$

[8668] To determine the coefficient l, we shall remark that t and t' *denote the degrees of the centigrade mercurial thermometer, counted from zero.* If we consider an invariable mass of air at the temperature zero, each degree of increment in its temperature increases equally its elastic force or pressure, the increment of pressure, corresponding to a degree of the thermometer, is
[8669] very nearly 0,00375 [8488]; so that if we put $\overline{(p)}$ *for the pressure, or the*
[8670] *elastic force of a mass of air at the temperature zero*, we may suppose that
[8670′] for each degree of the thermometer the pressure increases by $\overline{(p)}.0,00375$; but this pressure is, by what precedes, equal to* $K\rho.(l+t)$; hence we
[8671] have $\overline{(p)} = K\rho l$. The increment of one degree in the temperature gives an increment in the pressure equal to $K\rho$, or to $K\rho l . \frac{1}{l}$; or lastly to
[8672] $\overline{(p)} . \frac{1}{l}$; putting this quantity equal to $\overline{(p)}.0,00375$ [8670], we obtain
[8673] $l = \frac{100000}{375} = 266{,}6 \ldots$ Therefore we shall have, upon the parallel of 50°,

[8674] $$r' = 18336^{\text{metres}} . \left(1 + \frac{(t+t')}{2} . 0{,}00375 \right) . \log . \frac{(p)}{p} ;$$

the pressures (p) and p are determined by the heights of the barometer;

[8671a] * (3991) Substituting in [8652] the value of q [8665], we get, at the lower station, $(p) = K\rho.(l+t)$; and when $t=0$, (p) becomes $\overline{(p)}$ [8649, 8670]; hence $\overline{(p)} = K\rho l$,
[8671b] as in [8671]. Moreover when $t=1$, the preceding value of (p) [8671a] becomes
[8671c] $(p) = K\rho l + K\rho = \overline{(p)} + K\rho$; therefore $(p) - \overline{(p)} = K\rho = \frac{\overline{(p)}}{l}$ [8671a], represents the increment of $\overline{(p)}$, arising from a variation of 1°, in the centigrade thermometer; being the same as in [8672]. Putting this quantity equal to $\overline{(p)}.0,00375$ [8670′], and then
[8671d] dividing by $\overline{(p)}$, we get l [8673]. Substituting this value of l, and that of $\frac{Kl}{(g)} = 18336^{\text{metres}}$ [8667], in [8666], it becomes as in [8674].

but we must reduce the mercury in the barometer to the same temperature. It has been found, by an accurate experiment, that mercury increases in bulk $\frac{1}{5412}$th part,* for each degree of the centigrade thermometer; therefore, in the station corresponding to the lower temperature, we must increase the observed height of the barometer, by as many times its $\frac{1}{5412}$th part, as there are degrees of difference in the temperatures of the mercury in the barometer at the two stations. Moreover, the temperature of the mercury in the barometer is not always exactly the same as that of the surrounding air; therefore we must use a thermometer attached to the barometer. Besides this correction for the temperature, there is also another which is required, in order to reduce the observed heights of the barometer to the gravity (g), corresponding to the lower station. The gravity at the superior station is $(g).\frac{a^2}{(a+r)^2}$ [8656]; therefore, by putting (h) and h for the observed heights of the barometer at the two stations, reduced to the same temperature, we find that these heights, reduced to the same gravity of the mercury, become (h) and $\frac{h}{\left(1+\frac{r}{a}\right)^2}$; hence we have,† [8675] [8676] [8677] [8678]

$$\log.\frac{(p)}{p} = \log.\frac{(h)}{h} + 2.\log.\left(1+\frac{r}{a}\right). \qquad [8679]$$

$\frac{r}{a}$ being a very small fraction, the hyperbolic logarithm of $1+\frac{r}{a}$ is very nearly equal to $\frac{r}{a}$; consequently its tabular logarithm is $\frac{r}{a}.0{,}4342945$; therefore we shall have, [8680]

$$\log.\frac{(p)}{p} = \log.\frac{(h)}{h} + \frac{r}{a}.0{,}868589. \qquad [8681]$$

The coefficient 18336$^{\text{metres}}$ *is exact only upon the parallel of* 50°; *it varies with the latitude, and is inversely as the gravity* (g) [8667]. If we put $[\overline{g}]$ [8681′] [8682]

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* (3992) The late experiments of Dulong and Petit [8490a], make this increment $\frac{1}{5550}$, instead of $\frac{1}{5412}$. [8675a]

† (3993) The pressures (p), p, are as the corrected heights of the barometer (h) and $h.\left(1+\frac{r}{a}\right)^{-2}$; hence we have $\frac{(p)}{p}=\frac{(h)}{h}.\left(1+\frac{r}{a}\right)^2$; whose logarithm is as in [8679]. This is easily reduced to the form [8681]. [8679a]

for the gravity at the equator, and ψ for the latitude corresponding to (g), we shall have, as in [2054, 1770*a*],*

[8683] $$(g)=[\underline{g}].\left\{1+\frac{0{,}004208}{0{,}739502}.\sin.^2\psi\right\}.$$

[8684] From this we readily see that the coefficient 18336^{metres}, corresponding to 50° of latitude, is for any other latitude ψ, equal to,†

[8684′] $$18336^{\text{metres}}.\{1+0{,}002845.\cos.2\psi\}.$$

This being supposed we shall have, to determine the heights by a barometer, the following formula ;‡

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* (3994) In [2054] we have, for the length of a pendulum vibrating in one second,

[8683*a*] $$0^{\text{metre}},739502.\left\{1+\frac{0{,}004208}{0{,}739502}.\sin.^2\psi\right\},$$

[8683*b*] in the latitude ψ. At the equator, where $\psi=0$, it becomes $0^{\text{metre}},739502$. These
[8683*c*] lengths are proportional to the gravities (g) and $[\underline{g}]$ [8682], respectively ; as we have seen in [1769″] ; hence we get the expression of (g) [8683]. For greater correctness, we
[8683*d*] may change the coefficient of $\sin.^2\psi$ into $\frac{0{,}20805}{39{,}01228}=0{,}005333$ [2056*o*, *p*], corresponding to the best observations of the pendulum.

† (3995) Putting for brevity $2b=\frac{0{,}004208}{0{,}739502}=0{,}005690$, we find that the factor of
[8684*a*] $[\underline{g}]$, in the second member of [8683], becomes $1+2b.\sin.^2\psi=1+b-b.\cos.2\psi$ [1] Int. ;
[8684*b*] hence the expression of (g) [8683], is $(g)=[\underline{g}].\{1+b-b.\cos.2\psi\}$. Substituting this in the factor $\frac{Kl}{(g)}$ [8666], we get,

[8684*c*] $$\frac{Kl}{(g)}=\frac{Kl}{(1+b).[\underline{g}]}.\left\{1-\frac{b}{1+b}.\cos.2\psi\right\}^{-1}=\frac{Kl}{(1+b).[\underline{g}]}.\left\{1+\frac{b}{1+b}.\cos.2\psi\right\}\text{ nearly.}$$

When $\psi=50°$, it becomes $\frac{Kl}{(1+b).[\underline{g}]}$; and if we put this factor equal to 18336^{metres}, as in [8667, 8681′], we get, for the general expression of this coefficient in any latitude, the same value as in [8684′]. For greater correctness, we may use,

[8684*d*] $$b=\tfrac{1}{2}\times 0{,}005333=0{,}002666 \quad [8683d],$$

for the coefficient of $\cos.2\psi$.

‡ (3996) Substituting in [8666] the value of $\frac{Kl}{(g)}$ [8667, 8684′] ; also for $\frac{(t+t')}{2l}$ its
[8685*a*] value deduced from [8673], namely $\frac{(t+t')}{2}.0{,}00375$; and for $\log.\frac{(p)}{p}$ its value [8681], it becomes,

Formula to find the height by a barometer.

$$r = 18336^{\text{m}}.\{1+0{,}002845.\cos.2\psi\}.\left\{1+\frac{(t+t')}{2}.0{,}00375\right\}.\left\{\left(1+\frac{r}{a}\right).\log.\frac{(h)}{h}+\frac{r}{a}.0{,}868589\right\}.$$ [8685]

It is sufficiently exact to substitute, in the second member of this equation, the value of r, computed upon the supposition that $r=0$, in the second member. We may also suppose, without any sensible error, that $a = 6366198^{\text{metres}}$ [8278]. The corrections depending upon the latitude and upon the variation of gravity, are very small; but as they really exist, it is best to notice them, so as to leave in the calculation no other imperfections than those which arise from the inevitable errors of observation; or from the effect of the unknown attractions of the mountains; or from the hygrometrical state of the air, which ought to be noticed; or finally from the error arising from the use of the hypothesis [8660], relative to the law of the diminution of the heat. We may satisfy in part for the state of the hygrometer, by increasing a little the coefficient 0,00375 in the term $\frac{1}{2}.(t+t').0{,}00375$, in the formula [8685]. For the aqueous vapor is lighter than the air [8526], and the increase of temperature increases the quantity of vapor, all other things being equal; so that we can very nearly satisfy the observations which have been made by the barometer, by changing $\frac{(t+t')}{2}.0{,}00375$ into $\frac{2.(t+t')}{1000}$, in the formula [8685];* and by this means it becomes,

[8686]

[8687]

[8688]

[8689]

$$r' = 18336^{\text{m}}.\{1+0{,}002845.\cos.2\psi\}.\left\{1+\frac{(t+t')}{2}.0{,}00375\right\}.\left\{\log.\frac{(h)}{h}+\frac{r}{a}.0{,}868589\right\}.$$ [8685b]

Dividing the first member of this expression by $1-\frac{r}{a}$, it becomes equal to r [8657]. In like manner we must divide the second member of [8685b] by the same divisor $1-\frac{r}{a}$, which is very nearly the same as to multiply it by $1+\frac{r}{a}$; then connecting this factor with the last factor of [8685b], we get, by neglecting terms of the order r^2, the expression [8685].

[8685c]

[8685d]

* (3997) The quantity of vapor, in the column of air of the height r, increases with the mean temperature of the air $\frac{1}{2}.(t+t')$ [8508]; producing a corresponding increment in the pressure, and in the term depending on $\frac{1}{2}.(t+t')$ in the formula [8685]; so that the numerical coefficient $\frac{1}{2}\times 0{,}00375$ [8685], is found by observation to become nearly equal to 0,002, as in [8690].

[8690a]

[8690b]

The formula [8690] may be reduced to English fathoms and to Fahrenheit's scale in the following manner. If t, t', be expressed in degrees of Fahrenheit's scale, the

[8690c]

The same formula [8690] corrected for the humidity.

$$r = 18336^{\text{met.}}.\{1+0{,}002845.\cos.2\psi\}.\left\{1+\frac{2.(t+t')}{1000}\right\}.\left\{\left(\frac{1+r}{a}\right).\log.\frac{(h)}{h}+\frac{r}{a}.0{,}868589\right\}.$$

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corresponding degrees of the centigrade thermometer will be $(t-32^{\circ}).\frac{100}{180}$, $(t'-32^{\circ}).\frac{100}{180}$, respectively. These are to be used instead of t, t', in [8690]; so that $t+t'$ will

[8690d] become $(t+t'-64^{\circ}).\frac{100}{180}$; and the factor $1+\frac{2.(t+t')}{1000}$ will change into,

[8690e] $$1+\frac{2.(t+t'-64)}{1000}\times\tfrac{100}{180}=\tfrac{1}{900}.\{836^{\circ}+t+t'\};$$

moreover, as a metre is equal to $0^{\text{fathom}},54681$.. [2017ρ], the factor,

[8690e'] $$18336^{\text{metres}}.\left\{1+\frac{2.(t+t')}{1000}\right\},$$

changes into,

[8690e''] $$\frac{18336^{\text{m}}\times 0{,}54681}{900}\times\{836^{\circ}+t+t'\}=11^{\text{fath.}},1404.\{836^{\circ}+t+t'\},$$

and the expression [8690] becomes, by using the corrected factor relative to 2ψ [8690e],

[8690f] $$r=11^{\text{fath.}},1404.\{1+0{,}002845.\cos.2\psi\}.\{836^{\circ}+t+t'\}.\left\{\left(1+\frac{r}{a}\right).\log.\frac{(h)}{h}+\frac{r}{a}.0{,}868589\right\}.$$

Finally, the correction $\frac{1}{5412}$, which is used in [8675], must be reduced to

[8690g] $\frac{1}{5412\times 1{,}8}=\frac{1}{9742}$, to correspond to Fahrenheit's scale; if we use $\frac{1}{5550}$ [8675a],

[8690h] instead of $\frac{1}{5412}$, the correction becomes $\frac{1}{5550}\times\frac{1}{1{,}8}=\frac{1}{9990}$, instead of $\frac{1}{9742}$.

CHAPTER V.

ON THE DESCENT OF BODIES FALLING FROM A GREAT HEIGHT.

15. A BODY, *beginning to fall from a state of rest at a point considerably elevated above the surface of the earth, will deviate sensibly from the vertical line, on account of the rotatory motion of the earth;* an accurate observation of this deviation will therefore be useful in rendering this motion manifest; and although the rotation of the earth is now established, with all the certainty which comports with the state of the physical sciences, yet a direct proof of this phenomenon must be interesting to mathematicians and astronomers. For the purpose of comparing the theory with such observations, *we shall here give the expression of the deviation of the body, to the east of the vertical, whatever be the figure of the earth, or the resistance of the air. We shall also show that the deviation is greatest at the equator,* [8736′, 8760, 8769]. The following symbols will be used: [8691]

Deviation to the east of the vertical.

At the equator there is no deviation.

Symbols.

X, Y, Z, are the rectangular co-ordinates of the point, from which the body begins to fall, at the commencement of the time t; [8692]

x, y, z, are the rectangular co-ordinates of the body, after falling from rest, during the time t. The *fixed* axis of x or X is the same as the axis of rotation of the earth. The *fixed* axis of y or Y is in the plane of the equator, and coincides with one of the principal axes of the earth, at the commencement of the time t; [8692′]

$r =$ the *primitive radius*, drawn from the centre of the earth to the point where the motion commences; [8693]

$r - \alpha s =$ the *variable radius*, drawn from the centre of the earth to the place of the body, at the end of the time t; [8694]

$\theta =$ the angle formed by the radius r and the axis of rotation x; [8695]

[8696] $\theta+\alpha u=$ the angle formed by the radius $r-\alpha s$ and the axis of rotation x, at the end of the time t;

[8697] $\varpi=$ the angle formed by the two planes xr, xy, intersecting each other in the earth's axis of rotation x. One of these planes xr passes through the axis of rotation x and the radius r at the commencement of the motion; the other plane is the *fixed* plane of xy;

[8698] $nt=$ the rotatory motion of the earth during the time t;

[8699] $nt+\varpi=$ the angle formed by the fixed plane xy, and the revolving plane xr; this last plane being that which passes through the *revolving radius* r and the fixed axis x, at the end of the time t;

[8700] $nt+\varpi+\alpha v=$ the similar angle, formed by the plane xy, with the plane passing through the fixed axis x, and the *variable radius* $r-\alpha s$, at the end of the time t;

[8701] $V=$ the sum of all the particles of the earth, divided by their distances from the falling body;

[8702] $\left(\frac{dV}{dx}\right)$, $\left(\frac{dV}{dy}\right)$, $\left(\frac{dV}{dz}\right)$, represent the forces acting upon the body in the directions parallel to the co-ordinates x, y, z, respectively, and tending to *increase* them [455‴].

[8702′] From this notation it follows, that after the body has been falling during the time t, the radius r changes into $r-\alpha s$; the angle θ changes into $\theta+\alpha u$; and the angle ϖ changes into $\varpi+\alpha v$. Then we shall have,*

[8703] $$X=r.\cos.\theta;$$

[8703′] $$Y=r.\sin.\theta.\cos.(nt+\varpi);$$

[8703″] $$Z=r.\sin.\theta.\sin.(nt+\varpi).$$ [Co-ordinates.]

[8704] $$x=(r-\alpha s).\cos.(\theta+\alpha u);$$

[8704′] $$y=(r-\alpha s).\sin.(\theta+\alpha u).\cos.(nt+\varpi+\alpha v);$$

[8704″] $$z=(r-\alpha s).\sin.(\theta+\alpha u).\sin.(nt+\varpi+\alpha v).$$

[8703a] * (3998) The notation [8692—8700], is precisely like that in [323ᵛ—324], *except in the sign of* s [8694]; $r-\alpha s$ being used instead of $r+\alpha s$ [323ᵛ], because the radius decreases as the body falls. This change being made in x, y, z [324], they become as
[8703b] in [8704—8704″]; and at the commencement of the motion, when s, u, v, vanish, these values change into those of X, Y, Z [8703—8703″].

*We shall notice the resistance of the air, supposing it to be represented by** [8705]
$\varphi.\left(\alpha s,\ \alpha.\frac{ds}{dt}\right)$, *and that the body falls from a state of rest.* For the relative velocity of the body through the air, considering the air as at rest, is evidently much greater in the direction of the radius r, than in the direction perpendicular to r; hence it will follow that the expression of this relative velocity is very nearly represented by $\alpha.\frac{ds}{dt}$. If, for greater simplicity, we put $r=1$, the relative velocity of the body, in the direction [8706]
θ, will be $\alpha.\frac{du}{dt}$ [2209b]; and the relative velocity in the direction ϖ, will [8706′]
be equal to $\alpha.\frac{dv}{dt}.\sin.\theta$ [2209c]; therefore, if we put for brevity,†

Function φ, expressing the resistance, supposing the body to fall from a state of rest.

S.

$$S=\frac{\varphi.\left(\alpha s,\ \alpha.\frac{ds}{dt}\right)}{\alpha.\frac{ds}{dt}};\qquad [8707]$$

---

* (3999) The body falls from rest very nearly in the direction of the radius r, through the space αs, in the time t; therefore its velocity, at the end of the time t, will be [8705a]
nearly represented by $\alpha.\frac{ds}{dt}$ [8706]. Now the resistance must be as a function of this [8705b]
velocity and of the density of the medium; moreover the density of the medium depends on the radius $r-\alpha s$, or αs; therefore the resistance must be a function of αs and [8705c]
$\alpha.\frac{ds}{dt}$; which is represented by $\varphi.\left(\alpha s,\ \alpha.\frac{ds}{dt}\right)$, in [8705].

† (4000) The radius $r-\alpha s$ [8694], at the end of the time t, will be varied by the quantity $-\alpha ds$, in the element of time dt; making the velocity, in the direction of this [8707a]
radius, equal to $-\alpha.\frac{ds}{dt}$, or $\alpha.\frac{ds}{dt}$, towards the centre of the earth. The velocity, in the direction of the meridian αu, is $\alpha.\frac{du}{dt}$ [2209b]; and in the direction $\alpha\varpi$, perpendicular [8707b]
to the meridian $\alpha.\frac{d\varpi}{dt}.\sin.\theta$ [2209c]. The sum of the squares of these three partial rectangular velocities is equal to the square of the whole velocity; and as the parts depending on du^2, dv^2, are extremely small in comparison with that depending on ds^2, [8707c]
we may consider the whole velocity to be very nearly equal to the part $\alpha.\frac{ds}{dt}$ [8707a], as
in [8706]. Now if we divide the whole resistance [8705] by the whole velocity [8707d]

Resistance of the air in the directions r, θ, ϖ.

we shall have, for the resistance of the air in the directions r, θ, ϖ, respectively,

[8708] $$S.\,\alpha.\frac{ds}{dt};$$ [Resistance in the direction of the radius r.]

[8709] $$-S.\,\alpha.\frac{du}{dt};$$ [Resistance in the direction of the arc of the meridian $\alpha\theta$.]

[8710] $$-S.\,\alpha.\frac{dv}{dt}.\sin.\theta.$$ [Resistance in the direction of the parallel of latitude $\alpha v.\sin\theta$.]

[8711] Then we shall have, by the principle of virtual velocities,*

$\alpha.\frac{ds}{dt}$, we shall obtain the quantity S [8707]; and multiplying it by the three partial

[8707e] velocities $\alpha.\frac{ds}{dt}$, $\alpha.\frac{du}{dt}$, $\alpha.\frac{dv}{dt}.\sin.\theta$ [8707a, b], we shall evidently obtain the three relative resisting forces [8708, 8709, 8710], which tend to *decrease* s, θ, ϖ; or, in other

[8707f] words, the forces tend to *increase* the radius r, and to *decrease* θ, ϖ; agreeably to the signs which are used in [8708—8710]. For convenience of reference we have inserted

[8707g] the symbol S, in the expressions [8708—8710], instead of its value [8707], which is used in the original work; having transposed the definition of S from [8711] to [8707].

* (4001) The principle of virtual velocities is expressed in the equation [37], which may be put under the following form;

[8712a]
$$0=\delta x.\frac{ddx}{dt^2}+\delta y.\frac{ddy}{dt^2}+\delta z.\frac{ddz}{dt^2} \qquad 1$$
$$-\{P.\delta x+Q.\delta y+R.\delta z\}. \qquad 2$$

[8712b] The first line of this expression is the same as in [8712 line 1]; the second line produces that in [8712 line 2], depending on the attractions of the earth; also that in [8712 line 3], depending on the resistance of the air. For we have seen, in [41], that the function

[8712c] $P.\delta x+Q.\delta y+R.\delta z$ can be reduced to the form $\Sigma.S.\delta s$, representing the sum of the products, formed by multiplying each force S, by the element of its direction δs. Now the

[8712d] attraction of the earth produces the forces $\left(\frac{dV}{dx}\right)$, $\left(\frac{dV}{dy}\right)$, $\left(\frac{dV}{dz}\right)$ [8702], in the directions x, y, z, respectively; and by multiplying these forces by the elements δx, δy, δz, then taking

[8712e] the sum of the products, they produce, in [8712a, line 2], the same terms as in [8712 line 2]. In like manner, if we multiply the forces [8708, 8709, 8710], depending on the resistance

[8712f] of the air, by the elements of their directions δr, $\delta\theta$, $\delta\varpi.\sin.\theta$, and take the sum of these products, they produce, in [8712a, line 2], the same terms as in [8712 line 3]; therefore

[8712g] the equation [8712] expresses truly the fundamental equation of the motion of the falling body, arising from the principle of virtual velocities.

$$
\begin{aligned}
0 = \delta x.\frac{ddx}{dt^2} + \delta y.\frac{ddy}{dt^2} + \delta z.\frac{ddz}{dt^2} \qquad & 1 \\
-\delta x.\left(\frac{dV}{dx}\right) - \delta y.\left(\frac{dV}{dy}\right) - \delta z.\left(\frac{dV}{dz}\right) \qquad & 2 \\
-S.\delta r.\alpha.\frac{ds}{dt} + S.\delta\theta.\alpha.\frac{du}{dt} + S.\delta\varpi.\sin.^2\theta.\alpha.\frac{dv}{d\theta}. \qquad & 3
\end{aligned}
\qquad [8712]
$$

The differential symbol δ, refers to the co-ordinates r, θ, ϖ; and x, y, z, are functions of these quantities. If we substitute, in [8712], the values of x, y, z [8704, &c.], we shall have, by neglecting terms of the order α^2*, [8713]

$$
\left.
\begin{aligned}
0 = \delta r.\left\{-\alpha.\frac{dds}{dt^2} - 2\alpha.nr.\sin.^2\theta.\frac{dv}{dt} - \alpha S.\frac{ds}{dt}\right\} \qquad & 1 \\
+r^2.\delta\theta.\left\{\alpha.\frac{ddu}{dt^2} - 2\alpha n.\sin.\theta.\cos.\theta.\frac{dv}{dt} + \alpha S.\frac{du}{dt}\right\} \qquad & 2 \\
+r^2.\delta\varpi.\left\{\alpha.\sin.^2\theta.\frac{ddv}{dt^2} + 2\alpha n.\sin.\theta.\cos.\theta.\frac{du}{dt} - \frac{2\alpha n.\sin.^2\theta}{r}.\frac{ds}{dt} + \alpha S.\sin.^2\theta.\frac{dv}{dt}\right\} \qquad & 3 \\
-\delta V - \tfrac{1}{2}n^2.\delta.\{(r-\alpha s)^2.\sin.^2.(\theta+\alpha u)\} \qquad & 4
\end{aligned}
\right\}. \quad (1) \qquad [8714]
$$

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* (4002) The complete variation of $-V$, considered as a function of x, y, z, is the same as the expression in [8712 line 2], which must therefore be equal to $-\delta V$. Substituting this in [8712], it becomes,

$$0 = \delta x.\frac{ddx}{dt^2} + \delta y.\frac{ddy}{dt^2} + \delta z.\frac{ddz}{dt^2} - \delta V + \alpha S.\left\{-\delta r.\frac{ds}{dt} + \delta\theta.\frac{du}{dt} + \delta\varpi.\sin.^2\theta.\frac{dv}{dt}\right\}. \qquad [8714a]$$

This may be reduced to the form [8714], in the same manner as [325] is deduced from [296]. For the equation [296] may be put under the form,

$$0 = \delta x.\frac{ddx}{dt^2} + \delta y.\frac{ddy}{dt^2} + \delta z.\frac{ddz}{dt^2} - \delta V + \frac{\delta p}{\rho}; \qquad [8714b]$$

which becomes identically the same as [8714a], by putting,

$$\frac{\delta p}{\rho} = \alpha S.\left\{-\delta r.\frac{ds}{dt} + \delta\theta.\frac{du}{dt} + \delta\varpi.\sin.^2\theta.\frac{dv}{dt}\right\}. \qquad [8714c]$$

Now the equation [325] is deduced from [296], by substituting the values [324], which are the same as those in [8704, 8704′, 8704″], changing the sign of s [8703a]; therefore [8714d] if we change the sign of s, in [325], and then substitute the value of $\frac{\delta p}{\rho}$ [8714c], it will give the reduced value of [8712], arising from the substitution of the values of x, y, z, [8704, &c.]; observing that in the terms $S.\frac{du}{dt}$, $S.\frac{dv}{dt}$, we may re-substitute $r^2 = 1$; [8714e] making them $r^2.S.\frac{du}{dt}$, $r^2.S.\frac{dv}{dt}$, respectively, on account of the smallness of S, du, dv. By this means the equation [325] becomes as in [8714].

[8714'] The equilibrium of the stratum of air, in which the body is situated, gives as in [326],*

[8715] $$0=\delta V+\tfrac{1}{2}n^2.\delta.\{(r-\alpha s)^2.\sin.^2(\theta+\alpha u)\}; \qquad (2)$$

[8716] provided the value of δr is made to correspond to the surface of the level stratum, in which the pressure is constant [1616′, &c.]. We shall suppose that at this surface we have,

[8717] $$r=\mathrm{a}+y;$$

[8718] *y being a function of* θ, ϖ, a, *and* a *being constant for the same stratum.* Now if we put,

[8719] $$Q=V+\tfrac{1}{2}n^2.\{(r-\alpha s)^2.\sin.^2(\theta+\alpha u\},$$

the equation [8715] will become,†

[8720] $$0=\left(\frac{dQ}{dr}\right).\left\{\left(\frac{dy}{d\theta}\right).\delta\theta+\left(\frac{dy}{d\varpi}\right).\delta\varpi\right\}+\left(\frac{dQ}{d\theta}\right).\delta\theta+\left(\frac{dQ}{d\varpi}\right).\delta\varpi.$$

Adding together the equations [8714, 8720], we obtain,‡

[8715a] * (4003) The equation of equilibrium [326] becomes as in [8715], by changing the sign of s, as in [8703a], which corresponds to a level surface; and δr must be taken to conform to it, as is observed in [8716].

† (4004) The complete variation of Q, considered as a function of r, θ, ϖ, is

[8720a] $$\delta Q=\left(\frac{dQ}{dr}\right).\delta r+\left(\frac{dQ}{d\theta}\right).\delta\theta+\left(\frac{dQ}{d\varpi}\right).\delta\varpi.$$

Now if we suppose δr to correspond to the level stratum, as in [8716], we shall have,

[8720b] from [8717], $\delta r=\delta y$, because a is constant for this stratum [8718]; and as y is a function of θ, ϖ, and the constant quantity a, we shall have, at this surface,

[8720c] $$\delta y=\left(\frac{dy}{d\theta}\right).\delta\theta+\left(\frac{dy}{d\varpi}\right).\delta\varpi;$$

substituting this for δr in [8720a], we get [8720]; which represents the complete variation

[8720d] of Q [8719], corresponding to the surface of equilibrium, or to the value of the function [8715]. We may also remark that, if the surface of level on the earth be considered as

[8720e] elliptical, the radius r, or the quantity y [8717], will vary only by quantities of the same order as this ellipticity, or $\frac{1}{300}$ part, in proceeding from the equator to the pole; and this

[8720f] variation must be wholly insensible, within the limits of the space αu or αv, passed over

[8720g] by the falling body; so that within these limits *we may consider* $\left(\frac{dy}{d\theta}\right)$, $\left(\frac{dy}{d\varpi}\right)$, *as constant;* and the same holds good relative to the values of y, in any one of the strata of

[8720h] equilibrium [8714′], through which the body falls. This principle is adopted in [8734].

[8721a] ‡ (4005) The lower line of [8714] is evidently equal to the general value of $-\delta Q$, [8719]; and we have shown, in [8720a], that this is equal to,

$$0=\delta r.\left\{-\alpha.\frac{dds}{dt^2}-2\alpha nr.\sin.^2\theta.\frac{dv}{dt}-\alpha S.\frac{ds}{dt}\right\} \qquad 1$$

$$+r^2.\delta\theta.\left\{\alpha.\frac{ddu}{dt^2}-2\alpha n.\sin.\theta.\cos.\theta.\frac{dv}{dt}+\alpha S.\frac{du}{dt}\right\} \qquad 2$$ [8721]

$$+r^2.\delta\varpi.\sin.\theta.\left\{\alpha.\sin.\theta.\frac{ddv}{dt^2}+2\alpha n.\cos.\theta.\frac{du}{dt}-\frac{2an.\sin.\theta}{r}.\frac{ds}{dt}+\alpha S.\sin.\theta.\frac{dv}{dt}\right\} \qquad 3$$

$$-\left(\frac{dQ}{dr}\right).\left\{\delta r-\left(\frac{dy}{d\theta}\right).\delta\theta-\left(\frac{dy}{d\varpi}\right).\delta\varpi\right\}. \qquad 4$$

We must now put the coefficients of the variations δr, $\delta\theta$, $\delta\varpi$, equal to [8722]
nothing, observing that $-\left(\frac{dQ}{dr}\right)$ expresses, as in [1815, &c.], *the force of gravity, which we shall denote by* g.* Then putting the radius r equal [8723]

$$-\left(\frac{dQ}{dr}\right).\delta r-\left(\frac{dQ}{d\theta}\right).\delta\theta-\left(\frac{dQ}{d\varpi}\right).\delta\varpi. \qquad [8721a']$$

This may be reduced, by adding to it the second member of [8720], which is equal to nothing; by this means it becomes,

$$-\left(\frac{dQ}{dr}\right).\left\{\delta r-\left(\frac{dy}{d\theta}\right).\delta\theta-\left(\frac{dy}{d\varpi}\right).\delta\varpi\right\}. \qquad [8721b]$$

This is the same as the lower line of [8721]. The other lines of [8714], namely lines 1, 2, 3, correspond respectively to the same lines in [8721]; so that [8721] represents the sum of [8714, 8720].

* (4006) In the case of nature, where $n=-2$ [1812‴], the expression of p [1815] becomes $p=-\left(\frac{dV}{dr}\right)-gr.(1-\mu^2)$; r being the radius of the spheroid, and g the centrifugal force [1814′], at the distance 1 from the axis of rotation. Now in [8698] the actual velocity of rotation of a particle of the earth, at the distance 1 from the axis, is n, *its centrifugal force* n^2 [54′]; and as $1-\mu^2=\sin.^2\theta$ [1616^xxi], the preceding expression of p becomes $p=-\left(\frac{dV}{dr}\right)-n^2r.\sin.^2\theta$. The partial differential of Q [8719], relative to r, gives $-\left(\frac{dQ}{dr}\right)=-\left(\frac{dV}{dr}\right)-n^2.(r-\alpha s).\sin.^2(\theta+\alpha u)$, or $-\left(\frac{dV}{dr}\right)-n^2r.\sin.^2\theta$, nearly; being the same as the value of p [8722c]; hence $p=-\left(\frac{dQ}{dr}\right)$; and if we change the gravity p [1814″], into g, to conform to the notation [8723], we shall have $-\left(\frac{dQ}{dr}\right)=g$, as in [8723]. Substituting this in [8721], and then putting the coefficients of $-\delta r$, $r^2\delta\theta$, $r^2\delta\varpi.\sin.\theta$, separately equal to nothing, we obtain the three equations [8724—8724″]; r being put equal to unity, as in [8706].

[8722a] [8722b] [8722c] [8722d] [8722e]

Fundamental differential equations in s, u, v.

to 1, which can be done without any sensible error, we shall obtain the three following equations;

$$\left.\begin{array}{ll}[8724] & 0=\alpha.\frac{dds}{dt^2}+2\alpha n.\sin.^2\theta.\frac{dv}{dt}+\alpha S.\frac{ds}{dt}-g; \\ [8724'] & 0=\alpha.\frac{ddu}{dt^2}-2\alpha n.\sin.\theta.\cos.\theta.\frac{dv}{dt}+\alpha S.\frac{du}{dt}-g.\left(\frac{dy}{d\theta}\right); \\ [8724''] & 0=\alpha.\sin.\theta.\frac{ddv}{dt^2}+2\alpha n.\cos.\theta.\frac{du}{dt}-2\alpha n.\sin.\theta.\frac{ds}{dt}+\alpha S.\sin.\theta.\frac{dv}{dt}-\frac{g}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right)\end{array}\right\}.\ (A)$$

The inspection of these equations shows, that the ratio of αs to αu, or of αs to αv, is of the same order as the ratio of unity to $\left(\frac{dy}{d\theta}\right)$, or $\left(\frac{dy}{d\varpi}\right)$.
[8725] Moreover, $\alpha.\frac{dv}{dt}$ is of the same order as* $gt.\left(\frac{dy}{d\varpi}\right)$; consequently
[8726] $2\alpha n.\sin.^2\theta.\frac{dv}{dt}$ is of the order $gnt.\left(\frac{dy}{d\varpi}\right)$. We shall use the following symbols;

[8727] The *unit of time* is one centesimal second, or the hundred thousandth part of a mean day;

[8727'] $n=$ the small angle described, by the rotatory motion of the earth, in one centesimal second of time;

[8728] $nt=$ the product of the small angle n, by the number of centesimal seconds t, during the fall of the body.

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* (4007) The centrifugal force depending on n [8722b], and the resistance of the
[8725a] air depending on S [8707—8710], being very small in comparison with the gravity g, [8723], it follows that n and S must be very small in comparison with g; and by neglecting these small quantities in the three equations [8724—8724''], they become,

[8725b] $$\alpha.\frac{dds}{dt^2}=g;\qquad \alpha.\frac{ddu}{dt^2}=g.\left(\frac{dy}{d\theta}\right);\qquad \alpha.\frac{ddv}{dt^2}=\frac{g}{\sin.^2\theta}.\left(\frac{dy}{d\varpi}\right)+\&c.$$

If we suppose, as in [8734], that g, $\left(\frac{dy}{d\theta}\right)$, and also $\left(\frac{dy}{d\varpi}\right)$, are constant, we shall get
[8725c] the expressions [8725d], by integrating [8725b] and taking the arbitrary constant quantities, so that the integrals may vanish when $t=0$;

[8725d] $$\alpha.\frac{ds}{dt}=gt;\qquad \alpha.\frac{du}{dt}=gt.\left(\frac{dy}{d\theta}\right);\qquad \alpha.\frac{dv}{dt}=\frac{gt}{\sin.^2\theta}.\left(\frac{dy}{d\varpi}\right)+\&c.$$

Again integrating with the same condition, we get,

[8725e] $$\alpha s=\tfrac{1}{2}gt^2;\qquad \alpha u=\tfrac{1}{2}gt^2.\left(\frac{dy}{d\theta}\right);\qquad \alpha v=\frac{\frac{1}{2}gt^2}{\sin.^2\theta}.\left(\frac{dy}{d\varpi}\right)+\&c.$$

From these last values we easily perceive the correctness of the remarks in [8725, &c.].

This number of seconds is always very small, so that the product nt is a fraction, which may be neglected in comparison with unity; we may therefore neglect the term $2\alpha n.\sin.^2\theta.\frac{dv}{dt}$, in the equation [8724], also the term $-2\alpha n.\sin.\theta.\cos.\theta.\frac{dv}{dt}$, in [8724']; and in like manner we may neglect the term $2\alpha n.\cos.\theta.\frac{du}{dt}$, in [8724'']. By this means the equations [8724—8724''] are reduced to the following forms; [8728'] [8729]

Approximate values.

$$0=\alpha.\frac{dds}{dt^2}+\alpha S.\frac{ds}{dt}-g; \qquad [8730]$$

$$0=\alpha.\frac{ddu}{dt^2}+\alpha S.\frac{du}{dt}-g.\left(\frac{dy}{d\theta}\right); \qquad [8731]$$

$$0=\alpha.\sin.\theta.\frac{ddv}{dt^2}-2\alpha n.\sin.\theta.\frac{ds}{dt}+\alpha S.\sin.\theta.\frac{dv}{dt}-\frac{g}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right). \qquad [8732]$$

S being a function of αs and $\alpha.\frac{ds}{dt}$ [8707]; the first of these equations gives αs in a function of the time t, as we shall see in [8751, 8784]. If we put,* [8732']

$$\alpha u=\alpha s.\left(\frac{dy}{d\theta}\right); \qquad [8733]$$

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* (4008) From the remarks in [8732'] it is evident, that the equation [8730] gives αs in a function of t; and by substituting this value of αs in the expression of S [8707], we get S, expressed in terms of t. With this value of S, and considering $\left(\frac{dy}{d\theta}\right)$ as constant [8734], we find that the equation [8731] may be reduced to a differential equation of the second order in u, containing only t, dt, and known quantities. We shall suppose its integral to give αu in a series [8733d], whose first term is the same as that retained by the author in [8733], and the remaining terms are quantities ascending according to the powers of t, connected with constant coefficients a, b, c, &c.; which are to be determined so as to satisfy the equation [8731] *for all values of* t. The first and second differentials of [8733d] give [8733e, f]; [8733a] [8733b] [8733c]

$$\alpha u= \alpha s.\left(\frac{dy}{d\theta}\right)+b+ct+et^2+ft^3+gt^4+\&c.; \qquad [8733d]$$

$$\alpha.\frac{du}{dt}=\alpha.\frac{ds}{dt}.\left(\frac{dy}{d\theta}\right)+\ldots.c+2et+3ft^2+4gt^3+\&c.; \qquad [8733e]$$

$$\alpha.\frac{ddu}{dt^2}=\alpha.\frac{dds}{dt^2}.\left(\frac{dy}{d\theta}\right)+\ldots\ldots+2.1e+3.2ft+4.3gt^2+\&c. \qquad [8733f]$$

[8734] we can satisfy the second of these equations, because g and $\left(\frac{dy}{d\theta}\right)$ *may be supposed to be constant* [8720g, h] *during the motion, on account of the smallness of the height from which the body falls, in comparison with the earth's radius.* This manner of satisfying the second equation is the only one which accords with the present question, in which u and $\frac{du}{dt}$ *vanish,*
[8735] *as well as* s *and* $\frac{ds}{dt}$, *at the origin of the motion.* Now if we suppose a plumb line of the length αs to be suspended from the point where the body begins to fall, it will deviate from the radius r towards the south, by the
[8736] quantity* $\alpha s.\left(\frac{dy}{d\theta}\right)$, which is equal to the quantity αu [8733]. *Therefore*

Substituting these in [8731], it becomes as in the following expression, which ought to be satisfied *for all values of* t;

[8733g]
$$0=\left(\frac{dy}{d\theta}\right).\left\{\alpha.\frac{dds}{dt^2}+\alpha S.\frac{ds}{dt}-g\right\}$$
$$+\alpha c.S+2\alpha e.\{1+St\}+3\alpha ft.\{2+St\}+4\alpha gt^2.\{3+St\}+\&c.$$

Now when $t=0$, we have by hypothesis [8735], $s=0$, $u=0$; and then [8733d]
[8733h] gives $b=0$. We also have, when $t=0$, $\frac{du}{dt}=0$, $\frac{ds}{dt}=0$ [8735]; substituting these in [8733e], we get $c=0$. The coefficient of $\left(\frac{dy}{d\theta}\right)$, in [8733$g$], vanishes by means of the equation [8730]; and since $c=0$, the equation [8733g] becomes, by dividing by α,

[8733i]
$$0=2e.\{1+St\}+3ft.\{2+St\}+4gt^2.\{3+St\}+\&c.$$

[8733k] When $t=0$, this becomes $0=2e$, or $e=0$. Substituting this in [8733i], and then dividing by t, we get,

[8733l]
$$0=3f.\{2+St\}+4gt.\{3+St\}+\&c.$$

[8733m] Again, using $t=0$, we get $f=0$. Substituting this in [8733l], and then dividing by t, we get $0=4g.\{3+St\}+\&c.$; which gives, in like manner, $g=0$, &c. Hence it
[8733n] appears that all the coefficients b, c, e, f, g, &c. of the expression of u [8733d], vanish, and it is reduced to its first term, being the same as that given by the author, in [8733].

* (4009) The direction of a plumb line is perpendicular to the surface of equilibrium,
[8736a] and the equation of this surface is represented by $r=a+y$ [8717], a being constant, [8718]; so that in proceeding along this surface, through an arc of the meridian $d\theta$, the
[8736b] increment of r will be $\left(\frac{dr}{d\theta}\right).d\theta=\left(\frac{dy}{d\theta}\right).d\theta$. Dividing this by the arc $d\theta$, we find that

the falling body is always upon the parallels of the points of the vertical which are at the same height as the body; so that it does not suffer any sensible deviation to the south of that line. [8736′]

To integrate the equation [8732], we shall put,

$$\alpha v.\sin.\theta = \frac{\alpha s}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right) + \alpha v'; \qquad [8737]$$

and we shall have,*

$$0 = \alpha.\frac{ddv'}{dt^2} + \alpha S.\frac{dv'}{dt} - 2\alpha n.\sin.\theta.\frac{ds}{dt}. \qquad [8738]$$

The body deviates to the east of the radius r *by the quantity* $\alpha v.\sin.\theta$ [8737], or $\frac{\alpha s}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right) + \alpha v'$; but the plumb line varies to the east of this radius [8739] by the quantity† $\frac{\alpha s}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right)$; $\alpha v'$ *is therefore the deviation of the body to* [8740] *the eastward of the vertical.*

Deviation of the body to the eastward of the vertical.

$\left(\frac{dy}{d\theta}\right)$ represents the tangent of the angle which this surface, in the direction of the meridian, forms with a line drawn perpendicular to r; or, in other words, the tangent of [8736c] the angle formed by the radius r and the vertical line. Multiplying this by the whole length of the plumb line αs, we obtain the deviation of this line in the direction of the meridian, towards the equator $=\alpha s.\left(\frac{dy}{d\theta}\right)$; being the same as the expression of αu, [8733, 8736]. [8736d]

* (4010) Substituting the assumed value of $\alpha v.\sin.\theta$ [8737], in [8732], we get,

$$0 = \frac{1}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right).\left\{\alpha.\frac{dds}{dt^2} + \alpha S.\frac{ds}{dt} - g\right\} + \alpha.\frac{ddv'}{dt^2} + \alpha S.\frac{dv'}{dt} - 2\alpha n.\sin.\theta.\frac{ds}{dt}; \qquad [8738a]$$

and as the coefficient of $\left(\frac{dy}{d\varpi}\right)$ vanishes by using [8730], the equation becomes as in [8738].

† (4011) Instead of proceeding, as in [8736*a*, *b*], through the arc of the meridian $d\theta$, if we proceed through the arc $d\varpi.\sin.\theta$ of the parallel of latitude, the increment of r or [8740a] y will be represented by $\left(\frac{dr}{d\varpi}\right).d\varpi = \left(\frac{dy}{d\varpi}\right).d\varpi$. Dividing this, as in [8736*b*], by the [8740b] described arc or base $d\varpi.\sin.\theta$, we obtain $\frac{1}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right)$ for the tangent of the angle, formed by that surface and the perpendicular to r, drawn in an east direction; or, in other words, it represents the tangent of the angle of deviation of the plumb line towards the east. [8740c] Multiplying this by the length of the vertical line αs, we get the whole deviation of this

Resistance proportional to the square of the velocity.

*We shall now suppose that the resistance of the air is proportional to the square of the velocity,** so that,

[8740′] $$S = \alpha . m . \frac{ds}{dt};$$

[8740″] *m being a coefficient depending upon the figure of the body and the density of the air.* This density varies with the height, but may in the present case be supposed constant, without any sensible error; we shall then have, as in [8741a],

[8741] $$0 = \alpha . \frac{dds}{dt^2} + \alpha^2 . m . \frac{ds^2}{dt^2} - g.$$

To integrate this equation we shall put,

[8742] $$\alpha s = \frac{1}{m} . \log . s';$$

and we shall have,†

[8743] $$0 = \frac{dds'}{dt^2} - mgs';$$

[8740d] line, equal to $\frac{\alpha s}{\sin . \theta} . \left(\frac{dy}{d\varpi}\right)$. Subtracting this from the whole deviation of the falling body, [8739], we find *that the deviation of the falling body to the eastward of the vertical line is represented by* $\alpha v'$, as in [8740].

* (4012) The resistance being as the square of the velocity, the function [8705] is

[8741a] expressed by $\varphi . \left(\alpha s,\ \alpha . \frac{ds}{dt}\right) = m . \left(\alpha . \frac{ds}{dt}\right)^2$; hence S [8707] becomes $S = \alpha . m . \frac{ds}{dt}$, as in [8740′]. Substituting this in [8730], it changes into [8741].

† (4013) The first and second differentials of [8742] give,

[8743a] $$\alpha . \frac{ds}{dt} = \frac{ds'}{ms' . dt};$$

[8743a′] $$\alpha . \frac{dds}{dt^2} = \frac{dds'}{ms' . dt^2} - \frac{ds'^2}{ms'^2 . dt^2}.$$

Substituting these in [8741], and neglecting the terms which mutually destroy each other, we get,

[8743b] $$0 = \frac{dds'}{ms' . dt^2} - g, \qquad \text{or} \qquad \frac{dds'}{dt^2} - mgs' = 0;$$

as in [8743]. This linear equation of the second order in s', is solved in the usual

[8743c] manner, by putting $s' = Ac^{bt}$; A, b, being constant quantities. For by substituting this assumed value of s' in [8743], and dividing by the common factor Ac^{bt}, we get

[8743d] $b^2 - mg = 0$; whence $b = \pm\sqrt{mg}$. Supposing A to correspond to the positive root

[8743e] $b = +\sqrt{mg}$, and taking another constant quantity B, to be used with the negative root

which gives by integration,

$$s' = A.c^{t.\sqrt{mg}} + B.c^{-t.\sqrt{mg}} \,; \qquad [8744]$$

c being the number whose hyperbolic logarithm is unity [8290'], and A, B, [8744']
two arbitrary constant quantities. To determine them we shall observe,
that we have $s=0$ when $t=0$ [8735], and then we also have $s'=1$, [8745]
[8742]. Substituting these values in [8744], we obtain,

$$A + B = 1. \qquad [8746]$$

Moreover $\alpha.\frac{ds}{dt}$ [8735], must vanish when $t=0$; therefore $\frac{ds'}{dt}$ [8743a] [8747]
must also vanish. Hence we obtain,

$$A - B = 0 \,; \qquad [8748]$$

therefore we shall have, as in [8745h], $A = B = \frac{1}{2}$; consequently, [8749]

$$\alpha s = \frac{1}{m}.\log.\left\{ \tfrac{1}{2}.c^{t.\sqrt{mg}} + \tfrac{1}{2}.c^{-t.\sqrt{mg}} \right\}. \qquad [8750]$$

Developing this equation in a series, according to the powers of t, we obtain,* Vertical descent.

$$\alpha s = \tfrac{1}{2}.gt^2 - \tfrac{1}{12}.mg^2.t^4 + \tfrac{1}{45}.m^2g^3.t^6 - \&c. \qquad [8751]$$

$b = -\sqrt{mg}$, we shall obtain the two terms of s' [8744]; which can be added together, as the equation is linear; and then the sum is the complete integral, since it has two [8743f] arbitrary constant quantities A, B. These quantities are determined, as in [8745, &c.], from the consideration that the body commences its motion when $t=0$. Then $t=0$ and $s'=1$ [8745], being substituted in [8744], give [8746]; also $t=0$ and $\frac{ds'}{dt}=0$, [8743g] [8747], being substituted in $\frac{ds'}{dt}$ [8744], give [8748]. The half sum, and the half difference of the two expressions [8746, 8748], give $A=\frac{1}{2}$, $B=\frac{1}{2}$; and by substitution in [8744], we obtain $s' = \frac{1}{2}.c^{t.\sqrt{mg}} + \frac{1}{2}.c^{-t.\sqrt{mg}}$; hence the value of αs [8742] becomes as [8743h] in [8750]. Finally, if we multiply the equation [8743a] by m, we find that its first [8743i] member becomes equal to S [8740']; hence we have $S = \frac{ds'}{s'dt}$, or $Ss' = \frac{ds'}{dt}$; which [8743k] will be used hereafter.

* (4014) We shall use for a moment the symbols,

$$z = t.\sqrt{mg}, \qquad x = \tfrac{1}{2}.z^2 + \tfrac{1}{24}.z^4 + \tfrac{1}{720}.z^6 + \&c.\,; \qquad [8751a]$$

then developing the quantities in the first members of [8751b, c], by means of [55, 56] Int., we get for their sum the expression [8751d].

To determine $\alpha v'$, we shall observe that we have,

[8752] $$\alpha.\frac{ds}{dt}=\frac{1}{m}.\frac{ds'}{s'dt} \quad [8743a];$$

therefore the differential equation in $\alpha v'$ [8738], becomes,*

[8753] $$\alpha s'.\frac{ddv'}{dt^2}+\alpha.\frac{ds'}{dt}.\frac{dv'}{dt}-\frac{2n}{m}.\frac{ds'}{dt}.\sin.\theta=0.$$

Hence we get, by integration,

[8754] $$\alpha s'.\frac{dv'}{dt}=\frac{2n}{m}.s'.\sin.\theta+C;$$

C being a constant quantity. To determine it, we shall observe that when

[8755] $t=0$, we have† $\frac{dv'}{dt}=0$; and then $s'=1$. Hence we obtain,

[8756] $$C=-\frac{2n}{m}.\sin.\theta;$$

[8751b] $$\tfrac{1}{2}.c^{t.\sqrt{mg}}=\tfrac{1}{2}.c^{z}=\tfrac{1}{2}.\{1+z+\tfrac{1}{2}.z^2+\tfrac{1}{6}.z^3+\tfrac{1}{24}.z^4+\&c.\};$$

[8751c] $$\tfrac{1}{2}.c^{-t.\sqrt{mg}}=\tfrac{1}{2}.c^{-z}=\tfrac{1}{2}.\{1-z+\tfrac{1}{2}.z^2-\tfrac{1}{6}.z^3+\tfrac{1}{24}.z^4-\&c.\};$$

[8751d] $$\tfrac{1}{2}.c^{t.\sqrt{mg}}+\tfrac{1}{2}.c^{-t.\sqrt{mg}}=1+\tfrac{1}{2}.z^2+\tfrac{1}{24}.z^4+\&c.=1+x.$$

Substituting this last expression in [8750], then developing by [58] Int. and making successive reductions, we finally get the value of αs [8751g], as in [8751];

[8751e] $$\alpha s=\frac{1}{m}.\log.(1+x)=\frac{1}{m}.\{x-\tfrac{1}{2}.x^2+\tfrac{1}{3}.x^3-\&c.\}$$

[8751f] $$=\frac{1}{m}.\{(\tfrac{1}{2}.z^2+\tfrac{1}{24}.z^4+\tfrac{1}{720}.z^6+\&c.)-\tfrac{1}{2}.(\tfrac{1}{2}.z^2+\tfrac{1}{24}.z^4+\&c.)^2+\tfrac{1}{3}.(\tfrac{1}{2}.z^2+\&c.)^3-\&c.\}$$

[8751g] $$=\frac{1}{m}.\{\tfrac{1}{2}.z^2-\tfrac{1}{12}.z^4+\tfrac{1}{45}.z^6-\&c.\}=\tfrac{1}{2}.gt^2-\tfrac{1}{12}.mg^2.t^4+\tfrac{1}{45}.m^2g^3.t^6-\&c.$$

* (4015) Multiplying [8738] by s', we get,

[8753a] $$0=\alpha s'.\frac{ddv'}{dt^2}+\alpha.Ss'.\frac{dv'}{dt}-2n.\alpha s'.\frac{ds}{dt}.\sin.\theta.$$

Substituting in the second term the value of Ss' [8743k], and in the third term the value

[8753b] of $\alpha.\frac{ds}{dt}$ [8752], it becomes as in [8753]. The integral of [8753] is as in [8754]; as we may easily prove by differentration, after transposing the last term.

† (4016) As the body begins to fall from a state of rest, we shall have, when $t=0$,

[8756a] $s=0$, $v=0$, $\frac{ds}{dt}=0$, $\frac{dv}{dt}=0$, &c., as in [8735, &c.]. Substituting these in the

[8756b] differential of [8737], we get, at the same time, $\frac{dv'}{dt}=0$. This value and that of $s'=1$,

therefore,

$$\alpha.\frac{dv'}{dt}=\frac{2n}{m}.\sin.\theta.\left(1-\frac{1}{s'}\right)=\frac{2n}{m}.\sin.\theta.\left\{1-\frac{2}{c^{t.\sqrt{mg}}+c^{-t.\sqrt{mg}}}\right\}.$$ [8757]

Again integrating and taking the arbitrary constant quantity, so that v' may vanish when $t=0$, we shall have,* [8758]

[8745], being substituted in [8754], we get $0=\frac{2n}{m}.\sin.\theta+C$, as in [8756]. Substituting this value of C in [8754], then dividing by s', we get the first expression in [8757]; and by using the value of s' [8743h], we obtain the second of the expressions [8757]. [8756c]

* (4017) Multiplying the last form of [8757] by dt, and integrating, we get,

$$\alpha v'=\frac{2n}{m}.t.\sin.\theta-\frac{4n.\sin.\theta}{m.\sqrt{mg}}.\int\frac{\sqrt{mg}.dt}{c^{t.\sqrt{mg}}+c^{-t.\sqrt{mg}}}.$$ [8759a]

In finding the integral of the term in the second member, under the sign $\int$, we shall use for brevity the symbols,

$$T=\frac{c^{\frac{1}{2}t.\sqrt{mg}}-c^{-\frac{1}{2}t.\sqrt{mg}}}{c^{\frac{1}{2}t.\sqrt{mg}}+c^{-\frac{1}{2}t.\sqrt{mg}}}=\frac{c^{t.\sqrt{mg}}-1}{c^{t.\sqrt{mg}}+1};$$ [8759b]

$$V=\text{arc.}(\text{tang.}T);$$ [8759c]

the second form of T being deduced from the first, by multiplying the numerator and denominator by $c^{\frac{1}{2}t.\sqrt{mg}}$. From this last form of T we easily get,

$$1+TT=\frac{2.\left(c^{2t.\sqrt{mg}}+1\right)}{\left(c^{t.\sqrt{mg}}+1\right)^2},\quad\text{and}\quad dT=\frac{2.\sqrt{mg}.dt.c^{t.\sqrt{mg}}}{\left(c^{t.\sqrt{mg}}+1\right)^2};$$ [8759d]

whence,

$$\frac{dT}{1+TT}=\frac{\sqrt{mg}.dt.c^{t.\sqrt{mg}}}{c^{2t.\sqrt{mg}}+1}=\frac{\sqrt{mg}.dt}{c^{t.\sqrt{mg}}+c^{-t.\sqrt{mg}}}.$$ [8759e]

Now the differential of $V=\text{arc.}(\text{tang.}T)$ [8759c], is $dV=\frac{dT}{1+TT}$ [51] Int.; substituting this in the first member of the preceding equation, and again integrating, we get $V=\int\frac{\sqrt{mg}.dt}{c^{t.\sqrt{mg}}+c^{-t.\sqrt{mg}}}$; hence the equation [8759a] changes into,

$$\alpha v'=\frac{2n}{m}.t.\sin.\theta-\frac{4n.\sin.\theta}{m.\sqrt{mg}}.V;$$ [8759f]

and by putting instead of V, its value [8759c], it becomes as in [8759], using for T the first expression [8759b]. It is not necessary to add any constant quantity to the integral [8759], because it makes $\alpha v'$ vanish when $t=0$, as in [8756a, 8737]. [8759g] [8759h]

Deviation on the parallel of latitude.

[8759] $$\alpha v' = \frac{2n}{m}.t.\sin.\theta - \frac{4n.\sin.\theta}{m.\sqrt{mg}}.\text{ang. tang.}\left\{\frac{c^{\frac{t}{2}.\sqrt{mg}} - c^{-\frac{t}{2}.\sqrt{mg}}}{c^{\frac{t}{2}.\sqrt{mg}} + c^{-\frac{t}{2}.\sqrt{mg}}}\right\};$$

and by reducing to a series,*

[8760] $$\alpha v' = \frac{ngt^3.\sin.\theta}{3}.\left\{1 - \tfrac{1}{4}mg.t^2 + \tfrac{61}{840}.m^2g^2.t^4 - \&c.\right\}.$$

[8760a] * (4018) Putting $\frac{1}{2}t.\sqrt{mg} = b.\sqrt{-1}$, we get, from [11, 12] Int.,

[8760b] $$c^{\frac{1}{2}t.\sqrt{mg}} - c^{-\frac{1}{2}t.\sqrt{mg}} = c^{b.\sqrt{-1}} - c^{-b.\sqrt{-1}} = 2.\sqrt{-1}.\sin.b;$$

[8760b′] $$c^{\frac{1}{2}t.\sqrt{mg}} + c^{-\frac{1}{2}t.\sqrt{mg}} = c^{b.\sqrt{-1}} + c^{-b.\sqrt{-1}} = 2.\cos.b.$$

Dividing the first of these expressions by the second, we obtain,

[8760c] $$\frac{c^{\frac{1}{2}t.\sqrt{mg}} - c^{-\frac{1}{2}t.\sqrt{mg}}}{c^{\frac{1}{2}t.\sqrt{mg}} + c^{-\frac{1}{2}t.\sqrt{mg}}} = \sqrt{-1}.\frac{\sin.b}{\cos.b} = \sqrt{-1}.\text{tang}.b.$$

Substituting, in the first member of this expression, its value T [8759b], and developing
[8760d] tang.b in its last member by means of [45] Int., we obtain the expression of T [8760f]; we have also, from [8759c, 48 Int.],

[8760e] $$V = \text{arc.(tang.}T) = T - \tfrac{1}{3}T^3 + \tfrac{1}{5}T^5 - \tfrac{1}{7}T^7 + \&c.$$

Now by neglecting the powers of b which exceed the seventh, we get, from [8760f], the other terms of this value of V, as in [8760f, g, h]; their sum gives V [8760k, l].

[8760f] $$T = \sqrt{-1}.\{b + \tfrac{1}{3}b^3 + \tfrac{2}{15}b^5 + \tfrac{17}{315}b^7 + \&c.\};$$

[8760g] $$-\tfrac{1}{3}T^3 = \sqrt{-1}.\{\ldots + \tfrac{1}{3}b^3 + \tfrac{5}{15}b^5 + \tfrac{77}{315}b^7 + \&c.\};$$

[8760h] $$+\tfrac{1}{5}T^5 = \sqrt{-1}.\{\ldots\ldots + \tfrac{3}{15}b^5 + \tfrac{105}{315}b^7 + \&c.\};$$

[8760i] $$-\tfrac{1}{7}T^7 = \sqrt{-1}.\{\ldots\ldots\ldots\ldots + \tfrac{45}{315}b^7 + \&c.\};$$

[8760k] $$V = \sqrt{-1}.\{b + \tfrac{2}{3}b^3 + \tfrac{10}{15}b^5 + \tfrac{244}{315}b^7 + \&c.\};$$

[8760l] $$V = b.\sqrt{-1}.\{1 + \tfrac{2}{3}b^2 + \tfrac{2}{3}b^4 + \tfrac{244}{315}b^6 + \&c.\}.$$

[8760m] Now from [8760a] we have $b.\sqrt{-1} = \frac{1}{2}t.\sqrt{mg}$, and $b^2 = -\frac{1}{4}t^2.mg$; substituting these in V [8760l], and then inserting the resulting value in the expression of $\alpha v'$, [8759f], we get,

$$\alpha v' = \frac{2n}{m}.t.\sin.\theta + \frac{4n.\sin.\theta}{m.\sqrt{mg}}.\tfrac{1}{2}t.\sqrt{mg}.\{-1 + \tfrac{2}{3}.(\tfrac{1}{4}mg).t^2 - \tfrac{2}{3}.(\tfrac{1}{4}mg)^2.t^4 + \tfrac{244}{315}.(\tfrac{1}{4}mg)^3.t^6 - \&c.\}$$

[8760n] $$= \frac{2n.\sin.\theta}{m}.t^3.\{\tfrac{2}{3}.(\tfrac{1}{4}mg) - \tfrac{2}{3}.(\tfrac{1}{4}mg)^2.t^2 + \tfrac{244}{315}.(\tfrac{1}{4}mg)^3.t^4 - \&c.\};$$

which is easily reduced to the form [8760].

The symbols which are used in the expressions of αs, $\alpha v'$ [8751, 8760, 8766], are collected in the following table; Symbols.

$t =$ the number of centesimal seconds of time from the commencement of the fall of the body [8728]; [8761]

$g =$ double the space which a falling body describes in the first second of time, by the force of gravity [8723]; [8761']

$nt =$ the angle of rotation of the earth, during t centesimal seconds of time [8728]; [8762]

$mg =$ a number depending on the resistance which the air produces in the motion of the body [8740']; [8762']

$h =$ the height fallen through in the time t. [8763]

To obtain the time t of the fall of the body, and its deviation towards the east, in terms of the height fallen through h, we have from [8750],* [8763']

$$2c^{mh} = c^{t.\sqrt{mg}} + c^{-t.\sqrt{mg}}. \qquad [8764]$$

Hence we deduce, Time of descent.

$$t = \frac{1}{\sqrt{mg}} \cdot \log.\tfrac{1}{2}.\{\sqrt{c^{mh}+1} + \sqrt{c^{mh}-1}\}^2; \qquad [8765]$$

and then,†

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* (4019) Substituting $\alpha s = h$ [8763, 8694], in the first member of [8750], then multiplying by m, and by $\log.c = 1$, we get, [8764a]

$$mh.\log.c = \log.c^{mh} = \log.\tfrac{1}{2}.\{c^{t.\sqrt{mg}} + c^{-t.\sqrt{mg}}\}, \quad \text{or} \quad c^{mh} = \tfrac{1}{2}.(c^{t.\sqrt{mg}} + c^{-t.\sqrt{mg}}), \qquad [8764b]$$

as in [8764]. Multiplying [8764] by $-c^{t.\sqrt{mg}}$, and adding $c^{2t.\sqrt{mg}} + c^{2mh}$ to the product, we get,

$$c^{2t.\sqrt{mg}} - 2c^{mh+t.\sqrt{mg}} + c^{2mh} = c^{2mh} - 1. \qquad [8764c]$$

The square root of [8764c] is $c^{t.\sqrt{mg}} - c^{mh} = \sqrt{c^{2mh}-1}$; whence $c^{t.\sqrt{mg}} = c^{mh} + \sqrt{c^{2mh}-1}$. [8764d]
The second member of this last expression is equal to $\tfrac{1}{2}.\{\sqrt{c^{mh}+1} + \sqrt{c^{mh}-1}\}^2$, as is easily proved by developing and reducing; hence we have,

$$c^{t.\sqrt{mg}} = \tfrac{1}{2}.\{\sqrt{c^{mh}+1} + \sqrt{c^{mh}-1}\}^2. \qquad [8764e]$$

Taking the logarithms of both sides of this equation, and dividing by $\sqrt{mg}$, we get t [8765].

† (4020) Adding ± 2 to both members of [8764], and extracting the square root, we get, [8766a]

[8766] $$\alpha v' = \frac{2n.\sin.\theta}{m.\sqrt{mg}}.\left\{\log.\tfrac{1}{2}.\{\sqrt{c^{mh}+1}+\sqrt{c^{mh}-1}\}^2 - 2.\text{ang.tang.}\left\{\frac{\sqrt{c^{mh}-1}}{\sqrt{c^{mh}+1}}\right\}\right\}.$$

The height h being given, and the time t determined by observation, we
[8767] may, from [8765], deduce the value of m. Then from [8766] we can ascertain the value of $\alpha v'$, or the deviation of the body to the east of the vertical. We can also determine m, by the figure and density of the body, with the experiments already made on the resistance of the air.

[8768] *In a vacuum, or, in other words, when* m [8740''] *is infinitely small, we shall have*,*

[8769] $$\alpha v' = \frac{2nh}{3}.\sqrt{\frac{2h}{g}}.\sin.\theta.$$

There have been made, in Italy and Germany, several experiments upon
[8769'] the fall of bodies, which agree with the preceding results. But these experiments, which require very great care, ought to be repeated with still greater accuracy.

[8766a'] $$\sqrt{2}.\sqrt{c^{mh}+1} = c^{\frac{1}{2}t.\sqrt{mg}} + c^{-\frac{1}{2}t.\sqrt{mg}}; \qquad \sqrt{2}.\sqrt{c^{mh}-1} = c^{\frac{1}{2}t.\sqrt{mg}} - c^{-\frac{1}{2}t.\sqrt{mg}}.$$

Dividing the second of these formulas by the first, we obtain,

[8766b] $$\frac{c^{\frac{1}{2}t.\sqrt{mg}} - c^{-\frac{1}{2}t.\sqrt{mg}}}{c^{\frac{1}{2}t.\sqrt{mg}} + c^{-\frac{1}{2}t.\sqrt{mg}}} = \frac{\sqrt{c^{mh}-1}}{\sqrt{c^{mh}+1}}.$$

Substituting this in the last term of [8759], and the value of t [8765] in the first term of its second member, we get [8766].

* (4021) We have, by developing as in [55, 56] Int.,

[8768a] $$c^{mh} = 1 + mh + \tfrac{1}{2}m^2h^2 + \&c.; \qquad c^{t.\sqrt{mg}} = 1 + t.\sqrt{mg} + \tfrac{1}{2}t^2.mg + \&c.;$$
$$c^{-t.\sqrt{mg}} = 1 - t.\sqrt{mg} + \tfrac{1}{2}t^2.mg - \&c.;$$

substituting these in [8764], and rejecting 2 from each member of the equation, we get $2mh + \&c. = t^2.mg + \&c.$ Dividing by m, we get $2h = t^2g +$ terms multiplied by m.
[8768b] Now putting $m=0$, we shall have $2h = t^2g$, or $t = \sqrt{\frac{2h}{g}}$; substituting this in [8760], after putting $m=0$, we get the expression of,

[8768c] $$\alpha v' = \tfrac{1}{3}ngt^3.\sin.\theta = \frac{2nh}{3}.\sqrt{\frac{2h}{g}}.\sin.\theta,$$

as in [8769].

Case where the body is projected upwards.

16. *We shall now consider the case in which the body has a projectile motion in space*; and we shall resume the equations [8724, &c.], supposing, [8770]

$$\alpha u = \alpha s.\left(\frac{dy}{d\theta}\right) + \alpha u' ; \qquad [8771]$$

$$\alpha v.\sin.\theta = \frac{\alpha s}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right) + \alpha v' \quad [8737] ; \qquad [8772]$$

so that $\alpha u'$ *and* $\alpha v'$ *will be the deviations of the body, from the vertical line passing through the point where the motion commences. The deviation* $\alpha u'$ *being in the direction of the meridian; and the deviation* $\alpha v'$ *being in the direction of the parallel of latitude.** [8773] Then neglecting the resistance of the air, we find that the equations [8724—8724″] will become of the following forms ;† [8773′]

Fundamental differential equations.

$$0 = \alpha.\frac{dds}{dt^2} + 2\alpha n.\sin.\theta.\frac{dv'}{dt} + 2\alpha n.\frac{ds}{dt}.\left(\frac{dy}{d\varpi}\right) - g ; \qquad [8774]$$

$$0 = \alpha.\frac{dds}{dt^2}.\left(\frac{dy}{d\theta}\right) + \alpha.\frac{ddu'}{dt^2} - 2\alpha n.\frac{\cos.\theta}{\sin.\theta}.\frac{ds}{dt}.\left(\frac{dy}{d\varpi}\right) - 2\alpha n.\cos.\theta.\frac{dv'}{dt} - g.\left(\frac{dy}{d\theta}\right) ; \qquad [8775]$$

$$0 = \frac{\alpha}{\sin.\theta}.\frac{dds}{dt^2}.\left(\frac{dy}{d\varpi}\right) + \alpha.\frac{ddv'}{dt^2} + 2\alpha n.\cos.\theta.\frac{ds}{dt}.\left(\frac{dy}{d\theta}\right) + 2\alpha n.\cos.\theta.\frac{du'}{dt}$$
$$- 2\alpha n.\sin.\theta.\frac{ds}{dt} - \frac{g}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right). \qquad [8776]$$

Multiplying [8774] by $-\left(\frac{dy}{d\theta}\right)$, and adding the product to [8775], we get [8778], neglecting terms arising from the product of $\alpha n.\frac{ds}{dt}$, $\alpha n.\frac{dv'}{dt}$, by $\left(\frac{dy}{d\theta}\right)$ or $\left(\frac{dy}{d\varpi}\right)$. In like manner, multiplying [8774] by $-\frac{1}{\sin.\theta}.\left(\frac{dy}{d\varpi}\right)$, and [8777] adding the product to [8776], we get [8779], neglecting terms arising from

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* (4022) The deviation, in the direction of the parallel of latitude, is $\alpha v'$ [8740, 8739]; and it is shown, in [8736, 8736′], that if $\alpha u = \alpha s.\left(\frac{dy}{d\theta}\right)$, the deviation of the body from [8773a] the parallel of latitude, or in the direction of the meridian, will vanish. The difference between this and the real value of αu [8771], is $\alpha u'$, which represents the actual deviation [8773b] in the direction of the meridian, as in [8773].

† (4023) Neglecting the resistance of the air, we shall have, as in [8705, 8707], $S=0$. Substituting this value of S, and those of αu, αv [8771, 8772], we find that the [8774a] equations [8724, 8724′, 8724″] become respectively as in [8774, 8775, 8776].

Approximate equations of a projectile.

the products of $\alpha n.\frac{ds}{dt}$, $\alpha n.\frac{dv'}{dt}$, by $\left(\frac{dy}{d\theta}\right)$ or $\left(\frac{dy}{d\varpi}\right)$. Finally, by neglecting similar terms in [8774], it changes into [8780].

[8778] $$0=\alpha.\frac{ddu'}{dt^2}-2\alpha n.\cos.\theta.\frac{dv'}{dt};$$

[8779] $$0=\alpha.\frac{ddv'}{dt^2}+2\alpha n.\cos.\theta.\frac{du'}{dt}-2\alpha n.\sin.\theta.\frac{ds}{dt};$$

[8780] $$0=\alpha.\frac{dds}{dt^2}+2\alpha n.\sin.\theta.\frac{dv'}{dt}-g.$$

Now integrating these equations, and *fixing the origin of the co-ordinates*
[8781] *αs, $\alpha u'$, $\alpha v'$, at the point where the motion commences, and the origin of the time t at the commencement of the motion,* we shall have,*

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* (4024) If we take the differential of [8779], and substitute in it the values of $\alpha.\frac{ddu'}{dt^2}$, $\alpha.\frac{dds}{dt^2}$, deduced from [8778, 8780], we get, after dividing by dt,

[8781a] $$0=\alpha.\frac{d^3v'}{dt^3}+4\alpha n^2.(\cos.^2\theta+\sin.^2\theta).\frac{dv'}{dt}-2gn.\sin.\theta.$$

Substituting $\cos.^2\theta+\sin.^2\theta=1$, multiplying by dt, integrating and adding the constant quantity $-4C'n^2$, we obtain,

[8781b] $$0=\alpha.\frac{d^2v'}{dt^2}+4\alpha n^2.v'-2gnt.\sin.\theta-4C'n^2.$$

This linear equation of the second degree, is the same as that which is solved in
[8781c] [865a, &c.], putting $y=\alpha v'$, $a=2n$, $\alpha Q=-2gnt.\sin.\theta-4C'n^2$, $b=C$, $\varphi=\varepsilon$, in order to conform to the present notation; and the integrations being performed in the manner pointed out in [865b], we get, by reduction,

[8781d] $$\alpha v'=-C\sin.(2nt+\varepsilon)+\frac{g.\sin.\theta}{2n}.t+C'.$$

Without going through the labor of these reductions, we may more easily verify this value of $\alpha v'$, by substituting it in the differential equation [8781b], which will vanish by this
[8781e] substitution. This value of $\alpha v'$ contains the three arbitrary constant quantities C, C', ε, but C, C', depend on each other. For av' vanishes when $t=0$ [8781], and then the
[8781f] value of $\alpha v'$ [8781d] becomes $0=-C.\sin.\varepsilon+C'$, or $C'=C.\sin.\varepsilon$. Substituting this in $\alpha v'$ [8781d], it becomes as in [8783], with the two arbitrary constant quantities C, ε, which are required for the complete integral. Multiplying [8778] by dt, integrating,
[8781f'] adding the constant quantity $B.\sin.\theta-2Cn.\cos.\theta.\sin.\varepsilon$, and using $\alpha v'$ [8783], we get successively,

Deviations.

$$\alpha u' = Bt.\sin.\theta + \tfrac{1}{2}gt^2.\sin.\theta.\cos.\theta + C.\cos.\theta.\{\cos.(2nt+\varepsilon)-\cos.\varepsilon\};$$ [8782]

$$\alpha v' = \frac{g.\sin.\theta}{2n}.t - C.\{\sin.(2nt+\varepsilon)-\sin.\varepsilon\};$$ [8783]

$$\alpha s = Bt.\cos.\theta + \tfrac{1}{2}gt^2.\cos.^2\theta - C.\sin.\theta.\{\cos.(2nt+\varepsilon)-\cos.\varepsilon\}.$$ [8784]

B, C, ε, being three arbitrary quantities, which depend on the initial velocity of the body in the direction of the three co-ordinates. [8785]

Vertical projection.

Suppose, for example, that the body is projected vertically upwards, with a velocity equal to K. The positive values of s being here counted downwards, we shall have, at the origin of the time t,* $\alpha.\frac{ds}{dt} = -K$. We [8786] [8787]

$$\alpha.\frac{du'}{dt} = 2n.\cos.\theta.\alpha v' + B.\sin.\theta - 2Cn.\cos.\theta.\sin.\varepsilon$$ [8781g]

$$= 2n.\cos.\theta.\left\{\frac{g.\sin.\theta}{2n}.t - C.\sin.(2nt+\varepsilon) + C.\sin.\varepsilon\right\} + B.\sin.\theta - 2Cn.\cos.\theta.\sin.\varepsilon$$

$$= B.\sin.\theta + gt.\sin.\theta.\cos.\theta - 2Cn.\cos.\theta.\sin.(2nt+\varepsilon).$$ [8781h]

Multiplying this last expression by dt, and integrating, we get $\alpha u'$ [8782]; the constant quantity $-C.\cos.\theta.\cos.\varepsilon$ being added, so as to make $\alpha u'$ vanish when $t=0$, as in [8781].

The second differential of [8783], divided by dt^2, gives $\alpha.\frac{ddv'}{dt^2} = 4Cn^2.\sin.(2nt+\varepsilon)$; [8781i] substituting this in [8779], and then dividing by $2n$, we get [8781l]. Substituting in its second member the expression [8781h], connecting the terms multiplied by C, and reducing by putting $1-\cos.^2\theta = \sin.^2\theta$, we get [8781n]. Dividing this by $\sin.\theta$, we obtain [8781o], whose integral gives αs [8784]; the constant quantity $+C.\sin.\theta.\cos.\varepsilon$ being added to the integral, so as to make αs vanish when $t=0$, as in [8781]. [8781k]

$$\alpha.\sin.\theta.\frac{ds}{dt} = 2Cn.\sin.(2nt+\varepsilon) + \alpha.\cos.\theta.\frac{du'}{dt}$$ [8781l]

$$= 2Cn.\sin.(2nt+\varepsilon) + \cos.\theta.\{B.\sin.\theta + gt.\sin.\theta.\cos.\theta - 2Cn.\cos.\theta.\sin.(2nt+\varepsilon)\}$$ [8781m]

$$= 2Cn.\sin.^2\theta.\sin.(2nt+\varepsilon) + B.\sin.\theta.\cos.\theta + gt.\sin.\theta.\cos.^2\theta;$$ [8781n]

$$\alpha.\frac{ds}{dt} = 2Cn.\sin.\theta.\sin.(2nt+\varepsilon) + B.\cos.\theta + gt.\cos.^2\theta.$$ [8781o]

* (4025) If the body be projected upwards, in the direction of the radius r, or $r-\alpha s$, with the velocity K, it will pass over the space $-\alpha ds = Kdt$, in the first moment of time dt, as in [8787], without altering the values of $\alpha u'$ or $\alpha v'$; so that at the commencement of the motion we shall have $\frac{du'}{dt}=0$; $\frac{dv'}{dt}=0$, as in [8787']. Now taking the differentials of $\alpha u'$, $\alpha v'$, αs [8782, 8783, 8784], dividing them by dt, and then substituting $t=0$, and the values [8787, 8787'], we obtain the equations [8788, 8789, 8790] respectively. [8787a] [8787b]

[8787'] shall also have at that origin, $\frac{du'}{dt}=0$; $\frac{dv'}{dt}=0$; therefore,

[8788] $$0=B.\sin.\theta-2Cn.\cos.\theta.\sin.\varepsilon;$$

[8789] $$0=\frac{g}{2n}.\sin.\theta-2Cn.\cos.\varepsilon;$$

[8790] $$-K=B.\cos.\theta+2Cn.\sin.\theta.\sin.\varepsilon.$$

Hence we deduce,*

[8791] $$C.\sin.\varepsilon=-\frac{K.\sin.\theta}{2n};$$

[8792] $$C.\cos.\varepsilon=\frac{g.\sin.\theta}{4n^2};$$

[8793] $$B=-K.\cos.\theta.$$

Deviations with a vertical projection.

which give,†

[8794] $$\alpha u'=-\sin.\theta.\cos.\theta.\left\{\frac{K}{2n}.(2nt-\sin.2nt)+\frac{g}{4n^2}.(1-2n^2t^2-\cos.2nt)\right\};$$

[8795] $$\alpha v'=\frac{\sin.\theta}{2n}.\left\{\frac{g}{2n}.(2nt-\sin.2nt)-K.(1-\cos.2nt)\right\};$$

[8796] $$\alpha s=-Kt+\tfrac{1}{2}gt^2+\frac{g.\sin.^2\theta}{4n^2}.\{1-2n^2t^2-\cos.2nt\}+\frac{K.\sin.^2\theta}{2n}.(2nt-\sin.2nt).$$

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* (4026) Dividing [8789] by $2n$, we get [8792]. Multiplying [8788] by $\sin.\theta$, and [8790] by $\cos.\theta$, then adding the two products, we get,

[8791a] $$-K.\cos.\theta=B.\{\sin.^2\theta+\cos.^2\theta\}=B,$$

as in [8793]. Substituting this value of B in [8788], we get,

[8791b] $$0=-K.\sin.\theta.\cos.\theta-2Cn.\cos.\theta.\sin.\varepsilon.$$

Dividing this by $-2n.\cos.\theta$, we obtain [8791].

† (4027) Substituting,

$\sin.(2nt+\varepsilon)=\sin.2nt.\cos.\varepsilon+\cos.2nt.\sin.\varepsilon$; $\cos.(2nt+\varepsilon)=\cos.2nt.\cos.\varepsilon-\sin.2nt.\sin.\varepsilon$,

[22, 24] Int., in [8782, 8783, 8784], together with the values [8791—8793], we obtain,

[8794a] $$\alpha u'=-Kt.\sin.\theta.\cos.\theta+\tfrac{1}{2}gt^2.\sin.\theta.\cos.\theta+\cos.\theta.\left\{\frac{g.\sin.\theta}{4n^2}.\cos.2nt+\frac{K.\sin.\theta}{2n}.\sin.2nt-\frac{g.\sin.\theta}{4n^2}\right\};$$

[8794b] $$\alpha v'=\frac{g.\sin.\theta}{2n}.t+\left\{-\frac{g.\sin.\theta}{4n^2}.\sin.2nt+\frac{K.\sin.\theta}{2n}.\cos.2nt-\frac{K.\sin.\theta}{2n}\right\};$$

[8794c] $$\alpha s=-Kt.\cos.^2\theta+\tfrac{1}{2}gt^2.\cos.^2\theta+\sin.\theta.\left\{-\frac{g.\sin.\theta}{4n^2}.\cos.2nt-\frac{K.\sin.\theta}{2n}.\sin.2nt+\frac{g.\sin.\theta}{4n^2}\right\}.$$

By a different arrangement of the terms, we may change [8794a] into [8794]; [8794b]
[8794d] into [8795]; and [8794c] into [8796]; observing that in [8794c] we must change $\cos.^2\theta$ into $1-\sin.^2\theta$.

Reducing these expressions to series, and neglecting quantities of the order n^2, we obtain,* [8796′]

$$\alpha u' = 0;$$ [8797]

$$\alpha v' = \tfrac{1}{3} nt^2.(gt-3K).\sin.\theta;$$ [8798]

$$\alpha s = -Kt + \tfrac{1}{2}gt^2.$$ [8799]

These expressions show that the deviation of the body, in the direction of the meridian $\alpha u'$, *is very small; and that it is only sensible in the direction of the parallel of latitude* $\alpha v'$. [8800] If we suppose $K=0$, in [8798], we shall have the same expression of the deviation as in [8768c]. If we suppose K to be given, and we wish to find the point where the body will strike the earth, [8801] we must put $\alpha s = 0$, whence we get $gt = 2K$; consequently,† [8801′]

$$\alpha v' = -\frac{4n.K^3.\sin.\theta}{3g^2}.$$ [8802]

Deviation on the parallel of latitude.

To reduce this formula to numbers, we shall observe that n is the angle described by the rotation of the earth in a centesimal second of time [8727′], and this angle is equal to $\frac{40''}{0{,}99727}$; because the duration of the sideral day [8803] is $99727''$; we must reduce it to parts of the radius, or, in other words, divide it by the radius in seconds $636620''$. g is double the space which [8804] gravity causes a heavy body to describe in the first centesimal second of its

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* (4028) We have, by developing as in [43, 44] Int.,

$$\sin.2nt = 2nt - \tfrac{4}{3}n^3t^3 + \&c.; \qquad \cos.2nt = 1 - 2n^2t^2 + \tfrac{2}{3}n^4t^4 - \&c.;$$ [8796a]

hence we deduce,

$$\frac{1}{2n}.\{2nt - \sin.2nt\} = \tfrac{2}{3}n^2t^3 - \&c.; \qquad \frac{1}{4n^2}.(1 - 2n^2t^2 - \cos.2nt) = -\tfrac{1}{6}n^2t^3 + \&c.;$$ [8796b]

so that if we neglect terms of the order n^2, as in [8796′], we shall find that both expressions in [8796b] will vanish; and then the equations [8794, 8796] will become as in [8797, 8799] respectively; moreover [8795] changes into [8798].

† (4029) Putting $\alpha s = 0$ in [8799], we get $gt = 2K$; substituting this in [8798] we obtain $\alpha v' = -\frac{1}{3}nt^2.K.\sin.\theta$; and by using $t = \frac{2K}{g}$ [8801′], it becomes as in [8802]. [8802a] Now the earth, by its diurnal motion, describes $4000000''$ in one sideral day of $99727''$ of the centesimal division; hence the arc n, corresponding to one second [8727′], is $\frac{40''}{0'',99727}$, or $n = \frac{40''}{0'',99727 \times 636620}$, in parts of the radius. Substituting this in [8802], [8802b] together with the values of g, K, θ [8804—8806], we get the expression of $\alpha v'$ [8807], or by reduction as in [8808]. [8802c]

fall [8761′]; and this space, in the latitude of Paris, is equal to $7^{\text{metres}},32214$.
[8805] If we suppose, for an example, that the velocity K is 500 metres per second, we shall have for Paris, whose latitude is $54^{\circ},2636$, θ equal to the
[8806] complement of this latitude, or $\theta = 45^{\circ},7364$. Hence we get,

[8807] $$\alpha v' = -\tfrac{4}{3}.500^{\text{metres}}.\left(\frac{500^{\text{m}},00000}{7^{\text{m}},32214}\right)^2.\frac{40''}{0,99727\times 636620''}.\sin.45^{\circ},7364\,;$$

whence we deduce,

[8808] $$\alpha v' = -128^{\text{metres}},9.$$

Deviation west of the place of projection. *This expresses the distance of the place, where the body falls upon the earth, to the west of the place of projection.* For the rotatory motion of the earth,
[8809] being from the west towards the east, the negative values of $\alpha v'$ are to be taken in the opposite direction, or from the east towards the west.

CHAPTER VI.

ON SOME CASES WHERE WE CAN RIGOROUSLY ASCERTAIN THE MOTIONS OF SEVERAL BODIES WHICH MUTUALLY ATTRACT EACH OTHER.

17. The problem of the motions of two bodies, mutually attracting each other, can be accurately solved, as we have seen in the second book, [531—534]; but when the system is composed of three or a greater number of bodies, the problem, in the present state of analysis, can be solved only by approximation. The following cases are however susceptible of a rigorous solution. [8810]

If we suppose the different bodies to be situated in the same plane, so that the resultant of the forces, acting upon each one of them, may pass through the centre of gravity of the system, and that the different resultants may be proportional to the respective distances of the bodies from this centre, then it is evident that if we impress upon the system an angular rotatory motion about its centre of gravity, so that the centrifugal force of each body may be equal to the force which attracts it towards that centre, all the bodies will continue to move in circles about that point, retaining, in relation to each other, the same relative positions, so that they will appear to describe circles about each other.* [8810′] First theorem. [8811] [8811′] [8812]

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* (4030) If we suppose all the bodies to have the same angular rotatory motion about the common centre of gravity, the centrifugal force of any one of them will be proportional to its distance from that centre, as in [54′]; and by hypothesis [8810′, &c.] the whole action of the bodies, upon any one of them, is reduced to a simple attraction towards the centre of gravity, with a force which is also proportional to the distance of the body from that centre. Now as both these forces are proportional to the distance from the centre of gravity, it is evident that we can adjust the rotatory motion, so that they may exactly balance each other, as in [8811′]. [8811a] [8811b] [8811c]

The bodies being in the preceding position, *if we suppose that the polygon,*
[8813] *at whose angles the bodies may be imagined to be placed, varies in any*
Second theorem. *manner, but always retaining a similar figure, it is evident that the law of*
[8814] *attraction, being supposed to be proportional to any power whatever of the distance, the resultants of the forces which act upon the bodies will be to each*
[8814′] *other, at all times, as the distances of the bodies from the centre of gravity of the system.** This being premised, we shall now suppose that all the bodies, when in a state of rest, are impressed at the same instant with
[8814″] velocities proportional to their distances from this centre, and in directions equally inclined to the radii drawn from this point to each of the bodies;

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* (4031) For the purpose of illustration, we shall suppose the attraction of the bodies
[8814a] upon each other to be as the power n of the distance; so that if, at the commencement of the time t, we represent the distances of the bodies m', m'', &c. from m, by s, s', &c.,
[8814b] the action of the bodies m', m'', &c. upon m, will be expressed by $m'.\mathrm{s}^n$, $m''.\mathrm{s}'^n$, &c.,
[8814c] in the direction of the lines s, s', &c. respectively. Moreover if we represent by r, r', r'', &c. the radii, drawn from the centre of gravity of the system to the bodies m, m', m'', &c. respectively, we shall have, according to the hypothesis assumed in [8814, &c.],
[8814d] Kr, Kr', Kr'', &c. for the resultants of all the forces of attraction acting upon these bodies respectively; K *being of the same magnitude for all the bodies at any one moment*
[8814e] *of time whatever.* In the hypothesis [8811, 8812], where the polygon does not vary, the value of K is also, at all times, invariable; but when the polygon varies in magnitude,
[8814f] but not in its figure, as in [8813, &c.], the value of K may also vary in successive moments of time, but at any particular instant it must, by hypothesis, have the same value
[8814g] for all the bodies m, m', m'', &c. Now if we suppose the figure of the polygon to vary as in [8813], so that every linear measure s, s', s'', &c., r, r', r'', &c., corresponding to the time $t=0$, may be increased in the ratio of a to 1; and we then represent the new values, corresponding to the time t, by the Italic letters s, s', s'', &c., r, r', r'', &c. respectively; we shall have,

[8814h] $$s=a\mathrm{s},\quad s'=a\mathrm{s}',\quad s''=a\mathrm{s}'',\ \&c.;\qquad r=a\mathrm{r},\quad r'=a\mathrm{r}'\quad r''=a\mathrm{r}'',\ \&c.$$

[8814i] In this case the action of the bodies m', m'', &c. upon m [8814b], will be changed into $m'.s^n$, $m''.s'^n$, &c.; or into the equivalent values $m'.a^n.\mathrm{s}^n$, $m''.a^n.\mathrm{s}'^n$, &c.; each of them having increased from its original value [8814b] in the ratio of a^n to 1; and as the
[8814k] resultants [8814d] must also increase in the same ratio, they will become $\mathrm{K}a^n.\mathrm{r}$, $\mathrm{K}a^n.\mathrm{r}'$, $\mathrm{K}a^n.\mathrm{r}''$, &c. respectively; so that if we put for symmetry $K=\mathrm{K}a^{n-1}$, and use
[8814l] the values of r, r', r'', &c. [8814h], these resultants will become Kr, Kr', Kr'', &c. respectively; or, in other words, they will be proportional to the augmented or new
[8814m] distances r, r', r'', &c. of the bodies from the centre of gravity of the system; it being evident that the position of this centre does not vary in consequence of these changes.
[8814n] These results are in conformity with the remarks of the author in [8814′, &c.].

then the polygons, formed at each moment by the right lines which connect these bodies, will be similar; the bodies will describe similar curves,* both [8815] about the centre of gravity of the system and about each other, and these curves will be of the same nature as that which a body attracted towards a [3815′] fixed point would describe.

To apply these theorems to an example, we shall consider three bodies, whose masses are m, m', m'', which attract each other, *according to the* [8816]

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* (4032) The velocity of projection of any one of the bodies, as for example that of [8815a] the body m, is by hypothesis proportional to its distance r, from the common centre of gravity of the system, at the time of the commencement of the motion; and as the angle of inclination of the line of projection with the radius, is the same for each of the bodies, it is evident that the area described in the first instant of time dt, will be proportional to r^2; [8815b] so that we may represent the double of this area by $\mathrm{c\,r}^2.dt$; c being the same for all the bodies m, m', m'', &c. Comparing this with the expression of the same area cdt [366], [8815c] we get $c=\mathrm{c\,r}^2$. Moreover the force φ [373], acting on the body m at the time t, is represented by Kr [8814l]. Substituting these values of c, φ, in the expression of dv [376], we get the value of dv [8815e]; and by accenting the letters we get the similar [8815d] expression of dv', corresponding to the motion of the body m'; observing that K is the same for both bodies [8814l];

$$dv=\frac{\mathrm{cr}^2.dr}{r.\sqrt{-\mathrm{c}^2\mathrm{r}^4-2r^2\int Krdr}};\qquad dv'=\frac{\mathrm{cr}'^2.dr'}{r'.\sqrt{-\mathrm{c}^2\mathrm{r}'^4-2r'^2\int Kr'dr'}}. \qquad [8815e]$$

We shall now suppose that at the commencement of the motion, when $t=0$, we have $\mathrm{r}'=b\,\mathrm{r}$, and that the angles v, v', commence at that time, so that their places of origin [8815e′] must be on the lines r, r', respectively; and we shall then compute, by means of the formulas [8815e], the relation of the arcs v, v', when the general value of r' is expressed by $r'=br$. In this case we find, by substituting $\mathrm{r}'=b\mathrm{r}$, $r'=br$, in the expression [8815f] of dv' [8815e], that the numerator and denominator can be divided by the constant quantity b^3; so that this value of dv' will become identically the same as that of dv, [8815g] [8815e], and we shall then have $dv=dv'$; or by integration $v=v'$; supposing, as in [8815h] [8815e'], that both angles commence at the origin of the time t. Hence we see that when $r'=br$, we shall have $v'=v$; therefore the figures described by the bodies m, m', will be similar; consequently the areas described by the radii vectores will be as r^2 to r'^2, or [8815i] as 1 to b^2; and as the areas described in the time dt, at the origin of the motion, are in the same ratio, the times of describing the equal angles v, v', by the bodies m, m', must [8815k] be equal. This is conformable to the remarks in [8815, &c.]. *What we have here stated relative to the paths of the bodies about the common centre of gravity being similar to* [8815l] *each other, may evidently be applied to the relative motions of the bodies about any one* [8815m] *of them, considered as at rest, since they must also be similar.*

[8816′] *function* $\varphi(r)$ *of the distance* r. We shall put x, y, for the co-ordinates of m, *referred to the plane which connects these bodies, and to the centre of gravity*
[8817] *of the system;* also x', y', for the co-ordinates of m', and x'', y'', for those of m''. Then the force acting upon m, parallel to the axis of x, *and drawing towards the centre of gravity*, is,*

[8818] $$m'.\frac{\varphi(s)}{s}.(x-x')+m''.\frac{\varphi(s')}{s'}.(x-x'');$$ [Force acting on m parallel to x.]

[8819] *s being the distance of m from m', and s' that of m from m''.* The force acting upon m parallel to the axis of y, is,

[8820] $$m'.\frac{\varphi(s)}{s}.(y-y')+m''.\frac{\varphi(s')}{s'}.(y-y'').$$ [Force acting on m parallel to y.]

Likewise the force acting upon m' parallel to the axis of x, is,

[8821] $$m.\frac{\varphi(s)}{s}.(x'-x)+m''.\frac{\varphi(s'')}{s''}.(x'-x'');$$ [Force acting on m' parallel to x.]

[8822] *s'' being the distance of m' from m''.* The force acting upon m' parallel to the axis of y, is,

[8823] $$m.\frac{\varphi(s)}{s}.(y'-y)+m''.\frac{\varphi(s'')}{s''}.(y'-y'').$$ [Force acting on m' parallel to y.]

Lastly, the forces acting upon m'', parallel to the axes of x'', y'', are respectively,

[8824] $$m.\frac{\varphi(s')}{s'}.(x''-x)+m'.\frac{\varphi(s'')}{s''}.(x''-x');$$ [Force acting on m'' parallel to x.]

[8825] $$m.\frac{\varphi(s')}{s'}.(y''-y)+m'.\frac{\varphi(s'')}{s''}.(y''-y').$$ [Force acting on m'' parallel to y.]

Now, in order that the resultant of the two forces which act upon m,

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* (4033) The action of m' upon m, in the direction s, is $m'.\varphi(s)$ [8816′]. Resolving
[8818a] it, in a direction parallel to the axis of x, it becomes $m'.\frac{\varphi(s)}{s}.(x'-x)$ in a direction opposite to the origin of the co-ordinates [393′], or to conform to the present hypothesis [8833], $m'.\frac{\varphi(s)}{s}.(x-x')$, *towards the origin.* In like manner the force of m'' upon m,
[8818b] is $m''.\frac{\varphi(s')}{s'}.(x-x'')$; the sum is as in [8818], which represents the whole force of m' and m'' upon m, resolved in a direction parallel to x. The other forces [8820—8825]
[8818c] are found in a similar manner; all these forces being supposed to tend towards the origin of the co-ordinates, as in [8833].

parallel to the axes of x and y, may pass through the centre of gravity of [8826]
the system, it is necessary that these forces should be in the ratio of x to
y;* therefore we shall have,

$$m'.\frac{\varphi(s)}{s}.(x-x')+m''.\frac{\varphi(s')}{s'}.(x-x'')=Kx; \qquad [8827]$$

$$m'.\frac{\varphi(s)}{s}.(y-y')+m''.\frac{\varphi(s')}{s'}.(y-y'')=Ky; \qquad [8828]$$

K being any quantity whatever. The force which attracts m towards the [8829]
centre of gravity, will be $K.\sqrt{x^2+y^2}$. We shall likewise have, by [8830]
considering the forces which act upon m',

$$m.\frac{\varphi(s)}{s}.(x'-x)+m''.\frac{\varphi(s'')}{s''}.(x'-x'')=K'x'; \qquad [8831]$$

$$m.\frac{\varphi(s)}{s}.(y'-y)+m''.\frac{\varphi(s'')}{s''}.(y'-y'')=K'y'; \qquad [8832]$$

which gives $K'.\sqrt{x'^2+y'^2}$ for the force which attracts m' *towards the centre* [8833]
of gravity of the system. In order that this force may be to that which acts [8834]
upon the body m, in the ratio of the distances of the two bodies from that
centre, it is necessary that we should have† $K=K'$; and as we must [8835]
apply the same result to the forces which act on m'', we shall have the
three following equations;

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* (4034) This is proved in [367a, &c.]. Now the two forces [8818, 8820], being to [8827a]
each other in the ratio of x to y, we may evidently represent them by Kx, Ky,
respectively, as in [8827, 8828]. The resultant of these two forces is $K.\sqrt{x^2+y^2}$ [3], [8827b]
as in [8830]. In like manner the forces acting on m' [8821, 8823], may be represented
by $K'x'$, $K'y'$, as in [8831, 8832]; and their resultant is $K'.\sqrt{x'^2+y'^2}$, as in [8833].

† (4035) The distances of the bodies m, m', m'', from the centre of gravity of the
system r, r', r'' [8844], are evidently represented in the notation [8816′, 8817], by [8836a]
$r=\sqrt{x^2+y^2}$, $r'=\sqrt{x'^2+y'^2}$, $r''=\sqrt{x''^2+y''^2}$. Hence the force acting upon m
[8830], is Kr; and that acting upon m' [8833], is $K'r'$; and by hypothesis these must [8836b]
be to each other as r to r' [8814′]; therefore we have $Kr : K'r' :: r : r'$; consequently
$K=K'$, as in [8835]; hence [8831] becomes as in [8836′]; moreover [8827] is the
same as [8836]. In the same manner we may prove that the forces acting on m'', give [8836c]
the similar equation [8836″]. The equations [8828, 8832], and the similar one for m'',
give the equations in y, y', y'', mentioned in [8837].

$$\left.\begin{array}{ll} [8836] & m'.\frac{\varphi(s)}{s}.(x-x')+m''.\frac{\varphi(s')}{s'}.(x-x'')=Kx; \\ [8836'] & m.\frac{\varphi(s)}{s}.(x'-x)+m''.\frac{\varphi(s'')}{s''}.(x'-x'')=Kx'; \\ [8836''] & m.\frac{\varphi(s')}{s'}.(x''-x)+m'.\frac{\varphi(s'')}{s''}.(x''-x')=Kx''; \end{array}\right\}.\ (a)$$

[8837] If we change in these equations x, x', x'', into y, y', y'', we shall obtain the three equations corresponding to the ordinates y, y', y''.

Multiplying [8836] by m, [8836′] by m', and [8836″] by m'', then taking the sum of these products, we get,*

[8838] $$0=mx+m'x'+m''x''.$$

[8839] This equation depends on the nature of the centre of gravity [124]; and by combining it with [8836], we get,†

[8840] $$x.\left\{m'.\frac{\varphi(s)}{s}+(m+m'').\frac{\varphi(s')}{s'}\right\}+m'x'.\left\{\frac{\varphi(s')}{s'}-\frac{\varphi(s)}{s}\right\}=Kx.$$

[8841] Therefore by supposing $s=s'$, we shall have,

[8842] $$K=(m+m'+m'').\frac{\varphi(s)}{s}.$$

[8843] Moreover if we suppose $s=s''$, we shall find that the equations [8836′, 8836″] give the same expression of K. Hence it follows that in the supposition of $s=s'=s''$, this expression will satisfy the equations [8836—8836″], and the similar equations in y, y', y'' [8837].

[8844] This being supposed, if we put r, r', r'', *for the respective distances of the bodies* m, m', m'', *from the centre of gravity of the system, the forces which attract these bodies towards that centre will be* Kr, Kr', Kr'', [8845] [8836*b*]; so that by impressing upon these bodies velocities proportional to

Case where three bodies are at equal distances from each other.

[8838*a*] * (4036) The terms of the first member of this sum mutually destroy each other; and those of the second member, being divided by the common factor K, become as in [8838]. This equation is the same as $0=\Sigma.mx$ [124], corresponding to the centre of gravity.

[8840*a*] † (4037) Multiplying [8838] by $\frac{\varphi(s')}{s'}$, and adding the product to the first member of [8836], we get [8840]. If we put $s=s'$, we shall have $\frac{\varphi(s')}{s'}=\frac{\varphi(s)}{s}$; and then [8840] becomes $x.\{m+m'+m''\}.\frac{\varphi(s)}{s}=Kx$. Dividing by x, we get K [8842].

r, r', r'', and with directions equally inclined to those radii, we shall have during the motion $s = s' = s''$;* or, in other words, *the three bodies will* [8846] *always form an equilateral triangle, by the lines which connect them with each other; and they will describe about each other, as well as about their* [8847] *common centre of gravity, perfectly similar curves* [8815m].

We shall now put,

X, Y, for the rectangular co-ordinates of the centre of gravity, referred to any point whatever as the origin; [8848]

x, y, for the rectangular co-ordinates of the body m, referred to the same origin; [8848']

x', y', for the rectangular co-ordinates of the body m', referred to the same origin; [8848'']

and so on for others; then we shall have, as in [129],†

$$X^2+Y^2 = \frac{\Sigma.m.(x^2+y^2)}{\Sigma.m} - \frac{\Sigma.mm'.\{(x-x')^2+(y-y')^2\}}{(\Sigma.m)^2}; \qquad [8849]$$

taking therefore the centre of the body m for the origin of the co-ordinates, [8850] which makes $x=0$, $y=0$, we shall have,

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* (4038) What we have stated in the notes [8814a, &c., 8815a, &c.], will serve to [8846a] explain the remarks of the author in [8844—8847]; observing that in this case we have, according to the notation in [8814a, &c.], $\mathrm{s} = \mathrm{s}' = \mathrm{s}''$; substituting these in [8814h], [8846b] we get $s = s' = s'' = a\mathrm{s}$.

† (4039) The bodies are supposed to move in the plane of xy [8810', 8816']; [8849a] therefore we shall have $Z=0$, $z=0$, $z'=0$, &c. in the formula [129]; and by neglecting these quantities, it becomes as in [8849]. Now it is evident, from the notation [8849b] [8848], that $\sqrt{X^2+Y^2}$ represents the distance of the centre of gravity from the origin of the co-ordinates, or from the body m [8850]; and this is represented by r in [8844]; [8849c] hence we have $r = \sqrt{X^2+Y^2}$, as in [8851]. The values s, s' [8852, 8853] correspond with those in [949'], using $z=0$, $z'=0$, &c. [8849a]; and by putting, as in [8850], $x=0$, $y=0$, and $s=s'=s''$ [8843], we get successively, [8849d]

$$x^2+y^2=0; \quad (x'-x)^2+(y'-y)^2 = x'^2+y'^2 = s^2; \quad (x''-x)^2+(y''-y)^2 = x''^2+y''^2 = s'^2 = s^2; \qquad [8849e]$$

$$\Sigma.m.(x^2+y^2) = m.(x^2+y^2)+m'.(x'^2+y'^2)+m''.(x''^2+y''^2) = (m'+m'').s^2; \qquad [8849f]$$

$$\Sigma.mm'.\{(x-x')^2+(y-y')^2\} = s^2.\Sigma.mm' = s^2.\{mm'+mm''+m'm''\}. \qquad [8849g]$$

Substituting [8851, 8849f, g] in [8849], it becomes as in [8854]; whence we easily [8849h] deduce s [8855].

[8851] $$X^2+Y^2=r^2;$$

[8852] $$(x'-x)^2+(y'-y)^2=s^2; \qquad [8849e]$$

[8853] $$(x''-x)^2+(y''-y)^2=s'^2=s^2, \&c.; \quad [8849e]$$

[8854] $$r^2=\frac{(m'+m'').s^2}{m+m'+m''}-\frac{(mm'+mm''+m'm'').s^2}{(m+m'+m'')^2}. \qquad [8849h]$$

Hence we deduce,

[8855] $$s=\frac{(m+m'+m'').r}{\sqrt{m'^2+m'm''+m''^2}}.$$

[8856] Substituting this value of s in the expression of the gravity $\varphi(s)$ [8816'], we obtain the law of gravitation of the body m, towards the centre of gravity of the system. The force which draws m towards that point, is

[8857] represented by Kr [8845], and we have $K=(m+m'+m'').\frac{\varphi(s)}{s}$ [8842]; hence this force will be,*

[8858] $$\sqrt{m'^2+m'm''+m''^2}.\varphi.\left(\frac{(m+m'+m'').r}{\sqrt{m'^2+m'm''+m''^2}}\right).$$ [Force drawing the body m towards the centre of gravity.]

We shall have, by means of the formula [376], the equation of the curve described by the body m about the same point; consequently those of the

[8859] curves described by the bodies m' and m''; since these three curves are similar to each other, with dimensions respectively proportional to r, r', r'' [8847].

[8860] *In the case of nature* $\varphi(s)=\frac{1}{s^2}$; *the force which draws* m *towards the centre of gravity of the system, will therefore be,*

[8861] $$\frac{(m'^2+m'm''+m''^2)^{\frac{3}{2}}}{(m+m'+m'')^2.r^2}.$$ [Force drawing the body m towards the centre of gravity, in the law of nature.]

[8862] *Therefore the three bodies will describe similar conic sections about the centre of gravity of the system* [8857d]; *forming always between each other an*

[8857a] * (4040) Multiplying [8842] by r, and substituting $\frac{(m+m'+m'').r}{s}=\sqrt{m'^2+m'm''+m''^2}$

[8857b] [8855], it becomes $Kr=\sqrt{m'^2+m'm''+m''^2}.\varphi(s)$, which is reduced to the form [8858], by the substitution of the value of s [8855]. If we now suppose, as in [8860], that

[8857c] $\varphi(s)=\frac{1}{s^2}$, the expression of the force acting on m [8858] becomes as in [8861], being inversely as the square of the distance r; and as this is the case of nature [380, &c.], the

[8857d] described curves will be conic sections [534', &c.], as is stated in [8862].

equilateral triangle, whose sides vary incessantly; and they may even become infinite, if the section be a parabola, or a hyperbola.

Case where the distances of the three bodies are unequal.

We shall now consider the case where the three quantities s, s', s'', are not all equal to each other, supposing, for an example, that s, s', are unequal; we shall also resume the equation [8840]; [8863]

$$x.\left\{m'.\frac{\varphi(s)}{s}+(m+m'').\frac{\varphi(s')}{s'}\right\}+m'x'.\left\{\frac{\varphi(s')}{s'}-\frac{\varphi(s)}{s}\right\}=Kx. \qquad [8864]$$

We shall have, between y and y', an equation similar to that in [8864] between x and x'; from these two equations we deduce,*

$$x:x'::y:y'; \qquad [8865]$$

therefore the bodies m, m', must be on a right line with the centre of gravity of the system, *so that the three bodies m, m', m'', must be situated on the same right line.* We shall now take, at any moment whatever, this right line for the axis of the abscisses, and we shall suppose the bodies to be arranged in the order m, m', m'', so that their common centre of gravity may fall between m and m'. We shall then put, [8866]

$$x'=-\mu x; \qquad x''=-Vx; \qquad [8867]$$

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* (4041) The equation [8864] may be put under the form [8864a]; and the similar one in y, under the form [8864b];

$$x.\left\{m'.\frac{\varphi(s)}{s}+(m+m'').\frac{\varphi(s')}{s'}-K\right\}=x'.m'.\left\{\frac{\varphi(s)}{s}-\frac{\varphi(s')}{s'}\right\}; \qquad [8864a]$$

$$y.\left\{m'.\frac{\varphi(s)}{s}+(m+m'').\frac{\varphi(s')}{s'}-K\right\}=y'.m'.\left\{\frac{\varphi(s)}{s}-\frac{\varphi(s')}{s'}\right\}. \qquad [8864b]$$

Dividing the second of these equations by the first, we get $\frac{y}{x}=\frac{y'}{x'}$; whence we easily deduce [8865]. Now $\frac{y}{x}$ evidently represents the tangent of the angle formed by the radius r and the axis of x; and in like manner $\frac{y'}{x'}$ represents the angle formed by the radius r' and the same axis of x' or x; therefore these tangents and the corresponding angles must be equal, so that the radii r, r', must fall on the same right line, passing through their origin of the centre of gravity of the system. Hence it must necessarily follow that the three bodies are situated in that right line, as in [8866]. We may remark that in [8880] the author supposes m to be the sun, m' the earth, and m'' the moon; so that if we take the centre of gravity of the system for the origin, and count the distance x of the sun from the origin as *positive*, we must take those of the earth x' and moon x'' *negative*, as in [8867]. [8864c] [8864d] [8864e]

[8868] and we shall also suppose that the law of attraction is as the power n of the distance, so that $\varphi(s)=s^n$. Then the equations [8836—8836″] will

[8868′] give, by observing that $s=x.(1+\mu)$, $s'=x.(1+V)$,*

[8869] $$K=x^{n-1}.\{m'.(1+\mu)^n+m''.(1+V)^n\};$$

[8870] $$\mu.\{m'.(1+\mu)^n+m''.(1+V)^n\}=m.(1+\mu)^n-m''.(V-\mu)^n.$$

We shall now put,

[8871] $$V-\mu=(1+\mu).z;$$

and from this we get,†

[8872] $$1+V=(1+\mu).(1+z);$$

consequently,

[8873] $$\mu.\{m'+m''.(1+z)^n\}=m-m''z^n;$$

but the equation [8838],

[8874] $$0=mx+m'x'+m''x'',$$

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* (4042) In the hypothesis that the bodies are situated in a right line, as in [8866], we have,

[8868a] $$s=x-x'; \qquad s'=x-x''; \qquad s''=x'-x''.$$

Substituting these values in [8836, 8836′, 8836″], we obtain,

[8868b] $$m'.\varphi(s)+m''.\varphi(s')=Kx; \qquad -m.\varphi(s)+m''.\varphi(s'')=Kx'; \qquad -m.\varphi(s')-m'.\varphi(s'')=Kx''.$$

If we now substitute the values of x', x'' [8867], in the expressions of s, s', s'' [8868a], we shall get, as in [8868′],

[8868c] $$s=x.(1+\mu); \qquad s'=x.(1+V); \qquad s''=x.(V-\mu).$$

[8868d] Substituting these in $\varphi(s)=s^n$, $\varphi(s')=s'^n$, $\varphi(s'')=s''^n$ [8868], we get,

[8868e] $$\varphi(s)=x^n.(1+\mu)^n; \qquad \varphi(s')=x^n.(1+V)^n; \qquad \varphi(s'')=x^n.(V-\mu)^n.$$

Now substituting the values of $\varphi(s)$, $\varphi(s')$ [8868e], in the first of the equations [8868b], and then dividing by x, we get the value of K [8869]; multiplying this by $x'=-\mu x$, [8867], we get the value of Kx', under the form,

[8868f] $$Kx'=-\mu x^n.\{m'.(1+\mu)^n+m''.(1+V)^n\}.$$

Substituting the values of Kx', $\varphi(s)$, $\varphi(s'')$ [8868f, e], in the second of the equations [8868b], and then dividing by $-x^n$, we get [8870].

† (4043) Adding $1+\mu$ to both members of [8871], we get [8872]. Substituting this

[8873a] in the first member of [8870], and $V-\mu$ [8871] in its second member, then dividing by

[8873b] $(1+\mu)^n$, we get [8873]. Substituting the values of x', x'' [8867] in [8874], and then dividing by x, we get [8875]; from this we deduce $V=\frac{m-m'.\mu}{m''}$; and by substitution

[8873c] in [8871], we obtain $\frac{m-m'.\mu}{m''}-\mu=(1+\mu).z$; which gives μ [8876].

gives,

$$0 = m - m'\mu - m''V. \qquad [8875]$$

Hence we deduce,

$$\mu = \frac{m - m''z}{m' + m''.(1+z)}; \qquad [8876]$$

therefore we shall have,*

$$(m - m''z).\{m' + m''.(1+z)^n\} = \{m' + m''.(1+z)\}.\{m - m''.z^n\}. \qquad [8877]$$

In the case of nature in which $n = -2$, this equation becomes,

$$0 = mz^2.\{(1+z)^3 - 1\} - m'.(1+z)^2.(1-z^3) - m''.\{(1+z)^3 - z^3\}; \qquad [8878]$$

an equation of the fifth degree, which must therefore have one real root. Now the supposition of $z = 0$, renders the second member negative, and on the contrary it becomes positive if $z = \infty$ [8877e]; therefore z must necessarily have a real positive value. [8879]

If we suppose m to be the sun, m' the earth, and m'' the moon, we shall have very nearly,† [8880]

$$z = \sqrt[3]{\frac{m' + m''}{3m}}; \qquad [8881]$$

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* (4044) Substituting μ [8876] in [8873], and multiplying by $m' + m''.(1+z)$, we get [8877]. In the case of nature, where $n = -2$, the equation [8877] becomes, by multiplying by $(1+z)^2.z^2$, [8877a]

$$(m - m''z).z^2.\{m'.(1+z)^2 + m''\} = \{m'.(1+z)^2 + m''.(1+z)^3\}.\{mz^2 - m''\}. \qquad [8877b]$$

Transposing all the terms to the second member, developing, neglecting the terms $\pm mm'.z^2.(1+z)^2$, which mutually destroy each other, dividing by m'', and putting in separate parts the terms multiplied by m, m', m'', it becomes as in [8878]. If we arrange the terms of this equation according to the powers of z, it becomes, [8877c]

$$0 = z^5.(m+m') + z^4.(3m+2m') + z^3.(3m+m') - z^2.(3m''+m') - z.(3m''+2m') - m' - m''. \qquad [8877d]$$

The second member of this expression is evidently negative when $z = 0$, and positive when $z = \infty$, as is observed in [8879]. [8877e]

† (4045) We have, in [4061], $m = 1$, $m' = \frac{1}{329630}$, also $m'' = \frac{m'}{68,5} = \frac{1}{329630 \times 68,5}$, [8881a]

[4631]. After substituting these values in [8877d], we shall see, by inspection, that this equation can be satisfied by supposing z to have a very small value. In this case we may neglect z^5, z^4, in comparison with z^3; and if we also neglect m' in comparison with m, m'' in comparison with m', we shall find that this equation will be very nearly represented by $z^3.3m - z^2.m' - z.2m' - m' - m'' = 0$. The second and third terms of this equation being very small, on account of the smallness of the coefficient m', we shall have very [8881b] [8881c]

[8882] which gives $z=\frac{1}{100}$ nearly. Hence it follows that if, at the origin of the world, the earth and moon were situated on a right line passing through the
[8882′] sun, at distances from it proportional to 1 and $1+\frac{1}{100}$, and at the same time were impressed with velocities proportional to their distances from the sun, and in parallel directions, the moon in her course would always continue in opposition to the sun; consequently one of these luminaries would be rising above the horizon, at the same moment the other was setting.
[8883] Moreover, at this distance of the moon from the earth, she could not be
[8883′] eclipsed;* so that during the night we should always have the benefit of the light of a full moon.

[8881d] nearly, by neglecting them, $z^3.3m-m'-m''=0$; whence we get the expression of z, [8881]; and by substituting in it the values of m, m', m'' [8881a], we get very nearly
[8881e] $z=\frac{1}{100}$; so that the neglected terms are only of the order $\frac{1}{100}$, in comparison with those which are retained. The expressions of s, s' [8868c], give $s:s'::1+\mu:1+V$;
[8881f] and by means of the value of $1+V$ [8872], it becomes $s:s'::1:1+z$; or by using the value of z [8881e], $s:s'::1:1+\frac{1}{100}$, as in [8882′].

* (4046) The moon's real distance from the earth is $\frac{1}{400}$ part of that of the sun,
[8882a] [5586], being only one fourth part of the distance $\frac{1}{100}$ computed in [8882′ or 8882]; and if the moon were placed at this last distance from the earth, her horizontal parallax would
[8882b] be reduced to one quarter part of its present value, or $\frac{1}{4}\times 3442^s,4=860^s,6=14^m\,20^s,6$, nearly [5605 line 1]; and as this is less than the sun's apparent semi-diameter, when
[8882c] viewed from the moon, which is nearly 16^m, it will follow that the moon cannot be
[8882d] eclipsed at the distance $\frac{1}{100}$, corresponding to the value of z, and to the relative situations of the bodies which are assumed in [8882, &c.].

Some calculations of a similar kind to those which have been treated of in this chapter,
[8882e] concerning the motion of a body which is drawn towards two *fixed* centres, have been given by Euler, in the Berlin Memoirs for 1760; also by La Grange, in his *Mécanique*
[8882f] *Analytique;* but much more fully by Legendre, in his *Exercices de Calcul Intégral*, Paris, 1817; and in his *Traité des Fonctions Elliptiques*, Paris, 1825. Several of these results are very curious, and they afford a very good specimen of the uses of the calculation of
[8882g] elliptical functions; but as the hypothesis of *fixed* points of attraction is not the case of nature, where every thing appears to be in motion, we have not thought it to be necessary to enter into any discussion of these methods; but refer the reader particularly to either
[8882h] of the above works of Legendre, for their full development; as also for some cases of attraction of two bodies, where the forces differ from that of the law of nature.

CHAPTER VII.

ON THE ALTERATIONS WHICH THE MOTIONS OF THE PLANETS AND COMETS MAY SUFFER BY THE RESISTANCE OF THE MEDIUMS THEY PASS THROUGH, OR BY THE SUCCESSIVE TRANSMISSION OF GRAVITY.

18. We *shall suppose that the sun is surrounded by a fluid, and that its resistance upon the motions of the planets and comets is required to be ascertained.* We have already paid some attention to the subject, in Chap. VI, Book VII; but we shall here examine it more attentively, and shall determine the variations in the orbits for any time whatever. We shall put, [8884]

Symbols.

$\varphi\left(\frac{1}{r}\right)$ for the density of the fluid at the distance r from the sun's centre; [8885]

ds for the element of the curve described by the planet in the time dt; [8885′]

$K.\varphi\left(\frac{1}{r}\right)\cdot\frac{ds^2}{dt^2}$ for the expression of the resistance which the planet suffers, in the direction of its motion;* [8886]

K is a constant coefficient depending on the figure of the planet and on its density; [8886′]

x, y, are the rectangular co-ordinates of the planet, taken in the plane of its orbit; their origin being the sun's centre. [8887]

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* (4017) The resistance is supposed to be proportional to the product of the density $\varphi\left(\frac{1}{r}\right)$, by the square of the velocity $\frac{ds^2}{dt^2}$, and may therefore be represented by [8886a] $K.\varphi\left(\frac{1}{r}\right)\cdot\frac{ds^2}{dt^2}$, as in [8886], in the direction of its motion. This can evidently be resolved [8886b] in directions parallel to the axes x, y, by multiplying it by $\frac{dx}{ds}$, $\frac{dy}{ds}$, respectively, as in [8888]. Putting these quantities negative, supposing them to decrease x, y, and then making them equal to the expressions [8889], we get the values [8890, 8891]. In a [8886c] similar manner we may find the values of $\left(\frac{dR}{dr}\right)$, $\left(\frac{dR}{dv}\right)$, corresponding to the polar [8886d]

[8887′] Then the resistance, resolved in directions parallel to the co-ordinates x, y, *and tending to decrease them*, are respectively,

[8888] $$K.\varphi\left(\frac{1}{r}\right).\frac{ds.dx}{dt^2}; \qquad K.\varphi\left(\frac{1}{r}\right).\frac{ds.dy}{dt^2};$$

[8889] Now we have represented in [1172′, &c.], by $-\left(\frac{dR}{dx}\right)$ and $-\left(\frac{dR}{dy}\right)$, *the*
[8889′] *forces which act upon the planet in the directions* x, y, *and tending to increase them;* we shall therefore have,

[8890] $$\left(\frac{dR}{dx}\right)=K.\varphi\left(\frac{1}{r}\right).\frac{ds.dx}{dt^2}; \qquad \text{[Resistance tending to decrease } x.]$$

[8891] $$\left(\frac{dR}{dy}\right)=K.\varphi\left(\frac{1}{r}\right).\frac{ds.dy}{dt^2}; \qquad \text{[Resistance tending to decrease } y.]$$

[8892] If we put *the sum of the masses of the sun and planet equal to unity,** *or* $\mu=1$ [914′], we shall have, from [1177, &c.],

[8893] $$d.\frac{1}{a}=2.\mathrm{d}R=2.\left(\frac{dR}{dx}\right).dx+2.\left(\frac{dR}{dy}\right).dy;$$

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co-ordinates r, v; which are used in the formulas [8010i, &c.]. In this case we see that
[8886e] the increments dx, dy, which form the sides of the right angle whose hypotenuse is ds, must be changed into two other rectangular increments dr and rdv; dr being in the direction of the radius, and rdv perpendicular to the radius; so that we shall have, as in
[8886f] [8903], $ds=\sqrt{dr^2+r^2dv^2}$; and then to obtain the partial differentials of R relative to dr or rdv, we have only to change dx into dr in [8890], and dy into rdv in [8891]; then multiplying this last equation by r, we finally obtain the following expressions, which will be used hereafter;

[8886g] $$\left(\frac{dR}{dr}\right)=K.\varphi\left(\frac{1}{r}\right).\frac{ds.dr}{dt^2}; \qquad \left(\frac{dR}{dv}\right)=K.\varphi\left(\frac{1}{r}\right).\frac{ds.r^2dv}{dt^2}.$$

The resistance of the ether *acts in the plane of the orbit;* therefore it cannot alter the
[8886h] position of this plane, as the author has observed in [8900′], so that the variations $d\gamma$, $d\theta$ [8010l, m] will vanish; and the second members of these equations will give,

[8886i] $$\left(\frac{dR}{d\theta}\right)=0; \qquad \left(\frac{dR}{d\gamma}\right)=0;$$

[8886k] neglecting the very small terms of a higher order, depending on the product of K, by the masses of the disturbing planets.

* (4048) Substituting in [1177] the value $\mu=1$ [8892], we get, as in [8893],
[8893a] $d.\frac{1}{a}=2.\mathrm{d}R$; now using the development of $\mathrm{d}R$ [916a], and neglecting the term
[8893b] multiplied by dz, because z and its differential vanish by taking the orbit of the plane for the plane of xy [8887], we obtain the last expression in [8893].

therefore we have,*

$$d.\frac{1}{a}=2\mathrm{d}R=2K.\varphi\left(\frac{1}{r}\right).\frac{ds^3}{dt^2}. \qquad [8894]$$

Now e being the ratio of the excentricity to the semi-major axis a, and ϖ the longitude of the perihelion, we shall have, as in [1176, &c.],† [8895]

$$d.(e.\sin.\varpi)=dx.\left\{x.\left(\frac{dR}{dy}\right)-y.\left(\frac{dR}{dx}\right)\right\}+(xdy-ydx).\left(\frac{dR}{dx}\right); \qquad [8896]$$

$$d.(e.\cos.\varpi)=dy.\left\{y.\left(\frac{dR}{dx}\right)-x.\left(\frac{dR}{dy}\right)\right\}-(xdy-ydx).\left(\frac{dR}{dy}\right); \qquad [8897]$$

therefore,

$$d.(e.\sin.\varpi)=\quad 2K.\varphi\left(\frac{1}{r}\right).\frac{ds.dx}{dt^2}.(xdy-ydx); \qquad [8898]$$

$$d.(e.\cos.\varpi)=-2K.\varphi\left(\frac{1}{r}\right).\frac{ds.dy}{dt^2}.(xdy-ydx). \qquad [8899]$$

Lastly we have, as in [1181],‡

$$dn=3an.\mathrm{d}R=3K.an.\varphi\left(\frac{1}{r}\right).\frac{ds^3}{dt^2}. \qquad [8900]$$

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* (4049) Substituting the values [8890, 8891], in the last expression of [8893], and then reducing by putting $dx^2+dy^2=ds^2$, we get, as in [8894], [8894a]

$$d.\frac{1}{a}=2K.\varphi\left(\frac{1}{r}\right).\frac{ds}{dt^2}.(dx^2+dy^2)=2K.\varphi\left(\frac{1}{r}\right).\frac{ds^3}{dt^2}. \qquad [8894b]$$

† (4050) Substituting the differentials of h, l [7847], in the first members of [7849, 7850], we get [8896, 8897] respectively. Now the values [8890, 8891] give, [8896a]

$$x.\left(\frac{dR}{dy}\right)-y.\left(\frac{dR}{dx}\right)=K.\varphi\left(\frac{1}{r}\right).\frac{ds}{dt^2}.(xdy-ydx); \qquad [8896b]$$

substituting this, together with the values [8890, 8891], in [8896, 8897], they become respectively as in [8898, 8899].

‡ (4051) Putting $\mu=1$ [8893a] in [1181], we get $dn=3an.\mathrm{d}R$, as in [8900]. [8899a]
Now from [8894] we have $\mathrm{d}R=\frac{1}{2}d.\frac{1}{a}=K.\varphi\left(\frac{1}{r}\right).\frac{ds^3}{dt^2}$; hence $dn=3an.K.\varphi\left(\frac{1}{r}\right).\frac{ds^3}{dt^2}$, [8899b]
as in [8900]. Substituting $\mu=1$ in $U^2=\mu$ [711′], we get $U=1$ for the velocity of a body revolving about the sun in a circular orbit, at the mean distance of the earth from the sun, taken as the unit of distance [711′]. [8899c] Substituting this in [712], we get [8899d]
$V=\sqrt{\frac{2}{r}-\frac{1}{a}}$ for the velocity V of a body revolving about the sun in an elliptical orbit, whose semi-major axis is a, and the radius vector r. These will be used hereafter. [8899e]

With the expressions [8894, 8898, 8899] *we can obtain the variations of the elements of the orbit, depending on the resistance of the medium; since this*
[8900′] *resistance does not alter the position of the plane of the orbit* [8886*h*, *i*].

We have, by means of [372, 1147, 378],*

[8901] $$xdy-ydx=r^2dv=dt.\sqrt{a.(1-e^2)};$$

[8902] $$r=\frac{a.(1-e^2)}{1+e.\cos.(v-\varpi)}.$$

Moreover we have, as in [8902*b*],

[8903] $$ds=\sqrt{dr^2+r^2dv^2}=r^2dv.\sqrt{\left(\frac{dr}{r^2dv}\right)^2+\frac{1}{r^2}};$$

hence we deduce,†

[8904] $$ds=\frac{r^2dv.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{1}{2}}}{a.(1-e^2)};$$

[8905] $$\frac{ds^3}{dt^2}=\frac{r^2dv.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{3}{2}}}{a^2.(1-e^2)^2};$$

therefore,

[8906] $$d.\frac{1}{a}=\frac{2K.\varphi\left(\frac{1}{r}\right).r^2dv.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{3}{2}}}{a^2.(1-e^2)^2}.$$

[8902*a*] * (4052) Comparing [372, 1057], and putting $\mu=1$ [8892], we get [8901]. The value of r [8902] is the same as that in [378]. We have in [8894*a*] $ds=\sqrt{dx^2+dy^2}$;
[8902*b*] and by substituting its value, deduced from the first equation [372], we get the first expression in [8903]; which is easily reduced to the second form in [8903].

† (4053) The expression of r [8902] gives, as in [1256],

[8904*a*] $$\frac{dr}{r^2dv}=\frac{e.\sin.(v-\varpi)}{a.(1-e^2)}; \qquad \frac{1}{r}=\frac{1+e.\cos.(v-\varpi)}{a.(1-e^2)}.$$

Substituting these in the last value of ds [8903], we get [8904]; observing that the numerator of this expression may be reduced, by putting,

[8904*b*] $$e^2.\sin.^2(v-\varpi)+\{1+e.\cos.(v-\varpi)\}^2=1+2e.\cos.(v-\varpi)+e^2.$$

[8904*c*] The cube of ds [8904], being divided by the square of $dt=\frac{r^2dv}{\sqrt{a.(1-e^2)}}$ [8901], gives [8905]; and by substituting this last value in [8894], we get [8906].

We shall suppose that the function,* [8906']

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* (4054) When the form of the function $\varphi\left(\frac{1}{r}\right)$ is known, we may make the developments of the first member of [8908], by processes similar to those which are employed in [964, &c.]; and by this means we can ascertain the values of A, B, C, &c. We may also use the method of definite integrals, in the following manner. Putting for brevity V equal to the first member of [8908], also $v-\varpi=\text{v}$; and, for the sake of symmetry, $A=A_0$; $Be=A_1$; $Ce^2=A_2$, &c.; we obtain the following expression of the function V; [8908a] [8908b] [8908c]

$$\text{V}=K.\varphi\left(\frac{1}{r}\right).r^2.\{1+2e.\cos.\text{v}+e^2\}^{\frac{1}{2}}=A_0+A_1.\cos.\text{v}+A_2.\cos.2\text{v}+A_3.\cos.3\text{v}+\&c. \qquad [8908d]$$

Multiplying this by $d\text{v}$ and integrating, we get,

$$\int\text{V}\,d\text{v}=A_0.\text{v}+A_1.\sin.\text{v}+\tfrac{1}{2}A_2.\sin.2\text{v}+\&c. \qquad [8908e]$$

This vanishes when $\text{v}=0$, and when $\text{v}=\pi$ it becomes $\int_0^\pi\text{V}\,d\text{v}=A_0.\pi$; dividing this by π, we get A_0 [8908k]. Multiplying [8908d] by $d\text{v}.\cos.\text{v}$, and reducing the second member by means of [20] Int., we obtain by integration, [8908f]

$$\int\text{V}.\cos.\text{v}.d\text{v}=\int\{A_0.\cos.\text{v}+\tfrac{1}{2}A_1.(1+\cos.2\text{v})+\tfrac{1}{2}A_2.(\cos.\text{v}+\cos.3\text{v})+\&c.\}.d\text{v} \qquad [8908g]$$

$$=A_0.\sin.\text{v}+\tfrac{1}{2}A_1.(\text{v}+\tfrac{1}{2}.\sin.2\text{v})+\tfrac{1}{2}A_2.(\sin.\text{v}+\tfrac{1}{3}.\sin.3\text{v})+\&c. \qquad [8908h]$$

The second member vanishes when $\text{v}=0$, and when $\text{v}=\pi$ it becomes $\frac{1}{2}A_1.\pi$; hence we have $\int_0^\pi\text{V}.\cos.\text{v}.d\text{v}=\frac{1}{2}A_1.\pi$; which gives A_1, as in [8908l]. In like manner, multiplying [8908d] by $\cos.n\text{v}$, n being a whole number, we get by integration the expression of $\frac{1}{2}A_n.\pi$; whence we deduce A_n [8908m]. [8908i]

$$A_0=\frac{1}{\pi}.\int_0^\pi\text{V}\,d\text{v}; \qquad [8908k]$$

$$A_1=\frac{2}{\pi}.\int_0^\pi\text{V}.\cos.\text{v}.d\text{v}; \qquad [8908l]$$

$$\vdots \qquad\qquad \vdots$$

$$A_n=\frac{2}{\pi}.\int_0^\pi\text{V}.\cos.n\text{v}.d\text{v}. \qquad [8908m]$$

If the integrals of the second members of these equations can be obtained *in finite terms, or with the assistance of circular arcs, logarithms, elliptical functions, or converging series*, we shall have the values of A_0, A_1, A_2, &c. If neither of these methods can be advantageously employed, we must ascertain the integrals by means of quadratures, as in [7929x, &c.]; supposing di to be changed into $d\text{v}$, and $y^{(i)}$ into $\text{V}.\cos.n\text{v}$, in finding the general value of A_n [8908m]. [8908n] [8908o]

As an example of this method, we shall put $\varphi\left(\frac{1}{r}\right)=\frac{1}{r^2}$; this being the hypothesis used by Encke [5667b], in calculating the perturbations of the comet which bears his name, in vol. 9, page 333, of the *Astronomische Nachrichten*. We shall investigate the formulas for the determination of the values of A_0, A_1, or A, B; these being the [8908p] [8908q]

[8907] $$K.\varphi\left(\frac{1}{r}\right).r^2.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{1}{2}},$$

[8908r] only coefficients of the series [8908d] which are required, when we restrict ourselves to the computation of the secular inequalities, as in [8909′, 8915, &c.]; and these secular inequalities are all which deserve attention, in the present imperfect state of our knowledge
[8908s] relative to the nature and density of the resisting medium; the periodical inequalities depending on this cause being comparatively unimportant. Now substituting
[8908t] $\varphi(r)=\frac{1}{r^2}$ [8908p] in [8908d], we get the following general expression of V;

[8908u] $$\mathrm{V}=K.\{1+2e.\cos.\mathrm{v}+e^2\}^{\frac{1}{2}}=A_0+A_1.\cos.\mathrm{v}+A_2.\cos.2\mathrm{v}+\&c.$$

[8908v] This last development may be obtained by the formulas [975″, 976, &c.], putting $\theta=\mathrm{v}$, $\alpha=-e$, $s=-\frac{1}{2}$; hence we get,

[8908w] $$\lambda=1+2e.\cos.\mathrm{v}+e^2; \quad [975'']$$

[8908x] $$\lambda^{\frac{1}{2}}=\{1+2e.\cos.\mathrm{v}+e^2\}^{\frac{1}{2}}=\tfrac{1}{2}b_{-\frac{1}{2}}^{(0)}+b_{-\frac{1}{2}}^{(1)}.\cos.\mathrm{v}+b_{-\frac{1}{2}}^{(2)}.\cos.2\mathrm{v}+\&c.; \quad [976]$$

[8908y] $$\mathrm{V}=K.\{1+2e.\cos.\mathrm{v}+e^2\}^{\frac{1}{2}}=K.\tfrac{1}{2}b_{-\frac{1}{2}}^{(0)}+K.b_{-\frac{1}{2}}^{(1)}.\cos.\mathrm{v}+\&c. \quad [8908u, x]$$

Comparing together the two last of the developments of V in [8908u, y], and resuming the values of A_0, B_1 [8908c], we easily deduce,

[8908z] $$A=A_0=K.\tfrac{1}{2}b_{-\frac{1}{2}}^{(0)}; \qquad Be=A_1=K.b_{-\frac{1}{2}}^{(1)}.$$

The values of $\frac{1}{2}b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$, are given in [989]; and by putting $\alpha=-e$, as in [8908v], we obtain,

[8909a] $$A=A_0=K.\tfrac{1}{2}b_{-\frac{1}{2}}^{(0)}=K.\left\{1+(\tfrac{1}{2})^2.e^2+\left(\frac{1.1}{2.4}\right)^2.e^4+\left(\frac{1.1.3}{2.4.6}\right)^2.e^6+\left(\frac{1.1.3.5}{2.4.6.8}\right)^2.e^8+\&c.\right\};$$

[8909b] $$B=A_1e^{-1}=Ke^{-1}.b_{-\frac{1}{2}}^{(1)}=K.\left\{1-\frac{1.1}{2.4}.e^2-\frac{1}{4}.\frac{1.1.3}{2.4.6}.e^4-\frac{1.3}{4.6}.\frac{1.1.3.5}{2.4.6.8}.e^6-\&c.\right\}.$$

If we neglect the square of e, as in [8921], these expressions will become $A=K$,
[8909c] $B=K$; being the same as those which are deduced from [8922, 8923], by putting $\varphi\left(\frac{1}{a}\right)=\frac{1}{a^2}$ [8908p], and $\varphi'\left(\frac{1}{a}\right)=\frac{2}{a}$ [8924].

[8909d] When e is large, the values of A, B, can be more easily obtained by the method of elliptical functions, with the tables published by Legendre, which give, by mere inspection, the values of A, B; and as this example shows the importance of these functions, we shall compute the formulas which are necessary in this calculation. For this purpose we shall
[8909e] put $c=\frac{2\sqrt{e}}{1+e}$, and shall use the symbols given in [82a, b], being the same as the values
[8909f] of $\Delta(c,\varphi)$, $\mathrm{F}(c,\varphi)$, $\mathrm{E}(c,\varphi)$, $\mathrm{F}^1(c)$, $\mathrm{E}^1(c)$ [8910i—h]. Substituting the first value of V [8908u] in [8908k], we get the first form of A_0 [8909h]; its second form is
[8909g] obtained by putting $\mathrm{v}=2\varphi$; its third by making $\cos.2\varphi=1-2.\sin.^2\varphi$; then substituting c, E^1 [8909e, f], we successively obtain,

is developed in a series, arranged according to the cosines of $(v-\varpi)$ and [8907′]

$$\mathcal{A}_0=\frac{K}{\pi}\cdot\int_0^{\pi}\{1+2e.\cos.\mathrm{v}+e^2\}^{\frac{1}{2}}.d\mathrm{v}=\frac{K}{\pi}\cdot\int_0^{\frac{1}{2}\pi}\{1+2e.\cos.2\varphi+e^2\}^{\frac{1}{2}}.2d\varphi \qquad [8909h]$$

$$=\frac{2K}{\pi}\cdot\int_0^{\frac{1}{2}\pi}\{(1+e)^2-4e.\sin.^2\varphi\}^{\frac{1}{2}}.d\varphi=\frac{2K}{\pi}.(1+e).\int_0^{\frac{1}{2}\pi}\left\{1-\frac{4e}{(1+e)^2}.\sin.^2\varphi\right\}^{\frac{1}{2}}.d\varphi \qquad [8909i]$$

$$=\frac{2K}{\pi}.(1+e).\int_0^{\frac{1}{2}\pi}\sqrt{1-c^2.\sin.^2\varphi}.\,d\varphi=\frac{2K}{\pi}.(1+e).\mathrm{E}^1(c)=\mathcal{A}. \quad [8908c]. \qquad [8909k]$$

Putting for a moment $X=1+2e.\cos.\mathrm{v}+e^2$, we get $\cos.\mathrm{v}=\frac{X-1-e^2}{2e}$, and [8909l]
$\mathrm{V}=KX^{\frac{1}{2}}$ [8908u]. Then from [8909h, k] we have,

$$\int_0^{\pi}\{1+2e.\cos.\mathrm{v}+e^2\}^{\frac{1}{2}}.\,d\mathrm{v}=\int_0^{\pi}X^{\frac{1}{2}}.d\mathrm{v}=2.(1+e).\mathrm{E}^1(c)\,; \qquad [8909l']$$

and [8908l] gives, by successive reductions,

$$\mathcal{A}_1=\frac{K}{\pi e}\cdot\int_0^{\pi}(X-1-e^2).X^{\frac{1}{2}}.d\mathrm{v}=\frac{K}{\pi e}\cdot\int_0^{\pi}X^{\frac{3}{2}}.d\mathrm{v}-\frac{K}{\pi e}.(1+e^2).\int_0^{\pi}X^{\frac{1}{2}}.d\mathrm{v} \qquad [8909m]$$

$$=\frac{K}{\pi e}\cdot\int_0^{\pi}X^{\frac{3}{2}}.d\mathrm{v}-\frac{2K}{\pi e}.(1+e).(1+e^2).\mathrm{E}^1(c). \qquad [8909n]$$

The term in [8909n], under the sign $\int$, may be simplified by substituting, as in [8909g], $\mathrm{v}=2\varphi$; whence we have, [8909o]

$$X^{\frac{1}{2}}=\{1+2e.\cos.2\varphi+e^2\}^{\frac{1}{2}}=\{(1+e)^2-4e.\sin.^2\varphi\}^{\frac{1}{2}}=(1+e).\sqrt{1-c^2.\sin.^2\varphi}=(1+e).\Delta\,; \qquad [8909p]$$

writing for brevity Δ, *instead of* $\Delta(c,\varphi)=\sqrt{1-c^2.\sin.^2\varphi}$ [8910i]. Substituting this, [8909p′]
and $d\mathrm{v}=2d\varphi$ [8909o], in [8909n], we get,

$$\mathcal{A}_1=\frac{2K}{\pi e}.(1+e)^3.\int_0^{\frac{1}{2}\pi}\Delta^3.d\varphi-\frac{2K}{\pi e}.(1+e).(1+e^2).\mathrm{E}^1(c). \qquad [8909q]$$

Now it is easy to prove, by differentiation, that we have generally, for any value of φ,

$$\int\Delta^3.d\varphi=\tfrac{1}{3}c^2.\sin.\varphi.\cos.\varphi.\Delta+\tfrac{4}{3}.(1-\tfrac{1}{2}c^2).\mathrm{E}(c,\varphi)-\tfrac{1}{3}.(1-c^2).\mathrm{F}(c,\varphi). \qquad [8909r]$$

For by taking its differential, dividing by $d\varphi$, and using the values $\mathrm{E}(c,\varphi)$, $\mathrm{F}(c,\varphi)$, [8909f], then writing the terms down in the order in which they occur, without any reduction, we obtain,

$$\Delta^3=\tfrac{1}{3}c^2.(\cos.^2\varphi-\sin.^2\varphi).\Delta-\tfrac{1}{3}c^4.\sin.^2\varphi.\cos.^2\varphi.\Delta^{-1}+\tfrac{4}{3}.(1-\tfrac{1}{2}c^2).\Delta-\tfrac{1}{3}.(1-c^2).\Delta^{-1}. \qquad [8909s]$$

Multiplying this by 3Δ, and substituting $\cos.^2\varphi=1-\sin.^2\varphi$, it becomes as in [8909t]. Connecting together the terms depending on Δ^2, we get [8909u]; substituting $\Delta^2=1-c^2.\sin.^2\varphi$, we obtain [8909v]; and by successive reductions, it changes into $3\Delta^4$, [8909w], as in the first member of [8909t]; therefore the differentials of both members of [8909r], are equal to each other;

$$3\Delta^4=c^2.(1-2.\sin.^2\varphi).\Delta^2+c^4.(-\sin.^2\varphi+\sin.^4\varphi)+(4-2c^2).\Delta^2-(1-c^2) \qquad [8909t]$$

$$=(4-c^2-2c^2.\sin.^2\varphi).\Delta^2+(c^2-1-c^4.\sin.^2\varphi+c^4.\sin.^4\varphi) \qquad [8909u]$$

$$=(4-c^2-2c^2.\sin.^2\varphi)+c^2.\sin.^2\varphi.(-4+c^2+2c^2.\sin.^2\varphi)+(c^2-1-c^4.\sin.^2\varphi+c^4.\sin.^4\varphi) \qquad [8909v]$$

$$=3-6c^2.\sin.^2\varphi+3c^4.\sin.^4\varphi=3.(1-c^2.\sin.^2\varphi)^2=3\Delta^4\,; \qquad [8909w]$$

[8907''] its multiples, in the following form ;

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observing that the expressions $\pm c^2$, $\pm c^4.\sin.^2\varphi$, in [8909v], mutually destroy each
[8909x] other. The first term of the second member of the integral [8909r], vanishes at the two limits $\varphi=0$, $\varphi=\frac{1}{2}\pi$, and the general integrals $\mathrm{E}(c,\varphi)$, $\mathrm{F}(c,\varphi)$, become $\mathrm{E}^1(c)$, $\mathrm{F}^1(c)$ [8910m, n], respectively ; therefore we have,

[8909y] $$\int_0^{\frac{1}{2}\pi}\Delta^3.d\varphi=\tfrac{4}{3}.(1-\tfrac{1}{2}c^2).\mathrm{E}^1(c)-\tfrac{1}{3}.(1-c^2).\mathrm{F}^1(c).$$

Substituting this in [8909q], and making a slight reduction, we obtain,

[8909z] $$A_1=\frac{2K}{\pi e}.\{(1+e)^3.\tfrac{4}{3}(1-\tfrac{1}{2}c^2)-(1+e).(1+c^2)\}.\mathrm{E}^1(c)-\frac{2K}{3\pi e}.(1+e)^3.(1-c^2).\mathrm{F}^1(c)\,;$$

[8910a] and since $c^2=\frac{4e}{(1+e)^2}$ [8909c], gives $1-\frac{1}{2}c^2=\frac{(1+e^2)}{(1+e)^2}$, $1-c^2=\frac{(1-e)^2}{(1+e)^2}$, it becomes,

[8910b] $$A_1=\frac{2K}{3\pi e}.(1+e).\{(1+e^2).\mathrm{E}^1(c)-(1-e)^2.\mathrm{F}^1(c)\}\,;$$

substituting this in $B=A_1.e^{-1}$ [8909b], we obtain,

[8910c] $$B=\frac{2K}{3\pi e^2}.(1+e).\{(1+e^2).\mathrm{E}^1(c)-(1-e)^2.\mathrm{F}^1(c)\}.$$

The process we have here used in finding A_0 or A_1 by a series [8909a, b], or by
[8910d] elliptical functions [8909k, 8910b], may be used when $\varphi\left(\frac{1}{r}\right)$ is supposed to be of a different form from that which is assumed in [8908p] ; as, for example, we may suppose,

[8910e] $$\varphi\left(\frac{1}{r}\right)=B_0+\frac{B_1}{r}+\frac{B_2}{r^2}+\frac{B_3}{r^3}+\ldots+\frac{B_m}{r^m}\,;$$

and by re-substituting the value of r [8902], we can reduce the expression of V to the last form in [8908d], and then A_0, A_1, may evidently be obtained by means of elliptical
[8910e'] functions. This calculation has been made by Plana, in a valuable memoir on the resistance of the ethereal medium, published in Zach's *Correspondance Astronomique*, &c., vol. 13, page 341, &c. Instead of finding the values of the differentials de, da, dn, in terms of the differential of the true anomaly dv, as La Place has done in [8909', 8915, &c.]. Plana gives them in terms of the differential of the excentric anomaly
[8910f] du, according to the method which is used by La Place, in computing the perturbations of a comet, in [7872, &c.]. We may remark that the values of A_3, A_4, &c. [8908d], may be derived from A_0, A_1, by a similar method to that which is used in [966, &c.].

The expressions of A, B [8909k, 8910c], contain the elliptical functions $\mathrm{F}^1(c)$, $\mathrm{E}^1(c)$,
[8910g] which require the computation of $c=\frac{2\sqrt{e}}{1+e}$ [8909e], from the given value of e. This may be avoided by reducing these functions to others, depending on $\mathrm{F}^1(e)$, $\mathrm{E}^1(e)$. In
[8910g'] making these reductions we shall use the method and notation of Legendre, putting as in [82a, b, &c.],

$$K.\varphi\left(\frac{1}{r}\right).r^2.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{1}{2}}=A+Be.\cos.(v-\varpi)+Ce^2.\cos.(2v-2\varpi)+\&c. \qquad [8908]$$

$$c'=\frac{2\sqrt{c}}{1+c}; \qquad \sin.(2\varphi'-\varphi)=c.\sin.\varphi; \qquad [8910h]$$

$$\Delta(c,\varphi)=\sqrt{1-c^2.\sin.^2\varphi}; \qquad \Delta(c',\varphi')=\sqrt{1-c'^2.\sin.^2\varphi'}; \qquad [8910i]$$

$$\mathrm{F}(c,\varphi)=\int\frac{d\varphi}{\sqrt{1-c^2.\sin.^2\varphi}}; \qquad \mathrm{F}(c',\varphi')=\int\frac{d\varphi'}{\sqrt{1-c'^2.\sin.^2\varphi'}}; \qquad [8910k]$$

$$\mathrm{E}(c,\varphi)=\int d\varphi.\sqrt{1-c^2.\sin.^2\varphi}; \qquad \mathrm{E}(c',\varphi')=\int d\varphi'.\sqrt{1-c'^2.\sin.^2\varphi'}; \qquad [8910l]$$

$$\mathrm{F}^1(c)=\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{1-c^2.\sin.^2\varphi}}; \qquad \mathrm{F}^1(c')=\int_0^{\frac{1}{2}\pi}\frac{d\varphi'}{\sqrt{1-c'^2.\sin.^2\varphi'}}; \qquad [8910m]$$

$$\mathrm{E}^1(c)=\int_0^{\frac{1}{2}\pi}d\varphi.\sqrt{1-c^2.\sin.^2\varphi}; \qquad \mathrm{E}^1(c')=\int_0^{\frac{1}{2}\pi}d\varphi'.\sqrt{1-c'^2.\sin.^2\varphi'}. \qquad [8910n]$$

Having found c', φ', from c, φ, by means of the assumed equations [8910h], we shall have the following integral formulas;

$$\mathrm{F}(c',\varphi')=\tfrac{1}{2}(1+c).\mathrm{F}(c,\varphi); \qquad [8910o]$$

$$\mathrm{E}(c',\varphi')=\frac{1}{1+c}.\mathrm{E}(c,\varphi)-\tfrac{1}{2}(1-c).\mathrm{F}(c,\varphi)+\frac{c}{1+c}.\sin.\varphi; \qquad [8910p]$$

$$\mathrm{F}^1(c')=(1+c).\mathrm{F}^1(c); \qquad [8910q]$$

$$\mathrm{E}^1(c')=\frac{2}{1+c}.\mathrm{E}^1(c)-(1-c).\mathrm{F}^1(c). \qquad [8910r]$$

For by using the symbol Δ for $\sqrt{1-c^2.\sin.^2\varphi}$, as in [8909p'], we shall have, from the second equation [8910h], and [23] Int., [8910r']

$$\Delta=\sqrt{1-(c.\sin.\varphi)^2}=\sqrt{1-\sin.^2(2\varphi'-\varphi)}=\cos.(2\varphi'-\varphi); \qquad [8910s]$$

$$\cos.2\varphi'=\cos.\{(2\varphi'-\varphi)+\varphi\}=\cos.(2\varphi'-\varphi).\cos.\varphi-\sin.(2\varphi'-\varphi).\sin.\varphi=\cos.\varphi.\Delta-c.\sin.^2\varphi. \qquad [8910t]$$

Substituting this last value of $\cos.2\varphi'$, in $2.\cos.^2\varphi'=1+\cos.2\varphi'$, $2.\sin.^2\varphi'=1-\cos.2\varphi'$, [1, 6] Int., we obtain,

$$2.\cos.^2\varphi'=1-c.\sin.^2\varphi+\cos.\varphi.\Delta; \qquad [8910u]$$

$$2.\sin.^2\varphi'=1+c.\sin.^2\varphi-\cos.\varphi.\Delta. \qquad [8910v]$$

We have, by developing as in [21, 31] Int., and using the values [8910h, s],

$$\sin.2\varphi'=\sin.(2\varphi'-\varphi+\varphi)=\sin.(2\varphi'-\varphi).\cos.\varphi+\cos.(2\varphi'-\varphi).\sin.\varphi; \quad \text{or,}$$

$$2.\sin.\varphi'.\cos.\varphi'=c.\sin.\varphi.\cos.\varphi+\Delta.\sin.\varphi=\sin.\varphi.(c.\cos.\varphi+\Delta). \qquad [8910w]$$

Substituting the value of c' [8910h], and that of $2.\sin.^2\varphi'$ [8910v], in the first member of [8910x], altering the arrangement of the terms and making successive reductions, it becomes as in [8910y];

$$(1+c).\sqrt{1-c'^2.\sin.^2\varphi'}=\sqrt{(1+c)^2-4c.\sin.^2\varphi'}=\sqrt{(1+c)^2-2c.(1+c.\sin.^2\varphi-\cos.\varphi.\Delta)} \qquad [8910x]$$

$$=\sqrt{\{(1-c^2.\sin.^2\varphi)+2c.\cos.\varphi.\Delta+c^2.(1-\sin.^2\varphi)\}}$$
$$=\sqrt{\{\Delta^2+2c.\cos.\varphi.\Delta+c^2.\cos.^2\varphi\}}=\Delta+c.\cos.\varphi. \qquad [8910y]$$

[8908′] A, B, C, being functions of e^2, we shall have, *by neglecting the*

The differential of $\sin.(2\varphi'-\varphi)=c.\sin.\varphi$ [8910*h*], gives, by using [8910*s*],

[8910*z*] $$(2d\varphi'-d\varphi).\Delta=cd\varphi.\cos.\varphi; \qquad \text{hence} \qquad d\varphi'=\frac{d\varphi}{2\Delta}.(\Delta+c.\cos.\varphi).$$

Substituting [8910*y*, *z*] in the expression of $F(c',\varphi')$ [8910*k*], we get,

[8911*a*] $$F(c',\varphi')=\int\frac{d\varphi'}{\sqrt{1-c'^2.\sin.^2\varphi'}}=\tfrac{1}{2}.(1+c).\int\frac{d\varphi}{\Delta}=\tfrac{1}{2}.(1+c).F(c,\varphi).$$

If we substitute the same expressions [8910*y*, *z*] in $E(c',\varphi')$ [8910*l*], and multiply it by $2(1+c)$, using $c^2.\cos.^2\varphi=c^2-c^2.\sin.^2\varphi=c^2-1+\Delta^2$ [8910*r*′], we get,

[8911*b*] $$2.(1+c).E(c',\varphi')=(2+2c).\textstyle\int d\varphi'.\sqrt{1-c'^2.\sin.^2\varphi'}=\displaystyle\int\frac{d\varphi}{\Delta}.(\Delta+c.\cos.\varphi)^2$$

[8911*c*] $$=\int\frac{d\varphi}{\Delta}.\{\Delta^2+2c.\cos.\varphi.\Delta+(c^2-1+\Delta^2)\}$$

[8911*d*] $$=\int\frac{d\varphi}{\Delta}.\{2\Delta^2+2c.\cos.\varphi.\Delta+c^2-1\}$$

$$=2.\textstyle\int d\varphi.\Delta+2c.\int d\varphi.\cos.\varphi-(1-c^2).\displaystyle\int\frac{d\varphi}{\Delta}$$

[8911*e*] $$=2E(c,\varphi)+2c.\sin.\varphi-(1-c^2).F(c,\varphi).$$

Dividing this by $2.(1+c)$, we get,

[8911*f*] $$E(c',\varphi')=\frac{1}{1+c}.E(c,\varphi)-\tfrac{1}{2}.(1-c).F(c,\varphi)+\frac{c}{1+c}.\sin.\varphi.$$

[8911*g*] If we put $\sin.\delta=c.\sin.\varphi$, we shall have generally $\sin.\delta$, independent of its sign, less than c; and as $c<1$, δ must be less than a right angle; and we shall have generally $2\varphi'-\varphi=\delta$, as is evident from [8910*h*]; hence we get $2\varphi'=\varphi+\delta$. Now when
[8911*h*] $\varphi=0$, or $\varphi=180^d$, δ vanishes [8911*g*], and we have $\varphi=2\varphi'$; so that if we take the
[8911*i*] integrals relative to φ', from $\varphi'=0$ to $\varphi'=90^d$, we must take those relative to φ from $\varphi=0$ to $\varphi=180^d$; or what is equivalent, we must take them from $\varphi=0$ to $\varphi=90^d$, and double the results; since it is evident, from a mere inspection of the formulas
[8911*i*′] [8910*k*—*n*], that when $\varphi=180^d$, we have $F(c,\varphi)=2F^1(c)$ and $E(c,\varphi)=2E^1(c)$; observing that the expression of Δ, and the elements of the integrals, are the same for
[8911*k*] $\varphi=90^d-\varphi_1$, or $\varphi=90^d+\varphi_1$; φ_1 being any angle less than 90^d. Putting therefore at first $\varphi=0$, $\varphi'=0$, in [8911*a*, *f*], and then $\varphi=180^d$, $\varphi'=90^d$, we get,

[8911*l*] $$F^1(c')=(1+c).F^1(c);$$

[8911*m*] $$E^1(c')=\frac{2}{1+c}.E^1(c)-(1-c).F^1(c).$$

[8911*m*′] If we now change c into $e_{,}$ the expression of c' [8910*h*] becomes equal to that of c, [8909*e*], in the notation used by La Place in this chapter. Making these changes in [8911*l*, *m*], we get,

*periodical functions,** [8908″]

$$\mathrm{F}^1(c) = (1+e).\mathrm{F}^1(e);$$ [8911n]

$$\mathrm{E}^1(e) = \frac{2}{1+e}.\mathrm{E}^1(e) - (1-e).\mathrm{F}^1(e).$$ [8911o]

Substituting the expressions [8911n, o] in [8909k, 8910c], we obtain, after making some slight reductions,

$$A = \frac{2K}{\pi}.\{2\mathrm{E}^1(e) - (1-e^2).\mathrm{F}^1(e)\};$$ [8911p]

$$B = \frac{2K}{\pi}.\left\{\frac{2.(1+e^2)}{3e^2}.\mathrm{E}^1(e) - \frac{2.(1-e^2)}{3e^2}.\mathrm{F}^1(e)\right\}.$$ [8911q]

If we take for an example the value of $e = 0{,}8446862 = \sin.57^d\,38^m\,18^s$ [4079m], [8911r]
corresponding to Encke's comet, we shall have, from Legendre's tables,

$$\mathrm{F}^1(e) = 2{,}09575; \qquad \mathrm{E}^1(e) = 1{,}23357.$$ [8911s]

Substituting these in [8911p, q], we get,

$$A = K \times 1{,}18838; \qquad B = K \times 0{,}90015;$$ [8911t]

which will be used hereafter, together with the value of $a = 2{,}224346$ [4079m], and [8911u]
e [8911r].

Finally, if we substitute the values of A, B [8911p, q] in [8908z], we get, by rejecting the common factor K,

$$b_{-\frac{1}{2}}^{(0)} = \frac{4}{\pi}.\{2\mathrm{E}^1(e) - (1-e^2).\mathrm{F}^1(e)\};$$ [8911v]

$$b_{-\frac{1}{2}}^{(1)} = \frac{4}{\pi}.\left\{\frac{(1+e^2)}{3e}.\mathrm{E}^1(e) - \frac{(1-e^2)}{3e}.\mathrm{F}^1(e)\right\};$$ [8911w]

so that we shall have $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$, by means of elliptical functions; and by a similar [8911x]
process we may obtain the general value of the coefficient $b_s^{(i)}$, by means of such functions. The values of $b_s^{(i)}$ may also be derived from those in [8911v, w], using the formulas [966, 975, &c.], and putting $\alpha = -e$ [8908v]. Thus from [992] we obtain,

$$b_{\frac{3}{2}}^{(0)} = \frac{4}{\pi.(1-e^2)^2}.\{2\mathrm{E}^1(e) - (1-e^2).\mathrm{F}^1(e)\};$$ [8911y]

$$b_{\frac{3}{2}}^{(1)} = \frac{-4}{\pi e.(1-e^2)^2}.\{(1+e^2).\mathrm{E}^1(e) - (1-e^2).\mathrm{F}^1(e)\}.$$ [8911y′]

In like manner the formulas [990, 991] give, by substituting the values [8911v, w],

$$b_{\frac{1}{2}}^{(0)} = \frac{4}{\pi}.\mathrm{F}^1(e);$$ [8911z]

$$b_{\frac{1}{2}}^{(1)} = \frac{4}{\pi e}.\{\mathrm{E}^1(e) - \mathrm{F}^1(e)\};$$ [8911z′]

and so on for other cases.

* (4055) Substituting $v - \varpi = \mathrm{v}$ [8908b], in [8908], and then multiplying by $\frac{2dv.(1+2e.\cos.\mathrm{v}+e^2)}{a^2.(1-e^2)^2}$, we get the expression of the second member of [8906]; hence we have, [8912a]

[8909] $$d.\frac{1}{a}=\frac{\{2A.(1+e^2)+2Be^2\}.dv}{a^2.(1-e^2)^2};\quad \text{or}$$

[8909′] $$da=-\frac{\{2A.(1+e^2)+2Be^2\}.dv}{(1-e^2)^2}.$$

Then we have, as in [371 or 927],

[8910] $$x=r.\cos.v;\qquad y=r.\sin.v;$$

hence we get,*

[8912b] $$d.\frac{1}{a}=\frac{2dv}{a^2.(1-e^2)^2}.\{(1+e^2).(A+Be.\cos.\text{v}+\&c.)+2e.\cos.\text{v}.(A+Be.\cos.\text{v}+\&c.)\}.$$

[8912c] If we neglect the periodical functions, as in [8908′], and put $2e.\cos.\text{v}\times Be.\cos.\text{v}=Be^2+\&c.$ [6] Int., we get [8909]; developing the differential, we obtain [8909′]. From [8919]

[8912d] we have $\log.n=-\frac{3}{2}.\log.a$, whose differential is $\frac{dn}{n}=-\frac{3}{2}.\frac{da}{a}$; substituting da [8909′], and multiplying by n, we obtain,

[8912e] $$dn=\frac{n.\{3A.(1+e^2)+3Be^2\}.dv}{a.(1-e^2)^2};$$

which will be used hereafter.

* (4056) The differential of x [8910] is as in the first of the expressions [8912f], which is easily reduced to its second form; then substituting the values [8904a], we get [8912g, h];

[8912f] $$dx=dr.\cos.v-rdv.\sin.v=-r^2dv.\left\{-\frac{dr}{r^2dv}.\cos.v+\frac{1}{r}.\sin.v\right\}$$

[8912g] $$=-\frac{r^2dv}{a.(1-e^2)}.\{-e.\sin.(v-\varpi).\cos.v+[1+e.\cos.(v-\varpi)].\sin.v\}$$

[8912h] $$=-\frac{r^2dv}{a.(1-e^2)}.\{\sin.v+e.[\sin.v.\cos.(v-\varpi)-\cos.v.\sin.(v-\varpi)]\}.$$

The coefficient of e between the parenthesis in this last expression, being reduced by [22] Int., becomes $\sin.\{v-(v-\varpi)\}=\sin.\varpi$; hence this value of dx becomes as in

[8912i] [8911]. We may find in a similar manner the differential of y [8910], which gives successively [8912k, l, m]. The last of these values is easily reduced to the form [8912], by using [24] Int.;

[8912k] $$dy=dr.\sin.v+rdv.\cos.v=r^2dv.\left\{\frac{dr}{r^2dv}.\sin.v+\frac{1}{r}.\cos.v\right\}$$

[8912l] $$=\frac{r^2dv}{a.(1-e^2)}.\{e.\sin.(v-\varpi).\sin.v+[1+e.\cos.(v-\varpi)].\cos.v\}$$

[8912m] $$=\frac{r^2dv}{a.(1-e^2)}.\{\cos.v+e.[\cos.v.\cos.(v-\varpi)+\sin.v.\sin.(v-\varpi)]\}.$$

This value of dy might also have been easily derived from that of dx, by decreasing

$$dx = -\frac{r^2dv}{a.(1-e^2)}.(\sin.v + e.\sin.\varpi); \qquad [8911]$$

$$dy = \frac{r^2dv}{a.(1-e^2)}.(\cos.v + e.\cos.\varpi). \qquad [8912]$$

From these we easily deduce,*

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the angles v, ϖ, by a right angle; for this does not alter the angle $v-\varpi$, or the value in [8904a]; but it changes the first expression of dx [8912f], into the first expression of dy, [8912k]; therefore by making the same changes in the values of v, ϖ, in dx [8911], we shall get dy [8912]. [8912n]

* (4057) Multiplying [8904] by $2K.\varphi\left(\frac{1}{r}\right)$, and substituting the developed value [8908], we get,

$$2K.\varphi\left(\frac{1}{r}\right).ds = \frac{2dv}{a.(1-e^2)}.\{A+Be.\cos.(v-\varpi)+Ce^2.\cos.(2v-2\varpi)+\&c.\}. \qquad [8913a]$$

Multiplying this by $xdy-ydx = r^2dv$, and by $\frac{1}{dt^2} = \frac{a.(1-e^2)}{r^4dv^2}$ [8901], we obtain,

$$2K.\varphi\left(\frac{1}{r}\right).\frac{ds}{dt^2}.(xdy-ydx) = \frac{2}{r^2}.\{A+Be.\cos.(v-\varpi)+Ce^2.\cos.(2v-2\varpi)+\&c.\}. \qquad [8913b]$$

The product of [8913b] by dx [8911], being substituted in [8898], gives [8913c]; and in like manner the product of [8913b] by $-dy$ [8912], being substituted in [8899], gives [8913d];

$$d.(e.\sin.\varpi) = -\frac{2dv}{a.(1-e^2)}.\{\sin.v+e.\sin.\varpi\}.\{A+Be.\cos.(v-\varpi)+Ce^2.\cos.(2v-2\varpi)+\&c.\}; \qquad [8913c]$$

$$d.(e.\cos.\varpi) = -\frac{2dv}{a.(1-e^2)}.\{\cos.v+e.\cos.\varpi\}.\{A+Be.\cos.(v-\varpi)+Ce^2.\cos.(2v-2\varpi)+\&c.\}. \qquad [8913d]$$

Multiplying together the factors in the second member of each of these expressions, substituting $\sin.v.\cos.(v-\varpi) = \frac{1}{2}.\sin.\varpi + \&c.$; $\cos.v.\cos.(v-\varpi) = \frac{1}{2}.\cos.\varpi + \&c.$ [18, 20] Int., and retaining only the terms which are independent of v, we obtain, from [8913c], the expression [8913]; also from [8913d], the expression [8914]. If we develop the differentials in the first members of [8913, 8914], they will become of the following forms respectively, using for brevity the symbol $C = \frac{(2A+B).edv}{a.(1-e^2)}$; [8913$e$] [8913$f$]

$$de.\sin.\varpi + ed\varpi.\cos.\varpi = -C.\sin.\varpi; \qquad [8913g]$$

$$de.\cos.\varpi - ed\varpi.\sin.\varpi = -C.\cos.\varpi. \qquad [8913h]$$

Multiplying [8913g] by $\sin.\varpi$, and [8913h] by $\cos.\varpi$, then adding the products, and using $\sin.^2\varpi+\cos.^2\varpi = 1$, we get $de = -C$, as in [8915, 8913f]. Again multiplying [8913g] by $\cos.\varpi$, and [8913h] by $-\sin.\varpi$, adding the products and making the same reduction, we get $d\varpi = 0$, as in [8916]. Finally as $d\gamma$, $d\theta$, vanish [8886h], the secular variations are reduced to the finding of those of a, e [8909', 8915], as in [8916']; those of n being derived from a [8912d]. [8913i] [8913k] [8913l]

[8913] $$d.(e.\sin.\varpi) = -\frac{(2A+B).edv.\sin.\varpi}{a.(1-e^2)};$$

[8914] $$d.(e.\cos.\varpi) = -\frac{(2A+B).edv.\cos.\varpi}{a.(1-e^2)}.$$

Hence we obtain, as in [8913*i*, *k*],

[8915] $$de = -\frac{(2A+B).edv}{a.(1-e^2)};$$

[8916] $$d\varpi = 0.$$

[8916′] Therefore the perihelion is immovable, and there is no variation except in the greater axis, and in the excentricity of the orbit [8913*l*].

Variations of the elements. Dividing the expression of de [8915] by that of da [8909′], and multiplying the quotient by da, we get,

[8917] $$de = \frac{(2A+B).e.(1-ee).da}{a.\{2A.(1+ee)+2Be^2\}}.$$

If we integrate* this differential equation, we shall obtain e in terms of a;

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* (4058) Instead of proceeding in this general manner, it will be found sufficient, for [8917*a*] all practical purposes, to integrate the expressions of da, de, dn [8909′, 8915, 8912*e*], for one complete revolution of the comet, or from $v=0$ to $v=2\pi$; supposing the elements of the orbit in the second members of these equations to be constant during this period, on account of the smallness of their variations. By this means we shall have, for [8917*b*] the corresponding variations δa, δe, δn, respectively, the following expressions;

[8917*c*] $$\delta a = -\frac{\{2A.(1+e^2)+2Be^2\}.2\pi}{(1-e^2)^2};$$

[8917*d*] $$\delta e = -\frac{\{2A+B\}.e.2\pi}{a.(1-e^2)};$$

[8917*e*] $$\delta n = \frac{n.\{3A.(1+e^2)+3Be^2\}.2\pi}{a.(1-e^2)^2}.$$

[8917*f*] Substituting in these formulas the values of A, B, a, e [8911*t*, *u*, *r*], corresponding to Encke's comet, and to the hypothesis which he has assumed for the resistance, we obtain,

[8917*g*] $$\delta a = -410{,}057.K;$$

[8917*h*] $$\delta e = -\ 27{,}290.K;$$

[8917*i*] $$\delta n = \quad 1{,}434.K.$$

[8917*i′*] If we substitute the values of A, B [8911*p*, *q*] in [8917*c*, *d*, *e*], and then put $e^2=1-b^2$, we shall obtain, by some slight reductions,

and by substituting this value of e in the expression of da [8909′], we shall [8918]
have, by integration, v in terms of a; or a in terms of v. [8918′]

$$\delta a = -8K.\left(1-\frac{8}{3b^2}\right).\mathrm{F}^1(e)+\tfrac{64}{3}.K.\left(\frac{1}{b^2}-\frac{2}{b^4}\right).\mathrm{E}^1(e); \quad [8917k]$$

$$\delta e = -4.\frac{b^2}{ae}.K.\left(2-\frac{8}{3b^2}\right).\mathrm{F}^1(e)+\tfrac{32}{3}.\frac{1}{ae}.K.\left(\tfrac{7}{4}-\frac{2}{b^2}\right).\mathrm{E}^1(e); \quad [8917k']$$

$$\delta n = \quad 12.\frac{n}{a}.K.\left(1-\frac{8}{3b^2}\right).\mathrm{F}^1(e)-32.\frac{n}{a}.K.\left(\frac{1}{b^2}-\frac{2}{b^4}\right).\mathrm{E}^1(e); \quad [8917l]$$

being the same as were found by Plana, in vol. 13, page 352, of Zach's Journal [8910e'], and used in Pontécoulant's *Théorie Analytique*, &c., vol. 3, page 288.

Plana, in the memoir we have just referred to, makes the numerical calculations of the values of δa, δe, in two particular cases. *First;* where the density is constant, or all the terms of the series [8910e] vanish except B_0. *Second;* where all the terms of the same series vanish except B_2, which is the same as Encke's hypothesis [8908p]; the numerical results of Plana, in this last case, agree very nearly with those in [8917g—i]. Encke uses the common method of quadratures [7929x, &c.], in finding the values of δa, δe, considering t as the independent variable quantity, and computing the co-ordinates at equal intervals of time. He found this method to be convenient, because these co-ordinates had been previously ascertained by him, in making the calculations of the perturbations of the comet by the action of the planets. The intervals he used were 4 days, when the comet was near its perihelion, and within the sphere of the attraction of Mercury; afterwards 12 days were taken, until the comet had passed the sphere of attraction of the Earth and Venus, when the intervals were augmented to 36 days. Encke remarks that the periodical inequalities arising from the resistance can be neglected, on account of their smallness [8908s]; that the secular inequalities of θ, γ, ϖ, vanish [8913l], and da depends on dn [8912d]; so that it is only necessary to use the method of quadratures in computing the two elements n, e; observing that the perturbations of the epoch ε may be included in the double integration which is required in computing $\int ndt$. We shall give the formulas used by Encke, and shall retain his notation, as it is often referred to in speaking of this comet. The symbols he uses are the following; [8917l'] [8917m] [8917m'] [8917n] [8917o] [8917p] [8917q]

The sun's mean distance from the earth is *the unit of distance;* [8917r]

The *unit of time* is a mean solar day; [8917s]

The *unit of velocity* is that by which a body describes the space 1 in the time 1; [8917t]

$k = 0{,}017202...$ [5988(9)] represents in these measures the velocity which the earth would have, if it described a circle about the sun, at the distance 1; being the same as U [711′]; so that we have, as in [711′], $\mu = k^2$; [8917u] [8917v]

$V = \frac{ds}{dt}$ = the velocity of the comet in its orbit, as in [709″]; so that we have [8917w]

$V^2 = \frac{ds^2}{dt^2} = k^2.\left\{\frac{2}{r}-\frac{1}{a}\right\}$ [711, 8917v]; [8917w']

[8919] To obtain the value of v in terms of t, we shall observe that, if we reject the periodical quantities, we shall have $dv=ndt$ [1038]; moreover

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[8917x] U represents the resistance which the comet would suffer in moving in the resisting medium, supposing the density of the medium to be uniformly the same as that at
[8917y] the distance 1 from the sun, and the velocity of the comet to be 1.

[8917z] Now the resistance being supposed proportional to the product of the square of the velocity
[8918a] by the density $\varphi\left(\frac{1}{r}\right)$, its general expression will be represented by $U.\varphi\left(\frac{1}{r}\right).V^2$, which is similar to the expression of La Place, $K.\varphi\left(\frac{1}{r}\right).\frac{ds^2}{dt^2}$ [8886], or as it may be
[8918b] written, $K.\varphi\left(\frac{1}{r}\right).V^2$ [8917w]; and by comparing them together, it would seem, at first view, that we should have K equal to U. But this is not the case, as will be evident by comparing the two expressions together, in the case where the velocity of the comet is equal to that of a body revolving about the sun in a circular orbit, at a distance equal to that of the mean distance of the earth from the sun, where $r=a=1$. In this case we
[8918c] have in Encke's notation $V=k$ [8917w'], and in La Place's notation $V=U=1$, [8899c]; substituting these values respectively in [8918a, b], and putting the two results equal to each other, we get $U.\varphi\left(\frac{1}{r}\right).k^2=K.\varphi\left(\frac{1}{r}\right)$; dividing this by $\varphi\left(\frac{1}{r}\right)$, we obtain
[8918d] $K=U.k^2=U\mu$ [8917v], which will be used hereafter.

Instead of computing the quantity de directly, Encke puts $e=\sin.\varphi$, $p=a.(1-e^2)$,
[8918e] also u equal to the excentric anomaly, as in [5985(1, 5, 9)]; and he then assumes the following values for the determination of dn, $d\varphi$;

[8918f] $$\frac{dn}{dt}=\quad 3U.k^4.\frac{1}{r^2.\sqrt{a}}.\left(\frac{2}{r}-\frac{1}{a}\right)^{\frac{3}{2}};$$

[8918g] $$\frac{d\varphi}{dt}=-2U.k^3.\frac{p.\cos.u}{r^3.\cos.\varphi}.\left(\frac{2}{r}-\frac{1}{a}\right)^{\frac{1}{2}}.$$

The expression [8918f] is easily deduced from that of dn [1181]; for by dividing it by dt,
[8918g'] we get $\frac{dn}{dt}=\frac{3an}{\mu}.\frac{\mathrm{d}R}{dt}$; substituting $\frac{\mathrm{d}R}{dt}=K.\frac{1}{r^2}.\left(\frac{ds}{dt}\right)^3$ [8899a, 8908p], it becomes
[8918h] $\frac{dn}{dt}=3.\frac{K}{\mu}.\frac{an}{r^2}.\left(\frac{ds}{dt}\right)^3$. Now we have in [5987(12)], by neglecting the mass m of the
[8918i] comet in comparison with that of the sun, $n=ka^{-\frac{3}{2}}$, whence $an=\frac{k}{\sqrt{a}}$; moreover $\frac{K}{\mu}=U$ [8918d]. Substituting these and the value of $\frac{ds}{dt}$ [8917w'], in the preceding expression of $\frac{dn}{dt}$ [8918h], it becomes as in [8918f]. The equation [8918g] is easily
[8918k] deduced from [1262]; for by dividing it by edt, and using the values of $\left(\frac{dR}{dv}\right)$, $\mathrm{d}R$,

we have, as in [1044''], $n = a^{-\frac{3}{2}}$; hence we deduce, [8919']

$$dt = a^{\frac{3}{2}}.dv.$$ [8920]

[8886g, 8918g'], also $\varphi\left(\frac{1}{r}\right) = \frac{1}{r^2}$, we get [8918$n$]; substituting $r^2dv = a^2ndt.\sqrt{1-e^2}$, [1260], and $\frac{K}{\mu} = \mathrm{U}$ [8918d], it changes into [8918o]. This is reduced to the form [8918p], by putting $a^3n^2 = k^2$, $p = a.(1-e^2)$ [8918i, e], and using the value of $\frac{ds}{dt}$ [8917w']. Dividing the expression of r [7855] by $\frac{1}{2}ra$, we get $\frac{2}{a} - \frac{2}{r} = -\frac{2e}{r}.\cos.u$; substituting this in [8918p], we get the first expression of [8918q]. Lastly, substituting the value of $\frac{ds}{dt}$ [8917w'], we get the second expression in [8918q]; [8918l] [8918m]

$$\frac{de}{dt} = \frac{an.\sqrt{1-e^2}}{\mu.e} \cdot \frac{K}{r^2} \cdot \frac{ds.r^2dv}{dt^2} - \frac{a.(1-e^2)}{\mu.e} \cdot \frac{K}{r^2} \cdot \frac{ds^3}{dt^3} \quad [8918n]$$

$$= \frac{\mathrm{U}}{r^2} \cdot \frac{ds}{dt} \cdot \frac{a.(1-e^2)}{e} \cdot \left\{ a^2n^2 - \frac{ds^2}{dt^2} \right\} = \frac{\mathrm{U}}{r^2} \cdot \frac{ds}{dt} \cdot \frac{p}{e} \cdot \left\{ \frac{a^3n^2}{a} - \frac{ds^2}{dt^2} \right\} \quad [8918o]$$

$$= \frac{\mathrm{U}}{r^2} \cdot \frac{ds}{dt} \cdot \frac{p}{e} \cdot \left\{ \frac{k^2}{a} - k^2.\left(\frac{2}{r} - \frac{1}{a}\right) \right\} = \frac{\mathrm{U}}{r^2} \cdot \frac{ds}{dt} \cdot \frac{p}{e} . k^2. \left\{ \frac{2}{a} - \frac{2}{r} \right\} \quad [8918p]$$

$$= -\frac{2\mathrm{U}}{r^3} \cdot \frac{ds}{dt} . pk^2.\cos.u = -\frac{2\mathrm{U}}{r^3} . k^3.p.\cos.u.\left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{1}{2}}. \quad [8918q]$$

Now the differential of $e = \sin.\varphi$ [8918e], is $de = d\varphi.\cos.\varphi$; substituting this in [8918q], and then dividing by $\cos.\varphi$, we get [8918g]. The two formulas of Encke [8918f, g], are equivalent to those of La Place in [8909', 8915]; $d\varphi$ taking the place of de, [8915, 8918e, q], and dn taking the place of $d.\frac{1}{a}$ [8909, 8918f, 8912d]. We may incidentally remark that if we wish to compute the perturbations by the method of La Place [7872, 7873, &c.], by expressing dn, $d\varphi$, &c. in terms of u and du, we may easily obtain the formulas which are to be used, by substituting $r = a.(1-e.\cos.u)$ [7855], and $ndt = du.(1-e.\cos.u)$ [7882], in the expressions of the differentials given in [8918f, g, &c.]. [8918r] [8918s] [8918t]

Encke computed the places of the comet in the observations in its several appearances, from 1786 to 1819, and compared the calculated with the observed places, so as to determine the corresponding differences, after allowing for the action of all the planets. He also estimated the effect of a resisting medium, and that of a change in Jupiter's mass, and then combined these results of the observations by the method of the least squares, [849k], so as to obtain the most probable estimate of the actual effect of this resisting medium, and the most probable correction to be applied to the value of Jupiter's mass, as we have already observed in [5980m]. The final results given by him, in vol. 9, page 342, of the *Astronomische Nachrichten*, is $\frac{1}{1053,924}$ for Jupiter's mass, and $\mathrm{U} = \frac{1}{890,852}$; [8918$u$] [8918$v$]

[8920′] Substituting for a, its value in terms of v [8918′], and then integrating, we shall have t, in terms of v; and reciprocally v in terms of t.

[8918v′] whence we have $K=\mathrm{U}.k^2=\frac{(0,017202)^2}{890,852}=\frac{1}{3010529}$ [8918d]. This value of K is about $\frac{1}{9}$ part of the earth's mass, or $\frac{1}{9}m''$ [4061]; therefore when $r=1$, and $V^2=\frac{2}{r}-\frac{1}{a}=\frac{3}{2}$ [8899d] nearly, the expression of the resistance $K.\frac{1}{r^2}.V^2$
[8918w] [8918b, 8908p], becomes nearly equal to $m''\times\frac{1}{9}\times\frac{3}{2}$, or $\frac{1}{6}m''$. Hence it follows that when the comet is at the same distance from the sun and earth, so that the three bodies are situated in the angular parts of an equilateral triangle, whose side is equal to unity, or to the mean distance of the earth from the sun, the action of the resisting medium upon the comet will retard its progress in the direction of the tangent of its path, with a force which
[8918x] is about one sixth part of the attraction of the earth upon the comet, drawing it towards the earth's centre. We must not however consider these results as having much claim to accuracy, since they are grounded wholly upon the arbitrarily assumed law of the density of the medium $\varphi\left(\frac{1}{r}\right)=\frac{1}{r^2}$ [8908p]; and the slightest attention will make it apparent,
[8918y] that, with a different law of the density, we might obtain a very different result; moreover the imperfections of the observations have a very important bearing on the whole calculation. However as this assumed law is a very natural one, and many observations
[8918z] have been combined, we may consider the result in [8918v′] as the nearest approach to the true value of the resistance K, which can be obtained, with our present knowledge of the path of the comet, and of the law of the density of the ethereal fluid.

If we substitute the value of K [8918v′] in [8917g, h, i], we shall get,

[8919a] $$\delta a=-0,0001362;$$

[8919b] $$\delta e=-0,0000091;$$

[8919c] $$\delta n=\ 0^s,0983;$$

[8919d] observing that we have multiplied the expression of δn by 206265^s, to reduce it to sexagesimal seconds. Hence it appears that the daily motion n of the comet, which is about 1070^s, is increased about a tenth of a second in one revolution of the comet; *therefore the time of its periodical revolution, which is a little over* 1200 *days* [4079m],
[8919e] *will be decreased about one tenth of a day in each revolution; and the mean distance* a *will be decreased about* $\frac{1}{16000}$ of its value a [8911u], in the same time. We may
[8919f] observe that the values of δa, δe, δn, given by Pontécoulant in vol. 3, page 297, of his
[8919g] *Théorie Analytique du Système du Monde*, are much too great, and must be multiplied by $k^2=\frac{1}{3379}$ nearly, to obtain the values given in [8919a, b, c]. This difference arises from
[8919h] his having used the value $K=\mathrm{U}$, instead of $K=\mathrm{U}k^2$ [8918d], in making the reductions from [8917g, h, i].

When the orbits have but little excentricity, we shall have, by neglecting the square of e,* [8921]

$$A = \quad Ka^2.\varphi\left(\frac{1}{a}\right);$$ [8922]

$$B = -Ka^2.\varphi\left(\frac{1}{a}\right) + Ka.\varphi'\left(\frac{1}{a}\right);$$ [8923]

$\varphi'\left(\frac{1}{a}\right)$ *being the differential of* $\varphi\left(\frac{1}{a}\right)$ *divided by the differential of* $\frac{1}{a}$. [8924]

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* (4059) Neglecting terms of the order e^2, we get, from [8902],

$$\frac{1}{r} = \frac{1}{a} + \frac{e}{a}.\cos.(v-\varpi);$$ [8921a]

hence we have,

$$\varphi\left(\frac{1}{r}\right) = \varphi.\left\{\frac{1}{a} + \frac{e}{a}.\cos.(v-\varpi)\right\}.$$ [8921b]

Developing this according to the powers of e, by Taylor's theorem [617], retaining only its first power, and using the same notation as in [8924], we get,

$$\varphi\left(\frac{1}{r}\right) = \varphi\left(\frac{1}{a}\right) + \frac{e}{a}.\cos.(v-\varpi).\varphi'\left(\frac{1}{a}\right).$$ [8921c]

The expression of r [8902] gives $r^2 = a^2.\{1-2e.\cos.(v-\varpi)\}$; the product of this, by the factor $\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{1}{2}}$ or $1+e.\cos.(v-\varpi)+\&c.$, gives, [8921d]

$$r^2.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{1}{2}} = a^2.\{1-e.\cos.(v-\varpi)+\&c.\}.$$ [8921e]

Multiplying together the two expressions [8921c, e], and then their product by K, we get, by neglecting e^2,

$$K.\varphi\left(\frac{1}{r}\right).r^2.\{1+2e.\cos.(v-\varpi)+e^2\}^{\frac{1}{2}} = Ka^2.\varphi\left(\frac{1}{a}\right)+K.\left\{-a^2.\varphi\left(\frac{1}{a}\right)+a.\varphi'\left(\frac{1}{a}\right)\right\}.e.\cos.(v-\varpi).$$ [8921f]

The first member of this expression is the same as that of [8908]; and its second member being compared with that in [8908], namely $A+B.e.\cos.(v-\varpi)$, gives the values of A, B; the terms independent of $\cos.(v-\varpi)$, give the expression of A [8922], and those containing $\cos.(v-\varpi)$ give the value of B [8923]. [8921g]

If we suppose, as in [8908p], $\varphi\left(\frac{1}{r}\right) = \frac{1}{r^2}$, we shall have, as in [8909c], [8921h]
$\varphi\left(\frac{1}{a}\right) = \frac{1}{a^2}$, $\varphi'\left(\frac{1}{a}\right) = \frac{2}{a}$ [8924]; and then [8922, 8923] give $A = K$, [8921i]
$B = -K+2K = K$. Hence the expressions of da, de [8925, 8926], in the case of a nearly circular orbit, become $da = -2K.dv$, $\frac{de}{e} = -\frac{3K}{a}.dv$; lastly the expression of e [8929] gives $e = qa^{\frac{3}{2}}$. [8921k]

Then we shall have,*

[8925] da.
$$da = -Kdv. \left\{ 2a^2.\varphi\left(\frac{1}{a}\right) \right\};$$

[8926] de.
$$\frac{de}{e} = -Kdv. \left\{ a.\varphi\left(\frac{1}{a}\right) + \varphi'\left(\frac{1}{a}\right) \right\}.$$

[8927] $\varphi\left(\frac{1}{a}\right)$ is always positive,† and $\varphi'\left(\frac{1}{a}\right)$ is also positive, if, as is natural to suppose, $\varphi\left(\frac{1}{r}\right)$ increases when the distance r from the sun decreases; *therefore at the same time that the planet approaches towards the sun, by the effect of the resistance of the medium, the orbit will become more circular.*

The orbit decreases and becomes more circular by the resistance.

[8928] The equations [8925, 8926] give,‡

[8929] e.
$$e = q.\sqrt{\frac{a}{\varphi\left(\frac{1}{a}\right)}};$$ [Excentricity of the planet's orbit.]

[8925a] * (4060) Neglecting the square of e, as in [8921], we get, from [8909'], $da = -2Adv$; substituting the value of A [8922], it becomes as in [8925].

[8925b] Dividing the expression of de [8915] by e, we get $\frac{de}{e} = -\frac{(2A+B)}{a}.dv$; and by using the values of A, B [8922, 8923], it becomes as in [8926].

[8927a] † (4061) The density corresponding to the distance a is $\varphi\left(\frac{1}{a}\right)$ [8885], which must evidently be positive. Now if the distance a *decreases*, so that $\frac{1}{a}$ *increases* and becomes $\frac{1}{a}+\delta$, δ being *positive*, the resistance will become $\varphi\left(\frac{1}{a}+\delta\right)$; and by developing it as in [8921c, &c.], neglecting δ^2, it becomes $\varphi\left(\frac{1}{a}\right)+\delta.\varphi'\left(\frac{1}{a}\right)$; therefore upon the hypothesis

[8927b] assumed in [8927], this must exceed the resistance corresponding to a; consequently $\delta.\varphi'\left(\frac{1}{a}\right)$ must be *positive;* and as δ is *positive*, we must have $\varphi'\left(\frac{1}{a}\right)$ positive, as in

[8927c] [8927]. Hence the values of da, $\frac{de}{e}$ [8925, 8926], are both *negative; so that while the mean distance a decreases, the excentricity e also decreases;* or, in other words, the

[8927d] orbit becomes more circular by means of the resistance, as in [8928].

[8929a] ‡ (4062) Multiplying the equation [8926] by $2da$, and dividing the product by [8925], we get, with successive reductions,

q being an arbitrary constant quantity. Hence we easily perceive that, while a decreases, and $\varphi\left(\frac{1}{a}\right)$ *increases, the value of the excentricity e will incessantly decrease.* [8930]

19. *If light consist in the vibrations of an elastic fluid, the preceding analysis will give the effect of its resistance upon the motion of the planets and comets. If it be an emanation from the sun, the same analysis will also give, with some slight modification, the effect of its resistance.* For we may transfer to the light, and in a contrary direction, the real motion of the planet, and then consider the planet as being at rest, which will not affect their mutual action. Then the light will act upon the planet, according to a direction a little inclined to its primitive path; and it will communicate to the centre of gravity of the planet, according to this last direction, *a force which may be reduced to two others; the one in the direction of the radius vector of the planet, the other in a direction contrary to that of the element of the path which it describes.* We shall put, [8931] Resistance of light considered as an emanation from the sun. [8932] [8933] [8934] Symbols.

$\theta =$ the velocity of light; [8935]

$\frac{ds}{dt} =$ the velocity of the planet in the direction of the arc ds of its orbit [8885′]; [8935′]

$(\rho) =$ the density of the sun's light at the distance from the sun which is taken for the unit of measure; [8936]

$\rho =$ the density of the sun's light at the distance r from the sun. [8936′]

$$2.\ \frac{de}{e}=\frac{da}{a}+\frac{da}{a^2}\cdot\frac{\varphi'\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right)}=\frac{da}{a}-d.\frac{1}{a}\cdot\frac{\varphi'\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right)} \qquad [8929a']$$

$$=\frac{da}{a}-\frac{d.\varphi\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right)}. \qquad [8929b]$$

The integral of this last expression is $\log.e^2=\log.a-\log.\varphi\left(\frac{1}{a}\right)+\log.q^2$; q^2 being an arbitrary constant quantity [8930]. Reducing this to natural numbers it becomes $e^2=\frac{aq^2}{\varphi\left(\frac{1}{a}\right)}$; whence we get e [8929]. [8929c]

[8937] Then as the intensity of the sun's light decreases as the square of the distance increases, we shall evidently have $\rho = \frac{(\rho)}{r^2}$. From this notation it
[8937'] follows, that the two forces spoken of in [8934] are to each other as the velocities θ and $\frac{ds}{dt}$, and they can therefore be represented by* $\frac{H.\theta}{r^2}$ and
[8938] $\frac{H}{r^2}\cdot\frac{ds}{dt}$. The first of these forces is in a direction contrary to the gravitation towards the sun, and follows the same law; it must therefore be confounded with the force of gravity towards the sun, and decreases it a little. The
[8938'] second force $\frac{H}{r^2}\cdot\frac{ds}{dt}$ is in a direction opposite to that of the planet's motion, and produces a resistance to that motion. Putting it equal to the expression of the resistance [8886], and dividing by $\frac{ds^2}{dt^2}$, we get,

[8939] $$K.\varphi\left(\frac{1}{r}\right) = \frac{H}{r^2.\frac{ds}{dt}}.$$

Substituting this in [8894, 8898, 8899], we get the three following equations respectively;

[8940] $$d.\frac{1}{a} = \frac{2H}{r^2}\cdot\frac{ds^2}{dt};$$

[8941] $$d.(e.\sin.\varpi) = \frac{2H}{r^2}\cdot\frac{dx}{dt}\cdot(xdy - ydx);$$

[8942] $$d.(e.\cos.\varpi) = -\frac{2H}{r^2}\cdot\frac{dy}{dt}\cdot(xdy - ydx).$$

[8938a] * (4063) The forces in question are to each other as the velocity of light θ [8935] to the velocity of the planet in its orbit $\frac{ds}{dt}$ [8935'], as is evident from what is said in [8934].
[8938b] Moreover these forces must increase with the density $\rho = \frac{(\rho)}{r^2}$ [8937], and they will therefore be proportional to $\theta.\frac{(\rho)}{r^2}$ and $\frac{ds}{dt}\cdot\frac{(\rho)}{r^2}$; and by multiplying them by a constant
[8938c] quantity $\frac{H}{(\rho)}$, they will become respectively equal to $\frac{H.\theta}{r^2}$ and $\frac{H}{r^2}\cdot\frac{ds}{dt}$, as in [8738]; H being a constant quantity depending on the figure of the planet, its density, &c.

These three equations become, by neglecting periodical quantities,*

$$d.\frac{1}{a}=\frac{2H.dv.(1+e^2)}{a^{\frac{3}{2}}.(1-e^2)^{\frac{3}{2}}};$$ [8943]

$$d.(e.\sin.\varpi)=-\frac{2He.dv.\sin.\varpi}{\sqrt{a.(1-e^2)}};$$ [8944]

$$d.(e.\cos.\varpi)=-\frac{2He.dv.\cos.\varpi}{\sqrt{a.(1-e^2)}}.$$ [8945]

Hence we deduce,† $d\varpi$.

$$d\varpi=0;$$ [8946]

de.

$$de=-\frac{2He.dv}{\sqrt{a.(1-e^2)}}=\frac{eda.(1-e^2)}{a.(1+e^2)};$$ [8947]

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* (4064) Substituting in [8940] the values of ds, dt [8904, 8904c], we get $d.\frac{1}{a}=\frac{2H.dv.\{1+2e.\cos.(v-\varpi)+e^2\}}{a^{\frac{3}{2}}.(1-e^2)^{\frac{3}{2}}}$; and by neglecting the periodical terms, it becomes [8943a] as in [8940]. Dividing the first and third expressions of [8901] by dt, we get $\frac{xdy-ydx}{dt}=\sqrt{a.(1-e^2)}$; and by multiplying by $\frac{2H}{r^2}$, we obtain,

$$\frac{2H}{r^2}\cdot\frac{1}{dt}\cdot(xdy-ydx)=\frac{2H.\sqrt{a.(1-e^2)}}{r^2}.$$ [8943b]

Multiplying this successively by dx, $-dy$ [8911, 8912], we get the second members of [8941, 8942], under the following forms;

$$d.(e.\sin.\varpi)=-\frac{2H.dv}{\sqrt{a.(1-e^2)}}.\{\sin.v+e.\sin.\varpi\};$$ [8943c]

$$d.(e.\cos.\varpi)=-\frac{2H.dv}{\sqrt{a.(1-e^2)}}.\{\cos.v+e.\cos.\varpi\};$$ [8943c']

and if we neglect the periodical terms $\sin.v$, $\cos.v$, they become as in [8944, 8945].

† (4065) If we put $C=\frac{2He.dv}{\sqrt{a.(1-e^2)}}$ in the second members of [8944, 8945], and develop the differentials in the first members, they will become respectively as in [8913g, h]; and from them we may deduce, as in [8913k, i], $d\varpi=0$, $de=-C$, as in [8946, 8947]. But from [8943] we have, [8946a]

$$dv=\frac{a^{\frac{3}{2}}.(1-e^2)^{\frac{3}{2}}}{2H.(1+e^2)}.d.\frac{1}{a}=-\frac{(1-e^2)^{\frac{3}{2}}.da}{2Ha^{\frac{1}{2}}.(1+e^2)};$$ [8946b]

substituting this in the first expression of de [8947], we get its second form. Lastly, multiplying the first and third forms of [8947] by $\frac{1+e^2}{e.(1-e^2)}$, we get [8948]. [8946c]

da. therefore we have, as in [8946c],

[8948] $$\frac{da}{a} = \frac{(1+ee).de}{e.(1-ee)}.$$

Integrating [8948] we obtain,*

[8949] $$\frac{e}{1-e^2} = aq\,; \qquad \text{or} \qquad a = \frac{e}{q.(1-e^2)};$$

[8950] *q being an arbitrary constant quantity.* Substituting this value of a in the first formula of [8947], we get,†

[8951] $$de = -2H.dv.\sqrt{qe}\,;$$

e. which gives, by integration,

[8952] $$e = (h - Hv.\sqrt{q})^2;$$

[8953] *h being an arbitrary quantity, representing the square root of* e *when* $v=0$.

Substituting in [8920] the values of a, dv [8949, 8951], we get,‡

[8954] $$dt = \frac{-ede}{2Hq^2.(1-e^2)^{\frac{3}{2}}};$$

and then by integration,

[8955] $$t = \varepsilon - \frac{1}{2Hq^2.\sqrt{1-e^2}};$$

[8955′] *ε being an arbitrary constant quantity.* Substituting for e its value [8952], we obtain,

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* (4066) The second member of [8948] may be reduced, so as to become

[8949a] $\frac{da}{a} = \frac{de}{e} + \frac{2ede}{1-e^2}$. Taking the integral, and adding $\log.q$ for the arbitrary constant

[8949b] quantity, we get $\log.a + \log.q = \log.e + \log.\frac{1}{1-e^2}$, or in natural numbers $aq = \frac{e}{1-e^2}$, as in [8949].

† (4067) Substituting $a.(1-e^2) = \frac{e}{q}$ [8949], in the first expression of de [8947], it

[8952a] becomes as in [8951]. Multiplying this by $\frac{1}{2}e^{-\frac{1}{2}}$, we get $\frac{1}{2}e^{-\frac{1}{2}}de = -H.dv.\sqrt{q}$;

[8952b] whose integral is $e^{\frac{1}{2}} = h - H.v.\sqrt{q}$, as in [8952].

‡ (4068) From [8951] we obtain $dv = -\frac{de}{2H.\sqrt{qe}}$; substituting this value of dv

[8954a] and that of a [8949], in dt [8920], we get $dt = -\left\{\frac{e}{q.(1-e^2)}\right\}^{\frac{3}{2}}.\frac{de}{2H.\sqrt{qe}}$, which is

[8954b] easily reduced to the form [8954]. Its integral is given in [8955], as is easily proved by taking its differential. Finally, substituting the value of e^2 [8952] in [8955], we get [8956].

$$t = \varepsilon - \frac{1}{2Hq^2.\sqrt{1-(h-Hv.\sqrt{q})^4}}. \qquad [8956]$$

Reducing this to a series, and determining ε so that v may commence with t, we shall have very nearly,* [8957]

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* (4069) Putting $t=0$ and $v=0$ in [8956], according to the hypothesis in [8957], we obtain $\varepsilon = \frac{1}{2Hq^2.\sqrt{1-h^4}}$; substituting this in [8956], and developing $(h-Hv.\sqrt{q})^4$, we obtain, by successive reductions, the following expressions of t; [8958a]

$$t = \frac{1}{2Hq^2.\sqrt{1-h^4}} - \frac{1}{2Hq^2.\sqrt{1-h^4}}.\left\{1 + \frac{4H.h^3.\sqrt{q}}{1-h^4}.v - \frac{6H^2.h^2.q}{1-h^4}.v^2 + \&c.\right\}^{-\frac{1}{2}} \qquad [8958b]$$

$$= \frac{1}{2Hq^2.\sqrt{1-h^4}}.\left\{\frac{2H.h^3.\sqrt{q}}{1-h^4}.v - \frac{3H^2.h^2.q}{1-h^4}.v^2 - \frac{6H^2.h^6.q}{(1-h^4)^2}.v^2 + \&c.\right\} \qquad [8958c]$$

$$= \frac{h^3}{q^{\frac{3}{2}}.(1-h^4)^{\frac{3}{2}}}.v - \frac{3H.(h^2+h^6)}{2q.(1-h^4)^{\frac{5}{2}}}.v^2 + \&c. \qquad [8958d]$$

Taking the differential of [8958d], and then putting $v=0$, we get, at the commencement of the time t [8957], $dt = \frac{h^3.dv}{q^{\frac{3}{2}}.(1-h^4)^{\frac{3}{2}}}$; but we have in this case $dt = \frac{dv}{n}$ [8919], hence [8958e]

$\frac{1}{n} = \frac{h^3}{q^{\frac{3}{2}}.(1-h^4)^{\frac{3}{2}}}$. Substituting this value of $\frac{1}{n}$ in the first term of the second member [8958f]

of [8958d], and $v=nt$ in its second term, we get, very nearly,

$$t = \frac{v}{n} - \frac{3H.(h^2+h^6)}{2q.(1-h^4)^{\frac{5}{2}}}.n^2t^2; \qquad [8958g]$$

multiplying by n, and transposing the last term, we obtain,

$$v = nt + \frac{3Hn.(h^2+h^6)}{2q.(1-h^4)^{\frac{5}{2}}}.n^2t^2. \qquad [8958h]$$

At the commencement of the time t, when $v=0$, the expression of e [8952] becomes $e=h^2$; and we may substitute this value of h^2 in the small secular term of v [8958h], [8958i]

depending on n^2t^2; and by this means it becomes $\frac{3Hn.(e+e^3)}{2q.(1-e^2)^{\frac{5}{2}}}.n^2t^2$; and by using the [8958k]

value of q [8949], it is reduced to $\frac{3H.na.(1+e^2)}{2.(1-e^2)^{\frac{3}{2}}}.n^2t^2$. Lastly, putting $na = a^{-\frac{1}{2}}$ [8919'] [8958l]

in the numerator, it changes into $\frac{3H.(1+e^2)}{2.(1-e^2)^{\frac{3}{2}}.\sqrt{a}}.n^2t^2$; and by this means the preceding expression of v becomes as in [8958]. The term depending on H is the secular equation [8959]; and by rejecting terms of the order e^2, it becomes,

$$\frac{3H.n^2t^2}{2.\sqrt{a}}. \qquad \text{[Secular equation of a planet from the impulse of the sun's light.]} \qquad [8958m]$$

[8958] Secular equation of a planet depends on H.

$$v = nt + \frac{3H.(1+ee)}{2.(1-ee)^{\frac{3}{2}}.\sqrt{a}}.n^2t^2;$$

n, e, a, correspond to the origin of the time t. The second term of the
[8959] *expression of v is the secular equation of the planet, depending on the impulse of the light.*

20. *We shall now determine the corresponding secular inequality of the*
[8960] *moon. We shall mark with one accent, for this satellite, the quantities we have denoted by H, a, n, for the planet, which we shall suppose to be the earth; and we shall put x', y', z', for the moon's co-ordinates referred to*
[8961] *the earth's centre; or $x+x'$, $y+y'$, $z+z'$, referred to the sun's centre; so that by putting the sun's distance from the moon equal to f, we shall have,***

[8962]
$$f^2 = (x+x')^2 + (y+y')^2 + (z+z')^2.$$

It is evident, from the preceding article, that the action of the sun's light produces upon the moon's centre, and in directions towards the origin of its co-ordinates, the forces,

[8963]
$$\frac{H'}{f^2}.\frac{(dx+dx')}{dt}; \qquad \frac{H'}{f^2}.\frac{(dy+dy')}{dt}; \qquad \frac{H'}{f^2}.\frac{(dz+dz')}{dt}.$$

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* (4070) Proceeding in the same manner as in [8937'], we evidently see that the expression of the resistance to the earth's motion, in the directions parallel to the axes
[8962a] x, y, z, are represented by the product of $\frac{H}{r^2}$, by the velocity of the earth, resolved in
[8962b] the directions of those axes, namely, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, respectively. Moreover the rectangular co-ordinates of the moon [8961], referred to the sun's centre, give her
[8962c] distance from the sun f, as in [8962], by the usual principles of orthographic projections;
[8962d] so that the quantity which corresponds to $\frac{H}{r^2}$ for the earth, will be $\frac{H'}{f^2}$ for the moon; and as the velocity of the moon, resolved in directions parallel to these axes, are
[8962e] $\frac{dx+dx'}{dt}$, $\frac{dy+dy'}{dt}$, $\frac{dz+dz'}{dt}$, respectively [8961], their products by the factor $\frac{H'}{f^2}$,
[8962f] will give, as in [8962a, b] for the earth, the resistances corresponding to the moon's motion, as in [8963]. To obtain the effect on the moon's motion, in her relative orbit about the earth, we must subtract the expressions [8965] from the corresponding ones in
[8962g] [8963], and we shall obtain the second members of the expressions [8966, 8967, 8968], representing the relative forces acting on the moon in the directions of the co-ordinates,
[8962h] and tending to *decrease* them. Now these forces are represented, as in [8889, &c.], by $\left(\frac{dR}{dx'}\right)$, $\left(\frac{dR}{dy'}\right)$, $\left(\frac{dR}{dz'}\right)$; hence we obtain the expressions [8966—8968].

We must subtract from these expressions the forces acting upon the earth's centre, and arising from the same cause, if we wish to obtain the relative motion of the moon about the earth's centre. Now these forces acting on the earth's centre, are represented, as in [8962a, b], by, [8964]

$$\frac{H}{r^2}\cdot\frac{dx}{dt}; \qquad \frac{H}{r^2}\cdot\frac{dy}{dt}; \qquad \frac{H}{r^2}\cdot\frac{dz}{dt}; \qquad [8965]$$

therefore we shall have in the present case, as in [8962g, h],

$$\left(\frac{dR}{dx'}\right)=\frac{H'}{f^2}\cdot\frac{dx'}{dt}+\left(\frac{H'}{f^2}-\frac{H}{r^2}\right)\cdot\frac{dx}{dt}; \qquad [8966]$$

$$\left(\frac{dR}{dy'}\right)=\frac{H'}{f^2}\cdot\frac{dy'}{dt}+\left(\frac{H'}{f^2}-\frac{H}{r^2}\right)\cdot\frac{dy}{dt}; \qquad [8967]$$

[Resistances in the directions x', y', z', and tending to decrease these co-ordinates.]

$$\left(\frac{dR}{dz'}\right)=\frac{H'}{f^2}\cdot\frac{dz'}{dt}+\left(\frac{H'}{f^2}-\frac{H}{r^2}\right)\cdot\frac{dz}{dt}. \qquad [8968]$$

Hence we get,*

$$\mathrm{d}R=\frac{H'}{f^2}\cdot\frac{(dx'^2+dy'^2+dz'^2)}{dt}+\left(\frac{H'}{f^2}-\frac{H}{r^2}\right)\cdot\frac{(dx.dx'+dy.dy'+dz.dz')}{dt}; \qquad [8969]$$

the characteristic d refers only to the co-ordinates of the moon's relative orbit. The secular equation of this satellite is as in [1195],† [8970]

$$\frac{3a'n'}{\mu}\cdot\int(dt.\int\mathrm{d}R). \qquad [8971]$$

[Secular equation of the moon.]

In this case μ *is the sum of the masses of the earth and moon.* If we neglect periodical quantities, we shall have very nearly,‡ [8972]

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* (4071) The symbol d affects only the co-ordinates of the disturbed orbit [916′, &c.], so that in this case it operates only on x', y', z'; therefore the complete differential of R, relative to the characteristic d, is represented by, [8969a]

$$\mathrm{d}R=\left(\frac{dR}{dx'}\right).dx'+\left(\frac{dR}{dy'}\right).dy'+\left(\frac{dR}{dz'}\right).dz'; \qquad [8969b]$$

and by substituting the values [8966—8968], it becomes as in [8969].

† (4072) Neglecting the square of the disturbing force, we may consider an constant in the value of ζ [1195], and we shall have $\zeta=\frac{3an}{\mu}.\int(dt.\int\mathrm{d}R)$; then accenting [8971$a$] the letters as in [8960], so as to correspond to the present case, it becomes as in [8971]; [8971b] μ being the sum of the masses of the earth and moon, as in [530$^{\text{iv}}$].

‡ (4073) The square of the moon's relative velocity about the earth, is represented, as in [39′], by $\frac{dx'^2+dy'^2+dz'^2}{dt}$. Now the arc dv', described by the moon in her relative [8973a]

[8973] $$\frac{H'}{f^2}\cdot\left(\frac{dx'^2+dy'^2+dz'^2}{dt}\right)=\frac{H'.a'^2}{a^2}\cdot n'^2dt.$$

Moreover we have very nearly,*

[8974] $$\frac{H'}{f^2}=\frac{H'}{r^2}\cdot\left\{1-\frac{2.(xx'+yy'+zz')}{r^2}\right\}.$$

[8975] *Then by taking the plane of the ecliptic for the fixed plane,* we shall have very nearly,†

[8976] $$x=a.\cos.nt; \qquad y=a.\sin.nt; \qquad z=0;$$

[8977] $$x'=a'.\cos.n't; \qquad y'=a'.\sin.n't.$$

[8977'] *Therefore by neglecting the periodical terms, we shall have,*‡

[8973b] orbit about the earth in the time dt, is $dv'=n'dt$ [8919, 8960], neglecting the excentricity; multiplying this by the mean distance of the moon from the earth a', and dividing by the time dt, we get the mean velocity of the moon in her relative orbit

[8973c] about the earth equal to $a'n'$; hence we have, by using the mean values, $\frac{dx'^2+dy'^2+dz'^2}{dt^2}=a'^2n'^2$. Substituting this in the first member of [8973], and putting

[8973d] for f its mean value, which is nearly equal to the distance of the earth from the sun a, it becomes as in the second member of [8973].

* (4074) Developing the terms in the second member of [8962], neglecting x'^2, y'^2, z'^2, on account of their smallness, and putting as in [914'], $x^2+y^2+z^2=r^2$, we get,

[8974a] $$f^2=r^2+2.(xx'+yy'+zz')=r^2.\left\{1+\frac{2}{r^2}\cdot(xx'+yy'+zz')\right\}.$$

Substituting this in the first member of [8974], it becomes as in the second member, neglecting x'^2, &c.

† (4075) The values [8976, 8977] are similar to those in [8910], as appears by

[8976a] substituting for r, r', v, v', their mean values a, a', nt, $n't$; and putting also $z=0$, because the ecliptic is taken for the fixed plane [8975].

[8978a] ‡ (4076) Putting $-H=-H'+(H'-H)$, in the last term of dR [8969], we get,

[8978b] $$\mathrm{d}R=\frac{H'}{f^2}\cdot\frac{(dx'^2+dy'^2+dz'^2)}{dt}+\left(\frac{H'}{f^2}-\frac{H'}{r^2}\right)\cdot\frac{(dx.dx'+dy.dy'+dz.dz')}{dt} \quad 1$$

$$+\frac{(H'-H)}{r^2}\cdot\frac{(dx.dx'+dy.dy'+dz.dz')}{dt}. \quad 2$$

Now the differentials of [8976, 8977] give,

[8978c] $dx=-andt.\sin.nt$; $dy=andt.\cos.nt$; $dx'=-a'n'.dt.\sin.n't$; $dy'=a'n'.dt.\cos.n't$;

hence we get, by using [24] Int. and putting $dz=0$,

$$\left(\frac{H'}{f^2}-\frac{H'}{r^2}\right).\frac{(dx.dx'+dy.dy'+dz.dz')}{dt}=-\frac{H'.a'^2}{a^2}.nn'.dt. \qquad [8978]$$

Hence we deduce, as in [8978i],

$$\mathrm{d}R=\frac{H'.a'^2}{a^2}.n'dt.(n'-n). \qquad [8979]$$

Thus the moon's secular equation, depending on the action of the light, will be,*

$$\frac{3H'.a'^3.n'^2}{2\mu.a^2}.(n'-n).t^2. \qquad [8980]$$

But we have $n'^2=\frac{\mu}{a'^3}$, and $n^2=\frac{1}{a^3}$ [605', 8971b, 8919']; hence this secular equation becomes, [8981]

$$\frac{3H'.n.(n'-n).t^2}{2.\sqrt{a}}. \qquad [8982]$$

[Moon's secular equation from the impulse of light.]

$dx.dx'+dy.dy'+dz.dz'=aa'.nn'.dt^2.\{\sin.nt.\sin.n't+\cos.nt.\cos.n't\}=aa'.nn'.dt^2.\cos.(n't-nt)$. [8978$d$]

Moreover the same values [8976, 8977] give,

$$xx'+yy'+zz'=aa'.\{\cos.nt.\cos.n't+\sin.nt.\sin.n't\}=aa'.\cos.(n't-nt)\,; \qquad [8978e]$$

substituting this in [8974], we obtain by transposition,

$$\frac{H'}{f^2}-\frac{H'}{r^2}=-\frac{2H'}{r^4}.(xx'+yy'+zz')=-\frac{2H'.aa'}{r^4}.\cos.(n't-nt). \qquad [8978f]$$

Multiplying together the expressions [8978d,f], then dividing the product by dt, and putting $\cos.^2(n't-nt)=\frac{1}{2}+\frac{1}{2}.\cos.(2n't-2nt)$, we get,

$$\left(\frac{H'}{f^2}-\frac{H'}{r^2}\right).\frac{(dx.dx'+dy.dy'+dz.dz')}{dt}=\frac{-H'.a^2a'^2.nn'}{r^4}.dt.\{1+\cos.(2n't-2nt)\}\,; \qquad [8978g]$$

moreover if we neglect the periodical terms, we may put $r=a$ in the second member of this expression, and then it becomes as in [8978]. Finally, the substitution of [8978d] in the term [8978b, line 2], shows that it is a periodical quantity; and by neglecting it, the expression of $\mathrm{d}R$ is reduced to the terms in the second member of [8978b, line 1]; and by substituting their values [8973, 8978], it becomes as in [8979]. [8978h] [8978i]

* (4077) The integral of [8979] gives $\int\mathrm{d}R=\frac{H'.a'^2}{a^2}.n'.(n'-n).t$; hence we have, [8980$a$]

$$\int dt.\int\mathrm{d}R=\frac{H'.a'^2}{a^2}.n'.(n'-n).\int tdt=\frac{H.a'^2.n'}{2a^2}.(n'-n).t^2\,; \qquad [8980b]$$

substituting this in the moon's secular equation [8971], it becomes as in [8980]. Now multiplying the expression of n'^2 [8981] by $1=na^{\frac{3}{2}}$ [8981], we get $n'^2=\frac{n\mu.a^{\frac{3}{2}}}{a'^3}$; [8980$c$] substituting this in [8980], it becomes as in [8982]. Comparing this with the secular equation of the earth [8983], we obtain their ratio, as in [8984].

Secular equations of the moon and earth from the impulse of light.

The secular equation of the earth becomes, by neglecting the square of the excentricity, as in [8958m],

[8983]
$$\frac{3H.n^2t^2}{2.\sqrt{a}};$$ [Secular equation of the earth from the impulse of light.]

[8984] *therefore the secular equation of the earth is to that of the moon as unity to* $\frac{(n'-n)}{n}\cdot\frac{H'}{H}$.

To obtain the ratio $\frac{H'}{H}$, we shall suppose that the action of the sun's light upon the earth or moon is proportional to the surface of the body,

[8985] which is the most natural hypothesis that can be made. We shall then have the forces which act upon the centres of these two last bodies, by dividing these actions respectively by the masses of the earth and moon; so that we shall have very nearly,*

[8986]
$$\frac{H'}{H}=\frac{\text{earth's mass}\times\text{moon's surface}}{\text{moon's mass}\times\text{earth's surface}}=\frac{\text{earth's mass}\times\text{square of moon's app. semi-diameter}}{\text{moon's mass}\times\text{square of moon's horizontal parallax}}.$$

We have seen in [8986a, 5739] that this quantity is represented by,

[8986′]
$$\frac{H'}{H}=\frac{1}{0,195804}=5,10715;$$

[8987] and we also have $\frac{n}{n'}=0,0748013$ [5117]; *hence it follows that the part of the secular equation of the earth arising from the impulse of the sun's*

[8987′] *light, is to the corresponding part of the moon's secular equation as* 1 *to* 63,169.

Effect of the decrease of the sun's mass by the emission of light.

21. *These secular equations depend upon the impulse of the sun's light; but if this light be an emanation from the sun, its mass will incessantly decrease; and there will be found in the earth's mean motion a secular*

[8988] *equation, which must be incomparably greater and have a contrary sign to that arising from the impulse of the sun's light.* It is easy to determine this new secular equation by the following considerations. If we notice

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[8986a] * (4078) The expression [8986] is the same as [5736], observing that in this chapter H, H', correspond respectively to H', H, in [5736]. Moreover we have

[8986b] $n=n'.0,0748013$, the coefficient of n' being the same as the value of m [5117]; from

[8986c] this we get $\frac{n'-n}{n}=\frac{1-0,0748013}{0,0748013}=12,3687$; multiplying it by $\frac{H'}{H}=5,10715$ [8986′], we get $\frac{n'-n}{n}\cdot\frac{H'}{H}=63,169$, as in [8987′].

only the diminution of the sun's mass, the earth will be constantly attracted towards its centre; therefore the principal of areas will give, as in [372, 366],

$$r^2 dv = c dt;$$ [8989]

c being always the same. Now if we neglect the square of the excentricity, we shall have $r^2 dv = a^2 n dt$;* therefore $a^2 n$ will be constant, although the sun's mass decreases incessantly; *so that if we put a_1 and n_1 for the values of a, n, at the origin of the time t, we shall have,* [8990] [8991]

$$a^2 n = a_1^2 n_1.$$ [8992]

We shall then observe that the centrifugal force is equal to the square of the velocity, divided by the radius [54']; therefore *by neglecting the excentricity of the orbit*, this force will be an^2; but it is equal and contrary to the sun's attractive force. Now this last force is equal to the sun's mass divided by the square of the distance of the planet from the sun; and if the sun's mass be 1 at the origin of the time t, and $1-\alpha t$ at the end of the time t, α being a very small constant coefficient, we shall have, as in [8990d], [8993] [8994]

$$an^2 = \frac{1-\alpha t}{a^2}; \qquad \text{or} \qquad a^3 n^2 = 1-\alpha t.$$ [8995]

This equation, being combined with that in [8992], gives, by observing that† $a_1^3 n_1^2 = 1$, [8996]

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* (4079) We have in [1057] $r^2 dv = a^2 n dt.\sqrt{1-e^2}$; and by neglecting e^2, it becomes as in [8990], $r^2 dv = a^2 n dt = c dt$ [8989]; hence $a^2 n = c$, which is constant, as in [8990]. Moreover if we neglect the excentricity, as in [8993], the velocity of the body will be an, as in [8973c, &c.]; hence the centrifugal force [54'] will be proportional to $\frac{(an)^2}{a}$ or an^2, as in [8993]; and in the present hypothesis of a circular orbit, this must be equal to the force of gravity $\frac{1-\alpha t}{a^2}$, as in [8995]. [8990a] [8990b] [8990c] [8990d]

† (4080) When $t=0$, we have $a=a_1$, $n=n_1$ [8991]; substituting these in $a^3 n^2 = 1-\alpha t$ [8995], we get $a_1^3 n_1^2 = 1$ [8996]. Dividing the square of [8992] by the second equation in [8995], we get $a = \frac{a_1}{1-\alpha t} . a_1^3 n_1^2 = \frac{a_1}{1-\alpha t}$; this value of a is the same as in [8997]. Again dividing [8992] by the square of [8997], we get [8998]. Multiplying [8998] by dt, neglecting α^2, and integrating, we get, [8996a] [8996b] [8996c]

$$\textstyle\int n dt = n_1 \int dt - 2\alpha n_1 \int t dt = n_1 t - \alpha n_1 t^2,$$ [8996d]

as in [8999]; hence the secular equation depending on t^2, is $-\alpha n_1 t^2$, as in [9000].

[8997] $$a = \frac{a_1}{1 - \alpha t};$$

[8998] $$n = n_1.(1 - \alpha t)^2.$$

The earth's mean longitude being $\int n dt$, we shall have, by neglecting the square of α,

[8999] $$n_1 t - \alpha n_1 t^2,$$

Secular equation. for its value. *Therefore the secular equation of the mean motion, depending on the diminution of the sun's mass, is,*

[9000] $$-\alpha n_1 t^2.$$ [Secular equation of the earth from the decrease of the sun's mass.]

We shall now compare this expression with the value of the secular
[9001] equation depending upon the impulse of the light [8983]. *If we put i for the ratio of the velocity of light to that of the earth in its orbit, we shall*
[9001′] *have ian for the velocity of light* [8990c]. The density of the light at
[9002] the point of space occupied by the earth being ρ, the loss of the sun's light, in the time dt, will be $ian.\rho.dt$, multiplied by the surface $4\pi a^2$ of the
[9003] sphere whose radius is a; therefore it will be $4\pi.i\rho.a^3 n.dt$; π being the semi-circumference of a circle whose radius is unity. Hence we shall have,*

[9004] $$\alpha = 4\pi.i\rho.a^3 n;$$

consequently the secular equation depending upon the diminution of the sun's mass, will be,

[9005] $$-4\pi.i\rho.t^2.$$ [Secular equation of the earth from the decrease of the sun's mass.]

[9006] If we put ε for the sun's parallax, expressed in parts of the radius, the surface of a great circle of the earth will be† $\pi.\varepsilon^2 a^2$. The light received

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[9004a] * (4081) The decrement of the sun's mass, in the time dt, is represented by αdt, [8994]; putting this equal to the expression which we have found in [9003], namely
[9004b] $4\pi.i\rho.a^3 ndt$, and then dividing by dt, we get α [9004]. Substituting this value of α in the earth's secular equation [9000], it becomes $-4\pi.i\rho.t^2.(a^3 nn_1)$; and as $a^3 nn_1$ is
[9004c] nearly equal to $a_1{}^3 n_1{}^2$, or 1 [8996], it may be represented very nearly by $-4\pi.i\rho.t^2$, as in [9005].

[9007a] † (4082) The sun's horizontal parallax being ε, and its distance from the earth a, we shall have εa for the earth's semi-diameter, nearly; consequently a great circle of the earth will be represented by $\pi.(\varepsilon a)^2$, or $\pi.\varepsilon^2 a^2$, as in [9006]. Multiplying this by
[9007b] the factor $ian.\rho.dt$ [9002], we get the quantity of light falling on this great circle in the time dt, $\pi\varepsilon^2.i\rho.a^3 n.dt$ [9007]; multiplying this by the velocity ian [9001′], we obtain
[9007c] its impulse [9008]. Dividing this by the earth's mass T [9010], and by dt, we obtain

by this great circle in the time dt, will be $\pi\varepsilon^2.i\rho.a^3n.dt$; and as this light moves with the velocity ian [9001'], its impulse, supposing it to be absorbed by the earth, will be, [9007]

$$\pi\varepsilon^2.i^2\rho.a^4n^2.dt; \qquad [9008]$$

which produces in the centre of the earth the force,

$$\frac{\pi\varepsilon^2.i^2\rho.a^4n^2}{T}; \qquad \text{[Accelerative force acting upon the earth's centre, by the impulse of light.]} \qquad [9009]$$

T being the earth's mass. This force is found, in [8938, 9007*d*], to be equal to $\frac{H}{a^2}.ian$; hence we shall have, [9010]

$$H=\frac{\pi\varepsilon^2.i\rho.a^5n}{T}. \qquad [9011]$$

Therefore the earth's secular equation [8983] becomes, by observing that $a^3n^2=1$ [8919', 9007*f*], [9012]

$$\frac{3\pi\varepsilon^2.i\rho}{2T}.t^2. \qquad \text{[Secular equation of the earth arising from the impulse of the sun's light.]} \qquad [9013]$$

The two secular equations [9005, 9013] *arising from the diminution of the sun's mass, and from the impulse of its light, are therefore to each other in the ratio of* -4 *to* $\frac{3\varepsilon^2}{2T}$, *or* -1 *to* $\frac{3\varepsilon^2}{8T}$. [9014]

If we suppose the sun's parallax to be 26",4205 [$=8^s$,56 sex. 5589], and the earth's mass equal to $\frac{1}{329630}$ [4061], we shall find that these two secular equations are to each other in the ratio of* -1 to 0,0002129. [9015]

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the accelerative force acting upon the earth's centre, as in [9009]. This force is represented in [8938] by $\frac{H.\theta}{r^2}$, or $\frac{H.\theta}{a^2}$ nearly; θ *being the velocity of light* [8935] *which is represented by* ian [9001']; so that this force is $\frac{H}{a^2}.ian$, as in [9010]. Putting this equal to the expression [9009], and then multiplying by $\frac{a}{in}$, we get H [9011]. Substituting this in [8983], we get the earth's secular equation from the impulse of the sun's light, $\frac{3\pi.\varepsilon^2.i\rho}{2T}.t^2.(a^3n^2)^{\frac{3}{2}}$; and as $(a^3n^2)^{\frac{3}{2}}=1$ [9012], it becomes as in [9013]. [9007*d*] [9007*e*] [9007*f*]

* (4083) If we suppose that the secular equation of the earth, arising from the diminution of the sun's mass, is represented by -1, that of the earth, depending upon the impulse of the sun's light, will be $\frac{3\varepsilon^2}{8T}$ [9014], or, [9015*a*]

The earth's secular equation, depending on the diminution of the sun's
[9016] mass, is to the moon's secular equation, depending upon the impulse of its light, as -1 to $0,01345$. Therefore a secular equation of the moon,
[9017] of $-1''$, depending upon this cause, will correspond to $74'',35$ in the earth's secular equation; and as we are sure, by observation, that the
[9018] earth's secular equation is not $18''$, it follows *that the impulse of the sun's light upon the moon, does not produce a secular equation of a quarter of a centesimal second.*

It follows, from the preceding analysis, that for the last two thousand
[9019] *years the sun's mass has not varied a two-millionth part;* for $-\alpha t \times n_1 t$ being the earth's secular equation [9000] depending on this cause, if we
[9020] suppose that t represents a number of sideral years, n_1 will be equal to 400°, and $-\alpha t$ [8994] will be the diminution of the sun's mass. Putting therefore $t = 2000$, *and supposing that q represents, in degrees, the*
[9021] *secular equation of the earth corresponding to 2000 years, we shall have,**

[9022] $$\alpha t = \frac{q}{800000}. \qquad \left[\text{Decrease of the sun's mass in 2000 years.}\right]$$

On the decrease of the sun's mass.

From observation we cannot suppose q to be equal or greater than $0^\circ,4$;
[9023] *therefore the decrease of the sun's mass αt is less than $\frac{1}{2000000}$ in 2000 years.*

Effect of the successive transmission of gravity.

[9024] 22. *If gravitation be produced by the impulse of a fluid directed towards the centre of the attracting body, the preceding analysis, relative to the impulse of the solar light, will give the secular equation depending on the successive transmission of the attractive force.* For it follows, from what has been said, that if we put g for the action of the attracting body, as for

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[9015b] $$\tfrac{3}{8}.(\sin.26'',4205)^2 \times 329630 = 0,0002129,$$

as in [9015]. Multiplying this by $63,169$ [8987'], we get the secular equation of the
[9015c] moon, depending on the impulse of the sun's light, equal to $0,01345$, as in [9016]; so that if this part of the secular equation of the moon be $-1''$, the corresponding part of the
[9015d] secular equation of the earth will be $\frac{1''}{0,01345} = 74'',35$, as in [9017].

* (4084) Putting the secular equation [9019] equal to $-q$, we get $-\alpha t \times n_1 t = -q$,
[9022a] or $\alpha t = \frac{q}{n_1 t}$; substituting $n_1 t = 400^\circ \times 2000 = 800000^\circ$, and $q = 0^\circ,4$ [9021, 9023], we get αt [9023].

example that of the sun, the secular equation of the attracted body, which we may suppose to be the earth, will be,*

$$\frac{3}{2}\cdot\frac{gt^2}{ai}.$$ [Secular equation of the attracted body from the successive transmission of gravity.] [9025]

For then we have, in [8938], $g=\frac{H.\theta}{a^2}=\frac{H.in}{a}$, which changes the secular equation $\frac{3H.n^2t^2}{2\sqrt{a}}$ [8983], into the preceding value; but g is equal to the centrifugal force, and this force is equal to an^2 [8993]; therefore the secular equation of the attracted body is, [9026] [9026′]

$$\frac{3}{2}\cdot\frac{n^2t^2}{i};$$ [Secular equation of the attracted body from the successive transmission of gravity.] [9027]

i being in this case the ratio of the velocity of the fluid which causes the gravitation to that of the attracted body. $i.$ [9027′]

If we apply this result to the moon, and put *Nt for the earth's mean sideral motion about the sun, t denoting a number of Julian years*, we shall have the moon's secular equation equal to,† $N, t.$ [9028]

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* (4085) The velocity of light is θ [8935], and its force of impulse $\frac{H.\theta}{r^2}$ [8938], or $\frac{H.\theta}{a^2}$ nearly; we have also found in [8983] that the corresponding secular equation of the earth, arising from this impulse, is $\frac{3H.n^2t^2}{2\sqrt{a}}$, as in [9026]. The same expression will answer for estimating the effect of the fluid producing gravitation, supposing θ to be its velocity, and putting as in [9007e], $\theta=i.an$; changing the definition of i from that in [9001] to that in [9027′], corresponding to the case now under consideration; the velocity of the earth being represented, as in [8990c], by an. Hence the force of impulse of the fluid [9025a] $\frac{H.\theta}{a^2}$ becomes $\frac{H.ian}{a^2}$, or $\frac{H.in}{a}$; putting this equal to g, as in [9026], we get $H=\frac{ga}{in}$. Substituting this in the secular equation [9025b], depending on the impulse of the sun's rays, it becomes $\frac{3g.t^2}{2ai}.(na^{\frac{3}{2}})$; and since $na^{\frac{3}{2}}=1$ [9012], it becomes as in [9025]. Substituting $g=an^2$ [9026′], in [9025], it becomes $\frac{3}{2}\cdot\frac{n^2t^2}{i}$, as in [9027]. [9025a] [9025b] [9025c] [9025d] [9025e] [9025f]

† (4086) If we wish to apply the formula [9027] to the moon's motion about the earth, we must put *n for the angular velocity of the moon about the earth*, supposing it to be at rest; the absolute velocity of the moon, in her relative orbit about the earth, equal [9028a]

[9029] $$\frac{3}{2i}\cdot\left(\frac{n}{N}\right)^2.N^2t^2.$$ [Secular equation of the moon, arising from the successive transmission of gravity.]

[9029′] Putting a' = the mean distance of the sun from the earth;

[9030] a = the mean distance of the moon from the earth;

[9030′] i' = the ratio of the velocity of the fluid, producing gravitation, to that of light;

then the aberration being supposed equal to $62'',5$, the secular equation of the moon will become,*

[9031] $$\frac{3}{2}\cdot\frac{a}{a'}\cdot\left(\frac{n}{N}\right)^3\cdot\frac{N^2t^2.\sin.62'',5}{i'}.$$ [Secular equation of the moon arising from the successive transmission of gravity.]

[9032] We have seen, in [5543], that the moon's secular equation is $31'',424757$ $[=10^s,181621]$, when $t=100$ [5543′]; therefore if we attribute it to the preceding cause, we shall have,

[9033] $$i'=\frac{3}{2}\cdot\frac{a}{a'}\times\left(\frac{n}{N}\right)^3.N^2.\frac{10000.\sin.62'',5}{31'',424757}.$$

Reducing this expression of i' to numbers, we shall find that the velocity

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[9028b] to unity; and the velocity of the fluid producing gravitation equal to i [9027′]; then the moon's secular equation, with these symbols, will be as in [9027], $\frac{3}{2i}.n^2t^2$, or $\frac{3}{2i}\cdot\left(\frac{n}{N}\right)^2.N^2t^2$, as in [9029]. Now if we suppose, as in [9030], that the moon's

[9028c] distance from the earth is a, and her mean angular velocity about the earth n, as in [9028a], *the moon's relative velocity about the earth will be* an. In like manner, the earth's distance from the sun being a' [9029′], and the earth's angular velocity about the

[9028d] sun N [9028], *the velocity of the earth, in its orbit about the sun, will be* $a'N$ [9028]. These values are used in the next note.

[9031a] * (4087) The aberration is $62'',5$, and its sine expresses the ratio of the velocity of the earth in its orbit $a'N$ [9028d] to that of light; therefore the velocity of light is equal to $\frac{a'N}{\sin.62'',5}$. Multiplying this by i', we get, according to the definition in [9030′], the

[9031b] velocity of the fluid producing gravitation equal to $\frac{i'.a'N}{\sin.62'',5}$; and this is to the velocity of the moon an [9028c], as $\frac{i'.a'N}{an.\sin.62'',5}$ to 1; but this ratio is expressed by i to 1, in

[9031c] [9027′]; hence we have $i=\frac{i'.a'N}{an.\sin.62'',5}$. Substituting this value of i in [9029], it becomes as in [9031]; putting this equal to $31'',424757$ [9032], corresponding to $t=100$, we get i' [9033].

of the fluid producing gravitation will be about seven millions of times greater than that of light;* and as it is certain that the moon's secular equation depends almost wholly upon the cause we have assigned in the sixth book, *we must suppose that the gravitating fluid has a velocity which is at least a hundred millions of times greater than that of light;* or at least we must suppose, in its action on the moon, that it has at least that velocity to counteract her gravity towards the earth. *Therefore mathematicians may suppose, as they have heretofore done, that the velocity of the gravitating fluid is infinite.* [9034] [9035]

It is evident that the earth's secular equation, depending on the successive transmission of gravity, is only one sixth part of the corresponding equation of the moon;† *therefore it must vanish, or be insensible.* [9036]

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* (4088) We have nearly $\frac{a}{a'}=\frac{1}{400}$ [5221]; $\frac{\mathcal{N}}{n}=0,0748$ [5117 line 1]; [9032a]
$\mathcal{N}=400°$, or in parts of the radius $\mathcal{N}=6,28...$; lastly, putting $\sin.62'',5=62'',5$, we find that the expression of i' [9033], becomes nearly,

$$i'=\frac{3}{2}\times\frac{1}{400}\times\left(\frac{1}{0,0748}\right)^3\times(6,28)^2\times 10000\times\frac{62'',5}{31'',4};$$ [9032b]

which exceeds seven millions.

† (4089) Using the symbol $\mathcal{N}$, as in [9028], we find that the secular equation of the earth [9027] becomes $\frac{3}{2}\cdot\frac{\mathcal{N}^2 t^2}{i}$, and $i=\frac{\text{velocity of the gravitating fluid}}{\text{velocity of the earth}}$ [9027']. Now the velocity of the gravitating fluid is $\frac{i'.a'\mathcal{N}}{\sin.62'',5}$ [9031b], and that of the earth $a'\mathcal{N}$ [9031a]; hence the preceding expression of i becomes $i=\frac{i'}{\sin.62'',5}$; and by substituting it in the secular equation [9036a], it becomes $\frac{3}{2}\cdot\frac{\mathcal{N}^2 t^2.\sin.62'',5}{i'}$; and this is to the moon's secular equation [9031] as 1 to $\frac{a}{a'}\cdot\left(\frac{n}{\mathcal{N}}\right)^3$, or as 1 to $\frac{1}{400}\times\left(\frac{1}{0,0748}\right)^3$ [9032a]; being nearly as 1 to 6, as in [9036]. [9036a] [9036b] [9036c] [9036d] [9036e]

CHAPTER VIII.

SUPPLEMENT TO THE THEORIES OF THE PLANETS AND SATELLITES.

23. We have given, in the sixth book, the numerical expressions of the inequalities of the planets. The care which had been taken not to omit any sensible inequality, authorized the belief that the tables of the planets' motions would be improved by the use of these formulas; and it became an object of interest with astronomers to apply them to this purpose. These hopes have been realized by the labors of Delambre, Bouvard, Lefrançais Lalande and Burckhardt, who have compared the theory with a very great number of observations, in order to deduce from them the elliptical elements of the orbits of the planets; and I have also reviewed,
[9037] with great care, the theory of their perturbations; so that by these united efforts we have obtained very accurate tables of the motions of the planets. This new examination of the theoretical results, has not indicated any sensible inequalities to be added to those which had been before determined, except in the motions of Jupiter and Saturn. The nearly commensurable ratio of their mean motions gives rise, as we have seen in the second and sixth books, to some very great variations in the elements of the orbits of these two planets, depending on inequalities whose periods exceed nine centuries. The variations of the excentricity and perihelion of Jupiter's orbit, depending on this cause, produce in the motion of Jupiter
[9038] a very sensible inequality [4394], whose argument is three times the mean motion of Jupiter *minus* five times that of Saturn. The similar variations in the excentricity and perihelion of Saturn, produce in the motion of Saturn a great inequality [4468], whose argument is twice the mean motion
[9039] of Jupiter *minus* four times that of Saturn. These two inequalities may in fact be considered as real equations of the centre, whose excentricity and perihelion vary with extreme slowness. Now the two great equations of

the centre of these two planets give rise to some very sensible inequalities; therefore, by substituting in the expressions of these inequalities, instead of these great equations of the centre, those which we have just mentioned, there will be produced some small similar inequalities, which may be of sufficient importance to be noticed; we shall therefore consider, in this point of view, the chief inequalities of Jupiter and Saturn, depending on the excentricities. [9040]

We have seen, in [4392], that the expression of δv^{iv} contains the following inequalities;

$$\left.\begin{array}{l} -138^s{,}373.\sin.(2n^{v}t-\ n^{iv}t+2\varepsilon^{v}-\ \varepsilon^{iv}-\varpi^{iv}) \\ +\ 56^s{,}634.\sin.(2n^{v}t-\ n^{iv}t+2\varepsilon^{v}-\ \varepsilon^{iv}-\varpi^{v}) \\ -\ 44^s{,}461.\sin.(3n^{v}t-2n^{iv}t+3\varepsilon^{v}-2\varepsilon^{iv}-\varpi^{iv}) \\ +\ 84^s{,}942.\sin.(3n^{v}t-2n^{iv}t+3\varepsilon^{v}-2\varepsilon^{iv}-\varpi^{v}) \end{array}\right\}; (A)$$ [9041] [9042] [9043] [9044]

they are the most important ones arising from the first power of the excentricities. The first and third depend on the equation of the centre of Jupiter, $+2e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv})$ [3834, 4390e, &c.]. We have seen, in [4394], that Jupiter's motion is subjected to the following inequality; [9045]

$$169^s{,}266.\sin.(n^{iv}t+\varepsilon^{iv}+55^d\,40^m\,49^s-5n^{v}t+2n^{iv}t-5\varepsilon^{v}+2\varepsilon^{iv}).$$ [9046]

This inequality may be considered as a second equation of the centre of Jupiter's orbit; whose excentricity and perihelion vary with extreme slowness, their variations depending upon those of the angle $5n^{v}t-2n^{iv}t$. This being premised, we shall put the inequality [9041] under the following form; [9046′] [9047]

$$-\frac{138^s{,}373}{2e^{iv}}.2e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}+2n^{v}t-2n^{iv}t+2\varepsilon^{v}-2\varepsilon^{iv}).$$ [9048]

If we substitute in it, instead of* $2e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv})$, the expression [9046],

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* (4090) The term $2e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv})$ is the most important part of the equation of the centre [3834]; and if we substitute instead of it the term [9046 or 9049], it will be equivalent to changing $2e^{iv}$ into $169^s{,}266$, and $-\varpi^{iv}$ into, [9049a]

$$55^d\,40^m\,49^s-5n^{v}t+2n^{iv}t-5\varepsilon^{v}+2\varepsilon^{iv};$$ [9049b]

and by this means [9048] changes into [9050]. In like manner we obtain [9053].

This short but indirect method of computing the small inequalities [9050, 9053, 9058, 9060 or 9061, &c.], of the order of the square of the disturbing [9049c]

[9049] $$169^s{,}266.\sin.(3n^{iv}t-5n^{v}t+3\varepsilon^{iv}-5\varepsilon^{v}+55^d\,40^m\,49^s),$$

we shall obtain the inequality,

[9050] $$-\frac{138^s{,}373}{2e^{iv}}\,.\,169^s{,}266.\sin.(n^{iv}t-3n^{v}t+\varepsilon^{iv}-3\varepsilon^{v}+55^d\,40^m\,49^s).$$

[9051] Putting in like manner the inequality of the expression [9043] under the form,

[9052] $$-\frac{44^s{,}461}{2e^{iv}}\,.\,2e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}+3n^{v}t-3n^{iv}t+3\varepsilon^{v}-3\varepsilon^{iv}),$$

we shall obtain, by substituting the same term [9046], the following inequality;

[9053] $$-\frac{44^s{,}461}{2e^{iv}}\,.\,169^s{,}266.\sin.(-2n^{v}t-2\varepsilon^{v}+55^d\,40^m\,49^s).$$

[9054] The terms [9042, 9044] depend on the equation of the centre of Saturn, $2e^{v}.\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v})$, and we shall put [9042] under the form,

[9055] $$\frac{56^s{,}634}{2e^{v}}\,.\,2e^{v}.(n^{v}t+\varepsilon^{v}-\varpi^{v}+n^{v}t-n^{iv}t+\varepsilon^{v}-\varepsilon^{iv}).$$

We find, in [4468], that Saturn's motion is subjected to the inequality,

[9056] $$-669^s{,}682.\sin.(n^{v}t+\varepsilon^{v}+56^d\,10^m\,57^s-5n^{v}t+2n^{iv}t-5\varepsilon^{v}+2\varepsilon^{iv}).$$

[9056′] *This may be considered as a second equation of the centre of Saturn*, whose excentricity and perihelion vary with extreme slowness, these variations being dependent upon that of the angle $5n^{v}t-2n^{iv}t$. Therefore by
[9057] substituting it for* $2e^{v}.\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v})$, in [9055], we shall obtain the following inequality;

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masses, must be considered as nothing more than a tolerably near approximation for obtaining their values, since several of the small parts of the general expressions of these
[9049d] terms are neglected; those parts only being retained which are derived from the variation of the first term of the equation of the centre of Jupiter, $2e^{iv}.\sin.(n^{iv}+\varepsilon^{iv}-\varpi^{iv})$ [9045], or that of Saturn, $2e^{v}.\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v})$ [9054]. We have already mentioned a
[9049e] somewhat similar defect in the abridged method of computing the small inequality in the motion of Mercury [3872, &c.]. Plana has noticed this imperfection in vol. 2, page 406, &c. of the Memoirs of the Astronomical Society of London, to which we may refer;
[9049f] since it is not necessary to go into any particular detail on the subject, taking into view the smallness of these inequalities [9061, &c.], and that the corrections to be made in them
[9049g] are of very little importance. Similar remarks may be made relative to the other inequalities, computed in this section of the work.

[9057a] * (4091) If we proceed in this case as in [9049a, b], we shall find, by comparing the expressions [9056, 9057], that we must change $2e^{v}$ into $-669^s{,}682$, and $-\varpi^{v}$

$$-\frac{56^s,634}{2e^{v}}.669^s,682.\sin.(n^{iv}t-3n^{v}t+\varepsilon^{iv}-3\varepsilon^{v}+56^d\,10^m\,57^s). \qquad [9058]$$

In like manner, by putting [9044] under the form,

$$\frac{84^s,942}{2e^{v}}.2e^{v}.\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v}+2n^{v}t-2n^{iv}t+2\varepsilon^{v}-2\varepsilon^{iv}), \qquad [9059]$$

we shall obtain, by the same substitution, the following inequality;

$$-\frac{84^s,942}{2e^{v}}.669^s,682.\sin.(-2n^{v}t-2\varepsilon^{v}+56^d\,10^m\,57^s). \qquad [9060]$$

Substituting in [9050, 9053, 9058, 9060] the values of e^{iv}, e^{v} [4080], we find that the four expressions [9041—9044] produce the following inequalities;*

$$\begin{array}{ll} 1^s,1809.\sin.(3n^{v}t-n^{iv}t+3\varepsilon^{v}-\varepsilon^{iv}-55^d\,40^m\,49^s); & 1 \\ +0^s,3794.\sin.(2n^{v}t+2\varepsilon^{v}-55^d\,40^m\,49^s); & 2 \\ +1^s,6352.\sin.(3n^{v}t-n^{iv}t+3\varepsilon^{v}-\varepsilon^{iv}-56^d\,10^m\,57^s); & 3 \\ +2^s,4525.\sin.(2n^{v}t+2\varepsilon^{v}-56^d\,10^m\,57^s). & 4 \end{array} \qquad [9061]$$

These inequalities are very small; but as they may be connected with others of similar forms, they will not render the tables more complicated, and will make them more accurate.

We have seen, in [3916, &c.], that the inequality of Jupiter [4394],

$$169^s,266.\sin.(3n^{iv}t-5n^{v}t+3\varepsilon^{iv}-5\varepsilon^{v}+55^d\,40^m\,49^s), \qquad [9062]$$

is the result of the variations in the equation of the centre and the perihelion, depending upon the angle $5n^{v}t-2n^{iv}t$. If we represent these variations by $\delta\varepsilon^{iv}$ and $\delta\varpi^{iv}$, the preceding inequality may be put, as in [3916], under the form, [9063]

$$2\delta e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv})-2e^{iv}.\delta\varpi^{iv}.\cos.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}). \qquad [9064]$$

The expression of Jupiter's true longitude in terms of its mean longitude,† [668], contains the two terms,

$$\tfrac{5}{4}.e^{iv2}.\sin.(2n^{iv}t+2\varepsilon^{iv}-2\varpi^{iv})+\tfrac{13}{12}.e^{iv3}.(3n^{iv}t+3\varepsilon^{iv}-3\varpi^{iv}), \qquad [9065]$$

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into $56^d\,10^m\,57^s-5n^{v}t+2n^{iv}t-5\varepsilon^{v}+2\varepsilon^{iv}$; substituting these in [9055], we get [9058]. [9057b]

* (4092) After making the substitution of e^{iv}, e^{v} [4080], we must divide by the radius in seconds 206265^s, and we shall obtain the numerical values [9061]. [9061a]

† (4093) Changing e into e^{iv}, and nt into $n^{iv}t+\varepsilon^{iv}-\varpi^{iv}$, to conform to the notation here used. [9064a]

which gives the following expression;*

[9066]
$$\left.\begin{array}{l} +\tfrac{5}{2}.e^{iv}.\left\{\begin{array}{r} \delta e^{iv}.\sin.(2n^{iv}t+2\varepsilon^{iv}-2\varpi^{iv}) \\ -e^{iv}\delta\varpi^{iv}.\cos.(2n^{iv}t+2\varepsilon^{iv}-2\varpi^{iv}) \end{array}\right\} \\ +\tfrac{13}{4}.e^{iv}.e^{iv}.\left\{\begin{array}{r} \delta e^{iv}.\sin.(3n^{iv}t+3\varepsilon^{iv}-3\varpi^{iv}) \\ -e^{iv}\delta\varpi^{iv}.\cos.(3n^{iv}t+3\varepsilon^{iv}-3\varpi^{iv}) \end{array}\right\} \end{array}\right\}.\ (Q) \qquad \begin{array}{r}1\\2\\3\\4\end{array}$$

[9067] The two first of these terms give the inequality depending upon† $4n^{iv}t-5n^{v}t+4\varepsilon^{iv}-5\varepsilon^{v}+45^{d}\,21^{m}\,44^{s}$, which we have determined in [9068] [4440]. If we represent by $p.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}+f)$, the inequality of Jupiter [9046], depending upon $3n^{iv}t-5n^{v}t$, we shall have,

[9069] $$f=2n^{iv}t-5n^{v}t+2\varepsilon^{iv}-5\varepsilon^{v}+\varpi^{iv}+55^{d}\,40^{m}\,49^{s};$$

[9070] $$2\delta e^{iv}=p.\cos.f; \qquad -2e^{iv}.\delta\varpi^{iv}=p.\sin.f; \qquad p=169^{s},226.$$

Hence the terms [9066 lines 3, 4] become,‡

[9071] $$\tfrac{13}{8}.e^{iv}.e^{iv}.p.\sin.(3n^{iv}t+3\varepsilon^{iv}-3\varpi^{iv}+f);$$

therefore by substituting f, p [9069, 9070], we get,

[9072] $$\tfrac{13}{8}.e^{iv}.e^{iv}.169^{s},266.\sin.(5n^{iv}t-5n^{v}t+5\varepsilon^{iv}-5\varepsilon^{v}-2\varpi^{iv}+55^{d}\,40^{m}\,49^{s}).$$

Reducing the coefficient of this expression to numbers, we obtain the

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[9065a] * (4094) Taking the variation of [9065], considering e^{iv}, ϖ^{iv}, as variable, it becomes as in [9066].

[9067a] † (4095) If we put $p=169^{s},266$, and use the value of f [9069], the term [9046] will become $p.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}+f)$, as in [9068]. Developing it by [21] Int. we get,

[9067b] $$p.\cos.f.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv})+p.\sin.f.\cos.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}).$$

[9067c] Putting this equal to the expression [9064], we obtain $2\delta e^{iv}=p.\cos.f$; $-2e^{iv}.\delta\varpi^{iv}=p.\sin.f$, as in [9070]. Substituting these values in [9066 lines 1, 2], they become,

[9067d]
$$\tfrac{5}{4}p.e^{iv}.\{\cos.f.\sin.(2n^{iv}t+2\varepsilon^{iv}-2\varpi^{iv})+\sin.f.\cos.(2n^{iv}t+2\varepsilon^{iv}-2\varpi^{iv})\}$$
$$=\tfrac{5}{4}p.e^{iv}.\sin.(2n^{iv}t+2\varepsilon^{iv}-2\varpi^{iv}+f), \quad [21] \text{ Int.}$$

Re-substituting the value of f [9069], it becomes,

[9067e] $$\tfrac{5}{4}p.e^{iv}.\sin.(4n^{iv}t-5n^{v}t+4\varepsilon^{iv}-5\varepsilon^{v}-\varpi^{iv}+55^{d}\,40^{m}\,49^{s}),$$

being of the same form as that in [9067], which is computed in [4440 or 4439] by a similar process, changing K [3827] into $-p$ [9068], and using ϖ^{iv} [4081] nearly.

‡ (4096) Substituting the values [9070] in the two terms [9066 lines 3, 4], we obtain,

[9072a] $$\tfrac{13}{8}e^{iv}.e^{iv}.p.\{\cos.f.\sin.(3n^{iv}t+3\varepsilon^{iv}-3\varpi^{iv})+\sin.f.\cos.(3n^{iv}t+3\varepsilon^{iv}-3\varpi^{iv})\},$$

which is easily reduced to the form [9071], by using [21] Int.

following inequality ;*

$$0^s{,}6358.\sin.(5n^{iv}t - 5n^{v}t + 5\varepsilon^{iv} - 5\varepsilon^{v} + 34^d\,58^m\,41^s). \qquad [9073]$$

The following inequality is given in [4438, 9073*b*],

$$-4^s{,}0247.\sin.(5n^{iv}t - 10n^{v}t + 5\varepsilon^{iv} - 10\varepsilon^{v} + 51^d\,21^m\,55^s). \qquad [9074]$$

We have seen, in [4006′, &c.], that in all the arguments of Jupiter and Saturn, where the coefficient of t is neither $5n^{v} - 2n^{iv}$, nor differs from it by n^{iv} for Jupiter, or n^{v} for Saturn, we must increase the mean longitudes $n^{iv}t + \varepsilon^{iv}$, $n^{v}t + \varepsilon^{v}$, counted from the fixed equinox of 1750, by their great inequalities depending upon $5n^{v}t - 2n^{iv}t$. If we wish to use the mean longitudes thus increased in the inequality of Jupiter [9062], [9075]

$$169^s{,}266.\sin.(3n^{iv}t - 5n^{v}t + 3\varepsilon^{iv} - 5\varepsilon^{v} + 55^d\,40^m\,49^s), \qquad [9076]$$

we may put q^{iv}, q^{v}, for these longitudes thus augmented, and then put this inequality under the following form ; [9076′]

$$169^s{,}266.\sin.\{3q^{iv} - 5q^{v} - (3p^{iv} + 5p^{v}) + 55^d\,40^m\,49^s\} ; \qquad [9077]$$

p^{iv} being the great inequality of Jupiter, and $-p^{v}$ that of Saturn. If we develop the preceding function, we shall get,† [9078]

$$\left.\begin{array}{lr} 169^s{,}266.\sin.(3q^{iv} - 5q^{v} + 55^d\,40^m\,49^s) & 1 \\ \quad -(3p^{iv} + 5p^{v}).169^s{,}266.\cos.(3q^{iv} - 5q^{v} + 55^d\,40^m\,49^s). & 2 \end{array}\right\} \qquad [9079]$$

Now we have very nearly, ‡

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* (4097) Substituting the values of ε^{iv} [4080], also $\varpi^{iv} = 10^d\,21^m\,04^s$ [4081], in [9072], it becomes as in [9073]. The inequality [9074] is the same as [4438]; it is printed with a different sign in the original work, but it is corrected in [4438]. This is hereafter combined with the term which is computed in [9083]. [9073*a*] [9073*b*]

† (4098) This development is made as in [60] Int., by putting,

$$z = 3q^{iv} - 5q^{v} + 55^d\,40^m\,49^s, \qquad \alpha = -(3p^{iv} + 5p^{v}), \qquad [9076a]$$

neglecting the square and higher powers of α; then multiplying by the coefficient $169^s{,}266$, we reduce the expression [9077] to the form [9079].

‡ (4099) If we put for brevity $T = 5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv} + 4^d\,21^m\,20^s$, and use the symbols p^{iv}, p^{v} [9078], we shall have very nearly,

$$p^{iv} = 1265^s{,}3.\sin.T \quad [4434] ; \qquad p^{v} = 2939^s{,}6.\sin.T \quad [4492] ; \qquad [9080a]$$

hence $3p^{iv} + 5p^{v} = 18493^s{,}9.\sin.T$, as in [9080]; the argument being taken the same as for Saturn in [4492], because this produces by far the greatest part of the coefficient in [9080]. Moreover the difference of the arguments in [4492, 4434], is not of much importance in the small inequalities which are computed in [9085, &c.]. Dividing the coefficient $18493^s{,}9$ by the radius in seconds 206265^s, and multiplying the result by [9080*b*] [9080*c*]

[9080] $$3p^{\text{iv}}+5p^{\text{v}}=18493^s,834.\sin.(5n^{\text{v}}t-2n^{\text{iv}}t+5\varepsilon^{\text{v}}-2\varepsilon^{\text{iv}}+4^d\,21^m\,20^s);$$

which gives,

[9081] $$\begin{array}{ll} -(3p^{\text{iv}}+5p^{\text{v}}).169^s,266.\cos.(3q^{\text{iv}}-5q^{\text{v}}+55^d\,40^m\,49^s) & 1 \\ =-7^s,5882.\left\{\begin{array}{l}\sin.(3q^{\text{iv}}-5q^{\text{v}}+5n^{\text{v}}t-2n^{\text{iv}}t+5\varepsilon^{\text{v}}-2\varepsilon^{\text{iv}}+60^d\,02^m\,09^s) \\ -\sin.(3q^{\text{iv}}-5q^{\text{v}}-5n^{\text{v}}t+2n^{\text{iv}}t-5\varepsilon^{\text{v}}+2\varepsilon^{\text{iv}}+51^d\,19^m\,29^s)\end{array}\right\}. & \begin{array}{l}2\\3\end{array} \end{array}$$

We may substitute in these two last terms, without any sensible error, q^{iv}
[9082] and q^{v}, for $n^{\text{iv}}t+\varepsilon^{\text{iv}}$ and $n^{\text{v}}t+\varepsilon^{\text{v}}$. The first term will then be confounded with Jupiter's equation of the centre. The second becomes very nearly equal to,

[9083] $$7^s,5882.\sin.(5q^{\text{iv}}-10q^{\text{v}}+51^d,\,19^m\,29^s);$$

and by connecting it with that in [9074], namely,

[9084] $$-4^s,0247.\sin.(5q^{\text{iv}}-10q^{\text{v}}+51^d\,19^m\,29^s),$$

we obtain the following result,

[9085] $$3^s,5635.\sin.(5q^{\text{iv}}-10q^{\text{v}}+51^d\,19^m\,29^s).$$

[9086] Thus we may use q^{iv} and q^{v}, instead of $n^{\text{iv}}t+\varepsilon^{\text{iv}}$ and $n^{\text{v}}t+\varepsilon^{\text{v}}$, in all Jupiter's inequalities except the great inequality.

We shall now consider the analogous inequalities in the motion of Saturn, which are much more sensible than those of Jupiter. To determine them we shall observe that we have found, in [4466], that the motion of Saturn contains the two following large inequalities, depending on the first power of the excentricities;

[9087] $$\left.\begin{array}{l}-182^s,069.\sin.(2n^{\text{v}}t-n^{\text{iv}}t+2\varepsilon^{\text{v}}-\varepsilon^{\text{iv}}-\varpi^{\text{v}}) \\ +417^s,058.\sin.(2n^{\text{v}}t-n^{\text{iv}}t+2\varepsilon^{\text{v}}-\varepsilon^{\text{iv}}-\varpi^{\text{iv}})\end{array}\right\}.\quad(B)\qquad\begin{array}{l}1\\2\end{array}$$

The first of these inequalities depends on the equation of the centre of

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$169^s,266$, it becomes $15^s,1765$; hence the first member of [9081] is,

[9080d] $$-15^s,1765.\sin.(5n^{\text{v}}t-2n^{\text{iv}}t+5\varepsilon^{\text{v}}-2\varepsilon^{\text{iv}}+4^d\,21^m\,20^s)\times\cos.(3q^{\text{iv}}-5q^{\text{v}}+55^d\,40^m\,49^s);$$

reducing this by means of [19] Int., we get the second member of [9081]; observing that the author has given the angle $51^d\,19^m\,29^s$ [9081 line 3] equal to
[9080e] $57^{\circ},0725=51^d\,21^m\,55^s$, being too great by $2^m\,26^s$, from the data he has used. This is corrected in the formulas [9083—9085]. Now substituting q^{iv}, q^{v}, according to the directions in [9082], we find that the second member of [9081] becomes,

[9080f] $$-7^s,5882.\sin.(q^{\text{iv}}+60^d\,02^m\,09^s)+7^s,5882.\sin.(5q^{\text{iv}}-10q^{\text{v}}+51^d\,19^m\,29^s);$$

of which the first may be combined with the equation of the centre, and the second is as in [9083].

Saturn $+2e^{v}.\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v})$, and it may be put under the following form;* [9088]

$$-\frac{182^{s},069}{2e^{v}}.2e^{v}.\sin.(n^{v}t+\varepsilon^{v}-\varpi^{v}+n^{v}t-n^{iv}t+\varepsilon^{v}-\varepsilon^{iv}). \qquad [9089]$$

The inequality in the motion of Saturn [9056],

$$-669^{s},682.\sin.(2n^{iv}t-4n^{v}t+2\varepsilon^{iv}-4\varepsilon^{v}+56^{d}\,10^{m}\,57^{s}), \qquad [9090]$$

which, as we have observed in [9056′], may be considered as a second equation of the centre, will therefore produce, by its substitution in [9089], the following term;

$$\frac{182^{s},069}{2e^{v}}.669^{s},682.\sin.(n^{iv}t-3n^{v}t+\varepsilon^{iv}-3\varepsilon^{v}+56^{d}\,10^{m}\,57^{s}). \qquad [9091]$$

The equation [9087 line 2] arises from the equation of the centre of Jupiter, and it may be put under the following form;

$$\frac{417^{s},058}{2e^{iv}}.2e^{iv}.\sin.(n^{iv}t+\varepsilon^{iv}-\varpi^{iv}+2n^{v}t-2n^{iv}t+2\varepsilon^{v}-2\varepsilon^{iv}). \qquad [9092]$$

The inequality of Jupiter,

$$+169^{s},266.\sin.(3n^{iv}t-5n^{v}t+3\varepsilon^{iv}-5\varepsilon^{v}+55^{d}\,40^{m}\,49^{s}), \quad [9046] \qquad [9093]$$

which, as we have seen in [9046′], is a second equation of the centre of Jupiter, will therefore produce, by its substitution in [9092], the following term;

$$\frac{417^{s},058}{2e^{iv}}.169^{s},266.\sin.(n^{iv}t-3n^{v}t+\varepsilon^{iv}-3\varepsilon^{v}+55^{d}\,40^{m}\,49^{s}). \qquad [9094]$$

Therefore the expressions [9087] produce, as in [9089c], the following inequalities;

$$-5^{s},2568.\sin.(3n^{v}t-n^{iv}t+3\varepsilon^{v}-\varepsilon^{iv}-56^{d}\,10^{m}\,57^{s}) \qquad [9095]$$

$$-3^{s},5594.\sin.(3n^{v}t-n^{iv}t+3\varepsilon^{v}-\varepsilon^{iv}-55^{d}\,40^{m}\,49^{s}). \qquad [9096]$$

The expression of Saturn's true longitude in terms of the mean longitude, contains the inequality,†

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* (4100) The calculation is here made in the same manner as for Jupiter, [9049a, b, &c.], by putting the inequality [9090] under the form, [9089a]

$$-669^{s},682.\sin.(n^{v}t+\varepsilon^{v}+56^{d}\,10^{m}\,57^{s}+2n^{iv}t-5n^{v}t+2\varepsilon^{iv}-5\varepsilon^{v}), \qquad [9089b]$$

which is similar to [9046]; and changing, as in [9049a, b], $2e^{v}$ into $-669^{s},682$, also $-\varpi^{v}$ into $56^{d}\,10^{m}\,57^{s}+2n^{iv}t-5n^{v}t+2\varepsilon^{iv}-5\varepsilon^{v}$; and then [9089] becomes as in [9091]. In like manner we get [9094] from [9092]. Reducing the expressions [9091, 9094] to numbers, they become as in [9095, 9096] respectively. [9089c]

† (4101) This calculation is made in the same manner as for Jupiter, in [9065, &c.]. The second term of [9065] is similar to that in [9097]; those in [9066 lines 3, 4] are [9097a]

[9097] $\frac{13}{12}.(e^{\text{v}})^3.(3n^{\text{v}}t+3\varepsilon^{\text{v}}-3\varpi^{\text{v}})$.

Therefore by putting δe^{v} and $\delta\varpi^{\text{v}}$ for the variations of the excentricity and perihelion, depending upon $5n^{\text{v}}t-2n^{\text{iv}}t$, we shall have the function,

[9098] $\frac{13}{4}.e^{\text{v}}.e^{\text{v}}.\{\delta e^{\text{v}}.\sin.(3n^{\text{v}}t+3\varepsilon^{\text{v}}-3\varpi^{\text{v}})-e^{\text{v}}.\delta\varpi^{\text{v}}.\cos.(3n^{\text{v}}t+3\varepsilon^{\text{v}}-3\varpi^{\text{v}})\}$; (O)

To obtain δe^{v} and $\delta\varpi^{\text{v}}$, we shall consider this inequality of Saturn,

[9099] $-669^s,682.\sin.(2n^{\text{iv}}t-4n^{\text{v}}t+2\varepsilon^{\text{iv}}-4\varepsilon^{\text{v}}+56^d\,10^m\,57^s)$, [9090]

and we shall suppose it to be produced by the variation of the equation of
[9100] the centre and perihelion, in the term $2e^{\text{v}}.\sin.(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}})$; and we shall then have for the expression of this inequality,

[9101] $2\delta e^{\text{v}}.\sin.(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}})-2e^{\text{v}}.\delta\varpi^{\text{v}}.\cos.(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}})$.

Hence it is evident that the function [9098] will become,

[9102] $-\frac{13}{8}.e^{\text{v}}.e^{\text{v}}.669^s,682.\sin.(2n^{\text{iv}}-2n^{\text{v}}t+2\varepsilon^{\text{iv}}-2\varepsilon^{\text{v}}-2\varpi^{\text{v}}+56^d\,10^m\,57^s)$.

This inequality, reduced to numbers, is equal to,

[9103] $-3^s,4402.\sin.(2n^{\text{iv}}t-2n^{\text{v}}t+2\varepsilon^{\text{iv}}-2\varepsilon^{\text{v}}-120^d\,7^m\,17^s)$.

[9104] We have in [4496] the following inequality, corrected as in [4495*a*—*d*];

[9105] $8^s,2645.\sin.(4n^{\text{iv}}t-9n^{\text{v}}t+4\varepsilon^{\text{iv}}-9\varepsilon^{\text{v}}+51^d\,49^m\,37^s)$.

[9106] We have seen, as in [9075, 9076′], that we must change, in all the inequalities of Saturn, $n^{\text{iv}}t+\varepsilon^{\text{iv}}$ into q^{iv}, and $n^{\text{v}}t+\varepsilon^{\text{v}}$ into q^{v}, excepting in the great inequality, and in the following;

[9107] $-699^s,682.\sin.(2n^{\text{iv}}t-4n^{\text{v}}t+2\varepsilon^{\text{iv}}-4\varepsilon^{\text{v}}+56^d\,10^m\,57^s)$. [9099]

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similar to [9098]; the term [9057] is the same as [9100]; the term [9062] is similar to [9099]; the terms [9064] are similar to [9101]. Now if we put,

[9097*b*] $f=2n^{\text{iv}}t-5n^{\text{v}}t+2\varepsilon^{\text{iv}}-5\varepsilon^{\text{v}}+\varpi^{\text{v}}+56^d\,10^m\,57^s$,

the expression [9090] will become, by using [21] Int.,

[9097*c*] $$-669^s,682.\sin.(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}}+f)=-669^s,682.\cos.f.\sin.(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}})$$
$$-669^s,682.\sin.f.\cos.(n^{\text{v}}t+\varepsilon^{\text{v}}-\varpi^{\text{v}}).$$

[9097*d*] Comparing this with [9101], we get $2\delta e^{\text{v}}=-669^s,82.\cos.f$; $2e^{\text{v}}\delta\varpi^{\text{v}}=669^s,82.\sin.f$. Substituting these in [9098], it becomes,

[9097*e*] $-\frac{13}{8}e^{\text{v}}.e^{\text{v}}.669^s,682.\{\sin.(3n^{\text{v}}t+3\varepsilon^{\text{v}}-3\varpi^{\text{v}}).\cos.f+\cos.(3n^{\text{v}}t+3\varepsilon^{\text{v}}-3\varepsilon^{\text{v}}).\sin.f\}$;

and by means of [21] Int. it is reduced to,

[9097*e*′] $-\frac{13}{8}e^{\text{v}}.e^{\text{v}}.669^s,682.\sin.(3n^{\text{v}}t+3\varepsilon^{\text{v}}-3\varpi^{\text{v}}+f)$.

Now re-substituting the value of f [9097*b*], it becomes as in [9102]; and by using the
[9097*f*] value of e^{v} [4080], it changes into [9103]; observing that the value of $\varpi^{\text{v}}=88^d\,9^m\,07^s$, [4081], gives $-2\varpi^{\text{v}}+56^d\,10^m\,57^s=-120^d\,7^m\,17^s$.

If we wish to use q^{iv} and q^{v} [9076′] in this last inequality, we must put it under the form,

$$-669^s{,}682.\sin.(2q^{iv}-4q^{v}-2p^{iv}-4p^{v}+56^d\,10^m\,57^s);$$ [9108]

p^{iv} and $-p^{v}$ [9078] being the two great inequalities of Jupiter and Saturn. The inequality [9108] becomes by development,*

$$\begin{array}{lr} -669^s{,}682.\sin.(2q^{iv}-4q^{v}+56^d\,10^m\,57^s) & 1 \\ +669^s{,}682.(2p^{iv}+4p^{v}).\cos.(2q^{iv}-4q^{v}+56^d\,10^m\,57^s); & 2 \end{array}$$ [9109]

and we have very nearly,

$$\begin{array}{lr} 669^s{,}682.(2p^{iv}+4p^{v}).\cos.(2q^{iv}-2q^{v}+56^d\,10^m\,57^s) & 1 \\ =23^s{,}1960.\left\{\begin{array}{l} \sin.(2q^{iv}-4q^{v}+5n^{v}t-2n^{iv}t+5\varepsilon^{v}-2\varepsilon^{iv}+60^d\,32^m\,17^s) \\ -\sin.(2q^{iv}-4q^{v}-5n^{v}t+2n^{iv}t-5\varepsilon^{v}+2\varepsilon^{iv}+51^d\,49^m\,37^s) \end{array}\right\}. & \begin{array}{r}2\\3\end{array} \end{array}$$ [9110]

We may, in these two last inequalities, change $n^{iv}t+\varepsilon^{iv}$ and $n^{v}t+\varepsilon^{v}$ into q^{iv} and q^{v}; then the first will be confounded with the equation of the centre of Saturn, and the second becomes, [9111]

$$-23^s{,}1960.\sin.(4q^{iv}-9q^{v}+51^d\,49^m\,37^s).$$ [9112]

Connecting it with that in [9105], namely,

$$8^s{,}2645.\sin.(4q^{iv}-9q^{v}+51^d\,49^m\,37^s),$$ [9113]

we obtain the following inequality,

$$-14^s{,}9315.\sin.(4q^{iv}-9q^{v}+51^d\,49^m\,37^s).$$ [9114]

We may thus use q^{iv} and q^{v}, instead of $n^{iv}t+\varepsilon^{iv}$ and $n^{v}t+\varepsilon^{v}$, in all the inequalities of Saturn, except in its great inequality. [9115]

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* (4102) This development is made as in [9076a], by using [60] Int., putting $z=2q^{iv}-4q^{v}+56^d,10^m\,57^s$, and $\alpha=-(2p^{iv}+4p^{v})$; then multiplying the developed value of $\sin.(z+\alpha)$, by $-669^s{,}682$ [9108]; hence we get [9109]. Now the values of p^{iv}, p^{v} [9080a], give $2p^{iv}+4p^{v}=14289^s.\sin.T$; substituting it in the term [9109 line 2], and dividing by the radius in seconds, which gives $\frac{669^s{,}682\times 14289^s}{206265^s}=46^s{,}3920$, it becomes, [9111$a$] [9111$b$]

$$\begin{array}{l} 46^s{,}3920.\sin.T.\cos.(2q^{iv}-4q^{v}+56^d\,10^m\,57^s) \\ =23^s{,}1960.\sin.(2q^{iv}-4q^{v}+56^d\,10^m\,57^s+T)-23^s{,}1960.\sin.(2q^{iv}-4q^{v}+56^d\,10^m\,57^s-T). \end{array}$$ [9111c]

Re-substituting the value of T [9080a], it takes the same form as in [9110 lines 2, 3]; and by using the values of q^{iv}, q^{v} [9111], it becomes,

$$23^s{,}1960.\sin.(q^{v}+60^d\,32^m\,17^s)-23^s{,}1960.\sin.(4q^{iv}-9q^{v}+51^d\,49^m\,37^s).$$ [9111d]

The first of these terms can be connected with the equation of the centre, and the second is as in [9112].

[9116] We must, for greater accuracy, increase q^{v} by the inequality [4468 line 6], namely,

[9117] $$31^{s},025.\sin.(3n^{vi}t - n^{v}t + 3\varepsilon^{vi} - \varepsilon^{v} - 85^{d}\,34^{m}\,12^{s}),$$

[9118] arising from the action of Uranus, and which must be applied to Saturn's mean motion, as we have seen in [4472, &c.].

If we connect the preceding inequalities with those which have been determined in the sixth book, we shall obtain the formulas of the true longitudes of Jupiter and Saturn. Bouvard has compared these formulas
[9119] with observation, by means of the oppositions of Jupiter and Saturn, which he has collected chiefly from those of Bradley and Maskelyne at Greenwich, and those made at the observatory in Paris, in late years. These observations were made with excellent transit instruments, and with the best mural quadrants, during an interval of more than fifty years. They furnish, by their accuracy, as well as by the greatness of the number of observations, the most accurate method of correcting the elements of the elliptical motion. From these sources were obtained, from 1747 to 1804
[9120] inclusively, fifty oppositions of Jupiter and fifty-four oppositions of Saturn, [9122c]. These have given one hundred and four equations of condition, between the corrections of the elliptical elements of the motions of the two planets; and as the value of Saturn's mass is somewhat uncertain, the correction depending upon it was introduced into these equations. It was soon discovered that the value of this mass, given in [4061], must be
[9121] decreased by $\frac{1}{20,232}$, which reduces it to $\frac{1}{3534,08}$, that of the sun being taken for unity [9122d, 4061c]. This important correction, which is evidently indicated by the preceding observations, and by those of Flamsteed, is one of the principal results of this improved theory. The accuracy of these formulas, and the great correctness of this set of observations, must give to this result a preference over those which are deduced from the elongations of the outer satellite, taking into consideration the extreme difficulty of observing these elongations, and the uncertainty of the estimated value of the ellipticity of the orbit of the satellite. The comparison of the formulas given by the theory, with the observed oppositions of Jupiter, has not indicated any correction in the value of its mass [4061].
[9122] In fact, if we consider Pound's observations, as given by Newton in his *Principia*, we shall see that they must give accurately the mass of Jupiter, whilst they will leave a little uncertainty in that of Saturn. Therefore

these formulas produce the same value of the mass of Jupiter, as those which are deduced from the observed elongations of the satellites; and it is curious to see the same result deduced from two methods so entirely different from each other.* I have endeavored to determine, in the same [9123] manner, the correction of the mass of Uranus, in which there is a greater degree of uncertainty than in the mass of Saturn. The observations have not indicated any sensible correction in the value of that mass; but its influence upon Saturn's motion is too small to trust wholly to this result. The oppositions I have just mentioned are very proper to determine the mean motions of Jupiter and Saturn; because the two great inequalities were at their *maximum* during the interval included by these observations,† and they must therefore have varied but little during that interval; so that [9124] the uncertainty which may still remain, as to the magnitude of these inequalities, could not have any sensible influence upon the mean motions deduced from these observations; and I have had the satisfaction of finding that the formulas represent, as well as could be expected, the ancient observations mentioned by Ptolemy, and also the Arabian observations. [9125]

We shall now give those formulas in which we have introduced the corrections of the elliptical elements of both planets, and the mass of Saturn,

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* (4103) Notwithstanding the confidence the author expresses relative to his estimate of the value of the mass of Jupiter [4061], it has been found necessary to increase it [9122a] about a seventieth part, in order to satisfy the corrected measures of the elongations of the satellites by Airy, and the perturbations of the planets Juno, Vesta, &c., as we have already observed in [5980*i*, &c.]; where it is shown that both these methods of ascertaining the mass of the planet, indicate an augmentation in the estimated value in [9122b] [4061]. We may also observe that Bouvard, in the second edition of his tables, published in 1821, uses 126 observations of Jupiter, and 129 observations of Saturn, taken in the oppositions to the sun, and in the quadratures from 1747 to 1814; and that the result of [9122c] these observations gives the masses of Jupiter and Saturn, which we have inserted [9122d] in [4061*d*].

† (4104) Substituting the values [9128, 9129] in the argument of the great inequality [9134], it becomes nearly,

$$5n^{\mathrm{v}}t-2n^{\mathrm{iv}}t+5\varepsilon^{\mathrm{v}}-2\varepsilon^{\mathrm{iv}}+4^{d}\,30^{m}\,30^{s}-t.78^{s}=73^{d}\,48^{m}\,24^{s}+t.23^{m}\,14^{s}. \qquad [9124a]$$

Substituting $t=-3$ and $t=54$, we find the arguments corresponding to 1747, 1804, are nearly 73^{d} and 94^{d}; and the sine of this argument passes the maximum during this [9124b] interval, as in [9124].

[9126] deduced from the equations of condition. *In these formulas* t *represents*
[9127] *any number of Julian years of* $365\frac{1}{4}$ *days, elapsed from the midnight at the*
Epoch of 1750. *commencement of the first of January*, 1750.

FORMULAS FOR THE HELIOCENTRIC MOTION OF JUPITER.

We have,*

Mean longitudes.

	[Mean longitudes in the year $1750+t$.]	Coefficients, January 1, 1800.	
		$\overset{d}{}\ \overset{m}{}\ \overset{s}{}$	d
[9128]	Jupiter, $n^{iv}t+\varepsilon^{iv} = 3^d\,45^m\,47^s{,}5 + t.30^d{,}348999$;	81 52 19,3	30,349084
[9129]	Saturn, $n^{v}t+\varepsilon^{v} = 231^d\,21^m\,53^s{,}9 + t.12^d{,}221421$;	123 05 29,4	12,221148
[9130]	Uranus, $n^{vi}t+\varepsilon^{vi} = 318^d\,34^m\,14^s{,}8 + t.\ 4^d{,}284639$.	173 30 16,4	4,284890

[9128a] * (4105) The coefficients of t [9128, 9129, 9130] agree nearly with those in [4077]. The values in [9132—9138′] agree nearly with those in [4407, 4408′, &c.], or may be
[9128b] easily deduced from them. The expressions of q^{iv}, q^{v} [9137, 9138, &c.], correspond nearly with [4434, 4436, 4491, 4494], changing the mass of Saturn, or the value of $1+\mu^{v}$,
[9128c] as in [9121, &c.]. The corrections in this theory, on account of the error in the signs of some of the terms, were made by the author after he had published the *fourth* volume, and were in fact subsequently printed, as an appendix to the *third* volume [5974, &c.].
[9128d] In consequence of this error of the signs, it becomes necessary to apply the corrections C^{iv}, C^{v}, &c. [4434, 4492], to the great inequalities given by the author in this chapter;
[9128e] it is also necessary to augment the perturbations of Jupiter, arising from the action of Saturn, on account of the increased value of Jupiter's mass [5980*m*, &c.]. This process, of correcting the elements of the orbits, may be considered as constantly in operation, on
[9128f] account of the additional observations which are daily obtained. For by combining these new observations with those previously known, we obtain more accurate means of ascertaining the correct values of the masses of the planets and the elements of their orbits.
[9128g] Pontécoulant, in his *Théorie Analytique du Système du Monde*, vol. 3, page 511, &c., has given the elements of the motions of Jupiter, Saturn and Uranus, in the same form as in this chapter; making the corrections and alterations indicated in [9128*c*, *d*], and putting the epoch [9127] fifty years in advance, as in Bouvard's new tables, so as to make it correspond to the midnight which separates December 31, 1799, from January 1, 1800.
[9128h] *In making these calculations he uses the same values of* a, a', a'', a''', a^{iv}, a^{v}, a^{vi}; n, n', n'', n''', n^{iv}, n^{v}, n^{vi}, *as are given by La Place in* [4079, 4077]; also those of
[9128i] $b_s^{(i)}$ and its differentials [4085—4227]. As it will be more easy for reference to have the perturbations free from the corrections μ^{v}, C^{iv}, C^{v} [9128*c*, *d*], we have inserted the coefficients for the epoch of 1800, as given by Pontécoulant, in the same lines with those
[9128k] given by La Place, in [9128—9149]; and it will be found upon comparison that they differ but very little from each other, except from the changes produced by μ^{v}, C^{iv}, C^{v}. For convenience in making these insertions, we have introduced, in [9133″—9135′], the

These three quantities are the mean longitudes of Jupiter, Saturn and Uranus, counted from the fixed equinox of 1750 [9127], *and reduced to the midnight at the beginning of January* 1, 1750. [9131]

We also have,

[Values corresponding to the year 1750 + t.]	Coefficients, January 1, 1800.			
	d m s	s	s	
$\varpi^{iv} = 10^d 20^m 23^s,1 + t.\,6^s,618603 + t^2.0^s,0002001;$	11 7 36	6,683122	−0,000235	[9132]
$\varpi^{v} = 88^d 09^m 55^s,9 + t.19^s,354599 + t^2.0^s,0001608;$	89 8 20	19,055044	0,000162	[9132′]
$\theta^{iv} = 97^d 56^m 25^s,4 + t.34^s,324429;$	98 25 45	36,582830		[9133]
$\theta^{v} = 111^d 29^m 42^s,2 + t.30^s,675510;$	111 56 07	31,32131		[9133′]
$T = 5n^{v}t - 2n^{iv}t + 5\varepsilon^{v} - 2\varepsilon^{iv};$				[9133″]
$P^{iv} = T + 4^d 30^m 30^s - t.78^s,49 + t^2.0^s,01228;$	3 40 59	−76,2770	−0,0012620	[9134]
$P^{v} = T + 4^d 32^m 45^s - t.77^s,06 + t^2.0^s,01178;$	3 38 32	−76,598	−0,001164	[9134′]
$A^{iv} = (1205^s,40 - t.0^s,03618 + t^2.0^s,0000349).\sin.P^{iv}$	s 1187,247	−0,04845	0,00000226	[9135]
$\quad -13^s,17.\sin.2P^{iv};$	−12,21854			
$A^{v} = (-2952^s,10 + t.0^s,0887 - t^2.0^s,0000821).\sin.P^{v}$	−2906,661	0,11411	−0,00000052	[9135′]
$\quad +(30^s,69 - t.0^s,0017).\sin.2P^{v}$	29,76156			
$\quad +31^s,03.\sin.(3n^{vi}t - n^{v}t + 3\varepsilon^{vi} - \varepsilon^{v} - 85^d 34^m 12^s).$	30,894		d m −87 28	

ϖ^{iv}, ϖ^{v}, *being the longitudes of the perihelion, and* θ^{iv}, θ^{v}, *the longitudes of the ascending nodes, counted from the same equinox* [9131], *and at the same epoch;* we shall then have, [9136]

$$q^{iv} = n^{iv}t + \varepsilon^{iv} + A^{iv}; \tag{9137}$$

$$q^{v} = n^{v}t + \varepsilon^{v} + A^{v}; \tag{9138}$$

$$q^{vi} = n^{vi}t + \varepsilon^{vi}. \tag{9138'}$$

The annual precession of the equinoxes being supposed equal to $50^s,1$, [3380*a*]; *the true longitude of Jupiter* v^{iv}, *reckoned upon its orbit and counted from the mean equinox, will be,* [9139]

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additional symbols T, P^{iv}, P^{v}, A^{iv}, A^{v}, which are not in the original work; by this means the formulas [9137, 9138] are given in an abridged form, instead of inserting in them the complete values of A^{iv}, A^{v}. [9128*l*]

Longitude of Jupiter.

[9140]

[Values corresponding to the year 1750+t.]	Coefficients, January 1, 1800.				
$v^{\mathrm{iv}} = q^{\mathrm{iv}} + t.50^s,1$	$50^s,2235$				1
$+\{+(19833^s,75+t.0^s,6269).\sin.(q^{\mathrm{iv}}-\varpi^{\mathrm{iv}})$	19862,80		$0^s,63242$		2
$+(595^s,69+t.0^s,0376).\sin.2.(q^{\mathrm{iv}}-\varpi^{\mathrm{iv}})$	597,55		0,03807		3
$+(24^s,81+t.0^s,0023).\sin.3.(q^{\mathrm{iv}}-\varpi^{\mathrm{iv}})$	24,92		0,00240		4
$+(1^s,18+t.0^s,0002).\sin.4.(q^{\mathrm{iv}}-\varpi^{\mathrm{iv}})$	1,19		0,00016		5
$+(0^s,06 \ldots\ldots).\sin.5.(q^{\mathrm{iv}}-\varpi^{\mathrm{iv}})\}$	0,06				6
$+\{-80^s,14.\sin.(q^{\mathrm{iv}}-q^{\mathrm{v}}-1^d\,09^m)$	−84,63		$-1^d\ 6^m$		7
$+198^s,81.\sin.(2q^{\mathrm{iv}}-2q^{\mathrm{v}}-1^d\,10^m)$	209,10		−1 10		8
$+16^s,23.\sin.(3q^{\mathrm{iv}}-3q^{\mathrm{v}})$	16,32				9
$+3^s,73.\sin.(4q^{\mathrm{iv}}-4q^{\mathrm{v}})$	3,75				10
$+1^s,68.\sin.(5q^{\mathrm{iv}}-5q^{\mathrm{v}}+11^d\,57^m)$	1,69		12 11		11
$+0^s,41.\sin.(6q^{\mathrm{iv}}-6q^{\mathrm{v}})$	0,41				12
$+0^s,17.\sin.(7q^{\mathrm{iv}}-7q^{\mathrm{v}})\}$	0,16				13
$+\{+(131^s,56+t.0^s,0066).\sin.(q^{\mathrm{iv}}-2q^{\mathrm{v}}-13^d\,18^m+t.15^s,26)$	132,39	$0^s,067$	$-12^d\ 20^m$	$15^s,7$	14
$+17^s,18.\sin.(2q^{\mathrm{iv}}-4q^{\mathrm{v}}+57^d\,12^m)$	17,29		57 12		15
$+3^s,39.\sin.(5q^{\mathrm{iv}}-10q^{\mathrm{v}}+51^d\,22^m)\}$	3,29		54 30		16
$+\{+(82^s,86-t.0^s,0045).\sin.(2q^{\mathrm{iv}}-3q^{\mathrm{v}}-61^d\,56^m+t.26^s,32)$	83,45	−0,005	−60 54	27,1	17
$-1^s,57.\sin.(4q^{\mathrm{iv}}-6q^{\mathrm{v}}+54^d\,26^m)\}$	−1,58		54 26		18
$+(160^s,93-t.0^s,0042).\sin.(3q^{\mathrm{iv}}-5q^{\mathrm{v}}+55^d\,41^m+t.50^s,51)$	161,15	0,0014	58 11	45,31	19
$-15^s,18.\sin.(3q^{\mathrm{iv}}-4q^{\mathrm{v}}-62^d\,49^m)$	−15,28		−61 46		20
$+12^s,18.\sin.(3q^{\mathrm{iv}}-2q^{\mathrm{v}}-8^d\,49^m)$	12,27		−9 35		21
$+9^s,42.\sin.(3q^{\mathrm{v}}-q^{\mathrm{iv}}+68^d\,12^m)$	9,50		69 45		22
$+\{+10^s,95.\sin.(q^{\mathrm{v}}+44^d\,57^m)$	11,05		43 56		23
$-5^s,15.\sin.(2q^{\mathrm{v}}+45^d\,42^m)\}$	−5,44		43 55		24
$+10^s,94.\sin.(4q^{\mathrm{iv}}-5q^{\mathrm{v}}+58^d\,1^m)$	11,12		59 23		25
$-5^s,10.\sin.(2q^{\mathrm{iv}}-q^{\mathrm{v}}+15^d\,25^m)$	−5,13		−17 5		26
$+1^s,21.\sin.(4q^{\mathrm{iv}}-3q^{\mathrm{v}}-2^d\,41^m)$	+1,21		−3 28		27
$-0^s,87.\sin.(5q^{\mathrm{iv}}-6q^{\mathrm{v}}+66^d\,9^m)$	−0,88		−65 7		28
$+1^s,00.\sin.(q^{\mathrm{iv}}+q^{\mathrm{v}}+45^d\,29^m)$	+0,74		+67 4		29
$+\{-1^s,05.\sin.(q^{\mathrm{iv}}-q^{\mathrm{vi}})$	−1,14				30
$+0^s,43.\sin.(2q^{\mathrm{iv}}-2q^{\mathrm{vi}})$	+0,59				31
$+0^s,05.\sin.(3q^{\mathrm{iv}}-3q^{\mathrm{vi}})\}$;	+0,05				32

q^{vi} being equal to $n^{vi}+\varepsilon^{vi}$.* In the preceding formula, we have included between a parenthesis all the arguments which can be reduced to the same table. The reduction to the true ecliptic is made by the usual methods, [675 or 3800]; it is here equal to, [9141]

$$-27^s{,}15.\sin.(2v^{iv}-2\theta^{iv}). \qquad [-27^s{,}15] \qquad [9142]$$

The radius vector of Jupiter r^{iv} is given by the following formula;

[Values corresponding to the year $1750+t$.]	Coefficients, Jan. 1, 1800.		
$r^{iv}=5{,}208735+t.0{,}0000003718$	5,208760	0,0000384	1
$+\Big\{+(-0{,}249994-t.0{,}00000789).\cos.(q^{iv}-\varpi^{iv})$	—0,250358	—0,0007964	2
$+(-0{,}006004-t.0{,}0000003718).\cos.2.(q^{iv}-\varpi^{iv})$	—0,006022	—0,0000384	3
$+(-0{,}000217-t.0{,}0000000206).\cos.3.(q^{iv}-\varpi^{iv})$	—0,000218	—0,0000021	4
$-0{,}000010.\cos.4.(q^{iv}-\varpi^{iv})\Big\}$	—0,000093	—0,0000000	5
$+\Big\{+0{,}000652.\cos.(q^{iv}-q^{v}-1^d\,21^m)$	0,000647	angles d m 1 21	6
$-0{,}002783.\cos.(2q^{iv}-2q^{v}-1^d\,02^m)$	—0,002771	1 02 omitted.	7
$-0{,}000287.\cos.(3q^{iv}-3q^{v})$	—0,000289		8
$-0{,}000074.\cos.(4q^{iv}-4q^{v})$	These three terms are omitted; also the coefficients of t in the arguments of the rest of the terms.		9
$-0{,}000026.\cos.(5q^{iv}-5q^{v})$			10 [9143]
$-0{,}000010.\cos.(6q^{iv}-6q^{v})\Big\}$			11
$+\Big\{-0{,}000264.\cos.(q^{iv}-2q^{v}-22^d\,24^m+t.18^s{,}8)$	—0,000272	d m —35 49	12 Radius vector of Jupiter.
$-0{,}000096.\cos.(2q^{iv}-4q^{v}+51^d\,04^m)\Big\}$	omitted.		13
$-0{,}000879.\cos.(2q^{iv}-3q^{v}-62^d\,50^m+t.26^s{,}2)$	—0,000919	—67 55	14
$-(0{,}002008-t.0{,}0000000502).\cos.\Big\{\begin{matrix}3q^{iv}-5q^{v}\\+55^d36^m+t.50^s{,}4\end{matrix}\Big\}$	—0,002020	58 7	15
$+0{,}000236.\cos.(3q^{iv}-4q^{v}-62^d\,09^m)$	0,000238	—61 12	16
$-0{,}000126.\cos.(3q^{iv}-2q^{v}-7^d\,35^m)$	—0,000129	—12 4	17
$+\Big\{-0{,}000068.\cos.(q^{v}+29^d\,13^m)$	—0,000065	29 13	18
$+0{,}000077.\cos.(2q^{v}+10^d\,55^m)\Big\}$	+0,000074	11 1	19
$+0{,}000095.\cos.(4q^{iv}-5q^{v}-14^d\,23^m)$	+0,000091	—14 23	20
$-0{,}000264.\cos.(5q^{v}-2q^{iv}-12^d\,09^m)$	—0,000292	—15 33	21

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* (4106) This value of v^{iv} is easily deduced from the formulas [4388—4446], corrected for the change in Saturn's mass [9121, &c.], and using the corrected values of e^{iv}, similar to [4407]. The elliptical values depending on $q^{iv}-\varpi^{iv}$, and its multiples, [9143a]

Lastly, the heliocentric latitude of Jupiter, above the true ecliptic, is given by the formula,

[Values correspooding to the year 1750 + t.]

[9144]

Latitude of Jupiter.

$$
\begin{aligned}
s^{\text{iv}} = & \,(4742^s,8 - t.0^s,22606).\sin.(v^{\text{iv}} - \theta^{\text{iv}}) \\
& + 0^s,63.\sin.(\,q^{\text{iv}} - 2q^{\text{v}} - 54^d\,16^m) \\
& + 1^s,06.\sin.(2q^{\text{iv}} - 3q^{\text{v}} - 54^d\,16^m) \\
& + 3^s,75.\sin.(3q^{\text{iv}} - 5q^{\text{v}} + 59^d\,30^m) \\
& - 0^s,53.\sin.(q^{\text{v}} + 54^d\,16^m).
\end{aligned}
$$

Coefficients, Jan. 1, 1800.

s	s	
4742,7	−0,22606	1
0,6351	d m −53 54	2
1,0711	−53 54	3
11,5986	+50 26	4
−0,9978	+53 54	5

FORMULAS OF THE HELIOCENTRIC MOTION OF SATURN.

The longitude of Saturn in its orbit v^{v}, counted from the mean equinox, is given by the following formula;*

[9143b] being found as in formula [668]. This has no difficulty except the length of the calculation, so that we shall not enter into any particular detail on the subject. The same may be observed relative to the radius vector [9143]; the chief term of its constant part, 5,208735, is given in [4451]; other terms of r are in [4389, 4393, 4448, 4450]. The [9143c] elliptical terms depending on $q^{\text{iv}} - \varpi^{\text{iv}}$, and its multiples, are found from [659], which, by means of the term $\frac{1}{2}e^2$, and its value, similar to that in [4407], increases the constant part before mentioned, and produces also the term $t.0,0000003718$. Lastly, we get the heliocentric latitude [9144] by connecting the inequalities [4457—4458] with the chief [9143d] term $\varphi.\sin.(v^{\text{iv}} - \theta^{\text{iv}})$, of its elliptical motion, φ being given in [4082]; observing that we may, when necessary, use reductions similar to those in [4017—4021].

* (4107) The formulas for the motion of Saturn [9145—9148], are found as in the last note for Jupiter; using the perturbations in longitude [4463—4505]. Those of the [9145a] radius vector are given in [4464, 4467, 4470, 4471, 4507—4510]; and those of the latitude, in [4511—4518]; the elliptical parts being found by the same formulas as for Jupiter.

[Values corresponding to the year 1750 + t.]	Coefficients, January 1, 1800.					
$v^{\text{v}} = q^{\text{v}} + t.50^s,1$	$\overset{s}{50,2235}$				1	
$+\left\{+(23219^s,52-t.1^s,2854).\sin.\ (q^{\text{v}}-\varpi^{\text{v}})\right.$	23154,40		$\overset{s}{-1,2783}$		2	
$+(\ 816^s,49-t.0^s,0905).\sin.2.(q^{\text{v}}-\varpi^{\text{v}})$	811,96		−0,0901		3	
$+(\ 39^s,81-t.0^s,0066).\sin.3.(q^{\text{v}}-\varpi^{\text{v}})$	39,48		−0,0066		4	
$+(\ 2^s,22-t.0^s,0005).\sin.4.(q^{\text{v}}-\varpi^{\text{v}})$	2,19		−0,0005		5	
$\left.+(\ 0^s,13\ .\ .\ .\ .\ .\ ).\sin.5.(q^{\text{v}}-\varpi^{\text{v}})\right\}$					6	
$+\left\{+\ 28^s,97.\sin.(\ q^{\text{iv}}-\ q^{\text{v}}+78^d\,03^m)\right.$	29,40		$\overset{d\ \ m}{77\ 45}$		7	
$-\ 29^s,91.\sin.(2q^{\text{iv}}-2q^{\text{v}}-\ 5^d\,42^m)$	−31,89				8	
$-\ \ 6^s,57.\sin.(3q^{\text{iv}}-3q^{\text{v}})$	−6,65				9	
$-\ \ 1^s,97.\sin.(4q^{\text{iv}}-4q^{\text{v}})$	−1,99				10	
$-\ \ 0^s,70.\sin.(5q^{\text{iv}}-5q^{\text{v}})$	−0,71				11	
$-\ \ 0^s,27.\sin.(6q^{\text{iv}}-6q^{\text{v}})$	−0,27				12	
$\left.-\ \ 0^s,12.\sin.(7q^{\text{iv}}-7q^{\text{v}})\right\}$	−0,12				13	
$+(-418^s,33-t.0^s,0221).\sin.(\ q^{\text{iv}}-2q^{\text{v}}-14^d\,49^m+t.13^s,50)$	−424,83	$\overset{s}{-0,0277}$	$\overset{d\ \ m}{-13\ 57}$	$\overset{s}{13,89}$	14	Longitude of Saturn.
$+(-669^s,68+t.0^s,0155).\sin.(2q^{\text{iv}}-4q^{\text{v}}+56^d\,11^m+t.49^s,50)$	−652,59	0,03817	59 34	−60,76	15	
$+(-\ 48^s,29+t.0^s,0004).\sin.(3q^{\text{v}}-\ q^{\text{iv}}+77^d\,50^m-t.34^s,55)$	−48,89	0,000366	78 4	−34,55	16	[9145]
$+(-\ 24^s,57+t.0^s,0044).\sin.(2q^{\text{iv}}-3q^{\text{v}}+14^d\,48^m+t.12^s,39)$	−24,37		$\overset{d\ \ m}{20\ 45}$		17	
$+11^s,28.\sin.(\ q^{\text{iv}}+85^d\,36^m)$	11,44		84 36		18	
$-14^s,93.\sin.(4q^{\text{iv}}-9q^{\text{v}}+51^d\,50^m)$	−13,12		55 55		19	
$+\ \ 4^s,90.\sin.(3q^{\text{iv}}-4q^{\text{v}}-62^d\,47^m)$	+4,966		−61 45		20	
$+\ \ 3^s,01.\sin.(2q^{\text{iv}}-\ q^{\text{v}}+31^d\,42^m)$	+3,06		+30 45		21	
$+\ \ 2^s,94.\sin.(3q^{\text{iv}}-5q^{\text{v}}+57^d\ \ 9^m)$	+2,97		59 12		22	
$+\ \ 1^s,42.\sin.(4q^{\text{iv}}-5q^{\text{v}}-62^d\,56^m)$	+1,44		−61 57		23	
[omitted] $\sin.(5q^{\text{iv}}-6q^{\text{v}}-61^d\,53^m)$	0,53		−61 53		24	
$+\left\{-\ \ 9^s,25.\sin.(\ q^{\text{v}}-\ q^{\text{vi}})\right.$	−10,07				25	
$+14^s,45.\sin.(2q^{\text{v}}-2q^{\text{vi}})$	+15,73				26	
$+\ \ 1^s,91.\sin.(3q^{\text{v}}-3q^{\text{vi}}-68^d\,27^m)$	2,06		−69 6		27	
$+\ \ 0^s,31.\sin.(4q^{\text{v}}-4q^{\text{vi}})$	0,34				28	
$\left.+\ \ 0^s,09.\sin.(5q^{\text{v}}-5q^{\text{vi}})\right\}$	omitted				29	
$+27^s,37.\sin.(2q^{\text{v}}-3q^{\text{vi}}+23^d\,56^m)$	29,63		24 31		30	
$+\ \ 9^s,86.\sin.(\ q^{\text{v}}-2q^{\text{vi}}+72^d\,12^m)$	10,71		73 11		31	
$+\ \ 1^s,52.\sin.(3q^{\text{v}}-2q^{\text{vi}}-88^d\,09^m)$	1,65		−89 8		32	
$+\ \ 1^s,36.\sin.(\ q^{\text{vi}}-41^d\,38^m)$	1,48		−42 36		33	

The reduction to the ecliptic is,

[9146] $$-97^s{,}83.\sin.(2v^{v}-2\theta^{v}). \qquad [-97^s{,}83]$$

The radius vector of Saturn r^{v} is given by the following formula;

[9147] Radius vector of Saturn.

[Values corresponding to the year 1750+t.]	Coefficients, January 1, 1800.		
$r^{v}=9{,}557833-t.0{,}00000167$	9,557777	0,0000167	1
$+\big\{+(-0{,}536467+t.0{,}00002963).\cos.(q^{v}-\varpi^{v})$	—0,534988	0,00002966	2
$+(-0{,}015090+t.0{,}00000167).\cos.(2q^{v}-\varpi^{v})$	—0,015005	0,00000167	3
$+(-0{,}000639+t.0{,}00000011).\cos.(3q^{v}-3\varpi^{v})$	—0,000634	0,00000011	4
$-0{,}000032.\cos.(4q^{v}-4\varpi^{v})$	—0,000032		5
$-0{,}000340.\cos.(q^{v}-10^d\,21^m)\big\}$	—0,000339	$-10^d\,21^m$	6
$+\big\{+0{,}00811.\cos.(q^{iv}-q^{v}+3^d\,58^m)$	0,00763	4 15	7
$+0{,}00138.\cos.(2q^{iv}-2q^{v})$	0,00140		8
$+0{,}00032.\cos.(3q^{iv}-3q^{v})$	0,00034		9
$+0{,}00010.\cos.(4q^{iv}-4q^{v})$	omitted		10
$+0{,}00004.\cos.(5q^{iv}-5q^{v})\big\}$			11
$+(0{,}00535+t.0{,}00000027).\cos.\left\{\begin{array}{l} q^{iv}-2q^{v} \\ -11^d\,58^m+t.14^s{,}7 \end{array}\right\}$	0,00542; $-11^d\,10^m$	- - - -; - - - -	12
$+(0{,}01520-t.0{,}00000034).\cos.\left\{\begin{array}{l} 2q^{iv}-4q^{v} \\ +56^d\,0^m+t.49^s{,}1 \end{array}\right\}$	0,01479; $59^d\,28^m$	—0,000000734; $-63^s{,}24$	13
$+0{,}00117.\cos.(3q^{v}-q^{iv}-90^d\,12^m)$	—0,00119	$-90^d\,12^m$	14
$-0{,}00138.\cos.(3q^{v}-2q^{iv}+23^d\,19^m)$	—0,00155	38 55	15
$-0{,}00022.\cos.(4q^{v}-3q^{iv}+61^d\,21^m)$	—0,00021	61 23	16
$+0{,}00352.\cos.(5q^{v}-2q^{iv}+13^d\,02^m)$	0,00095	32 32	17
$+\big\{+0{,}00015.\cos.(q^{v}-q^{vi})$	0,00016		18
$-0{,}00040.\cos.(2q^{v}-2q^{vi})$	—0,00043		19
$-0{,}00005.\cos.(3q^{v}-3q^{vi})\big\}$	omitted		20
$-0{,}00061.\cos.(2q^{v}-3q^{vi}+23^d\,44^m).$	—0,00066	+23 44	21

The heliocentric latitude of Saturn, above the true ecliptic, is,

[Values corresponding to the year 1750 + t.]

$$s^{\mathrm{v}} = + \left\{ \begin{array}{l} (8990^{s},42 - t.0^{s},15514).\sin.(v^{\mathrm{v}} - \theta^{\mathrm{v}}) \\ -0^{s},71.\sin.(3v^{\mathrm{v}} - 3\theta^{\mathrm{v}}) \end{array} \right\}$$

$$+ \left\{ \begin{array}{l} -3^{s},14.\sin.(\ q^{\mathrm{iv}} - 2q^{\mathrm{v}} - 54^{d}\,16^{m}) \\ -9^{s},16.\sin.(2q^{\mathrm{iv}} - 4q^{\mathrm{v}} + 59^{d}\,30^{m}) \end{array} \right\}$$

$$+ 0^{s},52.\sin.(2q^{\mathrm{iv}} - 3q^{\mathrm{v}} - 54^{d}\,16^{m})$$

$$+ 1^{s},79.\sin.(\ q^{\mathrm{iv}} + 54^{d}\,16^{m})$$

$$-0^{s},66.\sin.(2q^{\mathrm{iv}} - 3q^{\mathrm{vi}} - 54^{d}\,9^{m}).$$

[9148]

Latitude of Saturn.

Coefficients, Jan. 1, 1800.

$8990^{s},3$	$-0^{s},155137$	1
omitted		2
$-3,18$	$-53^{d}\ 54^{m}$	3
$-9,31$	$60\ 24$	4
$0,53$	$-53\ 54$	5
$1,81$	$53\ 54$	6
$-0,67$	$-54\ \ 8$	7

The 104 oppositions before mentioned, are represented by these formulas with a remarkable degree of accuracy.* The greatest error does not amount

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* (4108) The method of quadratures by single integrals, which is found so useful in computing the perturbations of the orbits of the small planets and comets [7929], and in the development of the function [8908*d*], may also be applied to many similar objects; as, for example, to that of the development of u, r, v, in terms of nt, as in [657, 659, 668], [9146*a*] and to other functions of the like nature. The same process, by means of a double integration relative to two unknown quantities, may be extended to cases of a much more complicated form; as, for example, to that of the development of the function R [957], in terms depending on the angle $i'n't - int + A$ [957$^{\mathrm{viii}}$], or to the expressions of δv, δr, [9146*b*] &c., like those in [3809, 3821, &c.], depending on the same angle, and to others of a like form. This method of double integration by quadratures, which has the important advantage of including all the powers and products of the excentricities in the values of the terms which occur under the signs of integration, has been used by Hansen in the computation of the perturbations of the planets Jupiter and Saturn, as we may see in his treatise on the subject, entitled *Untersuchung üeber die gegenseitigen Störungen des Jupiters und Saturns*, which gained the prize of the Academy of Arts and Sciences of [9146*c*] Berlin, in 1830. The manner of making such calculations has since been treated of by Poisson, in the *Connaissance des tems* for 1836, and also by Pontécoulant, in the work mentioned in [8919*g*]. This way of computing the inequalities of the motions of Jupiter [9146*d*] and Saturn, is so entirely different from that of La Place, that it becomes a very important means of verification, not only of the great inequalities, but also of such others as require considerable care and attention in making the developments, when we wish to include [9146*e*] every sensible quantity. We shall therefore explain the principles of this calculation, without entering into any numerical details, which have no other difficulty than their great labor. We shall also give, as examples of the process of single integration, the developments of u, r, v, mentioned in [9146*a*]; observing that a similar method was [9146*f*] used by the early geometricians, in treating of the problem of the three bodies, in the middle of the eighteenth century.

[9148′] to twelve sexagesimal seconds; and it is not twenty years since the errors of

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If we put for abridgment $nt=\mathrm{t}$, we shall have, as in [605′, 606], for the elements of the elliptical motion, the following formulas, the longitudes being counted from the perihelion, by supposing $\varepsilon=0$, $\varpi=0$, $\mu=1$; [9146g]

[9146h] $$na^{\frac{3}{2}}=1\,;$$

[9146i] $$nt=u-e.\sin.u=\mathrm{t}\,;$$

[9146k] $$r=a.(1-e.\cos.u)\,;$$

[9146l] $$\mathrm{tang}.\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}.\mathrm{tang}.\tfrac{1}{2}u.$$

These elements are developed, in [657, 659, 668], into series of the following forms;

[9146m] $$u-\mathrm{t}=\quad+A_1.\sin.\mathrm{t}+A_2.\sin.2\mathrm{t}+A_3.\sin.3\mathrm{t}\ \ldots\ +A_i.\sin.i\mathrm{t}+\&c.\,;$$

[9146n] $$r=B_0+B_1.\cos.\mathrm{t}+B_2.\cos.2\mathrm{t}+B_3.\cos.3\mathrm{t}\ \ldots\ +B_i.\cos.i\mathrm{t}+\&c.\,;$$

[9146o] $$v-\mathrm{t}=\quad+C_1.\sin.\mathrm{t}+C_2.\sin.2\mathrm{t}+C_3.\sin.3\mathrm{t}\ \ldots\ +C_i.\sin.i\mathrm{t}+\&c.$$

Multiplying the second of these equations by $\cos.i\mathrm{t}$, and the two others by $\sin.i\mathrm{t}$, then integrating from $\mathrm{t}=0$ to $\mathrm{t}=\pi$, we get, as in [8908m, k],

[9146p] $$A_i=\frac{2}{\pi}.\int_0^{\pi}(u-\mathrm{t}).\sin.i\mathrm{t}.d\mathrm{t}\,;$$

[9146q] $$B_i=\frac{2}{\pi}.\int_0^{\pi}r.\cos.i\mathrm{t}.d\mathrm{t}\,;$$

[9146r] $$C_i=\frac{2}{\pi}.\int_0^{\pi}(v-\mathrm{t}).\sin.i\mathrm{t}.d\mathrm{t}\,;$$

[9146s] $$B_0=\frac{1}{\pi}.\int_0^{\pi}r d\mathrm{t}.$$

These last expressions contain the three variable quantities t, u, v, and they may be easily reduced, so as to have only the excentric anomaly u, and its differential, with the constant elements a, e. For we have generally,

[9146t] $$\int(u-\mathrm{t}).\sin.i\mathrm{t}.d\mathrm{t}=-\frac{1}{i}.(u-\mathrm{t}).\cos.i\mathrm{t}+\frac{1}{i}.\int(du-d\mathrm{t}).\cos.i\mathrm{t}\,;$$

[9146u] as is easily proved by taking its differential and reducing. Now $u-\mathrm{t}$ vanishes when $\mathrm{t}=0$, or $\mathrm{t}=\pi$, as is evident from [9146m]; therefore if we take the integral [9146t]

[9146v] between these limits, the term without the sign $\int$ will vanish. Moreover the term $-\frac{1}{i}.\int\cos.i\mathrm{t}.d\mathrm{t}=-\frac{1}{i^2}.\sin.i\mathrm{t}$, vanishes at the same limits, observing that the least

[9146w] value of i, in [9146m], is $i=1$; hence the integral [9146t] becomes,

[9146x] $$\int_0^{\pi}(u-\mathrm{t}).\sin.i\mathrm{t}.d\mathrm{t}=\frac{1}{i}.\int_0^{\pi}\left(\frac{du}{d\mathrm{t}}\right).\cos.i\mathrm{t}.d\mathrm{t}\,;$$

[9146y] and by substituting it in [9146p], we get [9147a]. In like manner we get [9147b] from

the best tables of Saturn sometimes exceeded thirteen hundred sexagesimal [9148″]

[9146r], or we may obtain it more simply by derivation from [9147a]; remarking that if we change u into v, in the value of A_i [9146p], it will become equal to that of C_i, [9146z]
[9146r]; and by making the same changes in A_i [9147a], we shall get C_i [9147b];

$$A_i = \frac{2}{i\pi}.\int_0^\pi \left(\frac{du}{dt}\right).\cos.it.dt;$$ [9147a]

$$C_i = \frac{2}{i\pi}.\int_0^\pi \left(\frac{dv}{dt}\right).\cos.it.dt.$$ [9147b]

The differential of the equations [9146i, l], give,

$$dt = (1-e.\cos.u).du; \qquad \text{or} \qquad du = \frac{dt}{1-e.\cos.u};$$ [9147c]

$$dv = \frac{\sqrt{1-e^2}}{1-e.\cos.u}.du = \frac{\sqrt{1-e^2}}{(1-e.\cos.u)^2}.dt;$$ [9147d]

observing that the first expression of dv [9147d], is easily deduced from the differential of the equation [7856], which is computed in [7884, &c.], and given in [7885], supposing [9147e]
u, e, ϖ, to be the variable quantities, and $d.(v-\varpi) = -d\varpi$; but to conform to the present question, we must put $de=0$, and $d.(v-\varpi)=dv$, which changes $-d\varpi$ into dv; by this means [7885] gives dv [9147d]. The second form of dv is easily deduced from [9147f]
the first, by using the value of du [9147c]. From [9147c, d] we obtain,

$$\left(\frac{du}{dt}\right) = \frac{1}{1-e.\cos.u}; \qquad \left(\frac{dv}{dt}\right) = \frac{\sqrt{1-e^2}}{(1-e.\cos.u)^2}.$$ [9147g]

Substituting the values [9147g], also $t = u - e.\sin.u$, $r = a.(1-e.\cos.u)$, $dt = (1-e.\cos.u).du$ [9146i, k, 9147c], in the expressions [9147a, b, 9146q, s], we [9147h]
obtain the following formulas, which contain no other variable quantity than the excentric anomaly u;

$$A_i = \frac{2}{i\pi}.\int_0^\pi .\cos.i(u-e.\sin.u).du;$$ [9147i]

$$B_i = \frac{2a}{\pi}.\int_0^\pi (1-e.\cos.u)^2.\cos.i(u-e.\sin.u).du;$$ [9147k]

$$C_i = \frac{2\sqrt{1-e^2}}{i\pi}.\int_0^\pi \frac{\cos.i.(u-e.\sin.u)}{1-e.\cos.u}.du;$$ [9147l]

$$B_0 = \frac{a}{\pi}.\int_0^\pi (1-e.\cos.u)^2.du = a.(1+\tfrac{1}{2}e^2);$$ [9147m]

the limits of these integrals relative to u, being the same as those for t; because when $t=0$ we have $u=0$, and when $t=\pi$ we have $u=\pi$, as is evident from the expression [9147n]
of t [9147h], or from that of $u-t$ [9146m]. The second form of B_0 [9147m] is easily [9147o]
deduced from the first, by putting $\cos.^2u = \frac{1}{2}+\frac{1}{2}.\cos.2u$, then developing and performing

[9148‴] seconds. These formulas represent also, with as great a degree of accuracy

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[9147p] the integrations. The general integral of the term under the sign f, in the second member of [9147k], is,

[9147q] $$\frac{1}{i}.(1-e.\cos.u).\sin.i(u-e.\sin.u)-\frac{e}{i}.f\sin.u.\sin.i(u-e.\sin.u).du;$$

[9147r] as is easily proved by taking its differential. Now at the limits $u=0$, $u=\pi$, the term without the sign f, in [9147q], vanishes; hence we find by substitution, that the integral [9147k], becomes,

[9147s] $$B_i=-\frac{2ae}{i\pi}.\int_0^{\pi}\sin.u.\sin.i(u-e.\sin.u).du.$$

The differential of [9147i] relative to e, gives,

[9147t] $$\left(\frac{dA_i}{de}\right)=\frac{2}{\pi}.\int_0^{\pi}\sin.u.\sin.i(u-e.\sin.u).du;$$

[9147u] comparing this with the value of B_i [9147s], we obtain, for all values of i except $i=0$, the following formula, as it is given by Poisson;

[9147v] $$B_i=-\frac{ae}{i}.\left(\frac{dA_i}{de}\right);$$

[9147w] whence we may very easily deduce the coefficient of cos.int, in the expression of r, [659], from that of sin.int in u [657]. We may also, by developing the integrals [9147i, k, l], obtain the expressions of u, r, v [657, 659, 668], arranged according to the

[9147x] powers of e, as we may see in the paper of Poisson, mentioned above; or we may find the corresponding numerical results by quadratures [7929x], when e is so large that the terms of the series do not approximate with sufficient rapidity to render the calculation easy.

We shall now show how a formula, adapted to double integrations, may be investigated,

[9147y] similar to that in [7929x] for single integrations. This is done by an extension of the method given in [7929f, &c.], for computing the area of a plane parabolic surface; to that

[9147z] of finding, by quadratures, the solid contents of a body whose upper surface is of a parabolic nature, its base a rectangular parallelogram, and its sides plane surfaces perpendicular to the base. We shall put as usual x, y, z, for the rectangular co-ordinates of the parabolic

[9148a] surface, the axes of x, y, being parallel to the sides of the base, and the axis of z

[9148b] perpendicular to the base, the origin of the co-ordinates being at one of the lower corners of the base. The ordinate z of the parabolic surface, is a function of x, y, which we

[9148c] may represent generally by $z=f(x,y)$, and we shall suppose the greatest values of x and y to be respectively $x=a$, $y=b$. Then we have by the usual formulas,

[9148d] $\int_0^a\int_0^b dx.dy.f(x,y)$, for the solid contents of the parabolic body; the integrals relative to

[9148e] x being taken from $x=0$ to $x=a$, and those relative to y, from $y=0$ to $y=b$. To obtain this integral by quadratures, we shall suppose the side a to be divided into m

[9148f] equal parts, each of them being represented by w, so that we shall have $mw=a$. In

[9148g] like manner, we shall suppose the side b to be divided into n equal parts w', making

as that of the observations themselves, the observations of Flamsteed, those [9149]

$nw' = b$. Through these points planes are to be drawn, perpendicular to the base, intersecting each other at right angles, and dividing the solid into mn parallelopipedons, [9148h]
each of them standing upon an equal base, whose area is represented by ww'; but with variable heights corresponding to the ordinate z, which is a function of x,y; so that if we [9148i]
represent by $z_{i,i'}$ the value of $z = f(x,y)$, corresponding to the ordinates $x = iw$, [9148k]
$y = i'w'$, we shall have the following table of the expressions of z, corresponding to the different values of i, i'; [9148l]

TABLE OF THE EXPRESSIONS OF z.

$i=0$	$i=1$		$i=m-1$	$i=m$	Values of i'	
$z_{0,0} = f(0,0)$	$z_{1,0} = f(w,0)$	...	$z_{m-1,0} = f(\overline{m-1}.w,0)$	$z_{m,0} = f(mw,0)$	0	1
$z_{0,1} = f(0,w')$	$z_{1,1} = f(w,w')$	...	$z_{m-1,1} = f(\overline{m-1}.w,w')$	$z_{m,1} = f(mw,w')$	1	2
$z_{0,2} = f(0,2w')$	$z_{1,2} = f(w,2w')$	...	$z_{m-1,2} = f(\overline{m-1}.w,2w')$	$z_{m,2} = f(mw,2w')$	2	3
⋮	⋮		⋮	⋮	⋮	⋮
$z_{0,n-1} = f(0,\overline{n-1}.w')$	$z_{1,n-1} = f(w,\overline{n-1}.w')$	...	$z_{m-1,n-1} = f(\overline{m-1}.w,\overline{n-1}.w')$	$z_{m,n-1} = f(mw,\overline{n-1}.w')$	$n-1$	n
$z_{0,n} = f(0,nw')$	$z_{1,n} = f(w,nw')$	...	$z_{m-1,n} = f(\overline{m-1}.w,nw')$	$z_{m,n} = f(mw,nw')$	n	$n+1$

[9148m]

Each of the $n+1$ horizontal lines of this table contains $m+1$ values of z, corresponding to m parallelopipedons, each of them standing upon a base which is equal to ww'. Thus if we take into consideration only the terms in the line marked 1, in the margin of [9148n]
[9148m], the co-ordinates will be $z_{0,0}$, $z_{1,0} \ldots . z_{m,0}$; and if we suppose these to be an increasing series, the sum S of the m corresponding parts or divisions of the parabolic [9148n′]
solid, will evidently *exceed* the expression in [9148p], and will be *less* than that in [9148q]; because the first series contains the *least* values of z, corresponding to an *inscribed* series of parallelopipedons, and the second series the *greatest* values of z, relative to a series of [9148o]
circumscribed parallelopipedons.

$$(z_{0,0} + z_{1,0} + z_{2,0} \cdot \cdot \cdot \cdot + z_{m-2,0} + z_{m-1,0}).ww'; \quad \text{[Inscribed parallelopipedons.]} \qquad [9148p]$$

$$(z_{1,0} + z_{2,0} + z_{3,0} \cdot \cdot \cdot \cdot + z_{m-1,0} + z_{m,0} \quad).ww'; \quad \text{[Circumscribed parallelopipedons.]} \qquad [9148q]$$

and it is evident, from the slightest consideration, that we shall obtain a more correct value of S by taking the mean of these two expressions, which gives very nearly, for S, the expression in [9148s, line 1]. In like manner, the expression in [9148m, line 2] produces [9148s, line 2]; that in [9148m, line 3] produces [9148s, line 3]; and so on to the line marked n; the whole sum being an approximate value of the solidity of the parabolic body; [9148r]

[9149′] of the Arabs, and those which are mentioned by Ptolemy. This agreement

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[9148s]

$$\begin{array}{lr}
(\tfrac{1}{2}z_{0,0} + z_{1,0} + z_{2,0} \ldots\ldots + z_{m-2,0} + z_{m-1,0} + \tfrac{1}{2}z_{m,0}).ww'; & 1 \\
+(\tfrac{1}{2}z_{0,1} + z_{1,1} + z_{2,1} \ldots\ldots + z_{m-2,1} + z_{m-1,1} + \tfrac{1}{2}z_{m,1}).ww'; & 2 \\
+(\tfrac{1}{2}z_{0,2} + z_{1,2} + z_{2,2} \ldots\ldots + z_{m-2,2} + z_{m-1,2} + \tfrac{1}{2}z_{m,2}).ww'; & 3 \\
\vdots & \\
+(\tfrac{1}{2}z_{0,n-2} + z_{1,n-2} + z_{2,n-2} \ldots\ldots + z_{m-2,n-2} + z_{m-1,n-2} + \tfrac{1}{2}z_{m,n-2}).ww'; & n-1 \\
+(\tfrac{1}{2}z_{0,n-1} + z_{1,n-1} + z_{2,n-1} \ldots\ldots + z_{m-2,n-1} + z_{m-1,n-1} + \tfrac{1}{2}z_{m,n-1}).ww'. & n
\end{array}$$

[9148t] Instead of commencing the calculation as in [9148n], at the top line of the table [9148m], and terminating in the line marked n, we may begin at the second line, marked 2, and terminate at the lower line, marked $n+1$. The effect of this change will be to form a second series of terms, similar to the first [9148s], and which can be deduced from it, by taking away from the first series its upper line,

[9148u] $$(\tfrac{1}{2}z_{0,0} + z_{1,0} + z_{2,0} \ldots\ldots + z_{m-2,0} + z_{m-1,0} + \tfrac{1}{2}z_{m,0}).ww',$$

and inserting, as a substitute, the $n+1$ line of the same table, namely,

[9148v] $$(\tfrac{1}{2}z_{0,n} + z_{1,n} + z_{2,n} \ldots\ldots + z_{m-2,n} + z_{m-1,n} + \tfrac{1}{2}z_{m,n}).ww'.$$

[9148w] This second series of terms may be considered as appertaining to a *circumscribed* series, and the first series [9148s] to an *inscribed* series of parallelopipedons; and if we take the mean of the two series, we shall obtain the following approximate value of the double integral $\int\!\int z\,dx.dy$; supposing the limits of the integrals, relative to x, to be from $x=0$ to $x=a=mw$; and those relative to y, to be from $y=0$ to $y=b=nw'$;

[9148x]

$$\begin{array}{rl}
\int_0^a\!\int_0^b z\,dx.dy = & \tfrac{1}{2}.(\tfrac{1}{2}z_{0,0} + z_{1,0} + z_{2,0} \ldots + z_{m-2,0} + z_{m-1,0} + \tfrac{1}{2}z_{m,0}).ww' \\
& + (\tfrac{1}{2}z_{0,1} + z_{1,1} + z_{2,1} \ldots + z_{m-2,1} + z_{m-1,1} + \tfrac{1}{2}z_{m,1}).ww' \\
& + (\tfrac{1}{2}z_{0,2} + z_{1,2} + z_{2,2} \ldots + z_{m-2,2} + z_{m-1,2} + \tfrac{1}{2}z_{m,2}).ww' \\
& \vdots \\
& + (\tfrac{1}{2}z_{0,n-2} + z_{1,n-2} + z_{2,n-2} \ldots + z_{m-2,n-2} + z_{m-1,n-2} + \tfrac{1}{2}z_{m,n-2}).ww' \\
& + (\tfrac{1}{2}z_{0,n-1} + z_{1,n-1} + z_{2,n-1} \ldots + z_{m-2,n-1} + z_{m-1,n-1} + \tfrac{1}{2}z_{m,n-1}).ww' \\
& + \tfrac{1}{2}(\tfrac{1}{2}z_{0,n} + z_{1,n} + z_{2,n} \ldots + z_{m-2,n} + z_{m-1,n} + \tfrac{1}{2}z_{m,n}).ww'.
\end{array}$$

[9148y] This integral will be so much the more accurate as the number of parts into which a, b, are divided is augmented; as is evident from the consideration that, by increasing the number of these elementary parallelopipedons, their sum will approximate more closely to the true form of the proposed parabolic solid. We must also take care to have the number of these sub-divisions great enough to enable us to dispense entirely with the consideration of the differences of every order; otherwise the calculation becomes very complicated.

[9148z] These principles are the same as those in single integrals by quadratures [8014v, &c.], and

proves the stability of the solar system, since Saturn, whose attraction [9149″]

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are so very obvious that it is unnecessary to enter into any further illustration or explanation of this part of the process.

As an example of this method, we shall now apply it to the development of the function R [957], upon which the perturbations of the planets depend. The general term of the development of R is given in [961], where the mean anomalies are represented by $nt+\varepsilon-\varpi$, $n't+\varepsilon'-\varpi'$, instead of being counted from the perihelion, as in [9146*i*, &c.]; so that if we put for brevity, [9149*a*]

$$\mathrm{v}=nt+\varepsilon-\varpi; \qquad \mathrm{v}'=n't+\varepsilon'-\varpi'; \qquad f=(g+i).\varpi+(g'-i').\varpi'+g''.\theta+g'''.\theta'; \qquad [9149b]$$

$$K_{i,i'}=H.e^{g}.e'^{g'}.(\text{tang}.\tfrac{1}{2}\varphi)^{g''}.(\text{tang}.\tfrac{1}{2}\varphi')^{g'''}.\cos.f; \qquad [9149c]$$

$$K'_{i,i'}=H.e^{g}.e'^{g'}.(\text{tang}.\tfrac{1}{2}\varphi)^{g''}.(\text{tang}.\tfrac{1}{2}\varphi')^{g'''}.\sin.f; \qquad [9149d]$$

we shall find that this general term of [961] may be put under the form,

$$R=H.e^{g}.e'^{g'}.(\text{tang}.\tfrac{1}{2}\varphi)^{g''}.(\text{tang}.\tfrac{1}{2}\varphi')^{g'''}.\cos.(i'\mathrm{v}'-i\mathrm{v}-f); \qquad [9149e]$$

and if we substitute,

$$\cos.(i\mathrm{v}'-i\mathrm{v}-f)=\cos.f.\cos.(i'\mathrm{v}'-i\mathrm{v})+\sin.f.\sin.(i'\mathrm{v}'-i\mathrm{v}), \quad [21]\ \text{Int.}$$

and use the symbols [9149*c*, *d*], it will become,

$$R=K_{i,i'}.\cos.(i'\mathrm{v}'-i\mathrm{v})+K'_{i,i'}.\sin.(i'\mathrm{v}'-i\mathrm{v}). \qquad [9149f]$$

We shall now proceed to show how the expressions of $K_{i,i'}$, $K'_{i,i'}$, can be obtained, by quadratures, for any integral values of i, i', positive or negative, including zero [954″]. [9149*g*]
Multiplying the expression of R [9149*f*] by $d\mathrm{v}'.\cos.(i\mathrm{v}'-i\mathrm{v})$, and reducing the products by [6, 31] Int., we obtain,

$$R\,d\mathrm{v}'.\cos.(i'\mathrm{v}'-i\mathrm{v})=\tfrac{1}{2}K_{i,i'}.d\mathrm{v}'+\tfrac{1}{2}K_{i,i'}.d\mathrm{v}'.\cos.2(i'\mathrm{v}'-i\mathrm{v})+\tfrac{1}{2}K'_{i,i'}.d\mathrm{v}'.\sin.2(i'\mathrm{v}'-i\mathrm{v}). \quad [9149h]$$

If we integrate this from $\mathrm{v}'=0$ to $\mathrm{v}'=2\pi$, the terms depending on $\cos.2(i'\mathrm{v}'-i\mathrm{v})$, or $\sin.2(i'\mathrm{v}'-i\mathrm{v})$, will be the same at both limits; they will therefore vanish. The same must also occur with all the other terms of the development of R, which will produce only quantities depending on sines and cosines of similar angles, and having the same values at the limits of the integral. Therefore the expression [9149*h*, &c.] will be reduced to its first term, $\int_0^{2\pi}R\,d\mathrm{v}'.\cos.(i'\mathrm{v}'-i\mathrm{v})=\tfrac{1}{2}K_{i,i'}.2\pi=\pi.K_{i,i'}$. Multiplying this by $d\mathrm{v}$, [9149*i*]
and integrating from $\mathrm{v}=0$ to $\mathrm{v}=2\pi$, it becomes equal to $2\pi^2.K_{i,i'}$; then dividing by $2\pi^2$, we get the value of $K_{i,i'}$ [9149*k*]. In like manner, by multiplying the value of R [9149*f*] by $d\mathrm{v}.d\mathrm{v}'.\sin.(i\mathrm{v}'-i\mathrm{v})$, and making the integrations in the same order as before, we obtain the value of $K'_{i,i'}$ [9149*l*];

$$K_{i,i'}=\frac{1}{2\pi^2}.\int_0^{2\pi}\!\!\int_0^{2\pi}R\,d\mathrm{v}.d\mathrm{v}'.\cos.(i'\mathrm{v}'-i\mathrm{v}); \qquad [9149k]$$

$$K'_{i,i'}=\frac{1}{2\pi^2}.\int_0^{2\pi}\!\!\int_0^{2\pi}R\,d\mathrm{v}.d\mathrm{v}'.\sin.(i'\mathrm{v}'-i\mathrm{v}). \qquad [9149l]$$

[9149‴] towards the sun is about one hundred times less than that of the earth

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[9149m] When $i'=0,\ i=0$, the expression of R [9149f] is reduced to its first term $R=K_{0,0}$; multiplying this by $dv.dv'$, and integrating between the same limits, we evidently get, by dividing by $4\pi^2$,

[9149n] $$K_{0,0}=\frac{1}{4\pi^2}\cdot\int_0^{2\pi}\int_0^{2\pi}R\,dv.dv';$$

[9149n′] which is just half the value which would be obtained, from the general expression of $K_{i,i'}$, [9149k], or from that of A [9149o], by putting $i=0,\ i'=0$.

In order to make the calculations relative to v, v′, more independent of each other, and in a convenient manner, we shall put,

[9149o] $$A=\frac{1}{2\pi^2}\cdot\int_0^{2\pi}\int_0^{2\pi}R\,dv.dv'.\cos.i\,v.\cos.i'v';$$

[9149p] $$B=\frac{1}{2\pi^2}\cdot\int_0^{2\pi}\int_0^{2\pi}R\,dv.dv'.\sin.i\,v.\sin.i'v';$$

[9149q] $$C=\frac{1}{2\pi^2}\cdot\int_0^{2\pi}\int_0^{2\pi}R\,dv.dv'.\cos.i\,v.\sin.i'v';$$

[9149r] $$D=\frac{1}{2\pi^2}\cdot\int_0^{2\pi}\int_0^{2\pi}R\,dv.dv'.\sin.i\,v.\cos.i'v'.$$

Substituting,

[9149s] $\cos.(i'v'-i\,v)=\cos.i\,v.\cos.i'v'+\sin.i\,v.\sin.i\,v'$; $\sin.(i'v'-i\,v)=\cos.i\,v.\sin.i'v'-\sin.iv.\cos.i'v'$, in [9149k, l], and using the values [9149o—r], we obtain,

[9149t] $$K_{i,i'}=A+B;$$

[9149u] $$K'_{i,i'}=C-D;$$

so that the problem is finally reduced to the computation, by quadratures, of the four functions $A,\ B,\ C,\ D$; which can be done in the following manner;

[9149v] Taking in the first place the value of A [9149o], and comparing it with that of the general integral [9148x], we find that we shall have,

[9149w] $$x=v;\qquad y=v';\qquad z=F(x,y)=\frac{1}{2\pi^2}\cdot R.\cos.i\,v.\cos.i'v'.$$

Now if we put for brevity,

[9149x] $$\mathrm{m}=\frac{2\pi}{m};\qquad \mathrm{n}=\frac{2\pi}{n};\qquad \alpha=i.\frac{2\pi}{m}=i\,\mathrm{m};\qquad \alpha'=i'.\frac{2\pi}{n}=i'\mathrm{n}.$$

[9149y] we must suppose, in the integrations relative to v, that the whole circumference 2π is divided into m equal parts, each of the value m, corresponding to the $m+1$ values of v, [9150a]; and, in the integration relative to v′, that the whole circumference is divided
[9149z] into n equal parts, each of the value n, corresponding to the $n+1$ values of v′ [9150b]; so that we shall have successively,

[9150a] $$v=0;\quad v=\mathrm{m};\quad v=2\mathrm{m};\ \ldots\ v=\lambda\mathrm{m};\ \ldots\ v=(m-1).\mathrm{m};\quad v=m\mathrm{m};$$

[9150b] $$v'=0;\quad v'=\mathrm{n};\quad v'=2\mathrm{n};\ \ldots\ v'=\lambda'\mathrm{n};\ \ldots\ v'=(n-1).\mathrm{n};\quad v'=n\mathrm{n};$$

towards the sun, has not however suffered, from the time of Hipparchus to [9150]

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supposing λ to be any integral number whatever, of the series $0,1,2\ldots m$; and λ' any integral number whatever of the series $0,1,2,3\ldots n$. [9150c]

We shall represent by $R_{\lambda,\lambda'}$ the value of the function R, when we substitute in it, for v, v', their values [9150a, b], corresponding to λ, λ', respectively, namely, $\mathrm{v}=\lambda\mathrm{m}$, $\mathrm{v}'=\lambda'\mathrm{n}$; which give $i\mathrm{v}=\lambda.i\mathrm{m}=\lambda\alpha$, $i'\mathrm{v}'=\lambda'.i'\mathrm{n}=\lambda'\alpha'$ [9149x]. Then the corresponding value of z [9149w] will be represented by, [9150c'] [9150d]

$$z_{\lambda,\lambda'}=\frac{1}{2\pi^2}.R_{\lambda,\lambda'}.\cos.\lambda\alpha.\cos.\lambda'\alpha'; \qquad [9150e]$$

and by putting $w=\frac{2\pi}{m}$, $w'=\frac{2\pi}{n}$, which give $\frac{ww'}{2\pi^2}=\frac{2}{mn}$, we find that the formula [9148x] will give the following value of A; [9150f]

$$\begin{array}{ll}
A=\frac{1}{mn}.\cos.0.\ \{\tfrac{1}{2}R_{0,0}.\cos.0+R_{1,0}.\cos.\alpha+R_{2,0}.\cos.2\alpha\ldots+R_{m-1,0}.\cos.(m-1).\alpha+\tfrac{1}{2}R_{m,0}.\cos.m\alpha\} & 1\\
+\frac{2}{mn}.\cos.\alpha'.\ \{\tfrac{1}{2}R_{0,1}.\cos.0+R_{1,1}.\cos.\alpha+R_{2,1}.\cos.2\alpha\ldots+R_{m-1,0}.\cos.(m-1).\alpha+\tfrac{1}{2}R_{m,1}.\cos.m\alpha\} & 2\\
+\frac{2}{mn}.\cos.2\alpha'.\{\tfrac{1}{2}R_{0,2}.\cos.0+R_{1,2}.\cos.\alpha+R_{2,2}.\cos.2\alpha\ldots+R_{m-1,2}.\cos.(m-1).\alpha+\tfrac{1}{2}R_{m,2}.\cos.m\alpha\} & 3\\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots & \\
+\frac{1}{mn}.\cos.n\alpha'.\{\tfrac{1}{2}R_{0,n}.\cos.0+R_{1,n}.\cos.\alpha+R_{2,n}.\cos.2\alpha\ldots+R_{m-1,n}.\cos.(m-1).\alpha+\tfrac{1}{2}R_{m,n}.\cos.m\alpha\}. & n+1
\end{array} \qquad [9150g]$$

This may be put under a more abridged form, by computing $n+1$ quantities, C_0, C_1, $C_2\ldots C_\lambda\ldots C_n$, like those whose general expression is given in [9150k], corresponding to any integral number λ, included in the series $0,1,2,3\ldots n$. Substituting these in [9150g], we get the value of A [9150m]. In like manner, by using the symbols S_0, S_1, $S_2\ldots S_\lambda\ldots S_n$, corresponding to the form [9150l], we obtain the value of B, [9150n], resulting from the expression [9149p]. In the same way we deduce the values of C, D [9150o, p], from those in [9149q, r] respectively; and then, by substitution in [9150q], we get $K_{i,i'}$, $K'_{i,i'}$; [9150h] [9150i]

$$C_\lambda=\tfrac{1}{2}R_{0,\lambda}.\cos.0+R_{1,\lambda}.\cos.\alpha+R_{2,\lambda}.\cos.2\alpha\ldots+R_{m-1,\lambda}.\cos.(m-1).\alpha+\tfrac{1}{2}R_{m,\lambda}.\cos.m\alpha; \qquad [9150k]$$

$$S_\lambda=\tfrac{1}{2}R_{0,\lambda}.\sin.0+R_{1,\lambda}.\sin.\alpha+R_{2,\lambda}.\sin.2\alpha\ldots+R_{m-1,\lambda}.\sin.(m-1).\alpha+\tfrac{1}{2}R_{m,\lambda}.\sin.m\alpha; \qquad [9150l]$$

$$A=\frac{2}{mn}.(\tfrac{1}{2}C_0.\cos.0+C_1.\cos.\alpha'+C_2.\cos.2\alpha'\ldots+C_{n-1}.\cos.(n-1).\alpha'+\tfrac{1}{2}C_n.\cos.n\alpha'); \qquad [9150m]$$

$$B=\frac{2}{mn}.(\tfrac{1}{2}S_0.\sin.0+S_1.\sin.\alpha'+S_2.\sin.2\alpha'\ldots+S_{n-1}.\sin.(n-1).\alpha'+\tfrac{1}{2}S_n.\sin.n\alpha'); \qquad [9150n]$$

$$C=\frac{2}{mn}.(\tfrac{1}{2}C_0.\sin.0+C_1.\sin.\alpha'+C_2.\sin.2\alpha'\ldots+C_{n-1}.\sin.(n-1).\alpha'+\tfrac{1}{2}C_n.\sin.n\alpha'); \qquad [9150o]$$

$$D=\frac{2}{mn}.(\tfrac{1}{2}S_0.\cos.0+S_1.\cos.\alpha'+S_2.\cos.2\alpha'\ldots+S_{n-1}.\cos.(n-1).\alpha'+\tfrac{1}{2}C_n.\cos.n\alpha'). \qquad [9150p]$$

$$K_{i,i'}=A+B; \qquad K'_{i,i'}=C-D. \qquad [9149t,u] \qquad [9150q]$$

[9150'] the present day, any sensible alteration from the action of bodies which

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The first term of the development of R, namely $K_{0,0}$, is obtained as in [9149n'], by
[9150r] putting $i=0$, $i'=0$, in the value of $\frac{1}{2}A$, deduced from [9150m]. In this case we have $\mathfrak{a}=0$, $\mathfrak{a}'=0$ [9149x]; and the general expression [9150k] becomes,

[9150s] $$C_{\lambda}=\tfrac{1}{2}R_{0,\lambda}+R_{1,\lambda}+R_{2,\lambda}\,\ldots\,+R_{m-1,\lambda}+\tfrac{1}{2}R_{m,\lambda};$$

and the value of $\frac{1}{2}A$, deduced from [9150m], gives,

[9150t] $$K_{0,0}=\frac{1}{mn}.(\tfrac{1}{2}C_0+C_1+C_2\,\ldots\,+C_{n-1}+\tfrac{1}{2}C_n).$$

This quantity produces a term in the secular inequalities.

In applying these formulas to numbers, Hansen recommends the use of some power of 2, for the value of m or n; as it tends to make the sines and cosines of several terms of
[9150u] the series [9150a, b] equal to each other, and affords several very peculiar and convenient methods of computation and verification. Thus, if we put $m=8$, we shall have $\mathrm{m}=45^d$; and the sines of the *nine* terms of the series [9150a], will be reduced to the *three* quantities 0, $\pm\sqrt{\frac{1}{2}}$, ± 1. He finds it to be sufficiently accurate to assume
[9150u'] $m=32$, $n=16$, in the very important and delicate computation of the great inequalities of Jupiter and Saturn. In making these calculations relative to R [9149a], it is convenient to take separately into consideration the two terms of which it is composed, namely,

[9150v] $$R=\frac{m'.(xx'+yy'+zz')}{r'^3}-\frac{m'}{\rho};\quad [5755]$$

because the part $-\frac{m'}{\rho}$, or rather $-\frac{1}{\rho}$, is the same for both planets; and we have seen, in [3832], that in calculating those great inequalities, the values of P, P' [3810],
[9150w] which take the place of $K_{i,i'}$, $K'_{i,i'}$, in the expression of R [9149f], will be the same whether we consider the action of m on m', or that of m' on m. Instead of the rectangular
[9150x] co-ordinates x, y, z, x', y', z', which occur in [9150v], we may use the polar co-ordinates r, v, r', v', and put, as in [5849], γ for the mutual inclination of the two orbits; taking the orbit of m for the plane of the co-ordinates, and the line of nodes of the orbit for the
[9150y] axis of x, or the origin of the angles v, v'. Then we have, in [5852], $\beta=2.\sin.^2\frac{1}{2}\gamma$; consequently,

[9150z] $$\beta.\sin.v.\sin.v'=2.\sin.^2\tfrac{1}{2}\gamma.\sin.v.\sin.v'=\sin.^2\tfrac{1}{2}\gamma.\cos.(v'-v)-\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v);$$

substituting this in [5853d, f], and making a slight reduction, we get,

[9151a] $$xx'+yy'+zz'=rr'.\cos.^2\tfrac{1}{2}\gamma.\cos.(v'-v)+rr'.\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v);$$

[9151b] $$\rho^2=r^2+r'^2-2rr'.\cos.^2\tfrac{1}{2}\gamma.\cos.(v'-v)-2rr'.\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v);$$

hence the expression of R [9150v] becomes, by still retaining for brevity the symbol ρ, [9151b],

[9151c] $$R=\frac{m'.r}{r'^2}.\{\cos.^2\tfrac{1}{2}\gamma.\cos.(v'-v)+\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v)\}-\frac{m'}{\rho}.$$

are foreign from our system. [9150″]

The angles v, v', are here counted from the line of nodes [9150y], which is equivalent to the supposition that the longitude of the node is nothing; but if we suppose this longitude to be represented, as in [8009s], by θ, each of the angles v, v', must be decreased by θ. This will not alter the angle $v'-v$, but will change $\cos.(v'+v)$ into $\cos.(v'+v-2\theta)$, and its differential relative to θ will be $2d\theta.\sin.(v'+v-2\theta)$. Hence it is evident that in finding the differential of either of the quantities [9151a, b, c], relative to θ, we may consider $\cos.(v'+v)$ as the only variable quantity, and put its differential equal to $2d\theta.\sin.(v'+v)$; so that we shall have, from [9151c], the expression [9151g]. Moreover the partial differential of R [9151c], relative to γ, gives [9151g'], observing that if we consider γ as the only variable quantity in the first member of [9151f], we shall have, [9151d] [9151e] [9151e']

$$d.\{\cos.^2\tfrac{1}{2}\gamma.\cos.(v'-v)+\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v)\}$$
$$=-d\gamma.(2.\sin.\tfrac{1}{2}\gamma.\cos.\tfrac{1}{2}\gamma).\{\tfrac{1}{2}.\cos.(v'-v)-\tfrac{1}{2}.\cos.(v'+v)\}=-d\gamma.\sin.\gamma.\{\sin.v.\sin.v'\}. \qquad [9151f]$$

The partial differential of R [9151c], relative to r, gives [9151h], by using [6089e]. In like manner we may obtain other similar differentials, relative to r, v', v, &c.;

$$\left(\frac{dR}{d\theta}\right)=\frac{m'.r}{r'^2}.2.\sin.^2\tfrac{1}{2}\gamma.\sin.(v'+v)-\frac{2m'.rr'}{\rho^3}.\sin.^2\tfrac{1}{2}\gamma.\sin.(v'+v); \qquad [9151g]$$

$$\left(\frac{dR}{d\gamma}\right)=\left\{-\frac{m'.r}{r'^2}+\frac{m'.rr'}{\rho^3}\right\}.\sin.\gamma.\sin.v.\sin.v'; \qquad [9151g']$$

$$r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right)=\frac{m'.r}{r'^2}.\{\cos.^2\tfrac{1}{2}\gamma.\cos.(v'-v)+\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v)\}$$
$$+\frac{m'.r}{\rho^3}.\{r-r'.\cos.^2\tfrac{1}{2}\gamma.\cos.(v'-v)-r'.\sin.^2\tfrac{1}{2}\gamma.\cos.(v'+v)\}. \qquad [9151h]$$

It is unnecessary to point out all the peculiarities of these calculations, with the methods of abridgment, and the checks to prove the accuracy of the numerical process, as they will occur very readily to an experienced computer; we shall therefore merely give a rapid sketch of the manner of proceeding, in computing the part of R depending upon P, P', [3810, &c.], corresponding to the angle $5n't-2nt$, or $5v'-2v$, upon which the great inequalities of Jupiter and Saturn chiefly depend. In this case we have $i=2$, $i'=5$, [9149f], and we must substitute these values in the expressions of A, B, C, D, [9149$o-r$], or in the equivalent formulas [9150$k-q$]. [9151i] [9151k]

Supposing now, as in [9150u'], that $m=32$, $n=16$, we shall have, as in [9149x], [9151l]

$$\mathrm{m}=\frac{360^d}{32}=11^d\,15^m;\quad \mathrm{n}=\frac{360^d}{16}=22^d30^m;\quad \alpha=2\mathrm{m}=22^d30^m;\quad \alpha'=5\mathrm{n}=112^d30^m; \qquad [9151m]$$

and the series of angles v, v' [9150a, b], will become,

$$\mathrm{v}=0;\quad \mathrm{v}=11^d\,15^m;\quad \mathrm{v}=22^d30^m;\ \ldots\ \mathrm{v}=348^d\,45^m;\quad \mathrm{v}=360^d; \qquad [9151n]$$
$$\mathrm{v}'=0;\quad \mathrm{v}'=22^d\,30^m;\quad \mathrm{v}'=45^d;\quad \mathrm{v}'=337^d\,30^m;\quad \mathrm{v}'=360^d. \qquad [9151o]$$

Each one of these values of v must be separately combined with each of the values of

[9150‴] 24. The principle which has conducted us, in the preceding article, to

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[9151p] v′, making in all $33\times 17=561$ combinations; and for every one of these combinations we must compute the corresponding value of R, which we may represent, as in
[9151q] [9150c′, &c.], by $R_{0,0}$, $R_{1,0}$, $R_{2,0}$, &c.; $R_{0,1}$, $R_{1,1}$, $R_{2,1}$, &c.; using for this purpose the most correct tables of the motions of these planets, in finding the values of
[9151r] r, r', &c. Then the values of C_0, $C_1 \ldots C_{16}$; S_0, $S_1 \ldots S_{16}$, must be calculated by means of the formulas [9150k, l], which become, in this case, of the following forms;

[9151s] $$C_\lambda=\tfrac{1}{2}R_{0,\lambda}.\cos.0+R_{1,\lambda}.\cos.\alpha+R_{2,\lambda}.\cos.2\alpha \ldots . +R_{31,\lambda}.\cos.31\alpha+\tfrac{1}{2}R_{32,\lambda}.\cos.32\alpha;$$

[9151t] $$S_\lambda=\tfrac{1}{2}R_{0,\lambda}.\sin.0+R_{1,\lambda}.\sin.\alpha+R_{2,\lambda}.\cos.2\alpha \ldots . +R_{31,\lambda}.\sin.31\alpha+\tfrac{1}{2}R_{31,\lambda}.\sin.32\alpha.$$

With these values we have, as in [9150m—q],

[9151u] $$A=\frac{2}{32\times 16}.\{\tfrac{1}{2}C_0.\cos.0+C_1.\cos.\alpha'+C_2.\cos.2\alpha' \ldots . +C_{15}.\cos.15\alpha'+\tfrac{1}{2}C_{16}.\cos.16\alpha'\};$$

[9151v] $$B=\frac{2}{32\times 16}.\{\tfrac{1}{2}S_0.\sin.0+S_1.\sin.\alpha'+S_2.\sin.2\alpha' \ldots . +S_{15}.\sin.15\alpha'+\tfrac{1}{2}S_{16}.\sin.16\alpha'\};$$

[9151w] $$C=\frac{2}{32\times 16}.\{\tfrac{1}{2}C_0.\sin.0+C_1.\sin.\alpha'+C_2.\sin.2\alpha' \ldots . +C_{15}.\sin.15\alpha'+\tfrac{1}{2}C_{16}.\sin.16\alpha'\};$$

[9151x] $$D=\frac{2}{32\times 16}.\{\tfrac{1}{2}S_0.\cos.0+S_1.\cos.\alpha'+S_2.\cos.2\alpha' \ldots . +S_{15}.\cos.15\alpha'+\tfrac{1}{2}S_{16}.\cos.16\alpha'\};$$

[9151y] $$K_{2,5}=A+B; \qquad K'_{2,5}=C-D;$$

[9151z] and by substituting the values $\alpha=22^d 30^m$, $\alpha'=112^d 30^m$ [9151m], we easily deduce the values of $K_{2,5}$, $K'_{2,5}$, which we shall hereafter represent simply by K, K', for the sake of brevity. These values being substituted in [9149f], give,

[9152a] $$R=K.\cos.(5v'-2v)+K'.\sin.(5v'-2v).$$

This may be reduced to a form which is similar to that in [3810], namely,

[9152b] $$R=m'.P.\cos.T+m'.P'.\sin.T.$$

[9152c] For if we put $T=5n't-2nt+5\epsilon'-2\epsilon$, and $w=5\varpi'-2\varpi$, the values of v, v′ [9149b],
[9152d] will give $i'v'-iv=(5n't-2nt+5\epsilon'-2\epsilon)-(5\varpi'-2\varpi)=T-w$; whence we shall have,

[9152d′] $$\cos.(5v'-2v)=\cos.w.\cos.T+\sin.w.\sin.T; \qquad \sin.(5v'-2v)=\cos.w.\sin.T-\sin.w.\cos.T.$$

Substituting these in [9152a], we get,

[9152e] $$R=(K.\cos.w-K'.\sin.w).\cos.T+(K.\sin.w+K'.\cos.w).\sin.T;$$

comparing together the two expressions of R [9152b, e], we obtain,

[9152f] $$m'P=K.\cos.w-K'.\sin.w; \qquad m'P'=K.\sin.w+K'.\cos.w.$$

[9152g] It is evident that we may apply the same method of integration, by quadratures, to any function which is derived from R, like those in [9151g, g′, h, &c.], by computing these quantities for the different values of v, v′ [9150a, b], and using them for $R_{0,0}$, $R_{1,0}$, &c., in the formulas [9150k—q]. Then having, as in [962, 1002, 954, 951, &c.],

[9152h] $$r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right); \quad r'.\left(\frac{dR}{dr'}\right)=a'.\left(\frac{dR}{da'}\right); \quad r.\left(\frac{dR}{dr}\right)+r'.\left(\frac{dR}{dr'}\right)=-R,$$

several sensible inequalities in the motions of Jupiter and Saturn, gives [9150iv]

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we shall get the values of $a.\left(\frac{dR}{da}\right)$, $a'.\left(\frac{dR}{da'}\right)$, &c.; or, in other words, we shall have the values of,

$$a^{\text{iv}}.\left(\frac{dK}{da^{\text{iv}}}\right), \quad a^{\text{iv}}.\left(\frac{dK'}{da^{\text{iv}}}\right), \quad a^{\text{v}}.\left(\frac{dK}{da^{\text{v}}}\right), \quad a^{\text{v}}.\left(\frac{dK'}{da^{\text{v}}}\right), \text{ \&c.};$$ [9152i]

and from these we may deduce the values of $\left(\frac{dP}{da^{\text{iv}}}\right)$, $\left(\frac{dP}{da^{\text{v}}}\right)$ [4422, 4475, &c.], by a process similar to that in [9152b,f]. In this way we may compute the values of the parts of δv, δr, &c. depending upon particular angles, like those in [3809, 3821, &c.], which include the most important parts of the great inequalities of Jupiter and Saturn. We may also apply this method to any of the integrals in [8010f—m]; as, for example, to those in [8010l, m], which can be obtained by the substitution of the values [9151g, g']; and so on for others. [9152k] [9152k']

To show the accuracy of this method of quadratures, with the values $m=32$, $n=16$, [9151l], applied to the computation of a few of the most important terms of the great inequalities of Jupiter and Saturn, we have inserted, in the table [9152p—s], the results of several calculations of Pontécoulant, of the same terms, by quadratures, and by the methods of approximation given by La Place, in the third volume of this work. The terms in [9152p] represent the part of the inequality of Jupiter δv^{iv} [4417], depending on P, P', but neglecting their first and second differentials dP, ddP, dP', ddP'. The terms in [9152r] represent the similar part of the inequality of Saturn δv^{v} [4483], depending on $P_{,}$, $P'_{,}$; but neglecting their differentials. The terms in [9152q] represent the inequality δv^{iv} [4422] relative to Jupiter; and that in [9152s] the similar expression of δv^{v} [4475], relative to Saturn; the fractions of a second, beyond the first decimal place, being neglected; [9152l] [9152m] [9152n] [9152o]

By quadratures.	By La Place's formulas.	
$\delta v^{\text{iv}} = \quad 1084^s{,}0.\sin.T - 64^s{,}9.\cos.T$;	$\delta v^{\text{iv}} = \quad 1089^s{,}8.\sin.T - 55^s{,}2.\cos.T$;	[9152p]
$\delta v^{\text{iv}} = - \quad 16^s{,}3.\sin.T + 5^s{,}8.\cos.T$;	$\delta v^{\text{iv}} = - \quad 16^s{,}3.\sin.T + 6^s{,}0.\cos.T$;	[9152q]
$\delta v^{\text{v}} = -2685^s{,}4.\sin.T + 160^s{,}9.\cos.T$;	$\delta v^{\text{v}} = -2678^s{,}5.\sin.T + 156^s{,}4.\cos.T$;	[9152r]
$\delta v^{\text{v}} = \quad 52^s{,}1.\sin.T - 13^s{,}6.\cos.T$;	$\delta v^{\text{v}} = \quad 49^s{,}4.\sin.T - 13^s{,}5.\cos.T$.	[9152s]

The near agreement of these results, obtained from such entirely different methods, is very satisfactory; not only from its indicating that the usual methods of development, by series, are sufficiently accurate for all the purposes of astronomy to which they have been usually applied, but from its affording an independent method of verifying all these results; and it is to be hoped that this process will be extensively employed. [9152t]

Instead of taking the mean anomalies v, v′ [9149b], for the variable angles in the expression of R [9149f, &c.], we may use any other quantities; as, for example, the [9152u]

also, in the moon's motion, a small equation which we shall now investigate,
[9150^v] using the same symbols as in the seventh book, where we have computed the following inequality in the moon's mean longitude $nt+e$, in terms of the true longitude v;*

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excentric anomalies u, u', which are found so useful in reducing the formulas [9147i—m] to functions of only one unknown quantity u. In this case we shall have, in like manner as in [9146i, 9149b],

[9152v] $$\mathrm{v}=u-e.\sin.u\,; \qquad \mathrm{v}'=u'-e'.\sin.u'\,;$$

[9152w] $$d\mathrm{v}=(1-e.\cos.u).du=ndt\,; \qquad d\mathrm{v}'=(1-e'.\cos.u').du'=n'dt.$$

Substituting these values of v, v′, dv, dv′, in terms of u, u', du, du', in [9149k, l, n], we get,

[9152x] $$K_{i,i'}=\frac{1}{2\pi^2}.\int_0^{2\pi}\int_0^{2\pi}R.\cos.\{i'(u'-e'.\sin.u')-i(u-e.\sin.u)\}.(1-e.\cos.u).(1-e'.\cos.u').du.du'\,;$$

[9152y] $$K'_{i,i'}=\frac{1}{2\pi^2}.\int_0^{2\pi}\int_0^{2\pi}R.\sin.\{i'(u'-e'.\sin.u')-i(u-e.\sin.u)\}.(1-e.\cos.u).(1-e'.\cos.u').du.du'\,;$$

[9152z] $$K_{0,0}=\frac{1}{4\pi^2}.\int_0^{2\pi}\int_0^{2\pi}R.(1-e.\cos.u).(1-e'.\cos.u').du.du'.$$

Developing these functions of u according to the powers and products of e, e', then
[9153a] performing the integrations relative to u, u', we shall obtain the values of $K_{i,i'}$, $K'_{i,i'}$, $K_{0,0}$. Now it is evident that R can be expressed in a function of the sines and cosines of angles of the form $i'u'-iu$; so that we shall have,

[9153b] $$R=\Sigma.A_{i,i'}.\cos.(i'u'-iu)+\Sigma.B_{i,i'}.\sin.(i'u'-iu)\,;$$

and by multiplying this by $du.du'.\cos.(i'v'-iv)$, or $du.du'.\sin.(i'v'-iv)$, and integrating, we get the values of $A_{i,i'}$, $B_{i,i'}$, in formulas similar to [9149k, l, n]. The function R, when developed in this way, has a peculiar and very remarkable property, that we can
[9153c] deduce from it directly the values of $\int dR$, $\int dt.\int dR$, and other partial differentials of the function $\int Rdt$, which enter into the expressions of the variations of the elements of the orbit, in [5873—5879, &c.], without being obliged to substitute the values of u, u',
[9153d] in functions of the time before the integrations; but this advantage is counterbalanced by the trouble of reducing the expressions of the elements which result from this process, in terms of the excentric anomaly, to those of the mean anomaly, which leads to very
[9153e] complicated calculations; for this reason we shall not enter into any further discussion of this method, which is in reality not so simple as the process already explained, in [9149a—n, &c.].

* (4109) We see in [4817, 4831, 4834], that $mv-c'mv$ represents the motion of the
[9154a] sun's perigee, so that $mv-c'mv+\varpi'$ represents the real place of the sun's perigee; and by subtracting it from the moon's longitude v, we get the moon's distance from the sun's
[9154b] perigee equal to $v-mv+c'mv-\varpi'$, being the argument of the inequality [9151]; moreover

$$-18^s{,}809.\sin.(v-mv+c'mv-\varpi'). \qquad [5220 \text{ line } 22]$$ [9151]

This inequality may be considered as a true equation of the moon's centre, referred to the sun's perigee; and it is analogous to the equation of the centre of the third satellite of Jupiter [7405 line 2], which corresponds to the perijove of the fourth satellite [7409]; it must therefore produce in the [9152] moon's motion an inequality similar to the evection, which will therefore be of the form,

$$K.\sin.\{2v-2mv-(v-mv+c'mv-\varpi')\}; \qquad [9154d]$$ [9153]

or,

$$K.\sin.(v-mv-c'mv+\varpi').$$ [9154]

To determine K, we shall observe that the coefficient $-18^s{,}809$, of the small equation of the centre, is to K, as -22677^s, the coefficient of the [9155] great equation of the centre, is to -4681^s, the coefficient of the evection; which gives,

$$K=-3^s{,}88.$$ [9156]

According to Burg, this coefficient is $-2^s{,}9$ [5578 line 12], which differs but little from the preceding result. If we adopt Burg's value of the [9157] coefficient of this small equation of the centre, $-13^s{,}5$ [5576 line 15], this difference will be decreased, and we shall have,

$$K=-2^s{,}78.$$ [9158]

$cv-\varpi$ represents the moon's distance from her perigee. Now the chief term of the moon's equation of the centre, is $-22677^s{,}5.\sin.(cv-\varpi)$ [5574], and the evection is [9154c] $-4681^s{,}5.\sin.\{2v-2mv-(cv-\varpi)\}$ [5575 line 3]; so that the argument of the evection is found, by subtracting the argument of the equation of the centre $cv-\varpi$ from $2v-2mv$. Therefore if we consider the inequality [9551] as an equation of the centre, whose argument is $v-mv'+c'mv-\varpi'$, the argument of the corresponding evection will be,

$$2v-2mv-(v-mv+c'mv-\varpi')=v-mv-c'mv+\varpi';$$ [9154d]

consequently this evection can be put under the same form as that given by the author in [9154]. If we suppose, as in [9155], that the coefficients of the inequalities [9551, 9554] [9154e] have the same ratio to each other as the numbers 22677^s, 4681^s [9155], we shall get the same value of K as in [9156]; and by using Burg's coefficient [9157], we shall obtain [9154f] the reduced value of K [9158]. This does not however express the complete value of K, but only that part of it which has the relation to the coefficient $18^s{,}809$ [9151], which is assumed by the author in [9155]; and the effect of the neglected terms might alter the [9154g] value essentially; as Plana has remarked in vol. 2, page 412, of the Memoirs of the Astronomical Society of London, where he reduces the value of K to $0^s{,}13$. We have not made any numerical calculations for finding the complete value of this coefficient, and [9157h] we shall merely observe that it is found in exactly the same way as that in [5220 line 22], [9154i] depending on $\sin.(v-mv+c'mv-\varpi')$.

CHAPTER IX.

ON THE MASSES OF THE PLANETS AND SATELLITES.

25. In the present state of astronomy, where both observation and theory have been carried to a great degree of perfection, one of the best methods of ascertaining the masses of the planets is to compare, with the analytical formulas of the perturbations, a very great number of observations, selected for that purpose in the most advantageous manner. In this way,
[9159] the masses of Venus, Mars, Saturn, the moon, and the four satellites of Jupiter, have been determined. We may even add thereto the mass of Jupiter; for by comparing the best observations of that planet, with the great inequalities his action* produces on the motion of Saturn, no sensible correction has been found in the value of the mass which is given by the
[9160] elongations of the satellites. It is evident that these masses will be known more accurately, in proportion as their effects are greater. We shall now collect together, so as to have at one view the values of these masses, determined by the methods we have just mentioned.

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* (4110) The mass of Jupiter now most generally used, is that found by Nicolai, from
[9159a] the perturbations of Juno; namely, $m^{\text{iv}}=\frac{1}{1053{,}924}$ [4061g]; that of Mars is $m'''=\frac{1}{2680337}$, given by Bessel, from a new discussion of the observations of Delambre. These are used by Pontécoulant, in the third volume of his *Théorie Analytique*, &c.,
[9159b] together with the values of m^{v}, m^{vi} [4061d]; he also puts $m=\frac{1}{1909706}$, $m'=\frac{1}{401839}$, $m''=\frac{1}{356354}$; which are very nearly the same as in [4061d]. In calculating the
[9159c] perturbations of all the planets, he uses La Place's values of a, a', a'', a''', a^{iv}, a^{v}, a^{vi}; n, n', n'', n''', n^{iv}, n^{v}, n^{vi} [4079, 4077]; but in an appendix to the third volume of the same work, he has applied some slight corrections to these quantities, to make them conform

VALUES OF THE MASSES OF VENUS, MARS, JUPITER AND SATURN, THAT OF THE SUN BEING TAKEN FOR UNITY.

Venus . . . . $m' = \frac{1}{356632}$; [4605] 1

Mars . . . . $m''' = \frac{1}{2546320}$; [4608] 2 [9161]

Jupiter . . . $m^{\text{iv}} = \frac{1}{1067,09}$; [4065, 9122] 3

Saturn . . . . $m^{\text{v}} = \frac{1}{3534,08}$. [9121] 4

[These values were altered by La Place, as in [4061d]. See also [9159f, g].]

Dividing these values respectively by those of the masses of Venus, Mars, Jupiter and Saturn, given in [4061], we obtain the values of the coefficients [9162]

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to the latest and most accurate astronomical tables. We have inserted these values in the following table, together with his latest computations of the other elements, adapted to the epoch corresponding to the midnight which separates December 31, 1799, from January 1, 1800. [9159d] [9159e]

TABLE OF THE VALUES OF THE ELEMENTS OF THE ORBITS OF THE PLANETS.

Names of the planets.	Masses of the planets, the sun's mass being 1.	Mean sideral motions in $365\frac{1}{4}$ days.	Mean distances of the planets from the sun.	Ratios of the excentricities to the mean distances, January 1, 1800.	
Mercury	$m = \frac{1}{1909706}$	$n = 5381016^s,53$	$a = 0,38709888$	$e = 0,2055149$	
Venus	$m' = \frac{1}{401839}$	$n' = 2106641^s,42$	$a' = 0,72333228$	$e' = 0,0068531$	
Earth	$m'' = \frac{1}{356354}$	$n'' = 1295977^s,37$	$a'' = 1,00000000$	$e'' = 0,01685359$	
Mars	$m''' = \frac{1}{2680337}$	$n''' = 689051^s,08$	$a''' = 1,52369210$	$e''' = 0,0933061$	[9159f]
Jupiter	$m^{\text{iv}} = \frac{1}{1053,924}$	$n^{\text{iv}} = 109256^s,59$	$a^{\text{iv}} = 5,20115524$	$e^{\text{iv}} = 0,0481621$	
Saturn	$m^{\text{v}} = \frac{1}{3512}$	$n^{\text{v}} = 43996^s,00$	$a^{\text{v}} = 9,53797320$	$e^{\text{v}} = 0,0561505$	
Uranus	$m^{\text{vi}} = \frac{1}{17918}$	$n^{\text{vi}} = 15425^s,49$	$a^{\text{vi}} = 19,18251740$	$e^{\text{vi}} = 0,0466108$	

Names of the planets.	Longitudes of the epochs, January 1, 1800.	Longitudes of the perihelia, Jan. 1, 1800; counted as in [8009r].	Inclinations of the orbits to the ecliptic, January 1, 1800.	Longitudes of the ascending nodes on the ecliptic, January 1, 1800.	
Mercury	$\varepsilon = 110^d\ 13^m\ 17^s,9$	$\varpi = 74^d\ 21^m\ 41^s$	$\varphi = 7^d\ 00^m\ 09^s$	$\theta = 45^d\ 57^m\ 39^s$	
Venus	$\varepsilon' = 145\ 56\ 52,1$	$\varpi' = 128\ 43\ 06$	$\varphi' = 3\ 23\ 29$	$\theta' = 74\ 52\ 39$	
Earth	$\varepsilon'' = 100\ 23\ 32,6$	$\varpi'' = 99\ 29\ 53$	$\varphi'' = 0\ 00\ 00$	$\theta'' = 00\ 00\ 00$	
Mars	$\varepsilon''' = 232\ 49\ 50,5$	$\varpi''' = 332\ 23\ 40$	$\varphi''' = 1\ 50\ 06$	$\theta''' = 48\ 00\ 26$	[9159g]
Jupiter	$\varepsilon^{\text{iv}} = 81\ 52\ 10,3$	$\varpi^{\text{iv}} = 11\ 07\ 36$	$\varphi^{\text{iv}} = 1\ 18\ 52$	$\theta^{\text{iv}} = 98\ 25\ 45$	
Saturn	$\varepsilon^{\text{v}} = 123\ 05\ 29,4$	$\varpi^{\text{v}} = 89\ 08\ 20$	$\varphi^{\text{v}} = 2\ 29\ 38$	$\theta^{\text{v}} = 111\ 56\ 07$	
Uranus	$\varepsilon^{\text{vi}} = 173\ 30\ 16,6$	$\varpi^{\text{vi}} = 167\ 30\ 24$	$\varphi^{\text{vi}} = 0\ 46\ 26$	$\theta^{\text{vi}} = 72\ 59\ 51$	

[9162′] $1+\mu'$ $1+\mu'''$, $1+\mu^{iv}$, $1+\mu^{v}$ [4230′], which occur in the formulas of the sixth book.

VALUE OF THE MOON'S MASS, THAT OF THE EARTH BEING TAKEN FOR UNITY.

[9163] $$\text{Moon's mass} = \frac{1}{68{,}5}. \quad [4631]$$

VALUES OF THE MASSES OF JUPITER'S SATELLITES, THAT OF JUPITER BEING TAKEN FOR UNITY.

[9164]

Mass of the First Satellite	= 0,0000173281;	[7162]	1
" Second Satellite	= 0,0000232355;	[7163]	2
" Third Satellite	= 0,0000884972;	[7164]	3
" Fourth Satellite	= 0,0000426591;	[7165]	4
Jupiter's mass	= 1.		5

[9165] All these values, which may be considered as very nearly exact, can be corrected, when, in the course of time, we shall have ascertained the secular variations of the orbits of the planets and satellites.

[9166] We have determined, in [4072], the earth's mass by means of the sun's parallax, and we have there remarked that the value of that mass must vary as the cube of the parallax, compared with the cube of the supposed parallax $8^s{,}8$. Hence it follows, *that a small error in the solar parallax is increased*
[9167] *threefold in its influence upon the value of the earth's mass.** It is therefore advantageous to determine this mass by the effects of its action on other bodies; and this action on the motions of Venus and Mars is sufficient to enable us to ascertain this mass, by means of a great number of observations,
[9168] selected under the most favorable circumstances. Then we shall have the sun's parallax so much the more accurately, as an error in the whole mass produces only a third part as much error in the parallax.

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* (4111) We shall represent the sun's horizontal parallax by $8^s{,}8.(1+\delta p)$, and the
[9167a] earth's mass by $\frac{(1+\delta m)}{329630}$; the mass corresponding to the parallax $8^s{,}8$, being $\frac{1}{329630}$, [4072]. Then we shall have, by the rule given in [4073],

[9167b] $$(8^s{,}8)^3 : \frac{1}{329630} :: (8^s{,}8.\overline{1+\delta p})^3 : \frac{1+\delta m}{329630};$$

whence we easily deduce $(1+\delta p)^3 = 1+\delta m$, or by developing, and neglecting the second and higher powers of δp, $1+3.\delta p = 1+\delta m$; whence $\delta m = 3.\delta p$, as in [9167].

ON ASTRONOMICAL TABLES.

26. Each observation of a planet determines its geocentric longitude and latitude; the differences between the observed longitudes and latitudes, and [9169]
those calculated by the preceding formulas, will give two equations of condition between the corrections of the elements of the elliptical motion, and the disturbing masses. We may form in this manner a great number of equations of condition, and from them we can deduce the values of these corrections; then using these corrected formulas, we can construct more accurate tables of the planet's motion. Subsequent observations being compared with the primitive formulas, will furnish new equations of condition, which we must connect with the preceding, and after a sufficiently long interval of time we can obtain a large number of additional equations; then we may again determine, by the whole body of these equations, both old and new, the corrections of the elliptical elements and of the masses; and we can by this means form a more accurate set of tables than those which we first used. Continuing this process, we may successively render these tables more and more complete. The same method can be used in [9170]
improving the tables of the satellites. Thus the labors of present astronomers, being constantly added to those which precede them, will finally give a great degree of accuracy to astronomical tables, and to the values of the elements upon which they are founded.

SUPPLEMENT

TO THE

TENTH BOOK OF THE MÉCANIQUE CÉLESTE.

ON CAPILLARY ATTRACTION.

We have considered, in the tenth book, the phenomena arising from the refractive power exerted by bodies upon light. This force is the result of the attraction of their particles; but the law of this attraction cannot be determined by the phenomena, because the only condition required is that it must be insensible at sensible distances. All the laws of attraction, where this [9171] condition is fulfilled, will satisfy equally well the various phenomena of refraction derived from observation. The most important of these observations is that of the constant ratio of the sine of refraction to the sine of incidence, in the passage of light through diaphanous bodies. It is in this case only that corpuscular attraction has been submitted to an accurate analysis. I shall now proceed to lay before mathematicians another case, which is still more remarkable, from the variety and singularity of the phenomena depending upon it, and from its being susceptible of an equally accurate analysis; the [9172] case alluded to is that of capillary attraction.* The effects of the refractive

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* (4112) This theory of capillary attraction was first published by La Place in 1806; and in 1807 he gave a supplement [9757—10520]. In neither of these works is the repulsive [9173a] force of the heat of the fluid taken into consideration, because he supposed it to be unnecessary [10508]. But in 1819 he observed, that this action could be taken into [9173b] account, by supposing the force $\varphi(f)$ [9229] to represent the difference between the attractive [9173c] force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$; [9173d] so that the combined action would be expressed by,

$$\varphi(f) = A(f) - R(f); \qquad [9173e]$$

[9173] power correspond to dynamics and to the theory of projectiles; those of capillary attraction correspond to hydrostatics and to the equilibrium of fluids,

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$A(f)$ and $R(f)$ being functions of f, which may be extremely great in comparison with
[9173f] their difference $\varphi(f)$; and it is evident that $\varphi(f)$ *may become negative*, if the repulsive force $R(f)$ exceed the attractive. In this case, if the density is uniform, the calculation
[9173f′] will not be in any wise altered, supposing the attractive force to decrease less rapidly than the repulsive, and to become the greatest of the two before it ceases to be a sensible magnitude.
[9173g] In 1830, Gauss published a work on capillary attraction entitled "*Principia generalia theoriæ figuræ fluidorum in statu æquilibrii, etc.*," where, by means of the principle of virtual velocities,
[9173h] he obtains the figure of the capillary surface, and other theorems as they are given by La Place in this volume, and he also gives a more complete demonstration of the constancy of the angle of contact of the fluid with the sides of the tube. Finally, M. Poisson, in 1831, published his "*Nouvelle Théorie de l'action capillaire, etc.*," where he expressly introduces into the formulas the consideration *of the change of density of the fluid at its surface and near the*
[9173i] *sides of the tube* in consequence of the corpuscular attraction. This circumstance had not been taken into account by La Place, in making his calculations, though he expressly mentions the subject in [10502′, &c.]. This change of density at an insensible distance from the surface and from the sides of the tube, arises from the great inequality in the corpuscular action
[9173i′] on opposite sides of any particle, in consequence of the very rapid change in the value of the function $\varphi(f)$, which expresses the corpuscular force; and in making the calculations of the effect of such a force, M. Poisson proceeds upon the principle that, in the state of
[9173k] equilibrium, each infinitely small stratum of the fluid is pressed equally, on its opposite sides, by a force which is equal to the repulsive force of the heat of the neighboring particles, decreased
[9173l] by their corpuscular attraction. In other words, any infinitely thin stratum may be considered as being supported by the part of the fluid on the one side, and compressed by the part situated on the opposite side; *and the degree of condensation will depend upon the*
[9173m] *magnitude of the compressing force.* At a sensible distance from the surface of the fluid, this compressing force arises from a stratum of the fluid adjacent to the infinitely thin
λ stratum, and which is of the same thickness in every direction; *the thickness being equal to*
[9173n] *the distance λ to which the corpuscular attraction extends; and for this reason the interior density of the fluid is constant*, neglecting the very small condensation arising from the action of gravity, which varies with the distance from the upper surface. But when this distance
[9173o] from the upper surface is less than the radius of the sphere of activity of the corpuscular
[9173p] attraction, the thickness of the stratum, situated above that which we have under consideration, will be less than this radius, and then the compressing force, arising from the superior stratum, decreases very rapidly with the distance from the surface, and vanishes entirely at the
[9173q] surface itself [10502′—10507], where this infinitely thin stratum suffers no other pressure
[9173r] than that of the atmosphere. Hence it follows, that the condensation of the fluid decreases
λ' according to some unknown law, as we approach towards the upper surface, and its density
[9173s] is very different at that surface from what it is at an extremely small distance λ' below it,

which are elevated or depressed according to laws that we shall now attempt to explain.

Clairaut is the first, and indeed the only person, who has hitherto reduced [9174] to a rigorous calculation, the phenomena of capillary tubes, in his treatise on the figure of the earth. After having shown, by reasons which apply equally well to all known systems, the vagueness and insufficiency of that of Jurin, [9175] he analyses accurately all the forces which may concur in elevating the water in a glass tube. But his theory, explained with all the elegance which characterizes his beautiful work, leaves yet without explanation the law of that ascension, which, by experiment, is in the inverse ratio of the diameter of the tube. This great mathematician contents himself with observing that [9176] there must be an infinite number of laws of attraction, which, by substitution in these formulas, will give that result. The knowledge of these laws is however the most delicate and important part of this theory: it is indispensably necessary to connect together the different phenomena of capillary attraction, [9177] and Clairaut himself would have seen the necessity of it, if he had attempted, for example, to extend his investigations from tubes to the capillary spaces

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where it becomes very nearly equal to that of the fluid in the interior of its mass; observing that, though the distance λ' is insensible, it may be supposed much greater than the distance [9173t] to which the corpuscular action of any one particle of the fluid upon any other particle of its [9173u] mass extends. Similar remarks may be made relative to the change of density in the fluid near the sides of the tube. We shall see in [9260*f*, &c. 9841*c*, &c.], that this change of density [9173v] produces a corresponding change in the value of the capillary intensity H [9262*e*]; but as this quantity can be found only from actual experiments, and not from the analytical expression [9173w] [9253′, &c.], it leaves the results of La Place's theory unimpaired in all the formulas depending on this quantity; and the calculation of the effects of the capillary attraction is in [9173x] almost every case the same as if the change of density had been noticed; the slight differences which occur will be pointed out, wherever they happen to be found, with the methods of [9173y] correction as in [9580*q*]; and we may in this connection observe, that the *first* or greatest force spoken of by the author in [9184] is that which corresponds to the interior of the fluid, as computed by him upon the supposition that the corpuscular action $\varphi(f)$ is always positive and the density uniform; it is also evident that this result might be very much modified near [9173z] the surface of the fluid, by taking into view the repulsive force of the heat, and the change of density near that surface. For, in a stratum of variable density, at an insensible distance from the surface, we may conceive of such an arrangement of the particles of the fluid, as will render this action very different from that which the author supposes; making it there either large or small, positive or negative, according to the nature of the functions which are [9174a] assumed, to express the corpuscular action [9173*e*] and the density in that part of the fluid.

included between parallel planes, in order to deduce from analysis, the ratio
[9178] of equality, indicated by experiment, between the ascent of the fluid in a cylindrical tube, and its ascension between two parallel planes, whose distance from each other is equal to the semi-diameter of the tube; which no person has yet attempted to explain. A long while ago, I endeavored in vain to determine the laws of attraction which would represent these phenomena; but some late researches have rendered it evident that the whole may be
[9179] represented by the same laws, which satisfy the phenomena of refraction; that is, by laws in which the attraction is sensible only at insensible distances; and from this principle we can deduce a complete theory of capillary attraction.

[9180] Clairaut supposes that the action of a capillary tube may be sensible upon the infinitely thin column, which passes through the axis of the tube. Upon this point I differ wholly from him, and think, with Hawksbee and other philosophers, that the capillary attraction is, like the force producing refraction,
[9181] and all the chemical affinities, sensible only at insensible distances. Hawksbee observed that in glass tubes, whether the glass is very thick, or very thin, the water rises to the same height, if the interior diameters are the same. Hence it follows that the cylindrical strata of glass, which are at a sensible distance from the interior surface, do not aid in raising the water, though in each one of these strata, taken separately, the fluid ought to rise above the level. It is not the interposition of the strata, which they include between them,
[9182] which prevents their action upon the water; for it is natural to suppose that the capillary attraction, like the force of gravity, is transmitted through other bodies; this attraction must therefore disappear solely by reason of the distance of the fluid from these strata; whence it follows that the attraction of the glass upon the water is sensible only at insensible distances.

Making use of this principle, I have determined the action of a fluid mass, terminated by a portion of a spherical concave or convex surface, upon a
[9183] column situated within it, contained in an infinitely narrow canal, and directed towards the centre of that surface. By this action, I mean the pressure which the fluid, contained in the canal, would exert, by means of the attraction of the whole mass, upon a plane base situated within the canal, perpendicular to its sides, and at any sensible distance from the surface: this base being taken for unity, I shall show that this action is less or greater than if the surface were plane; *less* if the surface be *concave* [9275]; *greater* if the surface be *convex*
[9184] [9276]. Its analytical expression is composed of two terms: the *first*, which is much greater than the *second* [9262a,c], denotes the action of a mass terminated by a plane surface; and *I think that upon this term depends the*

suspension of the mercury, in a barometrical tube, at a height two or three times greater than that which is produced by the pressure of the atmosphere; also the [9185]
refractive power of diaphanous bodies, cohesion, and in general the chemical affinities. The second term denotes the part of the action, depending on the spherical form of the surface; or in other words, the action of the meniscus, included between that surface and the plane which touches it. This action is to [9186]
be added to, or subtracted from, the preceding one, according as the surface is convex or concave. It is inversely proportional to the radius of the spherical [9187]
surface; for it is evident that the less this radius is, the greater will be the meniscus, near the point of contact. This second term produces the capillary [9188]
action, which differs therefore from the chemical affinities corresponding to the first term.

From these results, relative to bodies terminated by sensible segments of a [9189]
spherical surface, I have deduced this general theorem [9302]. "*In all the* General theorem.
laws which render the attraction insensible at sensible distances, the action of a body terminated by a curve surface, upon an infinitely narrow interior canal, [9190]
which is perpendicular to that surface, at any point whatever, is equal to the half sum of the actions upon the same canal, of two spheres which have the same radii as the greatest and the least radii of curvature of the surface at that [9191]
point." By means of this theorem, and of the laws of the equilibrium of fluids, we can determine the figure which a fluid mass must have, when it is included within a vessel of a given figure, and acted upon by gravity. It depends upon an equation of partial differentials of the second order [9318], whose integral cannot be obtained by any known methods. If the figure be of revolution, [9192]
this equation is reduced to common differentials [9324], and may be integrated by a very approximate method, when the surface is very small. By this means I shall prove that, in tubes of a very small diameter, the surface of the fluid will approximate the more towards the form of a spherical segment, as the diameter of the tube shall be decreased [9342, &c.]. If these segments be [9193]
similar in different tubes of the same matter, the radii of their surfaces will be in the direct ratio of the diameters of the tubes. Now this similarity of the spherical segments will appear evident by considering that *the distance, at which the action of the tube ceases to be sensible, is imperceptible; so that if,* [9194]
by means of a very powerful microscope, we should be able to make it appear equal to a millimetre, it is probable that the same magnifying power would give [9195]
to the diameter of the tube an apparent length of several metres. The surface of the tube may therefore be considered as very nearly a plane surface, for an [9195']

extent which is equal to that of the sphere of its sensible activity; the fluid will therefore be elevated or depressed near that surface, in almost the same manner as if it were a plane. Beyond this point, the fluid will be subjected only to the
[9196] *force of gravity and its own action on its particles; its surface will be very nearly that of a spherical segment, of which the extreme tangent planes, being those of the fluid surface at the limits of the sensible sphere of activity of the tube, will be very nearly, in the different tubes, equally inclined to their sides;*
[9197] *whence it follows that all these segments will be similar.*

The comparison of these results gives the true cause of the elevation, or depression, of fluids, in capillary tubes, in the inverse ratio of their diameters. If, in the axis of a glass tube, we imagine an infinitely narrow canal to be placed, which, after being continued downwards, in a vertical
[9198] direction, a little below the bottom of the tube, is then turned in a horizontal direction below the tube, and afterwards in a vertical direction upwards, until it meets the horizontal surface of the water in the vessel, where the lower extremity of the tube is immersed; the action of the water in the tube, upon this canal, will be less on account of the concavity of its surface, than the action of the water in the vessel upon the same canal; therefore the fluid must
[9199] rise in the tube to compensate for this difference; and as it is, by what has been said, in the inverse ratio of the diameter of the tube, the elevation of the fluid above its level must follow the same law.

If the interior surface of the fluid be convex, which is the case with mercury in a glass tube, the action of the fluid upon the canal will be greater
[9200] than that of the fluid in the vessel; the fluid must therefore sink in the tube, in proportion to this difference, and the depression will therefore be in the inverse ratio of the diameter of the tube.

Therefore the attraction of a capillary tube has no other influence upon the elevation or depression of the fluid which it contains, than that of determining
[9201] *the inclination of the first tangent planes of the interior fluid surface, situated very near to the sides of the tube; and it is upon this inclination that the concavity or convexity of the surface depends, as well as the magnitude of its radius.* The friction of the fluid against these sides, may increase or diminish, a little, the curvature of its surface, as we see daily in the barometer; and then
[9202] the capillary effects increase or diminish in the same ratio. These effects increase in a very sensible manner, by the combined forces arising from the concavity of one surface and the convexity of the other surface. We shall see hereafter [9717, &c.], that we may thus raise the water in capillary tubes to a

greater height above its level, than when they are dipped into a vessel filled with that fluid.

The differential equation of the surface of a fluid included within capillary tubes of revolution [9383], leads to this general result, namely; that if, in a cylindrical tube, we introduce a cylinder, having the same axis as the tube, [9203]
and with such a diameter that the interval or space between its surface and the interior surface of the tube may be very small, the fluid will ascend, in the space between these tubes, to the same height as in a tube whose radius is equal to the interval between them [9410]. If we suppose the radii of the tube and cylinder to be infinite, it will correspond to the case of a fluid, included between two vertical and parallel planes which are very near to each other. The preceding result is verified at that limit, by experiments [9204]
[9658, &c.] formerly made in presence of the Royal Society of London, and under the inspection of Newton, who has quoted them in his Optics, an admirable work, where that profound genius has put forth many original ideas, elevated far above the science of his time, and which have been confirmed by modern chemistry. M. Hauy has consented, at my request, to make some experiments near the other limit, with tubes and cylinders of very small [9205]
diameters, and he has found the preceding result to be equally correct at this limit, as at the first.

The phenomena observed in a drop of fluid, in motion, or suspended in equilibrium, either in a conical capillary tube, or between two planes a little inclined to each other, are very proper for verifying this theory. A small [9206]
column of water in a conical tube, open at both ends, and supported horizontally, flows towards the vertex of the tube: and we easily perceive that this ought to take place. For the surface of the fluid column is concave at both its extremities; but the radius of the surface which is nearest the vertex, is less than the radius of the other surface nearest to the base; therefore the action of the fluid upon its own particles, is the least on the side nearest the vertex, consequently the column must tend towards that part. If the fluid be [9207]
mercury, then its surface will be convex, and its radius will, in like manner, be less near the vertex than near the base; but, on account of its convexity, the action of the fluid upon its particles will be greatest towards the summit, and the column must therefore tend towards the base of the tube.

We may balance this action, by the weight of the column itself, and keep it suspended in equilibrium, by inclining the axis of the tube to the horizon. A very simple calculation shows, that, if the length of the column be small, [9208]

the sine of the inclination of the axis is then, very nearly, in the inverse ratio of the square of the distance from the middle of the column to the vertex of the cone [9475]. A similar result takes place, if we put a drop of fluid between two planes, which are inclined to each other by a very small angle,
[9209] and are in contact at their horizontal borders [9550]. These results are entirely conformable to experiment, as we may see in Newton's Optics (question 31). This great mathematician endeavored to explain them; but his explanation, when compared with that which is here given, shows the advantages of an accurate mathematical theory.

By calculation we also find, that the sine of the inclination of the axis of
[9210] the cone to the horizon, is then very nearly equal to a fraction, having for its denominator the distance from the middle of the drop to the vertex of the cone, and for its numerator the height to which the fluid will rise in a cylindrical tube whose diameter is equal to that of the cone in the middle of the column [9474]. If the two planes, enclosing a drop of the same fluid, are inclined to each other by an angle, which is equal to the angle formed by the axis of
[9211] the cone and one of its sides, the plane which bisects the angle formed by the two preceding planes, must have the same inclination to the horizon, as the axis of the cone, for the drop to remain in equilibrium. Hawksbee made very carefully an experiment of this kind [9709], which we shall hereafter compare with the preceding theorem; the very little difference which is
[9212] found, between this experiment and the theorem, is an incontestible proof of its accuracy.

The theory furnishes an explanation, and the measure, of a singular phenomenon noticed in experiments; namely, that, whether the fluid be elevated or depressed, between two vertical and parallel planes, dipped into the fluid by their lower extremities, these planes have a tendency to approach towards each other. It is shown by analysis [9580, &c.], that, if the fluid be elevated
[9213] between them, each plane suffers a pressure, tending inwardly, and equal to that of a column of the same fluid, having for its height the half sum of the elevations of the fluid above the level, at the points of contact of the interior and exterior surfaces of the fluid with the plane, and for its base the part of the plane which is included between the two horizontal lines drawn through these points. If the fluid is depressed between these planes, each of them, in like manner, will suffer a pressure tending inwards, and equal to a column of the
[9214] same fluid, having for its height the half sum of the depressions below the level of the points of contact of the interior and exterior surfaces of the

fluid with the plane, and for its base the part of the plane included between the two horizontal lines drawn through those points* [9586].

The concavity or convexity of the surface of a fluid, included within capillary spaces, has heretofore been considered as nothing more than a secondary effect of the capillary action, and not as the principal cause of such phenomena; so [9215]
that but little attention has been paid to the curvature of these surfaces; but as the preceding theory makes the phenomena depend chiefly on the curvature, it becomes interesting to determine it. *Several experiments made very carefully, by M. Haüy, indicate that, in glass capillary tubes of a very small diameter, the* [9216]
concave surface of water and oil, and the convex surface of mercury, differ but very little from that of a hemisphere.

Clairaut made this singular remark; namely, that, if the law of the attraction of the matter of the tube upon the fluid, differs only by its intensity from the law of the attraction of the fluid upon its own particles, the fluid will rise above [9217]
the level, so long as the intensity of the first of these attractions shall exceed the half of the intensity of the second. *If the intensity of the first of these attractions* [9217′]
be exactly equal to half† *of the intensity of the second, it will be easy to show that the surface of the fluid in the tube will be horizontal, and that it will not rise above the level. If these two intensities be equal, the surface of the fluid in the tube will be concave, and of a hemispherical form; then the* [9218]
fluid will be elevated in the tube. If the intensity of the attraction of the

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* (4113) We shall see in [9580q, &c.] that the values of the pressures here given require some modification, when the angles formed by the vertical planes and the tangents of the surfaces of the fluid near to them are different, on different sides of the same plane; as [9214a]
sometimes happens on account of the planes being more or less moistened on the one side than on the other.

† (4114) This ratio of the intensities of the attraction of the tube and fluid, when the [9218a]
surface is horizontal, is computed upon the supposition that the density of the fluid is uniform, and that it does not vary near the surface of this fluid or near the sides of the tube. Now [9218b]
this is not the case, as the author himself has remarked in [10502′]; therefore this demonstration is defective, as will more fully be seen in [9596l]; but it is probable, however, that this ratio [9218c]
is nearly correct. We shall also see in [9626n, &c. 9655a, &c.], that the remarks made in [9218], for a concave hemisphere, and in [9219], for a convex hemisphere, are very nearly correct. Lastly, what is stated in [9219′], for concave or convex segments, may be considered [9218d]
as very nearly correct, except that in [9219′] we must not suppose that the ratio of the intensities at the limit of the surfaces, is strictly equal to one *half*, as the author asserts, [9218e]
though it is probably very nearly equal to it [9596k, &c.]

[9219] *tube be nothing, or insensible, the surface of the fluid in the tube will be convex and hemispherical; and then the fluid will be depressed.* Between these two limits, the surface of the fluid will be that of a spherical segment; and it will be concave or convex, according as the intensity of the attraction of the matter of
[9219′] the tube upon the fluid shall be greater or less than the *half* of the attraction of the fluid upon its own particles [9587—9655, 9218*e*].

If the intensity of the attraction of the tube upon the fluid exceeds that of the attraction of the fluid upon its own particles, *it appears to me probable that*
[9220] *then the fluid, by attaching itself to the tube, forms an interior tube, which produces alone the elevation of the fluid,* whose surface is concave and hemispherical. We have reason to suppose, that this is the case with water, and some kinds of oil, in a glass tube.

The case of a fluid which rises up between vertical planes, forming with each other a very small angle, or that where a fluid flows out from a capillary siphon, affords several phenomena which are merely corollaries of this theory. In general, if we take the trouble to compare the numerous experiments of
[9221] observers upon capillary action, we shall find that the results, obtained in these experiments, when they have been made with the proper precautions, may be deduced from the theory; not by vague, and always uncertain considerations, but by a series of geometrical reasoning, which seems to leave no doubt about the truth of the theory. I hope that this application of analysis to one of the most curious objects of physics, may interest mathematicians, and excite them to increase more and more these applications, which unite the advantage of confirming physical theories and improving analysis itself, by requiring new processes of calculation.

SECTION I.

THEORY OF CAPILLARY ATTRACTION.

1. We shall suppose a vase $ABCD$ (fig. 112,) to be filled with water, as high as AB, and that a glass capillary tube, $NMEF$, open at both ends, has its lower end immersed in the water, which will [9222]
rise up into the tube to O, and the surface will form a concave figure MON; O being the lowest point of this surface. We shall also suppose that an infinitely narrow canal $OZRV$, composed of a single filament of water, passes through the point O and the axis of the tube; then it is evident, from the principles of capillary attraction, which we have just explained, that the action of the water below the horizontal line IOK, will be the same upon the column OZ, as the action of the water in the vase upon the column VR. But the meniscus $MIOKN$ will act upon the column OZ *upwards*, and will therefore tend to raise the fluid. Hence it follows, that, in the state of equilibrium, the water in the canal $OZRV$ must be elevated higher in the tube than in the vase, so that it may compensate, by its weight, for the action of the meniscus.

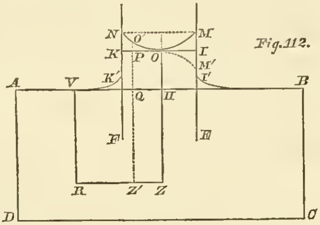

Fig. 112.

The law of this ascent, in tubes of different diameters, depends upon the attraction of the meniscus; and in this case, as in the theory of the figure of the planets, there is a mutual relation between the figure and the attraction of the [9223]
body, which renders the calculation difficult. In the investigation of this subject, *we shall consider the action of a body of any figure, upon a column of fluid, contained in an infinitely narrow canal, drawn perpendicularly to its surface, and whose base we shall take for unity.* [9223′]

We shall suppose, in the first place, that the body is a sphere, and we shall ascertain its action upon a fluid, contained in a canal, situated without the sphere, [9224]
and perpendicular to its surface. For this purpose we shall resume the analysis

[9225] given in [470^vi, &c.]; putting r for the distance of the attracted point, from the centre of a spherical stratum, whose radius is u, and thickness du; also θ for the
[9226] angle which the radius u forms with the right line r, and ϖ for the angle which the plane passing through the lines u, r, makes with a fixed plane passing through the right line r; then the element of the spherical stratum will be $u^2du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta$
[9227] [470^xiii]. Now if we put f for the distance of this element from the attracted point, which we shall suppose to be situated without the stratum, we shall have, as in [471],

[9228] $$f^2 = r^2 - 2ru \,.\, \cos.\theta + u^2 = (r - u \,.\, \cos.\theta)^2 + (u \,.\, \sin.\theta)^2, \text{ or}$$
[9228'] $$r = u \,.\, \cos.\theta + \sqrt{f^2 - u^2 \,.\, \sin.^2\theta}.$$

We shall represent the law of attraction by

[9229] $\varphi(f)$ = the attraction of a particle at the distance f; this attraction being insensible when f has a sensible value.*

[9229'] The action of the element of the stratum $u^2du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta$ [9227], upon the attracted point, being resolved in a direction parallel to r, *and tending towards the centre of the stratum*, will be represented as in [472], by

[9230] $$u^2du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\, \frac{(r - u \,.\, \cos.\theta)}{f} \,.\, \varphi(f).$$

We have, as in [473],

[9231] $$\frac{r - u \,.\, \cos.\theta}{f} = \left(\frac{df}{dr}\right);$$

substituting this in [9230], we get the following expression, being the same as in [474];

[9232] $$u^2du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\, \left(\frac{df}{dr}\right) \,.\, \varphi(f); \quad (a.)$$ [Attraction in the direction r.]

We shall now put,

[9232'] $$c = \int_0^\infty df \,.\, \varphi(f);$$
[9233] $$\Pi(f) = \int_f^\infty df \,.\, \varphi(f)$$
[9233'] $$= c - \int_0^f df \,.\, \varphi(f);$$

[9233''] $\Pi(f)$ *being a positive quantity, which decreases with extreme rapidity, as f increases,*† *so as to become insensible, when f has a sensible value* [*the density*

[9229a] * (4115) When the repulsive force of the heat of the particles is taken into consideration, the function $\varphi(f)$ will represent the excess of the attractive forces over the repulsive forces; and if the repulsive forces are the greatest, the function $\varphi(f)$ will become negative.

[9233a] † (4116) The value of $\Pi(f)$ [9233'], and that of $\psi(f)$ [9241], may be much affected by
[9233b] the change in the value of $\varphi(f)$ [9173e], arising from the consideration of the repulsive force of heat; and we must always keep in mind, that the author, in all his calculations, supposes the density of the fluid to be constant.

being supposed constant]. Then the expression of the attraction [9232] is evidently equal to the coefficient of dr, in the differential of the following function, taken relative to r; [9234]

$$u^2du\,.\,d\varpi\,.\,d\theta\,.\,\sin.\theta\,.\,\{c-\Pi(f)\}.$$ [9234′]

Now as c [9232′] is a constant quantity, wholly independent of the variable distance f, which occurs in r [9228′], we may neglect it, in finding the differential of [9234′], relative to r, and then this differential becomes equal to the coefficient of dr, in the differential of the following expression, taken relative to r; [9235]

$$-u^2du\,.\,d\varpi\,.\,d\theta\,.\,\sin.\theta\,.\,\Pi(f).$$ [9235′]

To extend this function to the whole stratum, we must first integrate it relative to ϖ, from $\varpi=0$ to $\varpi=2\pi$; π *being the ratio of the semi-circumference of a circle to its radius;* and then the integral becomes [9235″] [9236]

$$-2\pi\,.\,u^2du\,.\,d\theta\,.\,\sin.\theta\,.\,\Pi(f).$$ [9237]

We must now integrate this expression, from $\theta=0$ to $\theta=\pi$. For this purpose we shall take the differential of f^2 [9228], relative to θ, and divide it by $2ru$; hence we shall obtain as in [476], [9238]

$$d\theta\,.\,\sin.\theta=\frac{fdf}{ru};$$ [9239]

substituting this in [9237], and integrating, we get,

$$-2\pi\,.\,u^2du\,.\,\textstyle\int d\theta\,.\,\sin.\theta\,.\,\Pi(f)=-2\pi\,.\,\dfrac{udu}{r}\,.\,\textstyle\int fdf\,.\,\Pi(f).$$ [9240]

We shall now put, in like manner as in* [9232′, 9233, 9233′],

$$c'=\textstyle\int_0^\infty fdf\,.\,\Pi(f);$$ [9240′]

$$\psi(f)=\textstyle\int_f^\infty fdf\,.\,\Pi(f)$$ [9241]

$$=c'-\textstyle\int_0^f fdf\,.\,\Pi(f):$$ [9241′]

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* (4117) If we represent by λ *the limit of the distance at which the corpuscular attraction ceases to be sensible,* we shall have $\varphi(\lambda)=0$, *and the elements of the integral expression* $c=\int_0^\infty df\,.\,\varphi(f)$ [9232′] *will vanish when* f *exceeds* λ, so that we may put $c=\int_0^\lambda df\,.\,\varphi(f)$, instead of $c=\int_0^\infty df\,.\,\varphi(f)$. In like manner the elements of the integral $\Pi(f)$ vanish when f exceeds λ; and similar remarks may be made relative to the integrals [9240′—9241′]. Hence it is evident that, in all these integrals, it is only necessary to take the second limit at the point where $f=\lambda$, instead of $f=\infty$. It is important to notice this circumstance, otherwise we could not suppose that $\Pi(f)$, $\psi(f)$, vanish when f exceeds λ, since there are innumerable forms of the function $\varphi(f)$ which would not satisfy this condition. Thus if the complete value of $\varphi(f)$ contain the term $\frac{a}{f^2}$, depending on the common action [9240a] [9240b] [9240c] [9240d]

[9241''] $\psi(f)$ *being a positive quantity, which decreases with extreme rapidity* [9240a—l] [*the density being supposed constant*]. We shall have, by observing that the
[9242] integral must be taken from $\theta=0$ to $\theta=\pi$ [476''], and that at these two
[9243] points, $f=r-u$, and $f=r+u$ [476''],

[9244] $$-2\pi\,.\,u^2du\,.fd\theta\,.\sin.\theta\,.\,\Pi(f)=-\frac{2\pi.udu}{r}\,.\,\{\psi(r-u)-\psi(r+u)\}.$$

[9244'] Taking the differential of this function relative to r, we get the coefficient of dr, which represents, as in [9234], the action of the stratum upon the attracted point. But if we wish to obtain the action of the stratum upon a fluid column situated upon the line r, supposing the end nearest the centre of the stratum
[9245] to be at the distance b from that centre, we must multiply this coefficient by dr, and take the integral of the product; which reproduces the preceding function; to this we must add a constant quantity, which must be determined so *that the*
[9246] *integral may commence with* $r=b$. Hence we obtain, for this integral, the following expression; *

[9240e] of gravity, it will produce in $fdf.\varphi(f)$ the term $-\frac{a}{f}+\frac{a}{b}$; supposing this term to vanish when f is equal to the constant quantity b; then, putting $f=\infty$, we get in $\int_b^\infty df.\varphi(f)$ the term
[9240f] $\frac{a}{b}$; and by changing the first limit b into f, we get in $\int_f^\infty df.\varphi(f)$, or in $\Pi(f)$ [9233], the term $\frac{a}{f}$. This term of $\Pi(f)$ produces in $\int fdf.\Pi(f)$, the part $\int fdf.\frac{a}{f}=\int adf=af$,
[9240g] which vanishes when $f=0$, and when $f=\infty$ it becomes infinite, making the quantity c' [9240'] infinite, at the limit $f=\infty$, instead of adding to c', the insensible quantity $a\lambda$, corresponding to the insensible limit of f in $\Pi(f)$.

[9240h] To give some idea of functions which have properties like those of $\varphi(f)$, $\Pi(f)$, without attempting to ascertain their actual forms, and merely for the purpose of illustration, the
[9240i] author, in [9790], puts $\varphi(f)=ac^{-if}$; where a is a constant factor, i a very great positive number, and hyp. log. $c=1$. In this case, $f=0$ gives $\varphi(f)=a$, and when $f=\infty$ we have $\varphi(f)=0$; moreover the function af^nc^{-if} which may be put under the form

[9240k] $$af^nc^{-if}=\frac{af^n}{c^{if}}=\frac{af^n}{1+if+\frac{1}{2}i^2f^2+\&c.\,\ldots\,+gi^nf^n+hi^{n+1}f^{n+1}+\&c.}$$

[55] Int., always vanishes when $f=\infty$, whatever be the positive integral exponent n, as is
[9240l] very evident from the last developed expression, where g, h, &c., are used, for brevity, for the numerical coefficients of the terms of the developed function c^f [55] Int.

* (4118) For brevity we shall put $\Gamma(r)$ equal to the second member of [9244], also
[9247a] $\left(\frac{d.\Gamma(r)}{dr}\right)=\Gamma'(r)$; then we shall have as in [9244'] $\Gamma'(r)$, for the action of a spherical

$$\frac{2\pi . udu}{b} . \{\psi(b-u) - \psi(b+u)\}$$ [Action of the stratum upon part of the external fluid column, supposing the density = 1.] [9247]

$$-\frac{2\pi . udu}{r} . \{\psi(r-u) - \psi(r+u)\}.$$

Now $\psi(b+u)$ is always an insensible quantity, when b has a sensible value; [9247′]
and if, as we shall suppose to be the case, r at the extremity of the column the
most distant from the stratum, exceeds b, by a sensible quantity, $\psi(r-u)$ will [9248]
be insensible, and much more so $\psi(r+u)$; the preceding function [9247] will [9248′]
therefore be reduced to the single term,

$$\frac{2\pi . udu}{b} . \psi(b-u),$$ [Action of the stratum upon the external fluid column, supposing the density = 1.] [9249]

which will therefore express the action of the stratum upon the fluid contained in an infinitely narrow canal, situated upon the line r, so that its nearest extremity is at the distance b from the centre of the stratum. This action is evidently [9250]
equal to the pressure which the fluid produces, in consequence of the attraction of the stratum, upon a plane base situated at that extremity, within the canal, and perpendicular to its direction, *this base being taken for unity* [9223′]. [9250′]

To obtain the action of the whole sphere whose radius is b, we shall suppose [9251]
$b-u=z$; and then this action will be equal to the integral*

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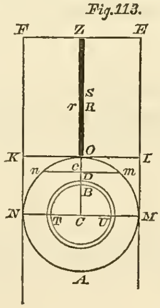
stratum BTU, fig. 113, whose centre is C, interior radius $CB=u$, exterior radius
$CD=u+du$, and density $=1$, *upon the fluid of the same density, situated in the* [9247b]
horizontal circular section of the infinitely narrow cylindrical tube at R,
at the distance $CR=r$ *from the centre* C; the axis of this tube OZ [9247c]
being on the continuation of the radius CO; the area of the circular
section of the base being supposed equal to unity [9223′]. Multiplying [9247d]
this base by the altitude $RS=dr$, of an infinitely small part of the
tube, we obtain the quantity of fluid it contains, which will therefore be [9247e]
represented by dr. Multiplying this by the action of the stratum $\Gamma'(r)$
[9247a], upon a particle of its mass, we get $dr . \Gamma'(r)$ for the action of [9247f]
the stratum upon the part RS of the fluid in the tube. Integrating
this, we get $\int dr . \Gamma'(r) = \Gamma(r) - \Gamma(b)$ [9247a]; the constant quantity [9247f′]
$-\Gamma(b)$ being added to make it vanish when $r=CO=b$ [9246].
This integral $\Gamma(r)-\Gamma(b)$ expresses the whole action of the stratum [9247g]
BTU, upon the fluid in the column OR; and it agrees with the
expression [9247]; the lower line being the same as $\Gamma(r)$ [9247a, 9244]; and the upper
line the same as $-\Gamma(b)$, which is deduced from $\Gamma(r)$, by changing its sign, and putting $r=b$. [9247h]

Fig. 113.



* (4119) To obtain the action of the whole sphere, we must evidently integrate the [9251a]
expression [9249] from $u=0$ to $u=b$, and if we change u into $b-z$, as in [9251], the

[9252] $$2\pi . \int_0^b \frac{(b-z)}{b} . dz . \psi(z);$$

and if we put

[9253] $$K = 2\pi . \int_0^b dz . \psi(z),$$

[9253'] $$H = 2\pi . \int_0^b zdz . \psi(z),$$

[Values of K, H, supposing the density of the fluid to be uniform, and equal to unity.]

the preceding action will become

[9254] $$K - \frac{H}{b}.$$

[Action of a sphere on an external column.]

[9255] *We must observe that K and H may be considered as independent of b; for, $\psi(z)$ being only sensible at insensible distances, it is a matter of indifference whether*
[9256] *we take the integrals [9253, 9253'] from $z=0$ to $z=b$, or from $z=0$ to $z=\infty$; so that we may here suppose K and H to correspond to these last limits.**

[9251b] corresponding limits, relative to z, will be $z=b$, $z=0$; so that the integral of [9249] will
[9251c] become $-2\pi . \int_b^0 \frac{(b-z).dz}{b} . \psi(z)$, or $2\pi . \int_0^b \frac{(b-z).dz}{b} . \psi(z)$; and, by separating the terms, it becomes $2\pi . \int_0^b dz . \psi(z) - \frac{1}{b} . 2\pi . \int_0^b zdz . \psi(z)$; then, substituting the integrals K, H,
[9251d] [9253, 9253'], it becomes $K - \frac{H}{b}$, as in [9254].

* (4120) In speaking of the second limit of z in these integrals, we must always keep in view the remark in [9240a—g], that it is not necessary to make $z=\infty$; but that, instead of
[9253a] it, we may put $z=\lambda$; λ *being of the same order as the radius of activity of the corpuscular*
[9253a'] *action;* so that, instead of the expressions $K = 2\pi . \int_0^\infty dz . \psi(z)$, $H = 2\pi . \int_0^\infty zdz . \psi(z)$, [9253—9256], we may substitute

[9253b] $$K = 2\pi . \int_0^\lambda dz . \psi(z),$$

[9253c] $$H = 2\pi . \int_0^\lambda zdz . \psi(z).$$

We shall hereafter want these integrals between the limits $z=0$, and $z=z$; and if we
[9253d] denote the corresponding values by the same letters K, H, with the index z below them, we shall have

[9253e] $$K_z = 2\pi . \int_0^z dz . \psi(z),$$

[9253f] $$H_z = 2\pi . \int_0^z zdz . \psi(z).$$

The author enters into a discussion of the relative values of K, $\frac{H}{b}$, in [10488—10499], always supposing the density of the fluid to be uniform; and by putting

[9253g] $$\frac{H}{b} = Ke,$$

he obtains a result, in [10496i], which may be expressed in the following manner;

[9253h] $e = \frac{1}{s} \times$ by the elevation of the fluid in a capillary tube whose radius is b;

the value of s is supposed by him to be very great, and by an estimate founded on the
[9253i] Newtonian theory of the emission of light, he makes it more than *ten millions of times the*

We may also observe that [in a fluid of uniform density] $\frac{H}{b}$ *is considerably less than* K, because the differential of the expression of $\frac{H}{b}$ [9253′] is equal to the differential of the expression of K [9253] multiplied by $\frac{z}{b}$; and as the factor $\psi(z)$, in these differentials, is sensible only when the value of $\frac{z}{b}$ is insensible, the integral $\frac{H}{b}$ must be considerably less than the integral K [9253g—m]. [9257] $\frac{H}{b}$ is much less than K.

The action of the whole sphere upon the fluid column which touches it, being $K-\frac{H}{b}$ [9254], *this quantity will also express the action of a sensible spherical segment, formed from the section of a sphere by a plane drawn perpendicular to the direction of the column;** for the part of the sphere situated beyond this plane, being at a sensible distance from the column, its action upon this column will be insensible; therefore $K-\frac{H}{b}$ [9254] will, for this reason, denote the action of any body terminated by the convex surface of a spherical segment whose radius is b, upon an external fluid column perpendicular to that surface. [9257′] Action of a spherical segment. [9258]

In the expression $K-\frac{H}{b}$, *K represents the action of a body terminated by* [9259]

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distance of the sun from the earth [10498′]; and though he considers such a large value as highly improbable, yet he infers, from this calculation, that s must be extremely great [10500]; and as the elevation of the fluid by the capillary action is small, it would follow that the value of *c must be an extremely minute quantity.* But we may observe, that, as the formula [9253h] is founded upon the hypothesis that the fluid, throughout its whole mass, is of the uniform density unity, it must, on this account, as well as on that of the uncertainty of the theory of the emission of light, upon which it depends, be considered as essentially defective and unsatisfactory; though it may serve the purpose of giving some idea of the vast difference there may be between the whole corpuscular attraction and that part of it which produces the capillary phenomena. [9253k] [9253l] [9253m]

* (4121) If this plane is represented in fig. 113, page 699, by $n\,c\,m$, drawn perpendicular to the axis $C\,c\,O\,Z$, so that the part c is at a sensible distance from the nearest point O of the tube, the action of the spherical segment $n\,O\,m$, upon the *external* column $O\,Z$ will be represented by $K-\frac{H}{b}$ [9254]; because the remaining part of the spherical mass $n\,N\,A\,M\,m$ is beyond the limits of any corpuscular action on the mass $O\,Z$ of the fluid in the tube. [9257a] [9257b]

[9259′] *a plane surface*;* for then, b being infinite, the term $\frac{H}{b}$ will disappear; *therefore*

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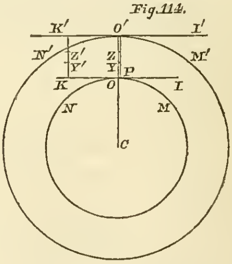
* (4122) The action of a spherical mass NOM, fig. 114, of a fluid of *uniform density*,
[9258a] upon an *external* cylindrical column OO' of the same fluid, whose axis OO' is on the
[9258b] continuation of the radius $CO=b$, and whose base O is taken for unity, has been
[9258c] computed in [9254 or 9258], and found to be equal to $K-\frac{H}{b}$. We may, in this hypothesis of a uniform density, suppose the height of the column OO' to be represented by the
[9258d] *extremely small insensible quantity* λ, which expresses the greatest distance or limit to which the corpuscular action extends, because then the particles of the fluid, situated above the
[9258e] point O' will be beyond the limits of the corpuscular action of the spherical fluid NOM, and
[9258f] we shall have $CO'=b+\lambda$. When b is infinite, the surface NOM coincides with the horizontal plane KOI, and the pressure of the column OO', on the base O, arising
[9258g] from the action of the fluid below the plane KOI, becomes equal to K [9258e]. The
[9258h] difference between the two expressions $K-\frac{H}{b}$ [9258c] and K [9258g], which is $\frac{H}{b}$, *evidently denotes, as in* [9260], *the action of the meniscus* $KOIMN$, on the same external column OO', in the direction $O'O$. Now if we draw through O' the horizontal
[9258i] plane $K'O'I'$, parallel to KOI, we shall have, as in [9258g, or 9268], K for the action of an indefinite mass of the same fluid, situated *above* the plane $K'O'I'$, upon the external
[9258k] column $O'O$, drawing *upwards* in the direction CO'; *and this quantity K also expresses*
[9258l] *the opposite or downward action of an indefinite mass of the fluid situated below the same plane* $K'O'I'$, upon the same column, because any point r of this column is attracted equally by the two masses in opposite directions, as the author observes in [9267], and which may be illustrated and rendered evident in the same manner as in [9258r′—s′]. Moreover by
[9258m] changing the radius $CO=b$ into $CO'=b+\lambda$, we get from [9258h] $\frac{H}{b+\lambda}$ for the action of the meniscus $K'O'I'M'N'$ upon the external column $O'O$, drawing upwards; and as
[9258m′] λ is an insensible quantity, relative to b [9195], we may neglect λ, and denote this action by $\frac{H}{b}$ *drawing upwards*, or
[9258n] $-\frac{H}{b}$ *drawing downwards*. Now the action of the whole mass below the plane $K'O'I'$, upon the column OO', is K [9258k]
[9258o] *drawing downwards*, and if we take away from this the action of the meniscus, we shall obtain the action of the spherical mass of fluid $N'O'M'$ upon the internal column OO'; therefore this action will be found by subtracting from K the
[9258p] expression $-\frac{H}{b}$; whence we get $K+\frac{H}{b}$ *for the action of the spherical mass* $N'O'M'$ upon the column OO', as in [9273].

Fig. 114.



Again, the action of the mass

this last term $\frac{H}{b}$ *denotes the action of the meniscus* $KOIMN$, fig. 112, page 695, [9260]

NOM upon the same column is $K-\frac{H}{b}$ [9258c]; subtracting this from the expression [9258p], we get $\frac{2H}{b}$ for the difference between the actions of the two spherical masses [9258q]
NOM, $N'O'M'$, upon the column OO'; or in other words $\frac{2H}{b}$ *expresses the action of the concentrical spherical stratum included between the surfaces* $N'O'M'$, NOM, upon the [9258r]
internal column OO'. This expression $\frac{2H}{b}$ vanishes when b is infinite, or when the concentrical surfaces change into the parallel planes KOI, $K'O'I'$; whence it follows that *the action of the fluid of uniform density included between the two parallel planes* KOI, $K'O'I'$, *upon the column* OO', *situated between them, is equal to nothing;* which is also easily [9258r']
proved by a geometrical method. For if, on the parallel and equal lines OO', KK', we take the four points Z, Y, Z', Y', so that $OZ=KZ'$, $OY=KY'$, and place two equal parts of the fluid at Z, Z', and two other equal parts at Y, Y'; it will be evident that the *upward* action of the particle Z' upon Y, will exactly balance the *downward* action of Y' upon Z; and this will happen throughout the whole space between the two parallel planes; [9258s]
therefore the action of the fluid between these planes upon the fluid in the column OO' must be equal to nothing; hence it is evident that the downward pressure of the fluid in the column $O'O$, at the base O, when the surface is the horizontal plane $K'O'I'$, must arise from the action of the fluid which is situated below the plane KI; and this action is represented by [9258s']
K [9259].

In the preceding calculations, when referring to fig. 114, we have supposed the surface of the fluid $N'O'M'$ to be convex; but if it be concave, as in fig. 112, we shall have, as in [9258n], $\frac{H}{b}$ for the action of the meniscus drawing *upwards*, or $-\frac{H}{b}$ for the action of the [9258t]
same meniscus drawing *downwards*, and, by adding this last expression to the quantity K [9258s'], which denotes the *downward* action of all the mass which falls below $K'O'I'$, [9258t']
we shall get $K-\frac{H}{b}$, for the *downward* action of the fluid, terminated by a concave spherical [9258u]
surface, upon the internal column OO', at its base O. We may here remark that the expression [9258u] may be derived from [9258p], by merely changing the sign of b, which represents the distance of the centre C of the spherical surface NOM from the point O; [9258v]
this centre being *below* the line KOI, in the case of a convex surface, corresponding to fig. 114, where we may consider the value of b as *positive;* but this centre falls above the line KOI, and on the side of the negative values of b, when the surface is *concave;* so [9258w]
that the sign of b must be different in the two cases.

In the second supplement to this work [9812—9845], the author gives another method [9258x]
of computing the formula [9294, or 9301], being essentially the same as that which is used by M. Poisson in pages 9—15 of his *Nouvelle Théorie*, &c. [9173i], where he obtains, [9258y]

[9261] formed by the spherical segment and the tangent of the surface, to raise the column OZ; therefore this action is inversely as the radius b of the surface MON, supposing it to be spherical.*

[9258z] in page 11 of his work, the same formula for a convex surface as that in [9301, or 9845]. The values of K, H, given by M. Poisson in pages 12, 14, in the hypothesis of a uniform density, and marked (1), (2), are

[9259a] $$K=\tfrac{2}{3}\pi\rho^2 . \int_0^\infty r^3 dr . \varphi(r),$$

[9259b] $$H=\tfrac{1}{4}\pi\rho^2 . \int_0^\infty r^4 dr . \varphi(r).$$

These expressions of K, H, appear to be different from those given by La Place in [9253, 9253′], but they are easily reduced to the same form, by putting the density

[9259c] $\rho=1$, and integrating by parts relative to z. For by this process [9253] becomes

[9259d] $K=2\pi . \int_0^b dz . \Psi(z)=2\pi b . \Psi(b)-2\pi . \int_0^b zdz . \left(\frac{d.\Psi(z)}{dz}\right)$; and as the limit b may be extended to $b=\infty$, the term $2\pi b.\Psi(b)$ will vanish [9240k]; so that we shall have $K=-2\pi . \int_0^b zdz . \left(\frac{d.\Psi(z)}{dz}\right)$.

[9259e] But from [9241′] we have $\left(\frac{d.\Psi(z)}{dz}\right)=-z . \Pi(z)$; hence $K=2\pi . \int_0^b z^2 dz . \Pi(z)$. Again

[9259f] integrating by parts, we get $K=\frac{2}{3}\pi b^3 . \Psi(b)-\frac{2}{3}\pi . \int_0^b z^3 dz . \left(\frac{d.\Pi(z)}{dz}\right)$; and, by rejecting, as

[9259g] before, the term depending on $b^3 . \Psi(b)$, it becomes $K=-\frac{2}{3}\pi . \int_0^b z^3 dz . \left(\frac{d.\Pi(z)}{dz}\right)$. Now

[9259h] substituting $\left(\frac{d.\Pi(z)}{dz}\right)=-\varphi(z)$ [9233′], we finally obtain $K=\frac{2}{3}\pi . \int_0^b z^3 dz . \varphi(z)$; being the

[9259i] same as in [9259a], changing the symbol z into r, and the limit b into ∞. By a similar process with the quantity H [9253′], and making the same substitutions as in [9259e, h], we get successively,

[9259k] $$H=2\pi . \int_0^b zdz . \Psi(z)=\pi b^2 . \Psi(b)-\pi . \int_0^b z^2 dz . \left(\frac{d.\Psi(z)}{dz}\right)=-\pi . \int_0^b z^2 dz . \left(\frac{d.\Psi(z)}{dz}\right)$$

[9259l] $$=\pi . \int_0^b z^3 dz . \Pi(z)=\tfrac{1}{4}\pi b^4 . \Pi(b)-\tfrac{1}{4}\pi . \int_0^b z^4 dz . \left(\frac{d.\Pi(z)}{dz}\right)=-\tfrac{1}{4}\pi . \int_0^b z^4 dz . \left(\frac{d.\Pi(z)}{dz}\right)$$

[9259m] $$=\tfrac{1}{4}\pi . \int_0^b z^4 dz . \varphi(z).$$

[9259n] Now putting $b=\infty$, and $z=r$, in [9259m], we get the same value of H, as that in [9259b], given by M. Poisson.

* (4123) In all the preceding calculations, the density of the fluid in the sphere $N'O'M'$,

[9260a] fig. 114, page 705, is supposed to be uniform, and it has been proved, in [9258p], that, in

[9260b] this case, the action of the sphere upon the internal column of the fluid $O'O$, at its base

[9260c] O, is represented by $K+\frac{H}{b}$, and by using $e=\frac{H}{Kb}$ [9253g], it becomes

[9260d] $$K+\frac{H}{b}=K+Ke; \qquad \text{[The density being supposed uniform.]}$$

[9260e] and this represents the downward pressure at O, in the column $O'O$; e being considered by La Place as a very small quantity [9253k].

[9260f] We shall now examine, in a brief manner, the effects of the variations of density [9173i, &c.],

We may here observe that the function K is analogous to that which is [9262]

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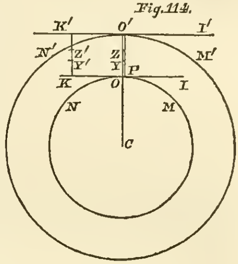
which occur in the surface of the fluid; that is, where the fluid is in contact with the air, or is wholly free from any pressure at its surface, as in a vacuum; and we shall again resume the subject in [9841c, &c. 9977r, &c.], after we have discussed the second method which the author has used in estimating the effects of the capillary action; [9260g] and we shall then give a more detailed computation of the effects of the change of the density in the fluid, near its surface and near the sides of the tube. We have not here attempted [9260g′] to do it, in so full a manner, because it requires the use of formulas, like those which are investigated in [9937, &c.], and which are too long to be repeated here, out of the regular course in which they have been given by the author.

Fig. 114.



The stratum of fluid near the surface, which is of a variable density, is of an insensible thickness, and if we suppose the point O to be situated beneath the lower surface of this stratum, and distant from it *by the insensible quantity* λ, which expresses the limit of the [9260h] distance to which the corpuscular action is sensible [9258d], *the whole length of the column* $OO'=\lambda'$, *will yet be of an insensible length.* We may consider the stratum of the fluid, [9260h′] contained between the two limiting spherical surfaces $N'O'M'$, NOM, to increase in density, from the point O', where it is nearly equal to nothing [10502′, &c.], to the lowest point [9260i] O, where it is very nearly equal to the internal density of the fluid, which we shall represent by unity; then, taking any point Z of the column $O'O$, and putting $OZ=z$, we may [9260k] suppose *the density D of the spherical stratum, of the thickness dz,* included between the [9260k′] surfaces which correspond to z and $z+dz$, to be a function of z; and the mass of the fluid contained in the part dz of the column OO', will be represented by the product of the two [9260l] quantities D and dz, which is Ddz, and this also will be a function of z. Moreover the [9260m] action of the whole spherical mass of the fluid $N'O'M'$, upon a particle at z, may *be represented by a function of z, which we shall denote by* $\Gamma(z)$. Multiplying this by the [9260n] mass Ddz of the column [9260m], we get $Ddz\,.\,\Gamma(z)$, for the action of the whole sphere $N'O'M'$ upon the part dz of the column $O'O$; and its integral $\int_0^{\lambda'} Ddz\,.\,\Gamma(z)$, or, as it may be written, $\int_0^{\infty} Ddz\,.\,\Gamma(z)$ [9240a], represents the whole action of the sphere $N'O'M'$ upon [9260o] the fluid of a variable density in the column $O'O$. When $b=\infty$, or $e=0$ [9260c], the [9260p] spherical surfaces $N'O'M'$, NOM, change into the horizontal planes $K'O'I'$, KOI, and [9260q] the pressure at the point O, in the case of the uniform density 1 [9260d], becomes equal to K, which is independent of $\frac{H}{b}$ or e [9260c]; and the slightest consideration will make [9260r] it manifest, that, when the density varies, in these *horizontal strata,* from the upper surface at O' towards the lower surface at O, the pressure at this last point, which we shall represent by K, will also be independent of e. Now in order to compare together the pressures at the [9260s] base O of the column $O'O$, in the case of a variable density D, with that which corresponds

[9262'] denoted by the same letter, in the theory of astronomical refraction explained

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to a constant density 1, the surfaces in both cases being supposed to be the horizontal planes $K'O'I'$, KOI, we shall suppose, in the first place, that the density D [9260k'], and the attraction $\Gamma(z)$ [9260n] correspond to the fluid of a variable density between these two
[9260t] parallel planes, and that $\Gamma(z)$ changes into $\Gamma_{,}(z)$, when the fluid is of the uniform density 1. Then we shall have, from [9260p, r, s],

[9260u] $$\int_0^\infty Ddz \,.\, \Gamma(z) = \mathrm{K}, \qquad \left[\begin{array}{c}\text{When the fluid in the column } O'O \text{ is of the variable density}\\ D\text{, and is terminated by the plane surface } K'O'I'.\end{array}\right]$$

[9260v] $$\int_0^\infty dz \,.\, \Gamma_{,}(z) = K. \qquad \left[\begin{array}{c}\text{When the fluid in the column } O'O \text{ is of the uniform density}\\ 1\text{, and is terminated by the plane surface } K'O'I'.\end{array}\right]$$

In the case of uniform density, the integral function $\int_0^\infty dz \,.\, \Gamma_{,}(z)$ is expressed by the symbol
[9260w] K [9260v], if $b=\infty$, or if the terminating surfaces be the planes $K'O'I'$, KOI; but the same integral function $\int_0^\infty dz \,.\, \Gamma_{,}(z)$ is expressed by $K+Ke$ [9260d], when b is of a finite magnitude, or, in other words, when the limiting planes $K'O'I'$, KOI, change into the
[9260x] limiting spherical surfaces $N'O'M'$, NOM, respectively. Hence it appears that, when the density of the fluid is constant, the effect of changing the infinite number of horizontal strata, parallel to $K'O'I'$, and of the thickness dz, into the same number of infinitely near
[9260y] concentrical spherical strata, of the thickness dz, and described about the centre C, is merely to add a term of the order e [9253k], to the function $\int_0^\infty dz \,.\, \Gamma_{,}(z)$; or, in other words, the function $\Gamma_{,}(z)$ may be supposed to be increased by a term of the order e. A similar change must evidently obtain in the expression with a variable density [9260u], when the
[9260z] horizontal strata are changed into a spherical form. For, as in the preceding case, $\Gamma_{,}(z)$ must be increased by a term of the order e; and the density D, which depends on the variations of the compressing force [9173m], will also vary in consequence of the change of pressure, in taking the spherical form, as has been already shown, when the density is uniform [9260d],
[9261a] where K changes into $K+Ke$, in consequence of this change of form; so that, by neglecting terms of the order e^2, which, according to observation, are not of sufficient importance to be retained, we may consider the function $D \times \Gamma(z)$ [9260u] as being augmented by a term
[9261b] of the order e, and this will increase the value of K, by a term of the same order as e, which we shall represent by Ke. Therefore the expressions of the pressure on the point O of the column $O'O$, will be finally reduced to the following forms, in the case of the spherical surfaces $N'O'M'$, NOM;

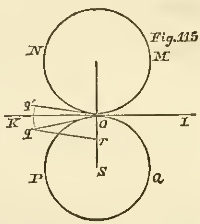
[9261c] $$\int_0^\infty dz \,.\, \Gamma_{,}(z) = K \,.\, (1+e); \qquad \text{[In the case of uniform density 1.]}$$

[9261d] $$\int_0^\infty Ddz \,.\, \Gamma(z) = \mathrm{K} \,.\, (1+\mathrm{e}). \qquad \text{[In the case of a variable density } D.]$$

Hence it appears that the only effect of the variableness of the density of the fluid is an
[9261e] alteration in the value of the symbols K and $\frac{H}{b}=Ke$ [9260e], which express the effect of the corpuscular and the capillary attraction, changing them respectively into K and $\frac{\mathrm{H}}{b}=\mathrm{Ke}$.
[9261f] This produces, however, no alteration in the calculations of the effects of the capillary attraction,
[9261g] because K, H, can be determined by observation alone, and not by analytical deductions, the
[9261h] law of attraction being unknown. Similar remarks may be made in case the surface is
[9261i] *concave*, and the effect will be as in [9258s—w], merely to change the signs of e, e.

in the tenth book * [8168]. [9262″]

2. *It is easy to deduce from what has been said, the action of a sphere* [*of uniform density*] *upon an infinitely narrow column of fluid, situated within the sphere, and perpendicular to its surface.* Suppose two equal spheres MON, POQ, fig. 115, to be in contact at the point O; that [9263]



IOK is a plane touching both spheres in that point, and that OS is the fluid column. The particle q of the lower meniscus $IOQPK$ will act upon the column OS, to elevate it. For if we draw the isosceles triangle Oqr, it is evident that the attractions of the particle q upon the part Or of the column mutually destroy each other; but the action of q upon rS tends to elevate the fluid, in the same manner as a point q' similarly situated in the upper meniscus $IOMNK$. *The two meniscuses act, therefore, with the same force to elevate the column of fluid; and we have seen, in the preceding article, that the action of the upper meniscus, in producing this elevation,* is $\frac{H}{b}$ [9260]; therefore this quantity also expresses the action of the lower meniscus. [9264] [9265] [9266]

Now the action of an indefinite mass above OS, and terminated by the plane IOK, is the same upon the column OS as that of a mass situated below, and terminated by the same plane; *for any point r of this column is equally attracted by both masses, but in opposite directions, since it is in equilibrium in virtue of these attractions* [9258*l*, &c.]. And as *K denotes, by the preceding article* [9259], [9267] [9267′] [9268]

Finally we may remark, that, in almost all the formulas of the author, where he uses the symbols K, H, as appertaining to the calculation of a fluid of uniform density, we can use the same formulas, supposing the fluid to vary in density near its surface, considering his values of K, H, to be the same as those we have named K, H, in the present article. For example, the general equation of the surface of the fluid [9318], corresponds to the case of a variable density, by using for H the value resulting from actual observation, which must necessarily be the same as H of the present article; and we shall use the author's formulas in this way throughout these notes. [9261*k*] [9261*l*] [9261*m*] [9261*n*]

* (4124) The quantity K, in the interior of the fluid mass, where the density is equal to unity, is supposed by La Place to be much greater than $\frac{H}{b}$ [9253*k*, *g*], and he seems to consider this value of K as a representation of the *corpuscular intensity* in the interior of the fluid mass; while H is proportional to that part of it upon which the *capillary intensity* depends [9360, &c.]. At an insensible distance from the surface of the fluid, the quantity K may be extremely small, or negative, as has been observed in [9174*a*]. [9262*a*] [9262*b*] [9262*c*] [9262*d*]

the action of the superior mass upon the column OS, it will also denote the
[9269] action of the inferior mass upon that column, drawing *downwards;* now this
[9270] action is composed of two parts, namely, that of the sphere POQ, and the action of the meniscus $IOQPK$; therefore, by putting S for the action of the sphere, and observing that the meniscus attracts the column *upwards* [9258*n*],
[9271] and that its action upon it is $\frac{H}{b}$, we shall have,

[9272] $$S-\frac{H}{b}=K;$$ [This is correct even when we notice the change in the density of the fluid near its surface.]

or, by transposition,

[9273] $$S=K+\frac{H}{b}.$$ [This is correct even when we notice the change of density in the fluid near its surface.]

Hence it follows, *that the action of a body terminated by a sensible portion of a convex spherical surface, upon a fluid column placed within it, and perpendicular*
[9274] *to the middle of that surface, is represented by* $K+\frac{H}{b}$, *drawing downwards from O to S.*

If the surface of the body, instead of being convex, is concave (as in fig. 112, page 695), *then the action of the mass $MEFN$ upon the canal OZ will be,*
[9274′] *as we have just seen, equal to* $K-\frac{H}{b}$ [9258*u*]; *therefore the action of a body*
Action of a spherical segment. *terminated by a sensible portion of a spherical surface* [9274′, 9274], *will be*

[9275] $K-\frac{H}{b}=$ the internal action of a *concave* spherical segment;

[9276] $K+\frac{H}{b}=$ the internal action of a *convex* spherical segment.

[These expressions are correct even when we notice the change of density near the surface of the fluid, as is seen in [9261*e*, &c.], or in [9842*d*, &c.]

3. *We may now determine generally the action of a body terminated by any*
[9277] *curve surface, upon a column situated within it, and contained in an infinitely narrow canal drawn perpendicular to any point of that surface.* If we suppose an ellipsoid to be drawn through this point, so as to touch the surface, its action will be very nearly the same as that of the solid, since, this action being supposed to extend only to insensible distances, *the meniscus which represents*
[9278] *the difference between the solid and the ellipsoid, will have no sensible action upon the column, at the points where these two bodies differ sensibly from each other.* We have seen in [9260], that the action of the meniscus which is
[9279] formed between the sphere and its tangent plane, is $\frac{H}{b}$, and that, [in the case of a fluid of uniform density,] it is of the order $\frac{z}{b}$ [9257], relative to the action K of this solid, z being equal to, or less than, the radius of the sphere of

sensible activity of the body. It is evident, for the same reason, that the *action of the meniscus which is the difference between the ellipsoid and the body, will be relative to the action* $\frac{H}{b}$ *of the order* * $\frac{z}{b}$, therefore it may be neglected in comparison with $\frac{H}{b}$; we shall therefore determine the action of this ellipsoid upon the column. One of the axes of this ellipsoid is in the direction of the column, and this axis we shall put equal to $2a$. If we suppose two planes to be drawn through this axis and the other two axes of the ellipsoid, their sections will be two ellipses, each of which will have $2a$ for an axis, and we shall represent the other two axes by $2a'$ and $2a''$. The radius of curvature of the first ellipsis, at the point of contact of the body with the ellipsoid, will be † $\frac{a'^2}{a}$, and that of the second at the same point will be $\frac{a''^2}{a}$; and by putting these radii equal to b, b', respectively, we shall have $b=\frac{a'^2}{a}$, $b'=\frac{a''^2}{a}$. Through the same point of contact and the axis $2a$, if we draw a plane, which is inclined by the angle θ to the plane passing through the two axes $2a$ and $2a'$, the section

[9280]
[9281]
[9282]
[9283]
[9284]

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* (4125) If we refer to the general expression of the value of z (9769), corresponding to the meniscus, we shall have, as in [9770], $z=A.x^2+\lambda.xy+B.y^2$, for the part of z [9280a] comprised in the ellipsoidal or parabolical part of the meniscus; and the remaining terms of z, namely, $C.x^3+D.x^2y+$ &c. [9769], for the part of z corresponding to the difference [9280b] between the meniscus and ellipsoid; and this last part may be considered as of the order $\frac{x}{a}$ relative to the first part, or of the order $\frac{\lambda}{a}$ [9240a]; therefore this part must be so very [9280c] small, that it may be neglected in comparison with the other part.

† (4126) In the ellipsis whose axes are $2a$, $2a'$, if we take a very small absciss z, counted from the vertex, corresponding to the coordinate y, we shall have, when z, y, are infinitely small, by the nature of the ellipsis [379c, 378s], $\frac{y^2}{2z}=p=\frac{a'^2}{a}$; and if we suppose r to be the [9283a] radius of curvature corresponding to this absciss and ordinate, we shall evidently have, from the properties of the circle, $2rz-z^2=y^2$, or, on account of the smallness of z, $\frac{y^2}{2z}=r$; hence [9283b] we have $r=\frac{a'^2}{a}$, as in [9283]. In like manner, we get $\frac{a''^2}{a}$ for the radius of curvature at the [9283c] vertex, in the ellipsis whose axes are $2a$, $2a''$. These axes are represented in [9283] by b, b', respectively; so that we shall have $b=\frac{a'^2}{a}$, $b'=\frac{a''^2}{a}$, *which are evidently the greatest* [9283d] *and the least radii of curvature of all the ellipses formed by the sections of the ellipsoid by a plane passing through the axis* a.

[9285] of the ellipsoid, by this last drawn plane, will be an ellipsis, having $2a$ for one of the axes;* and the other semi-axis, which we shall denote by A, will be given by the following formula;

[9286] $$A^2=\frac{a'^2a''^2}{a'^2.\sin.^2\theta+a''^2\cos.^2\theta}, \text{ or } \frac{1}{A^2}=\frac{1}{a''^2}.\sin.^2\theta+\frac{1}{a'^2}.\cos.^2\theta.$$

The radius of curvature of this ellipsis at the point of contact, being

[9287] represented by B, we have $B=\frac{A^2}{a}$, or $\frac{1}{B}=a.\frac{1}{A^2}$ [9284e]; and, by substituting [9286, 9283], we successively obtain,

[9288] $$\frac{1}{B}=a.\left\{\frac{1}{a''^2}.\sin.^2\theta+\frac{1}{a'^2}.\cos.^2\theta\right\}=\frac{1}{b'}.\sin.^2\theta+\frac{1}{b}.\cos.^2\theta.$$

The action of an infinitely small portion of the ellipsoid formed by the plane

[9289] which passes through the axes $2a$ and $2A$, and by another plane which is inclined to this by the angle $d\theta$, and passes through the axis $2a$, will be very nearly the same as that of the like portion of a sphere whose radius is B; and

[9290] as the action of this sphere is, by what has been said [9273], equal to $K+\frac{H}{B}$,

[9291] that of the infinitely small portion in question will be† $\frac{1}{2\pi}.d\theta.\left\{K+\frac{H}{B}\right\}$; therefore the whole action of the ellipsoid upon the canal will be

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* (4127) Changing α, β, γ, into a, a', a'', to conform to the present notation, we get

[9284a] the equation of an ellipsoid [1428b], under the form $\frac{x^2}{a^2}+\frac{y^2}{a'^2}+\frac{z^2}{a''^2}=1$; x, y, z, being the rectangular coordinates of the surface of the ellipsoid, whose semi-axes, parallel to those coordinates, are a, a', a'', respectively; the centre of the ellipsoid being the origin of the coordinates. If we now suppose a plane to be drawn through the axis $2a$, so as to form the angle θ, with the axis of y, the section of the ellipsoid will be an ellipsis whose axes are $2a$, $2A$ [9285]; and the coordinates of the extremities of the semi-axis A will evidently be

[9284b] $x=0$, $y=A.\cos.\theta$, $z=A.\sin.\theta$. Substituting these in the equation of the ellipsoid

[9284c] [9284a], we get $\frac{A^2.\cos.^2\theta}{a'^2}+\frac{A^2.\sin.^2\theta}{a''^2}=1$; which being multiplied by $\frac{a'^2a''^2}{a'^2.\sin.^2\theta+a''^2.\cos.^2\theta}$ gives [9286]. The radius of curvature, corresponding to this ellipsis, at the point of contact,

[9284d] is found, as in [9283], by changing, in the expression of b, a' into A; by this means it becomes, as in [9287], $B=\frac{A^2}{a}$, or $\frac{1}{B}=a.\frac{1}{A^2}$. Substituting the expression [9286], we

[9284e] get the first value of $\frac{1}{B}$ [9288]. The second of those expressions is deduced from the first, by using the values of b, b' [9283].

[9292a] † (4128) The whole action of a sphere of the radius B being $K+\frac{H}{B}$ [9290], that of

$$\frac{1}{2\pi}\cdot\int d\theta\cdot\left\{K+\frac{H}{b}\cdot\cos.^2\theta+\frac{H}{b'}\cdot\sin.^2\theta\right\};$$ [9292]

the integral being taken from $\theta=0$ to $\theta=2\pi$, which gives for that action the following expression; [9293]

$$K+\tfrac{1}{2}H\cdot\left\{\frac{1}{b}+\frac{1}{b'}\right\}.$$ [Action of an ellipsoid, which is correct even when the density of the fluid is variable near its surface.] [9294]

If the surface be concave, we must suppose b *and* b' *negative. If it be in part concave, and in part convex, like the circumference of a pulley, we must suppose the radius of curvature corresponding to the convex part to be positive, and that corresponding to the concave part to be negative.* [9294'] [9295]

Putting B, B', *for the radii of curvature of the sections of the surface of the body by two planes inclined to each other by a right angle, we shall have, by what has been said* [9288], B, B' [9296]

$$\frac{1}{B}=\frac{1}{b'}\cdot\sin.^2\theta+\frac{1}{b}\cdot\cos.^2\theta;$$ [9297]

hence we find, by changing θ into $\frac{\pi}{2}+\theta$, which changes B into B', [9298]

$$\frac{1}{B'}=\frac{1}{b'}\cdot\cos.^2\theta+\frac{1}{b}\cdot\sin.^2\theta;$$ [9299]

consequently *

$$\frac{1}{B}+\frac{1}{B'}=\frac{1}{b}+\frac{1}{b'}.$$ [9300]

Therefore the preceding action [9294] may also be put under the following form;

$$K+\frac{H}{2B}+\frac{H}{2B'};$$ [Action of any body upon an internal column, the rectangular radii of curvature being B, B'. This is correct even when we notice the change of density near the surface of the fluid.] [9301]

that is, *the action of a body, of any convex form, upon a fluid contained in an infinitely narrow canal, perpendicular to any point whatever of its surface, is equal to the half sum of the actions of two spheres whose radii are respectively equal to the following ones, namely, the radius of curvature of any section of the* General theorem. [9302]

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a section of the sphere corresponding to the angle $d\theta$, will be found by multiplying it by $\frac{d\theta}{2\pi}$, as in [9291]. Substituting the variable value of $\frac{1}{B}$ [9288], and prefixing the sign of integration, we obtain the whole action of the ellipsoid as in [9292]. This is reduced to the form [9294], by observing that we have, as in [1544a], $\int_0^{2\pi}d\theta\cdot\cos.^2\theta=\pi$; $\int_0^{2\pi}d\theta\cdot\sin.^2\theta=\pi$; also $\int_0^{2\pi}d\theta=2\pi$. [9292$b$] [9292$c$]

* (4129) Adding together the expressions [9297, 9299], and putting $\sin.^2\theta+\cos.^2\theta=1$, we get [9300]; substituting this in [9294], we get [9301]. [9300a]

surface by a plane drawn perpendicular to the surface through that point, and the radius of curvature of the section formed by a plane perpendicular to the
[9303] *preceding plane.**

4. *We shall now determine the form of the surface of water included within*
[9304] *a tube of any form.* We may, in this investigation, as is well known, use either the principle of a curvilinear canal terminated at two points of this surface, or the principle of the perpendicularity of the force at the surface. In the present question, the first of these principles has a great advantage over the second,
[9305] because it requires only the determination of the two actions K and $\frac{H}{2}\cdot\left(\frac{1}{b}+\frac{1}{b'}\right)$ [9294], and in fact only the last of these two forces, since the first K disappears from the equation of the surface [9318], as we shall soon see. Although *the force which produces this second action, is, at the surface, incomparably greater*
[9306] *than gravity; yet, as it acts only upon an insensible interval, its action upon a fluid column of a sensible length, may be compared with the force of gravity upon the column.* But if we wish to make use of the principle of the perpendicularity of the resultant of all the forces, at the surface, we must consider
[9307] not only the action which produces the forces K and $\frac{H}{2}\cdot\left(\frac{1}{b}+\frac{1}{b'}\right)$, which are perpendicular to the surface, but also the force of gravity, and the force which arises from the attraction of the meniscus corresponding to the difference
[9308] between the ellipsoid and the body; for, although the action of this part upon a fluid column is insensible, because it acts sensibly only at an insensible
[9308'] distance, yet it is of the same order as gravity. On account of the difficulties in estimating all these forces and their directions, it is much more convenient to use the principle of the equilibrium of the canals.†

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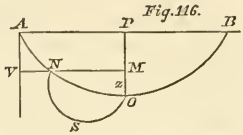
* (4130) The actions of the two spheres whose radii are B, B', are respectively, as in
[9301a] [9273], $K+\frac{H}{B}$, $K+\frac{H}{B'}$, whose half sum is $K+\frac{H}{2B}+\frac{H}{2B'}$; being the same as the action of the ellipsoid of curvature [9301], as in [9302—9303]. If the surface be *concave,* we must put b, b', negative, as in [9294']; consequently B, B', [9297, 9299], will be
[9301b] negative, and then the action [9301] will become $K-\frac{1}{2}H.\left\{\frac{1}{B}+\frac{1}{B'}\right\}$.

† (4131) La Place has used this second method in his supplement to this theory
[9309a] [9812—9845], and it serves to prove, *a posteriori,* the identity of the two results.

We shall suppose O, fig. 116, to be the lowest point of the surface AOB of the water contained in the tube, and we shall use the following symbols; [9308″]



z, the vertical coordinate OM; [9309]

x, y, the horizontal rectangular coordinates of any point N of the surface; [9309′]

R the greatest, and R' the least, radius of curvature of the surface, at the point N;* then R, R', will be the roots of the following quadratic equation; [9310]

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* (4132) We have x, y, z, for the rectangular coordinates of the point N of the surface for which the radius of curvature is to be computed; and we shall suppose that, for another extremely near point of the surface, these coordinates become respectively $x+h$, $y+k$, $z+l$; then, from Taylor's theorem [610], we shall obtain the value of $z+l$ [9310e], *neglecting terms of the third order*, which are not wanted in the present computation. We shall now suppose that the same symbols, accented, namely, x', y', z', p′, q′, r′, s′, t′, l', correspond to the osculatory sphere at the point N, without altering the increments h, k; hence we shall have the expression of $z'+l'$ [9310f], which is similar to that of $z+l$. Lastly, by putting α, β, γ, for the coordinates of the centre of the osculatory sphere, parallel respectively to the axes of x, y, z, also R for the radius of the sphere, we shall have the equation of this osculatory spherical surface at N, as in [9310g], being conformable to [19e]; [9310a] [9310b] [9310c] [9310d]

$$z+l=z+(\mathrm{p}h+\mathrm{q}k)+\tfrac{1}{2}(\mathrm{r}h^2+2\mathrm{s}hk+\mathrm{t}k^2); \qquad [9310e]$$

$$z'+l'=z'+(\mathrm{p}'h+\mathrm{q}'k)+\tfrac{1}{2}(\mathrm{r}'h^2+2\mathrm{s}'hk+\mathrm{t}'k^2); \qquad [9310f]$$

$$R^2=(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2. \qquad [9310g]$$

Taking the differentials of the equation [9310g], considering R, α, β, γ, as constant, we get, by successive reductions, the following values of p', q', r', s', t', relative to the spherical surface;

$$\mathrm{p}'=\left(\frac{dz'}{dx'}\right)=-\frac{(x'-\alpha)}{z'-\gamma}; \qquad \mathrm{q}'=\left(\frac{dz'}{dy'}\right)=-\frac{(y'-\beta)}{z'-\gamma}; \qquad [9310h]$$

$$\mathrm{r}'=\left(\frac{ddz'}{dx'^2}\right)=-\frac{1}{z'-\gamma}-\frac{(x'-\alpha)^2}{(z'-\gamma)^3}=\frac{-(1+\mathrm{p}'^2)}{z'-\gamma}; \qquad [9310i]$$

$$\mathrm{s}'=\left(\frac{ddz'}{dx'dy'}\right)=\frac{(x'-\alpha)\cdot\mathrm{q}'}{(z'-\gamma)^2}=\frac{(y'-\beta)\cdot\mathrm{p}'}{(z'-\gamma)^2}=-\frac{\mathrm{p}'\mathrm{q}'}{z'-\gamma}; \qquad [9310k]$$

$$\mathrm{t}'=\left(\frac{ddz'}{dy'^2}\right)=-\frac{1}{z'-\gamma}-\frac{(y'-\beta)^2}{(z'-\gamma)^3}=\frac{-(1+\mathrm{q}'^2)}{z'-\gamma}. \qquad [9310l]$$

Now if we compare the expressions [9310e, f], we shall find that, when $h=0$, $k=0$, we must put $z'=z$, to make the spherical surface pass through the point N, whose coordinates are x, y, z. Moreover if we put p′ = p, q′ = q, we shall find that the terms $\mathrm{p}h+\mathrm{q}k$, [9310l′]

[9311] $R^2.(rt-s^2)-R.\sqrt{1+p^2+q^2}.\{(1+q^2)r-2pqs+(1+p^2).t\}+(1+p^2+q^2)^2=0;$

in which the symbols are,

[9312] $$p=\left(\frac{dz}{dx}\right); \qquad q=\left(\frac{dz}{dy}\right);$$

[9313] $$r=\left(\frac{ddz}{dx^2}\right); \qquad s=\left(\frac{ddz}{dxdy}\right); \qquad t=\left(\frac{ddz}{dy^2}\right).$$

$p'h+q'k$ [9310e, f], depending on the first power of h, k, become equal to each other;

[9310m] hence the values [9310h] become, for this point N of the curve surface, where $z'=z$, $y'=y$, $x'=x$,

[9310n] $$p'=p=-\frac{(x-\alpha)}{z-\gamma}; \qquad q'=q=-\frac{(y-\beta)}{z-\gamma};$$

and it is very evident that these equations express the condition that the spherical surface is tangent to the proposed surface at the point N, whose coordinates are x, y, z. Substituting

[9310o] the values [9310l′—n] in [9310f], and subtracting the result from [9310e], we get [9310p], which is reduced to the form [9310q], by putting $k=mh$;

[9310p] $$l-l'=\tfrac{1}{2}.\left\{\left(r+\frac{1+p^2}{z-\gamma}\right).h^2+2\left(s+\frac{pq}{z-\gamma}\right).hk+\left(t+\frac{1+q^2}{z-\gamma}\right).k^2\right\}$$

[9310q] $$=\tfrac{1}{2}h^2.\left\{r+\frac{1+p^2}{z-\gamma}+2m.\left(s+\frac{pq}{z-\gamma}\right)+m^2.\left(t+\frac{1+q^2}{z-\gamma}\right)\right\}.$$

Now if we put the coefficient of $\frac{1}{2}h^2$ equal to nothing, we shall have the equations [9310r], and then $l-l'=0$; so that the osculatory sphere at N will coincide with the proposed surface, as far as terms of the second order in h, k, inclusively;

[9310r] $$0=r+\frac{1+p^2}{z-\gamma}+2m.\left(s+\frac{pq}{z-\gamma}\right)+m^2.\left(t+\frac{1+q^2}{z-\gamma}\right), \text{ or}$$

$$z-\gamma=-\left\{\frac{1+p^2+2pqm+(1+q^2)m^2}{r+2sm+tm^2}\right\}.$$

For brevity we shall put this value of $z-\gamma$ equal to M, and then we shall have, from [9310n, g],

[9310s] $$z-\gamma=M; \qquad y-\beta=-Mq; \qquad x-\alpha=-Mp; \qquad R=M\sqrt{1+p^2+q^2}.$$

[9310t] If we give to m various values, there will arise an infinite number of values of R; and the maximum or minimum of these values will be found by putting $\left(\frac{dR}{dm}\right)=0$, or $\left(\frac{dM}{dm}\right)=0$;

[9310u] because the factor $\sqrt{1+p^2+q^2}$, which occurs in the value of R [9310s], does not contain m. Now the expression of $z-\gamma$ or M [9310r], being multiplied by $r+2sm+tm^2$

[9310v] gives $(r+2sm+tm^2)M=-\{1+p^2+2pqm+(1+q^2)m^2\}$; and by putting for brevity

[9310w] $1+p^2+Mr=L$, $pq+Ms=D$, $1+q^2+Mt=N$, it becomes $L+2Dm+Nm^2=0$. Then taking its differential relative to m, dividing by $2dm$, and putting $\left(\frac{dM}{dm}\right)=0$, we

Therefore we shall have*

$$\frac{1}{R}+\frac{1}{R'}=\frac{(1+q^2).r-2pqs+(1+p^2).t}{(1+p^2+q^2)^{\frac{3}{2}}}. \qquad [9314]$$

This being premised, if we imagine any infinitely narrow canal NSO (fig. 116, page 713), to be formed, we must have, by the law of equilibrium of the fluid contained in this canal,†

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get $D+Nm=0$, or $m=-\frac{D}{N}$; substituting this in the last equation in [9310w], we get $L-\frac{2D^2}{N}+\frac{D^2}{N}=0$, or $NL-D^2=0$; resubstituting the values of N, L, D [9310x]
[9310w], we obtain $(1+p^2+Mr).(1+q^2+Mt)-(pq+Ms)^2=0$. Developing this, and arranging according to the powers of M, we finally obtain

$$M^2(rt-s^2)+\{(1+q^2)r-2pqs+(1+p^2)t\}.M+1+p^2+q^2=0. \qquad [9310y]$$

Multiplying this by $1+p^2+q^2$, and substituting $M.\sqrt{1+p^2+q^2}=R$ [9310s], we get [9311].

* (4133) Dividing the equation [9311] by $R^2.(1+p^2+q^2)^2$, we get

$$\frac{1}{R^2}-\frac{\{(1+q^2)r-2pqs+(1+p^2)t\}}{(1+p^2+q^2)^{\frac{3}{2}}}.\frac{1}{R}+\frac{rt-s^2}{(1+p^2+q^2)^2}=0. \qquad [9314a]$$

Putting for brevity n equal to the coefficient of $-\frac{1}{R}$, and n' for the term which is independent of R, we get $\frac{1}{R^2}-n.\frac{1}{R}+n'=0$, which is a quadratic equation in $\frac{1}{R}$, [9314b]
whose two roots are $\frac{1}{R}=\frac{1}{2}n+\sqrt{\frac{1}{4}n^2-n'}$, $\frac{1}{R'}=\frac{1}{2}n-\sqrt{\frac{1}{4}n^2-n'}$, and their sum is [9314$c$]
$\frac{1}{R}+\frac{1}{R'}=n$; n being the coefficient of $-\frac{1}{R}$ in [9314a]; and by substituting it, we get [9314].

† (4134) The point N, fig. 116, is situated above the level of the point O by the quantity $OM=z$, and the gravity of the particles contained in this column is equal to [9315a]
$g\times z$, or gz. This being added to the corpuscular action at N, namely $K-\frac{H}{2}\left(\frac{1}{R}+\frac{1}{R'}\right)$ [9315b]
[9317], ought to balance the corpuscular action at O, or $K-\frac{H}{2}\left(\frac{1}{b}+\frac{1}{b'}\right)$ [9301b, 9300]; this agrees with [9315]. Rejecting the terms K, which mutually destroy each other, in each member of the equation, then multiplying by $-\frac{2}{H}$, and substituting the value of [9315c]
$\frac{1}{R}+\frac{1}{R'}$ [9314], it becomes as in [9318]. We have already remarked, in [9261m], that this equation is the same whether we consider the fluid as variable in density near its surface, [9315d]
or homogeneous, using the value of H depending on observation, when applying it to the case of nature.

[9315] $$K-\frac{H}{2}\cdot\left(\frac{1}{R}+\frac{1}{R'}\right)+gz=K-\frac{H}{2}\cdot\left(\frac{1}{b}+\frac{1}{b'}\right);$$

b, b'.
[9316] *b and b' being the greatest and the least radii of curvature* [9283d] *of the*
g. *surface at the point O, and g the force of gravity.* For the action of the fluid upon the canal, at the point N, is, by what has been said [9301b],
[9317] $K-\frac{H}{2}\cdot\left\{\frac{1}{R}+\frac{1}{R'}\right\}$; moreover the height of the point N above O is z [9309].

The preceding equation gives, by substituting the value of $\frac{1}{R}+\frac{1}{R'}$ [9314],

[9318] General equation of the surface of the fluid.

$$\frac{(1+q^2)\cdot r-2pqs+(1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}=\frac{1}{b}+\frac{1}{b'}+\frac{2gz}{H}. \qquad (a)$$

[General differential equation of the concave surface of a fluid in a capillary tube. This is correct even when we notice the change of density of the fluid near the surface and sides of the tube.]

This equation is of the second order of partial differentials; and, by integrating it, we shall have two arbitrary functions, which we must determine by the equation of the surface of the sides of the tube in which the fluid is contained,
[9319] and by the inclination of the extreme planes of the surface of the fluid; this inclination, as we have seen, being the same for all the planes [9197].

[9319'] *When the surface is formed by the revolution of a curve about the axis z, the preceding equation can be reduced to common differentials; for z then becomes a function of* $\sqrt{x^2+y^2}$; and if we put

[9320] $$u=\sqrt{x^2+y^2},$$

we shall have*

[9321a] * (4135) z being a function of u, and $u=\sqrt{x^2+y^2}$, we have $dz=\left(\frac{dz}{du}\right)\cdot du$, and $ddz=\left(\frac{ddz}{du^2}\right)\cdot du^2+\left(\frac{dz}{du}\right)\cdot ddu$. From these general values of dz, ddz, we easily obtain the partial differentials relative to x, y, as follows;

[9321b] $$\left(\frac{dz}{dx}\right)=\left(\frac{dz}{du}\right)\cdot\left(\frac{du}{dx}\right);\quad \left(\frac{dz}{dy}\right)=\left(\frac{dz}{du}\right)\cdot\left(\frac{du}{dy}\right);\quad \left(\frac{ddz}{dx^2}\right)=\left(\frac{ddz}{du^2}\right)\cdot\left(\frac{du}{dx}\right)^2+\left(\frac{dz}{du}\right)\cdot\left(\frac{ddu}{dx^2}\right);$$

[9321b'] $$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddz}{du^2}\right)\cdot\left(\frac{du}{dx}\right)\cdot\left(\frac{du}{dy}\right)+\left(\frac{dz}{du}\right)\cdot\left(\frac{ddu}{dxdy}\right);\quad \left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{du^2}\right)\cdot\left(\frac{du}{dy}\right)^2+\left(\frac{dz}{du}\right)\cdot\left(\frac{ddu}{dy^2}\right).$$

[9321c] Now having $u=\sqrt{x^2+y^2}$, we get $\left(\frac{du}{dx}\right)=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{u}$; $\left(\frac{du}{dy}\right)=\frac{y}{\sqrt{x^2+y^2}}=\frac{y}{u}$; substituting these in $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ [9321b], they become as in [9321]. Again from
[9321d] $\left(\frac{du}{dx}\right)=\frac{x}{u}$, we get $\left(\frac{ddu}{dx^2}\right)=\frac{1}{u}-\frac{x}{u^2}\cdot\left(\frac{du}{dx}\right)=\frac{1}{u}-\frac{x^2}{u^3}=\frac{y^2}{u^3}$; and in like manner by
[9321e] changing x into y, and the contrary, $\left(\frac{ddu}{dy^2}\right)=\frac{x^2}{u^3}$; also, $\left(\frac{ddu}{dxdy}\right)=-\frac{x}{u^2}\cdot\left(\frac{du}{dy}\right)=-\frac{xy}{u^3}$.

$$\left(\frac{dz}{dx}\right)=\frac{x}{u}\cdot\frac{dz}{du}; \qquad \left(\frac{dz}{dy}\right)=\frac{y}{u}\cdot\frac{dz}{du}; \qquad [9321]$$

$$\left(\frac{ddz}{dx^2}\right)=\frac{x^2}{u^2}\cdot\frac{ddz}{du^2}+\frac{y^2}{u^3}\cdot\frac{dz}{du}; \qquad [9321']$$

$$\left(\frac{ddz}{dxdy}\right)=\frac{xy}{u^2}\cdot\frac{ddz}{du^2}-\frac{xy}{u^3}\cdot\frac{dz}{du}; \qquad [9322]$$

$$\left(\frac{ddz}{dy^2}\right)=\frac{y^2}{u^2}\cdot\frac{ddz}{du^2}+\frac{x^2}{u^3}\cdot\frac{dz}{du}. \qquad [9323]$$

Therefore the preceding equation [9318] becomes*

$$\frac{\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}=\frac{2}{b}+\frac{2gz}{H}; \qquad (b) \qquad [9324]$$

General differential equation of the concave surface of a fluid in a capillary tube, formed by the revolution of a curve about the axis of z; which is correct even when we notice the change of density near the surface and sides of the tube.

Substituting these in the preceding values of $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$ [9321b, b′], they become as in [9321′—9323]. [9321f]

* (4136) We shall put for brevity $\frac{dz}{du}=P$, $\frac{ddz}{du^2}=Q$; then, by comparing the expressions of p, q, r, s, t [9312, 9313], with their values in [9321—9323], we get [9323a]

$$\mathrm{p}=\frac{x}{u}\cdot P; \qquad \mathrm{q}=\frac{y}{u}\cdot P; \qquad [9323b]$$

$$\mathrm{r}=\frac{x^2}{u^2}\cdot Q+\frac{y^2}{u^3}\cdot P; \qquad \mathrm{s}=\frac{xy}{u^2}\cdot Q-\frac{xy}{u^3}\cdot P; \qquad \mathrm{t}=\frac{y^2}{u^2}\cdot Q+\frac{x^2}{u^3}\cdot P. \qquad [9323c]$$

Substituting these in the first member of [9323e], we get its second member; and as the terms depending on P^2Q mutually destroy each other, it may be put under the form [9323f]; and as $y^2+x^2=u^2$, $y^4+2x^2y^2+x^4=u^4$ [9320], this may be successively reduced to the forms [9323g, h]; [9323d]

$$(1+\mathrm{q}^2)\mathrm{r}-2\mathrm{pqs}+(1+\mathrm{p}^2)\cdot\mathrm{t}=\left(1+\frac{y^2}{u^2}\cdot P^2\right)\cdot\left(\frac{x^2}{u^2}\cdot Q+\frac{y^2}{u^3}\cdot P\right)-\frac{2xy}{u^2}\cdot P^2\cdot\left(\frac{xy}{u^2}\cdot Q-\frac{xy}{u^3}\cdot P\right)$$
$$+\left(1+\frac{x^2}{u^2}\cdot P^2\right)\cdot\left(\frac{y^2}{u^2}\cdot Q+\frac{x^2}{u^3}\cdot P\right) \qquad [9323e]$$

$$=\frac{Q}{u^2}\cdot(y^2+x^2)+\frac{P}{u^3}\cdot(y^2+x^2)+\frac{P^3}{u^5}\cdot(y^4+2y^2x^2+x^4) \qquad [9323f]$$

$$=Q+\frac{P}{u}+\frac{P^3}{u}=Q+\frac{1}{u}\cdot P\cdot(1+P^2) \qquad [9323g]$$

$$=\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right). \qquad [9323h]$$

for, at the point O, b is equal to b', when the surface is of revolution. In case
[9325] the surface is a circular crown, b and b' being unequal, $\frac{2}{b}$ *will then denote the sum of the two fractions which have unity for their numerators, and for*
[9325'] *denominators respectively the greatest and the least of the radii of curvature, at the lowest point of the surface.* We may also observe, that, in the equation

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The second member of this last equation is the same as the numerator of the first member of [9324], and its first member is the same as the numerator of the first member of [9318]. The values of p, q, &c. [9323b, d, a], give

[9323i] $$1+\mathrm{p}^2+\mathrm{q}^2=1+\frac{P^2}{u^2}\cdot(x^2+y^2)=1+P^2=1+\frac{dz^2}{du^2};$$

hence the denominator $(1+\mathrm{p}^2+\mathrm{q}^2)^{\frac{3}{2}}$ of the first member of [9318] is represented by
[9323k] $\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$, as in [9324]. Again, as the surface is supposed, in [9319'], to be formed by
[9323l] the revolution of a curve about the axis of z, we shall have $b=b'$; hence $\frac{1}{b}+\frac{1}{b'}$ [9318] becomes $\frac{2}{b}$, as in [9324].

The form of the equation [9324] may be simplified, as the author has shown, in [10164, 10264, &c.], by changing the origin of the ordinate z, from the lowest point of the surface O, to the level of the surface of the fluid in the vase; and as the difference of these
[9323m] levels is q [9353], we may put $z=\mathrm{z}-q=\mathrm{z}-\frac{H}{bg}$ [9354], the new coordinate being represented by z. In this case, the second member of [9324] becomes $\frac{2}{b}+\frac{2g}{H}\cdot\left(\mathrm{z}-\frac{H}{bg}\right)$, or simply $\frac{2g\mathrm{z}}{H}$; and the whole equation is reduced to the form

[9323n] $$\frac{\frac{dd\mathrm{z}}{du^2}+\frac{1}{u}\cdot\frac{d\mathrm{z}}{du}\cdot\left(1+\frac{d\mathrm{z}^2}{du^2}\right)}{\left(1+\frac{d\mathrm{z}^2}{du^2}\right)^{\frac{3}{2}}}=\frac{2g}{H}\cdot\mathrm{z}.$$

[9323o] Instead of using the symbol $\mathfrak{a}$ [9328], we may put, as M. Poisson has done, $\frac{g}{H}=\frac{1}{\alpha^2}$; and we shall have

[9323p] $$\frac{g}{H}=\mathfrak{a}=\frac{1}{\alpha^2}, \quad \text{or} \quad \mathfrak{a}\alpha^2=1;$$

and then the equation [9323n] becomes

[9323q] $$\frac{\frac{dd\mathrm{z}}{du^2}+\frac{1}{u}\cdot\frac{d\mathrm{z}}{du}\cdot\left(1+\frac{d\mathrm{z}^2}{du^2}\right)}{\left(1+\frac{d\mathrm{z}^2}{du^2}\right)^{\frac{3}{2}}}=\frac{2\mathrm{z}}{\alpha^2},$$

[9323r] which is the same form as is given by M. Poisson, in page 108 of his *Nouvelle Théorie, &c.*, changing u into t, z into z, and α into a, to conform to his notation.

[9324], we shall have *

$$\frac{\frac{ddz}{du^2}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}=\frac{1}{R}; \qquad [9326]$$

[R, R', are the greatest and least radii of curvature of the point of the surface whose coordinates are x, y, z.] Radii of curvature R, R'.

$$\frac{\frac{1}{u}\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\frac{1}{R'}; \qquad [9326']$$

$R=$ the radius of curvature of the surface, at a point whose coordinates are u, z, and in a plane passing through the axis of revolution; [9327]

$R'=$ the radius of curvature drawn perpendicular to the surface, at the same point, and continued till it meets the axis of revolution.† [9327']

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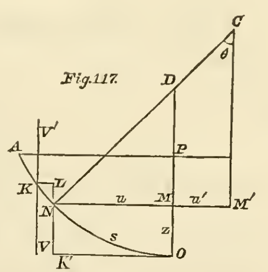
* (4137) It is shown in [1584c], that, if a, b, are rectangular coordinates of a plane curve, whose arc is s, and radius of curvature r, we shall have $\frac{1}{r}=\frac{dda}{ds.db}$, ds being constant [1584c—d]. To conform to the present notation, we must put $a=z$, $b=u$, $r=R$; and then $\frac{1}{R}=\frac{ddz}{ds.du}$. This may be put under the form $\frac{1}{R}=\frac{1}{du}\cdot d.\left(\frac{dz}{ds}\right)$, without considering any differential as constant. If we now suppose du to be constant, and substitute for ds its value $\pm\sqrt{dz^2+du^2}$, we shall get [9327a] [9327b] [9327c]

$$\frac{1}{R}=\pm\frac{1}{du}\cdot d.\left(\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}\right). \qquad [9327d]$$

Developing this differential, it becomes

$$\frac{1}{R}=\pm\frac{\frac{ddz}{du^2}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}, \qquad [9327e]$$

as in [9326], neglecting the double sign, which may be supposed to be included in the radical $\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$. [9327f]



† (4138) In the annexed figure 117, which is similar to fig. 116, we have $OM=z$, $MN=u$ [9320]; we shall also put the arc $ON=s$, and the differentials of these quantities are $NL=dz$, $KL=du$, $NK=ds$. Now if we draw ND perpendicular to the arc NK, and continue it, to meet the axis OM, in D, the line ND will represent the radius R' [9327'], and we shall have, in the similar triangles NLK, NMD, $NL(=dz):NK(=ds)::NM(=u):ND(=R')$; [9328a] [9328b] [9328c]

We shall now put

[9328] $$\frac{g}{H}=\alpha, \quad \text{or} \quad g=H\alpha;$$

and then we shall have by integration,*

[9329] $$\frac{u\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\frac{u^2}{b}+2\alpha\,.\,\int zudu+\text{constant}.$$

[9330] The integral $\int zudu$ being supposed to commence with u, the constant quantity will be nothing. If we now put

[9331] $$u'=u+\frac{2\alpha b}{u}\,.\int zudu,$$

the preceding equation will give †

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hence $R'=u\,.\,\frac{ds}{dz}$; and by substituting $ds=\sqrt{dz^2+du^2}$ [9327c,] we get

[9328d] $$R'=u\,.\,\frac{\sqrt{dz^2+du^2}}{dz}=u\,.\,\frac{du}{dz}\,.\,\sqrt{1+\frac{dz^2}{du^2}},$$

which gives $\frac{1}{R'}$, as in [9327'].

* (4139) Multiplying [9324] by udu, and using α [9328], it becomes

[9329a] $$\frac{\frac{uddz}{du}+dz\,.\left(1+\frac{dz^2}{du^2}\right)}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}=\frac{2udu}{b}+2\alpha\,.\,zudu.$$

Integrating this, we get [9329], as is easily proved by differentiation.

† (4140) Putting in [9329] the constant quantity $=0$, and multiplying by $\frac{b}{u}$, we get

[9332a] $$\frac{b\,.\,\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=u+\frac{2\alpha b}{u}\,.\int zudu,$$

which, in [9331], is put $=u'$. Squaring this expression, and multiplying by $1+\frac{dz^2}{du^2}$, we get

[9332b] $b^2\,.\,\frac{dz^2}{du^2}=u'^2\,.\left(1+\frac{dz^2}{du^2}\right)$, or $bdz=u'\sqrt{du^2+dz^2}$; whence we obtain $\frac{dz^2}{du^2}=\frac{u'^2}{b^2-u'^2}$ whose square root, being multiplied by du, gives [9332].

$$dz = \frac{u'.du}{\sqrt{b^2-u'^2}}.$$ [9332]

In the case of $\alpha=0$, we shall have $u'=u$ [9331], and then from [9332] we get* [9333]

$$z=b-\sqrt{b^2-u^2};$$ [Approximate equation or equation of the spherical surface of the fluid.] [9334]

consequently,

$$\frac{2\alpha b}{u}.\int zudu = \frac{\alpha b}{u}.\{bu^2+\tfrac{2}{3}.(b^2-u^2)^{\frac{3}{2}}-\tfrac{2}{3}b^3\}.$$ [9335]

The differential of the second member of this equation is, as in [9336b, c], easily reduced to the form

$$d.\left\{\frac{2\alpha b}{u}.\int zudu\right\} = \frac{\alpha b^2du.(3u^2+2b^2)}{3u^2} - \frac{2\alpha b.du}{3u^2}.\sqrt{b^2-u^2}.(b^2+2u^2).$$ [9336]

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* (4141) Putting $\alpha=0$ in [9331], we get $u'=u$; then [9332] becomes $dz=\frac{udu}{\sqrt{b^2-u^2}}$, whose integral is as in [9334]. Substituting this value of z in $\frac{2\alpha b}{u}.\int zudu$, [9336$a$] it becomes $\frac{\alpha b}{u}.\int\{2budu-2udu.\sqrt{b^2-u^2}\}$, whose integral is as in [9335], always completing the integrals so that they may vanish when $z=0$ and $u=0$ [9330]. By altering the arrangement of the terms in [9335], we have

$$\frac{2\alpha b}{u}.\int zudu = \frac{\alpha b^2}{3u}.(3u^2-2b^2)+\frac{2\alpha b}{3u}.(b^2-u^2)^{\frac{3}{2}};$$ [9336b]

and its differential gives [9336]; observing that the two portions into which the second member of [9336b] is divided produce respectively the two portions of the second member [9336c] of [9336].

We may here observe, that the numerical value of the quantity $\alpha=\frac{g}{H}$ [9328], which is neglected in [9333], is given in [10303, &c.], and from this we obtain $\frac{1}{\alpha}=15^{\text{mi.}}$ for [9336d] water; the radius of the tube, used in making the experiments, being rather less than $1^{\text{mi.}}$ [9336e] [10305, 10309]; and in such experiments, we may consider z, u, b, as being less than a millimetre, which we shall take for the unit of measure. In this case, a quantity of the order αbzu, or αbu^2, will be less than $\frac{1}{15}$; hence we perceive that the equation [9334] must [9336f] differ but very little from the truth; and this equation is evidently that of a circle whose radius is b, sine u, and versed sine z. [9336g]

If we suppose, in fig. 117, page 719, a tangent OK' to be drawn through the point O, parallel and equal to MN, also the line NK' parallel and equal to MO, so as [9336h] to make it similar to fig. 112, page 695, it will be evident that the volume of the annulus described by the revolution of the figure $OK'N$ about the axis OM will be represented by $2\pi.\int zudu$. For the circumference described by the line NK', in [9336i]

[9337] If we neglect quantities of the order α^2, we may change u' [9331] into u, in the function [9336]; and then we shall have, by taking the differential of the value of u' [9331],*

[9338] $$du = du'.(1-\alpha b^2) + \frac{2\alpha b}{3u'^2}.du'.\{(b^2+2u'^2).\sqrt{b^2-u'^2}-b^3\},$$

which gives

[9339] $$dz = \frac{u'du'.(1-\alpha b^2)}{\sqrt{b^2-u'^2}} + \frac{2\alpha b.du'}{3u'}.\left\{b^2+2u'^2-\frac{b^3}{\sqrt{b^2-u'^2}}\right\}.$$

[9340] Now putting $u'=b.\sin.\theta$, we shall have †

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its revolution about the axis MO, is $2\pi u$; multiplying this by $NK'=z$, and by [9336k] $LK=du$, we get $2\pi.zudu$ for the element of the volume, whose integral is as in [9336i]. If we suppose ON to be a circular arc described with the radius b and the centre D, the integral $2\pi.\int zudu$ will be of the same form as that in [9335], and may be derived from it, by merely multiplying [9335] by $\pi.\frac{u}{\alpha b}$, which gives

[9336l] $$2\pi.\int_0^u zudu = \pi.\{bu^2+\tfrac{2}{3}(b^2-u^2)^{\frac{3}{2}}-\tfrac{2}{3}b^3\}.$$

If we suppose the angle formed by the vertical line NK', and the tangent to the curve [9336m] NO, at the point N, to be represented by ϖ, we shall have the angle $DNM=\varpi$, and $NM=DN.\cos.DNM$; or in symbols $u=b.\cos.\varpi$, whence $b^2-u^2=b^2.\sin.^2\varpi$. Substituting these values in [9336l], we get

[9336n] $$2\pi.\int_0^u zudu = \pi.b^3.\{\cos.^2\varpi+\tfrac{2}{3}.\sin.^3\varpi-\tfrac{2}{3}\}$$ [Volume of a spherical annulus.]

[9336o] $$= \pi.\frac{u^3}{\cos.^3\varpi}.\{\cos.^2\varpi+\tfrac{2}{3}.\sin.^3\varpi-\tfrac{2}{3}\}.$$

This expression will be of use hereafter.

* (4142) Transposing the last term of [9331], and then taking its differential, we get [9338a] $du=du'-d.\left\{\frac{2\alpha b}{u}.\int zudu\right\}$. Substituting the differential [9336], and changing in it u into u', as in [9337], it becomes

[9338b] $$du = du' - \frac{\alpha b^2.du'.(3u'^2+2b^2)}{3u'^2} + \frac{2\alpha b.du'}{3u'^2}.\sqrt{b^2-u'^2}.(b^2+2u'^2),$$

which is easily reduced to the form [9338]. Multiplying this by $\frac{u'}{\sqrt{b^2-u'^2}}$, it gives the value of $\frac{u'du}{\sqrt{b^2-u'^2}}$, or dz [9332], as in [9339].

† (4143) This value of u' gives

[9341a] $$\sqrt{b^2-u'^2}=b.\cos.\theta, \qquad du'=bd\theta.\cos.\theta, \qquad \frac{u'du'}{\sqrt{b^2-u'^2}}=bd\theta.\sin.\theta;$$

$$\frac{dz}{b} = d\theta \,.\, \sin.\theta \,.\, (1 - \alpha b^2) + \frac{2\alpha b^2}{3} \,.\, d\theta \,.\, \left\{ \sin.2\theta - \frac{\sin.\frac{1}{2}\theta}{\cos.\frac{1}{2}\theta} \right\},$$ [9341]

which gives by integration, General value of z.

$$\frac{z}{b} = (1 - \alpha b^2) \,.\, (1 - \cos.\theta) + \frac{\alpha b^2}{3} \,.\, (1 - \cos.2\theta) + \frac{4\alpha b^2}{3} \,.\, \log.\cos.\tfrac{1}{2}\theta.$$ [9342]

Putting *the semi-diameter of the tube equal to* l, and observing that this semi-diameter is very nearly equal to the extreme value of u, because the extreme planes of the surface of the segment under consideration are, as we have before shown [9196, &c.], at an insensible distance from the tube, we shall have for the extreme value* of u' l. [9343] [9343′] [9344] Value of u'.

$$u' = l + \alpha \,.\, b^2 l - \tfrac{2}{3}\alpha \,.\, \frac{b^4}{l} + \tfrac{2}{3}\alpha \,.\, \frac{b^4}{l} \,.\, \cos.^3\theta',$$ [9345]

in which we have put

θ' = *the extreme value of* θ *near the side of the tube; and this value is the complement of the angle which the extreme parts of the curve* AOB (fig. 116, page 713) *make with the vertical sides of the tube;*† [being the same as the angle $90^d - \varpi$, in the notation [9892]. θ' [9346] [9346′]

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substituting these in [9339], it becomes

$$dz = bd\theta \,.\, \sin.\theta \,.\, (1 - \alpha b^2) + \frac{2\alpha b}{3} \,.\, \frac{d\theta \,.\, \cos.\theta}{\sin.\theta} \,.\, \left\{ b^2 + 2b^2 \,.\, \sin.^2\theta - \frac{b^2}{\cos.\theta} \right\};$$ [9341b]

or, by reduction and dividing by b,

$$\frac{dz}{b} = d\theta \,.\, \sin.\theta \,.\, (1 - \alpha b^2) + \frac{2\alpha b^2}{3} \,.\, d\theta \,.\, \left\{ 2\sin.\theta \,.\, \cos.\theta - \frac{(1 - \cos.\theta)}{\sin.\theta} \right\},$$ [9341c]

which, by using [31, 42′, 34′], Int., becomes as in [9341]. Its integral, taken so as to vanish when $z = 0$, or $u = 0$, becomes as in [9342]; observing that $u = 0$ gives $u' = 0$ [9331, 9330], and then also $\theta = 0$ [9340].

* (4144) Substituting in [9331] the value of the integral [9335], we get

$$u' = u + \frac{\alpha b}{u} \,.\, \{ bu^2 + \tfrac{2}{3} \,.\, (b^2 - u^2)^{\frac{3}{2}} - \tfrac{2}{3} b^3 \}.$$ [9345a]

Now u' differs from u [9345a] only by quantities of the order α, and if we neglect α^2, we may put $u = b \,.\, \sin.\theta$ [9340], or $(b^2 - u^2)^{\frac{1}{2}} = b \,.\, \cos.\theta$, in the terms multiplied by α, [9345b] in the preceding value of u'; and as the extreme values of u, θ, are $u = l$, $\theta = \theta'$ [9343, 9346], we shall have, for the extreme value of u, the expression [9345].

† (4145) Using fig. 117, page 719, where the line NDC is drawn perpendicular to KN, so as to make $NC = b$; also CM' and $V'KV$ perpendicular to the horizontal [9346a]

We then have for the extreme value of u',

[9347] $$u' = b \,.\, \sin.\theta'.$$

By comparing these two values of u', we shall obtain,

[9348] $$b = \frac{l}{\sin.\theta'} + \frac{\alpha.b^2 l}{\sin.\theta'} - \tfrac{2}{3} \,.\, \frac{\alpha.b^4}{l\,.\sin.\theta'} + \tfrac{2}{3} \,.\, \frac{\alpha b^4.\cos.^3\theta'}{l\,.\sin.\theta'};$$

Value of b. or, very nearly,*

[9349] $$b = \frac{l}{\sin.\theta'} + \frac{\alpha\,.\,l^3}{\sin.^3\theta'} - \tfrac{2}{3} \,.\, \frac{\alpha\,.\,l^3}{\sin.^5\theta'} + \tfrac{2}{3} \,.\, \frac{\alpha l^3\,.\cos.^3\theta'}{\sin.^5\theta'},$$

Value of z near the tube. which gives for the extreme value of z,†

[9350] $$z = l\,.\,\text{tang.}\tfrac{1}{2}\theta' \,.\, \left\{ 1 - \tfrac{2}{3}\alpha \,.\, \frac{l^2.(1-\cos.^3\theta')}{\sin.^4\theta'} \right\} + \frac{2\alpha\,.\,l^3}{3\,.\sin.\theta'} + \frac{4\alpha\,.\,l^3}{3\,.\sin.^3\theta'} \,.\, \text{log.cos.}\tfrac{1}{2}\theta',$$

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line NMM'; we shall have, as in [9332b],

[9346b] $$u' = b \,.\, \frac{dz}{\sqrt{du^2+dz^2}} = b \,.\, \frac{NL}{NK} = b\,.\cos.KNL = b\,.\cos.CNM' = b\,.\sin.NCM' = b\,.\sin.\theta \text{ [9340]}.$$

[9346c] Hence we have $NCM' = \theta$, or $NKL = \theta$; so that θ represents the complement of the angle NKV, which the vertical line KV makes with the arc KN; and at the side of the tube where θ becomes θ' [9346], this angle will be the complement of the angle which the arc KN
[9346d] makes with the side of the tube, supposing it to be vertical; and then the general value of u' [9346b] becomes $u' = b\,.\sin.\theta'$, as in [9347]. Putting this equal to the value of u'
[9346e] [9345], and then dividing by $\sin.\theta'$ we get b [9348]. In the second supplement [9892], the author uses the angle ϖ instead of θ', supposing ϖ to be the angle formed by the extreme
[9346f] side of the curve at A, with the line drawn parallel to DO, in a downward direction; so that we shall have $\varpi = 90^d - \theta'$.

[9348a] * (4146) Neglecting terms of the order α, we get $b = \frac{l}{\sin.\theta'}$, [9348]; and if we neglect terms of the order α^2, we may substitute this value of b in the terms of [9348] containing the factor α; hence we get [9349].

† (4147) We have in [9342], for the extreme value of z, corresponding to $\theta = \theta'$,

[9350a] $$z = b\,.\,(1-\cos.\theta') - \alpha b^3\,.\,(1-\cos.\theta') + \tfrac{1}{3}\alpha b^3\,.\,(1-\cos.2\theta') + \tfrac{4}{3}\alpha b^3\,.\,\text{log.cos.}\tfrac{1}{2}\theta'.$$

[9350a'] Substituting, in the first term of this expression, the value of b [9349], and in the rest of the terms $b = \frac{l}{\sin.\theta'}$, we get, by neglecting quantities of the order α^2,

[9350b] $$z = \left\{ \frac{l}{\sin.\theta'} + \frac{\alpha l^3}{\sin.^3\theta'} - \tfrac{2}{3} \,.\, \frac{\alpha l^3}{\sin.^5\theta'} + \tfrac{2}{3} \,.\, \frac{\alpha l^3.\cos.^3\theta'}{\sin.^5\theta'} \right\} .\,(1-\cos.\theta')$$
$$- \alpha\,.\,(1-\cos.\theta')\,\frac{l^3}{\sin^3.\theta'} + \tfrac{1}{3}\alpha\,.\,(1-\cos.2\theta')\,.\,\frac{l^3}{\sin.^3\theta'} + \tfrac{4}{3}\alpha\,.\,\frac{l^3}{\sin.^3\theta'}\,.\,\text{log.cos.}\tfrac{1}{2}\theta'.$$

and *

$$\frac{1}{b}=\frac{\sin.\theta'}{l}\cdot\left\{1-\frac{\alpha.l^2}{\sin.^2\theta'}\cdot\left(1-\tfrac{2}{3}\cdot\frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right)\right\}.$$ [9351] Value of $\frac{1}{b}$

It is evident that these expressions of z and $\frac{1}{b}$ hold good when the surface of the fluid is convex; only z must then be counted downwards, from the most elevated point of the surface.† [9352]

The second and fifth terms of these terms, having the factor $\frac{\alpha.l^3}{\sin.^3\theta'}$, mutually destroy each other; the rest may be arranged in the following order;

$$z=l.\frac{(1-\cos.\theta')}{\sin.\theta'}-\frac{2\ l^3.(1-\cos.^3\theta')}{3.\sin.^4\theta'}\cdot\frac{(1-\cos.\theta')}{\sin.\theta'}+\frac{\alpha.l^3}{3.\sin.\theta'}\cdot\frac{(1-\cos.2\theta')}{\sin.^2\theta'}+\frac{4\alpha l^3}{3.\sin.^3\theta'}.\log.\cos.\tfrac{1}{2}\theta'.$$ [9350c]

Now we have, as in [42', 1], Int.,

$$\frac{1-\cos.\theta'}{\sin.\theta'}=\tan g.\tfrac{1}{2}\theta', \qquad \frac{1-\cos.2\theta'}{\sin.^2\theta'}=\frac{2.\sin.^2\theta'}{\sin.^2\theta'}=2;$$ [9350d]

and by substitution in [9350c], we get [9350].

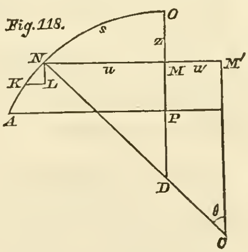
* (4148) The expression of b [9349] may be put under the form

$$b=\frac{l}{\sin.\theta'}\cdot\left\{1+\frac{\alpha.l^2}{\sin.^2\theta'}\cdot\left(1-\tfrac{2}{3}\cdot\frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right)\right\}.$$ [9351a]

Dividing unity by this, and neglecting terms of the order α^2, we get [9351].

† (4149) If we examine the calculation [9328—9351], we shall find that it will not be [9352a]
altered if we suppose the surface to be convex, as in fig. 118, instead of concave, as in fig. 117, page 719. We may also consider the subject in the following manner: Suppose the [9352b]
curvature in fig. 117 to decrease gradually till the point O falls in P, and then to become convex as in fig. 118. In this case, the positive value of z will decrease to nothing, and then become negative; the angle $NCM'=\theta$ will also [9352c]
decrease to 0, and then become negative; and in the same manner the quantity $\frac{1}{b}$ will decrease to 0, and then [9352c']
become negative. Hence we see that, when the surface

Fig. 118.



changes from concave to convex, the signs of the quantities $z, \theta, \theta', \frac{1}{b}$ or b, will change; and [9352d]
by this means the signs of every one of the terms, in both members of the equations [9350, 9351], will be changed; and, as this does not alter the values of the equations, we may suppose the expressions of z, $\frac{1}{b}$ [9350, 9351], to conform either to a concave or to a [9352e]
convex surface.

5. *We shall now consider the capillary tube $MNFE$*, fig. 112, page 695. The action of the meniscus $MIOKN$ to elevate the fluid of the canal OZ, is, by

[9353] q [9260], equal to $\frac{H}{b}$. *If we put q for the elevation of the point O above the level of the fluid in the vessel $ABCD$*, we shall have, by the preceding articles,*

[9354] $$\frac{H}{b}=gq, \quad \text{or} \quad q=\frac{H}{gb}=\frac{1}{\alpha b} \quad [9328].$$

Therefore, by substituting for $\frac{1}{b}$ its value, found in [9351], we shall have very nearly,

[9355] Elevation q of a fluid in a tube whose radius is l.
$$q=\frac{H}{g}\cdot\frac{\sin.\theta'}{l}\cdot\left\{1-\frac{\alpha l^2}{\sin.^2\theta'}\cdot\left(1-\tfrac{2}{3}\cdot\frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right)\right\}.$$

To determine α, we shall observe that $\alpha=\frac{g}{H}$ [9328], and we have very

[9356] nearly, $q=\frac{H.\sin.\theta'}{gl}$; hence we get †

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* (4150) The capillary action at O is $K-\frac{H}{b}$ [9254]; and if we add to this the

[9355a] gravity of the column OZ, fig. 112, page 695, which is equal to $g\times OZ$, we get the whole action at Z, in the column OZ, equal to $K-\frac{H}{b}+g\times OZ$. In like manner, from

[9355b] [9259], we have for the action at R in the column VR, the expression $K+g\times VR$. Putting these two expressions equal to each other, because the columns ought to balance each other,

[9355c] also observing that $OZ=VR+q$, we get $K-\frac{H}{b}+g\times(VR+q)=K+g\times VR$; whence we easily deduce $\frac{H}{b}=gq$, or $q=\frac{H}{g}\cdot\frac{1}{b}$ [9354]. Now substituting the value

[9355d] of $\frac{1}{b}$ [9351], we get [9355]. If we neglect terms of the order α, it becomes as in [9360];

[9355e] and, by putting $\theta'=90^d-\varpi$, as in [9346'], we get very nearly $q=\frac{H}{g}\cdot\frac{\cos.\varpi}{l}$, which will be of use hereafter.

† (4151) Neglecting terms of the order α, we get from [9355] $q=\frac{H.\sin.\theta'}{gl}$, as in

[9357a] [9356]; hence we deduce $\alpha=\frac{g}{H}=\frac{\sin.\theta'}{lq}$ [9357]; and if we neglect α^2, we may substitute this value of α, in [9355], and then it will become as in [9358]. If we suppose

[9357b] $\frac{l}{q}$ to be very small, we need retain only the first term of q [9358], and then it will become

[9357c] $q=\frac{2H.\sin.\theta'}{g}\cdot\frac{1}{2l}$, and by [9359] the coefficient of $\frac{1}{2l}$ is constant, so that q becomes, as in

[9357d] [9360], inversely as the diameter $2l$ of the tube. If we substitute $\alpha=a^{-2}$ [9323p], in

[9357e] [9357], we get $lq=a^2.\sin.\theta'$; substituting $\theta'=\frac{1}{2}\pi$, corresponding to a glass tube with

[9357f] water, or alcohol [9362'], it becomes $lq=a^2$, which will be of use hereafter.

$$\alpha = \frac{\sin.\theta'}{lq};$$ [9357]

consequently,

$$q = \frac{H}{g} \cdot \frac{\sin.\theta'}{l} \cdot \left\{ 1 - \frac{l}{q \cdot \sin.\theta'} \cdot \left(1 - \tfrac{2}{3} \cdot \frac{(1 - \cos.^3\theta')}{\sin.^2\theta'} \right) \right\}.$$ [Elevation q of a fluid in a tube whose radius is l.] [9358]

$\frac{H}{g}$ is a constant quantity [9253′, 9255], whatever be the diameter $2l$ of the tube; and θ' is, as we have seen [9346, 9197], a quantity independent of that diameter;* moreover if l is very small, the fraction $\frac{l}{q}$ may be neglected in comparison with unity; therefore we shall have very nearly, [9359] [9359′]

$$q = \frac{H}{g} \cdot \frac{\sin.\theta'}{l} = \frac{\text{constant}}{2l};$$ [Approximate elevation q of a fluid in a tube whose radius is l.] [9360]

that is, *the elevation of the fluid is very nearly in the inverse proportion of the diameter of the tube, which is conformable to experiment.* [9361]

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* (4152) We have seen in [9346c], that $\theta = \text{angle } NKL = 90^d - \text{angle } NKV$, fig. 117, page 719, supposing KV to be a vertical line drawn parallel to the inner surface of the cylindrical tube, and NK an infinitely small arc of the curve surface $AKNO$. The extreme value of the angle θ, at the border A of the vertical tube, is represented by θ' in [9346], and by $90^d - \varpi$ in [9346′]; and as it has been shown, in [9197], that this angle $\varpi = 90^d - \theta'$ is very nearly constant, it must be independent of the diameter of the tube, as in [9359]. We may remark that the analysis in § 4, 5, is not restricted to the case of a cylindrical tube, but may be extended to any tube whose inner surface is a figure of revolution about the vertical axis z. For example, if we suppose that the figure is a frustum of an *inverted cone*, whose side is inclined by the very small angle i to the vertical axis z; the angle ϖ, which is formed near A, by *the side of the tube and the tangent of the surface* of the fluid, will remain unaltered; but the angle which is included between the *vertical line and this tangent* will be augmented by the angle i, and will become $i + \varpi$; so that, instead of having $\varpi = 90^d - \theta'$, as in [9359c], we shall have $\varpi + i = 90^d - \theta'$, or $\theta' = 90^d - \varpi - i$, always supposing that θ' represents the extreme value of the angle NKL at the point A. Substituting this value of θ' in the formulas [9345, &c.], we shall get the values corresponding to a tube of the form of an inverted frustum of a cone, and by changing the sign of i, we get the corresponding values for the frustum of a *direct cone*. These results may be extended to a capillary tube formed by any figure of revolution about the vertical axis z, by referring the case to that of the osculatory cone having the same vertical axis z, and touching the surface of the tube at the point A, supposing the side of this osculatory cone to form the angle i with the vertical axis z. Now it is found by experiment, that, if water, alcohol, &c., be elevated in a glass capillary tube, we shall have $\varpi = 0$ [9631, 9743, &c.], and then the general value of $\theta' = 90^d - \varpi - i$ [9359g], will become $\theta' = 90^d - i$. [9359a] [9359b] [9359c] [9359d] [9359e] [9359f] [9359g] [9359h] [9359i] [9359k] [9359k′]

To estimate the degree of approximation which we obtain by putting
[9362] $q = \frac{H}{g} \cdot \frac{\sin.\theta'}{l}$ [9360], we shall suppose θ' *to be equal to a quarter of the*
[9362'] *circumference, which appears to be the case with water or alcohol in a glass*
[9363] *capillary tube;* the term which we then neglect is* $-\frac{l}{3}$, or $-q \cdot \frac{l}{3q}$. If we suppose l to be equal to a millimetre, or the diameter of the tube equal to two millimetres, we shall have by observation, as we shall see hereafter [9675],
[9364] relative to water in a glass tube, $q = 6^{\text{mi.}},784$; consequently the fraction
[9365] $\frac{l}{3q}$ then becomes $\frac{1}{20{,}352}$; therefore it may be neglected in comparison with unity. In a tube of less diameter, this fraction *decreases in proportion to the*
[9366] *square of*† l, because q varies inversely as l. Hence we see that, in a capillary

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[9359l] Substituting this in [9351], and neglecting the very small terms depending on α, we get
[9359m] $\frac{1}{b} = \frac{\cos.i}{l}$, or $b = l \,.$ secant i, nearly; so that, if we suppose l to be given, we shall find that, as i increases, the radius of curvature b of the surface will increase; consequently the
[9359n] curvature of the surface will decrease. When $i = 0$, the formula [9359m] gives $b = l$, and the surface is nearly of the form of a concave hemisphere. When $i = 90^{d}$, we shall
[9359o] have $\theta' = 0$ [9359k']; then $z = 0$ [9350], $\frac{1}{b} = 0$ [9351], and the surface ONA will be nearly horizontal. If i exceed 90^{d}, the expression of b [9359m] will become
[9359p] negative, and the surface convex; moreover as i increases, the convexity will increase, till
[9359q] $i = 180^{d}$, and then $b = -l$ nearly [9359m], so that the surface becomes a convex hemisphere nearly. We shall have occasion to treat of a case of this kind in [10012a, &c.],
[9359r] where the fluid is supposed to flow over the top of a capillary tube.

* (4153) The part of q [9358], neglected in [9360], is

[9363a] $$-\frac{H.\sin.\theta'}{gl} \cdot \left\{ \frac{l}{q.\sin.\theta'} \cdot \left(1 - \tfrac{2}{3} \cdot \frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right) \right\};$$

and, by putting for the factor $\frac{H.\sin.\theta'}{gl}$ its approximate value q [9360], it becomes
[9363b] $-\frac{l}{\sin.\theta'} \cdot \left(1 - \tfrac{2}{3} \cdot \frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right)$; and since $\theta' = 90^{d}$ [9362] makes $1 - \tfrac{2}{3} \cdot \frac{(1-\cos.^3\theta')}{\sin.^2\theta'} = \tfrac{1}{3}$,
[9363c] this part becomes simply $-\frac{l}{3} = -q \cdot \frac{l}{3q}$ [9363]; and, by using q [9364], it changes into $-\frac{q}{20{,}352}$, so that the part of q, neglected in [9360], is about one twentieth part.

† (4154) The chief term of q [9360] is the same as the first term, or first factor, of the
[9366a] general value of q [9355]; and the remaining terms of q [9355], which are neglected in [9360], have the factor l^2, contained in the part between the braces, as in [9365].

tube, we may suppose, without any sensible error, as in [9360],

$$q = \frac{\text{constant}}{2l};$$ [Approximate value of the elevation of a fluid in a capillary tube whose radius is l.] [9367]

that is, *the height of the fluid above the level is in the inverse ratio of the diameter of the tube.* [9368]

Elevation of the fluid is inversely as the diameter of the tube.

If the surface of the fluid within the tube be convex, we may suppose, as before, that an infinitely narrow canal is drawn through the axis of the tube, then bent under it, and finally turned upwards, till it comes to the surface of the fluid contained in the vessel. In this case, the action of the fluid in the tube [9369] upon the included canal will be, by [9276], equal to $K + \frac{H}{b}$. The action [9370] of the fluid in the vessel upon the outer branch of the canal, will be equal to K [9259]. But if we put q for the elevation of the exterior fluid above the [9371] fluid in the interior branch of the canal, we must add to the action K the weight gq;* therefore we shall have, by the condition of the equilibrium of the [9371′]

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* (4155) Proceeding as in [9355a, &c.] for a concave surface, we must add to the quantities $K + \frac{H}{b}$, and K [9370, 9371], the values $g \times OZ$, $g \times VR$, fig. 112, page [9372a] 695, representing respectively the gravities of the columns OZ, VR; and as these sums must be equal to each other, we shall have $K + \frac{H}{b} + g \times OZ = K + g \times VR$. Now in [9372$b$] this case, the fluid in the tube is depressed by the quantity q [9371], so that $VR = q + OZ$; and by substituting it in the preceding expression, we get

$$K + \frac{H}{b} + g \times OZ = K + gq + g \times OZ. \qquad [9372c]$$

If we reject $g \times OZ$ from both members, it becomes as in [9372]; and if we also reject K, and divide by g, we shall obtain q [9373], which is of the same form as [9354]. Now [9372d] substituting $\frac{1}{b}$ [9351], it becomes as in [9355], which may be easily reduced to the form [9358], agreeing with that in [9374]: finally, an approximate expression of this equation may be obtained, by retaining only the first or most important term of its second member, as in [9379]. [9372e] We have supposed, in [9372a], that the density of the fluid is 1; but if its density be D, we must change g into gD, in [9372a], to get the gravities of the columns OZ, VR, and the [9372e'] same change must be made in the rest of the calculations.

When the fluid is water or alcohol, which completely wets the tube, the surface of the water in the tube becomes hemispherical [9216, &c.]; and then we have $\theta' = 90^d$ [9346], [9372f] consequently $\cos.\theta' = 0$, $\sin.\theta' = 1$. Substituting these in the expression of the elevation or depression q of the fluid in the tube [9358 or 9374], it becomes $q = \frac{H}{gl} \cdot \left\{ 1 - \tfrac{1}{3} \cdot \frac{l}{q} \right\}$. [9372$g$]

fluid contained in the canal,

[9372] $$K+gq=K+\frac{H}{b},$$

which gives

[9373] $$q=\frac{H}{gb};$$

consequently

[9374] $$q=\frac{H}{g}\cdot\frac{\sin.\theta'}{l}\cdot\left\{1-\frac{l}{q\,.\,\sin.\theta'}\cdot\left(1-\tfrac{2}{3}\cdot\frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right)\right\}.$$

Depression q in a tube whose radius is l.

[9372g′] If we put, as in [9323o], $\frac{H}{g}=\alpha^2$, we shall get $q=\frac{\alpha^2}{l}-\frac{\alpha^2}{3q}$, or $q^2-\frac{\alpha^2}{l}\cdot q=-\frac{\alpha^2}{3}$;

[9372h] whence $q=\frac{\alpha^2}{2l}\cdot\left\{1+\sqrt{1-\tfrac{4}{3}\cdot\frac{l^2}{\alpha^2}}\right\}$; and, by developing it according to the powers of $\frac{1}{\alpha^2}$, neglecting terms of the order $\frac{1}{\alpha^4}$, on account of their smallness, we shall obtain

[9372i] $$q=\frac{\alpha^2}{l}-\tfrac{1}{3}l-\frac{1}{9}\cdot\frac{l^3}{\alpha^2}.$$

M. Poisson has computed from the fundamental equation [9324], or rather from the equivalent equation [9323q], a formula similar to [9372i]; and if we change his symbol h

[9372k] into q, and α into l, substituting also $\tfrac{1}{3}(\text{hyp.log.}4-1)=0{,}129$, to conform to the present notation, the formula given in page 112 of his *Nouvelle Théorie, &c.*, will become

[9372l] $$q=\frac{\alpha^2}{l}-\tfrac{1}{3}l+0{,}129\,.\frac{l^3}{\alpha^2}.$$

Comparing the formulas [9372i, l], we find that the two first terms agree, but the third term,

[9372m] which is of the order $\frac{l^3}{\alpha^2}$, differs, as might be expected, because La Place avowedly neglects terms of that order, which are extremely small; as appears from the experiments of Gay-

[9372n] Lussac, quoted by M. Poisson in his work, and also by La Place in [10303—10311]. For

[9372o] we have in the first experiment [10305, &c.], $q=23^{\text{mi.}},1634$, $l=0^{\text{mi.}},6472$; and from this M. Poisson deduces the value of $\alpha^2=15{,}1299$ square millimetres, or $\alpha=3^{\text{mi.}},88972$. In

[9372p] a second experiment of Gay-Lussac, the radius of the tube was $l=0^{\text{mi.}},9519$ [10309]; with this and the preceding value of α, M. Poisson computes from the formula [9372l]

[9372q] $q=15^{\text{mi.}},5829$. A similar process with the formula of La Place [9372h], gives $q=15^{\text{mi.}},5729$ nearly. The actual elevation found by Gay-Lussac was $15^{\text{mi.}},5861$ [10310], agreeing remarkably well with these calculations. The difference between the observation and either of the calculations is far within the limits of the errors to which such observations are liable;

[9372r] because they sometimes differ very much from each other, as is shown in [10322a—g, &c.]. If we use the preceding values $l=0^{\text{mi.}},6472$, $\alpha=3^{\text{mi.}},88972$, we shall get, for the third term of M. Poisson's formula [9372l], the expression $0{,}129\,.\frac{l^3}{\alpha^2}=0^{\text{mi.}},0023$; and for that

[9372s] of La Place's formula [9372i], $-\frac{1}{9}\cdot\frac{l^3}{\alpha^2}=-0^{\text{mi.}},0019$; now terms of this order may be

Hence it follows, that, in very slender tubes, *the depression q of the fluid within the tube below the level of the exterior fluid, is inversely proportional to the diameter of the tube $2l$,* which is confirmed by experiment. [9375]

If the tube be inclined to the horizon, the surface of the included fluid will be very nearly the same as if the tube were vertical; it will be, in both cases, very nearly that of a spherical segment whose axis is that of the tube; because the action of gravity only introduces into the result of the calculation some terms multiplied by* α; and we have just seen, that, with very slender tubes, these [9376] [9377]

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considered as wholly undeserving of notice in experiments of this nature, where the errors of the observation may amount to several millimetres. Hence we may conclude that it is wholly unnecessary to carry on the development farther than La Place has done, and we may therefore restrict ourselves to the degree of accuracy he has attained in the formulas [9358, 9374]; and instead of the formula [9372i], we may write simply, [9372t] [9372u]

$$q=\frac{\alpha^2}{l}-\tfrac{1}{3}l, \qquad [9372v]$$

for the elevation of a fluid in a capillary tube whose radius is l. If we correct the observed elevation q, by adding to it one third of the radius of the tube l, and call the corrected value $q_{,}$, we shall have, from [9372v], [9372w] $q_{,}$

$$q_{,}=q+\tfrac{1}{3}l=\frac{\alpha^2}{l}. \qquad [9372x]$$

Multiplying this by $2l$, and resubstituting $\alpha^2=\alpha^{-1}$ [9323p], we get [9372x']

$$\frac{2}{\alpha}=2\alpha^2=2lq_{,}=(\text{the diameter of the tube})\times(\text{by the corrected elevation } q_{,}). \qquad [9372y]$$

If we substitute $\alpha^2=\frac{H}{g}$ [9372g'] in [9372x], it becomes $q+\frac{1}{3}l=\frac{H}{gl}$, the density being 1; but if we suppose the density to be D, we must change g into gD, as in [9372e'], and we shall have $q+\frac{1}{3}l=\frac{H}{gDl}$, or [9372$y'$]

$$D\,.\,(q+\tfrac{1}{3}l)=\frac{H}{gl}; \qquad [9372z]$$

the same change of g into gD being made in [9323p], we shall get

$$H=\frac{gD}{\alpha}=gD\alpha^2. \qquad [9372z']$$

These formulas will be of use hereafter.

* (4156) If the tube be inclined to the horizon by the angle V, as in [9467], the effect will be to resolve the vertical gravity gz [9315b], into $gz\,.\,\sin.V$, in the direction of the axis of the tube z; and this will change g into $g\,.\,\sin.V$, in the equations [9315, 9318, &c.]; so that the term $\frac{2gz}{H}$, in the second member of [9318], will be changed [9379a] [9379b]

[9378] terms may be neglected. Therefore, *putting q for the vertical height of the fluid above the level, or its depression below the level,* we shall always have, from [9360, 9374, 9375],

[9379] $$q = \frac{H}{g} \cdot \frac{\sin.\theta'}{l},$$ [Approximate elevation or depression q, in a tube whose radius is l.]

which agrees with observation.

6. *We may extend the preceding analysis to the case in which a cylindrical tube*
[9380] *has a cylinder of the same matter inserted within it, both having the same axis.* Elevation between two cylindrical surfaces.
The fluid will rise in the space included between the inner sides of the tube and the outer surface of the cylinder; and, if this space is capillary, we may determine the equation of the surface of the included fluid in the following manner.

We shall resume the differential equation [9324]. The term $\frac{2}{b}$ of its
[9381] second member here denotes the sum of two fractions [9325′], which have each unity for a numerator, and for a denominator the greatest or the least of the radii of curvature of the surface of the fluid, at the lowest point, which is the
[9382] origin of* z; this equation gives by integration, as in [9329],

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[9379c] into $\frac{2gz}{H}.\sin.V$, or $\frac{2gz}{H} - \frac{2g}{H}.(1-\sin.V).z$; consequently that second member will be
[9379d] decreased by the term $\frac{2g}{H}.(1-\sin.V).z$, or $2\alpha.(1-\sin.V).z$ [9328], of the order α, as in [9377]; and we have seen, in [9365, 9336d, &c.], that terms of this kind are so small, that they have but very little effect on the result.

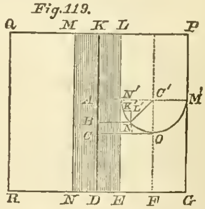

Fig. 119.

* (4157) Let $PQRG$, fig. 119, be the tube, $LMNE$ the
[9383a] cylinder, KD their common axis, $N'N_{\prime}OM'$ the surface of the fluid between the tube and cylinder, O its lowest point, and $N_{\prime}$ any other point corresponding to the coordinates z, u. These coordinates are found in the figure by drawing OC, $N_{\prime}B$, perpendicular to the axis of the tube KD; and then we have
[9383b] $CB=z$, $BN_{\prime}=u$, $DE=l$, $DG=l'$ [9385, 9388.]
[9383c] Through $N_{\prime}$ draw the radius $N_{\prime}C'$, perpendicular to the surface at the point $N_{\prime}$, and continue it till it meet, in C', the line FOC',
[9383d] drawn through O parallel to the axis DK. Then the angle $N_{\prime}C'O=\theta$ represents the acute angle which the small arc $N_{\prime}K'$ makes with either of the horizontal lines $BN_{\prime}$, $K'L'$;
[9383d′] the similar angle at the side of the tube at N', or at the cylinder M', being denoted by θ', in conformity with the symbols adopted in [9346, 9328a, b]. Now if we take the infinitely

$$\frac{u \cdot \frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}} - 2\alpha . \int zudu = \frac{u^2}{\mathrm{b}} + \text{constant.}$$ [Equation of the surface of a fluid between two cylinders.] [9383]

To determine the constant quantity, we shall observe that, at the point where the fluid touches the surface of the cylinder, we shall have, as in [9383g], [9383']

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small arc $N_{,}K'$, and draw $N_{,}L'$, $L'K'$, respectively parallel to OC', OC, we shall have [9383e]
the angle $N_{,}K'L' = N_{,}C'O = \theta$, and its sine is represented by $\frac{\text{tang.}\theta}{\sqrt{1+\text{tang.}^2\theta}}$; but
$\text{tang.}\theta = \frac{N_{,}L'}{L'K'}$, $N_{,}L' = dz$, $L'K' = -du$; hence $\text{tang.}\theta = -\frac{dz}{du}$; consequently [9383f]

$$-\sin.\theta = \frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}},$$ [9383g]

and at the point N', where θ changes into θ', this becomes as in [9384].

We shall now suppose that b represents the radius of curvature of the arc $M'ON'$ at its lowest point O; and b' the radius of curvature of the arc drawn through O, perpendicular to the plane of the figure; then we shall have for the terms depending on b, b', in the second [9383h]
member of the differential equation of the surface [9318], the quantity $\frac{1}{b}+\frac{1}{b'}$. If we [9383i]
suppose these radii to be equal to each other, and to a quantity which, for distinction, we shall
denote by b, we shall have, $b = b' = \mathrm{b}$, and $\frac{1}{b}+\frac{1}{b'} = \frac{2}{\mathrm{b}}$, as in [9325]; and this [9383k]
expression $\frac{2}{\mathrm{b}}$ is used in the reduced form of the differential equation of the surface [9324]. [9383l]
Therefore, if we wish to apply the equation [9324] to the present problem, where b, b', have [9383m]
different values, we must change $\frac{2}{\mathrm{b}}$ into $\frac{1}{b}+\frac{1}{b'}$; observing that, to avoid confusion in the symbols, we have inserted the symbol b in [9381—9404], instead of b, which is used by the author. [9383n] Now the surface of the fluid being formed by the revolution of the curve $M'ON'$ about the axis DK, the point O must describe a horizontal circle; hence it is
evident that the radius $b' = \infty$; substituting this in the equation [9383k], we get $\frac{1}{b} = \frac{2}{\mathrm{b}}$, [9383o]
or $\mathrm{b} = 2b$; which expresses the value of b, to be used in all the formulas where it occurs [9383p]
in [9381—9404]. If we now substitute the expression of the first member of [9384] in [9383q]
[9383], we shall get $-u.\sin\theta' - 2\alpha.\int zudu = \frac{u^2}{\mathrm{b}} + \text{constant}$, the integral being supposed to [9383r]
commence, as in [9385, 9383'], at the point where the fluid touches the tube; then, putting,
as in [9385], $u = l$, we get the value of the constant quantity [9386], and by substituting [9383s]
it in the formula [9383], we finally obtain [9387].

[9384]
$$\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=-\sin.\theta',$$

the negative sign being prefixed to $\sin.\theta'$, because at that point $\frac{dz}{du}$ is a
[9385] negative quantity. If we commence the integral $\int zudu$ at that point, and put
$l.$ *l for the radius of the cylinder*, or the value of u at that point, we shall have

[9386]
$$\text{constant}=-l\,.\sin.\theta'-\frac{l^2}{\mathrm{b}},$$

which gives

[9387]
$$\frac{u\,.\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}-2\alpha\,.\int zudu=\frac{u^2-l^2}{\mathrm{b}}-l\,.\sin.\theta'.$$

[9388] We shall at first suppose that $\alpha=0$; and we shall put *l' for the radius of*
$l'.$ *the inner surface of the tube*; l' will be the value of u at the point where the fluid touches the side of the tube. At this point we have *

[9389]
$$\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\sin.\theta';$$

therefore, at this point, we shall have

[9390]
$$l'.\sin.\theta'=\frac{l'^2-l^2}{\mathrm{b}}-l\,.\sin.\theta',$$

which gives

[9391]
$$\frac{1}{\mathrm{b}}=\frac{\sin.\theta'}{l'-l}.$$

This being premised, we shall have †

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* (4158) This is the same as [9384], except in its sign, which is taken positive, because
[9389a] dz, du, are both positive on the arc OM', or near the point M'. Multiplying this by the value of u near the point M', namely l', and substituting the result in [9387], we get [9390], by neglecting α, as in [9388]. Transposing the last term of [9390], we get
[9389b] $(l'+l).\sin.\theta'=\frac{l'^2-l^2}{\mathrm{b}}$; dividing by l'^2-l^2, or $(l'+l).(l'-l)$, we get [9391].

† (4159) Neglecting the term of [9387] depending on α, substituting $\frac{1}{\mathrm{b}}$ [9391], and
[9392a] reducing, it becomes as in [9392]. Squaring this, and multiplying by $(l'-l)^2.\left(1+\frac{dz^2}{du^2}\right)$,

$$\frac{u\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\frac{(u^2-ll')}{l'-l}\cdot\sin.\theta';$$ [9392]

whence we deduce

$$dz=\frac{(u^2-ll')\cdot du\cdot\sin.\theta'}{\sqrt{(l'-l)^2\cdot u^2-(u^2-ll')^2\cdot\sin.^2\theta'}},$$ [9393]

an equation whose integral depends on the rectification of a conic section.

If we do not suppose $\alpha=0$, we shall have *

$$\frac{1}{\mathrm{b}}=\frac{\sin.\theta'}{l'-l}-\frac{2\alpha}{(l'-l)\cdot(l'+l)}\cdot\int zudu,$$ [9394]

the integral $\int zudu$ being taken from $u=l$ to $u=l'$; we also have † [9395]

$$\int zudu=\tfrac{1}{2}u^2z-\tfrac{1}{2}\cdot\int u^2dz.$$ [9396]

If we neglect quantities of the order α, we may substitute the value of dz [9393] in the second member of [9396], and we shall get

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we easily get dz^2, whose square root gives dz [9393], which is of the form

$$dz=\frac{(\alpha_{\prime}+\beta_{\prime}u^2)\cdot du}{\sqrt{\alpha'+\gamma'u^2+\varepsilon'u^4}},$$ [9392b]

treated of by Le Gendre in Vol. I., page 3 or 4, of his *Fonctions Elliptiques*, where it is shown to be dependent on the rectification of the conic sections; but as this integral is not required in the course of the work, it was thought to be unnecessary to enter into the details of the reductions, which have no other difficulty than the mere labor of the computations. [9392c]

* (4160) By neglecting α, as in [9388], we have reduced the first member of [9390] to the form $l'.\sin.\theta'$ [9390]; if we now retain the term $-2\alpha.\int zudu$ [9387], the equation [9390] becomes $l'.\sin.\theta'-2\alpha\int zudu=\frac{l'^2-l^2}{\mathrm{b}}-l.\sin.\theta'$. Transposing $l.\sin.\theta'$, and dividing by l'^2-l^2, or $(l'+l)\cdot(l'-l)$, it becomes as in [9394], or, as it may be written, [9394a]

$$\frac{1}{\mathrm{b}}=\frac{\sin.\theta'}{l'-l}\cdot\left\{1-\frac{2\alpha}{(l+l')\cdot\sin.\theta'}\cdot\int zudu\right\}.$$ [9394b]

† (4161) Integrating $zudu$ by parts, we obtain [9396], as is easily seen by differentiation; substituting in this equation the value of dz [9393], we get [9397]; multiplying this by $-\frac{2\alpha}{(l'+l)\cdot(l'-l)}$, and substituting the product in [9394], it becomes as in [9398], the constant quantity $\sin.\theta'$, in the numerator, being brought from under the sign of integration. [9398a] This expression of $\frac{1}{\mathrm{b}}$ [9398] is evidently correct in terms of the order α; and by neglecting terms of that order, it becomes as in [9401]. [9398b]

[9397] $$fzudu = \tfrac{1}{2}u^2.\int\frac{(u^2-ll').du.\sin.\theta'}{\sqrt{(l'-l)^2.u^2-(u^2-ll')^2.\sin.^2\theta'}} - \tfrac{1}{2}.\int\frac{(u^2-ll').u^2du.\sin.\theta'}{\sqrt{(l'-l)^2.u^2-(u^2-ll')^2.\sin.^2\theta'}}.$$

Therefore we shall have, by neglecting quantities of the order α^2,

[9398] $$\frac{1}{\mathrm{b}}=\frac{\sin.\theta'}{l'-l}.\left\{1-\frac{\alpha u^2}{(l'+l)}.\int_l^{l'}\frac{(u^2-ll').du}{\sqrt{(l'-l)^2.u^2-(u^2-ll')^2.\sin.^2\theta'}}+\frac{\alpha}{(l'+l)}.\int_l^{l'}\frac{(u^2-ll').u^2du}{\sqrt{(l'-l)^2.u^2-(u^2-ll')^2.\sin.^2\theta'}}.\right.$$

[9399] These integrals can be determined by approximation only; but it will suffice here to observe that, α being very small, when the distance between the tube and cylinder is very small, we may, without any sensible error, neglect the [9400] terms multiplied by α, as we have seen in [9359′, 9366, &c.], that it may be done in a very narrow tube. Then we shall have very nearly,

[9401] $$\frac{1}{\mathrm{b}}=\frac{\sin.\theta'}{l'-l}.$$

7. We shall now suppose that an infinitely narrow canal is drawn through [9402] the lowest point of the surface of the fluid included within the capillary space, parallel to the axis of the tube, and passing below it, and then is bent upwards till it meets the surface of the fluid in which the tube is dipped. The action of the [9403] included fluid upon the canal, will be* $K-\frac{H}{\mathrm{b}}$; because $\frac{2}{\mathrm{b}}$ being, by what has been said in [9381], the sum of the two fractions which have unity for a numerator, and for denominators the greatest and the least radii of curvature of the surface at its lowest point, the action of the included fluid will be, by [9404] the theorem [9294, 9294′], $K-\frac{1}{2}H.\frac{2}{\mathrm{b}}$. Therefore this action will be, by the preceding article [9401],

[9405] $$K-\frac{H.\sin.\theta'}{l'-l}.$$

[9403a] * (4162) The term $\frac{2}{\mathrm{b}}$ [9324, 9325], is equivalent to, and used instead of, $\frac{1}{b}+\frac{1}{b'}$ [9383n]; b and b' being, as in [9383h], the greatest and the least radii of curvature; so that the action of the fluid between the tube and cylinder is, as in [9315b], $K-\frac{1}{2}H.\left(\frac{1}{b}+\frac{1}{b'}\right)$, or, according to the present notation [9381],

[9403b] $$K-\tfrac{1}{2}H.\frac{2}{\mathrm{b}},\ \text{or}\quad K-\frac{H}{\mathrm{b}},$$

[9403c] as in [9403, 9404]. Substituting the value of $\frac{1}{\mathrm{b}}$ [9401], it becomes as in [9405]. We may remark that the canal spoken of in [9402, &c.] is similar to that which is explained by fig. 112, page 695, to which we may refer for illustration.

If we put q' *for the elevation of the fluid in the interior branch of the canal,* q'.
above the level of the fluid in the vessel, and add gq' to the preceding action, [9406]
the sum will be in equilibrium with the action K of the fluid in the vessel upon [9407]
the canal; therefore we shall have *

$$K+gq'-\frac{H.\sin.\theta'}{l'-l}=K,$$ [9408]

which gives

$$q'=\frac{H}{g}\cdot\frac{\sin.\theta'}{l'-l}.$$ [Elevation q' of a fluid between two cylindrical surfaces, whose radii are l', l.] [9409]

Elevation of a fluid between two cylindrical surfaces.

We have found, in [9379], that *the elevation of a fluid above its level, in a tube whose radius is* $l'-l$, *is equal to this value of* q'; *therefore the fluid* [9410]
ascends in the capillary space to the same height as in a tube whose radius is equal to the width of that space.

If the surface of the fluid be convex, the preceding expression of q' [9409]
will denote the depression of the fluid below the level, and the fluid will then sink [9411]
in the capillary space, in the same manner as in a tube whose radius is equal to the width of that space.†

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* (4163) The calculation in [9408] is made in the same manner as for a tube in [9355a, &c.], and requires no particular explanation. From this we easily deduce the value [9408a]
of q' [9409].

† (4164) This will be evident by making the calculation in like manner as in [9372—9409],
the effect being merely to change the signs of q', θ', as in [9352d, &c.], which does not alter [9410a]
the value of q' [9409]. Therefore the elevation q' of the fluid when the surface is concave, [9410b]
or its depression when the surface is convex, is given very nearly by the formula [9409],
being, as in [9410], *the same as the elevation or depression in a tube whose radius is equal to* [9410c]
the width of the capillary space $M'N'$, fig. 119, page 732. In all the calculations of § 6, 7
[9380—9412], it is supposed that the tube and cylinder are of the same substance, or that [9410d]
the angle θ' [9346] is the same for both of them; *so that the fluid is either elevated by the* [9410e]
combined action of the tube and cylinder, or is depressed by the combined action of both of them. In this case, if the distance $M'N'$ or EG be small in comparison with α, the section [9410f]
of the surface $M'ON'$ may be supposed to coincide very nearly with the circle of curvature
at the point O, whose radius is b, as in the similar case [9336g] relative to the surface in [9410g]
a capillary tube, as is evident from the equation [9401]. For if we put for brevity
$l'-l=2\lambda$, and substitute $\mathrm{b}=2b$ [9383p] in [9401], we shall get, by a slight reduction,
$b.\sin.\theta'=\lambda$; which gives the same expression of λ as would be found if we were to [9410h]
suppose $M'ON'$ to be a circular arc, described with the radius $C'O=b$, making the
angle $OC'N'=OC'M'=\theta'$, and its sine $EF=FG=\lambda$. [9410i]

Instead of supposing the angle θ' to be the same at the points N' and M', we shall now

If we suppose the radii of the tube and cylinder to be infinite, we shall obtain the result of the capillary action upon a fluid between two vertical and parallel
[9412] *planes situated very near to each other; therefore the preceding theorem holds good in this case. We shall, however, investigate this theorem, in the following article, by a particular analysis.*

[9413] 8. We shall suppose, in fig. 116, page 713, *that AOB is the section of the surface of the fluid included between the two planes*, by a vertical plane drawn

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[9410k] take θ' for its value at the point N', and $\theta_{,}$ for the corresponding angle at the point M';
[9410l] putting also $EF=u'$, $FG=u_{,}$, $u'+u_{,}=2\lambda$, and considering $M'ON'$ as a circular arc whose radius is b, as in [9410g]. Then we shall have, in the same hypothesis relative to the nature of the fluids, as in [9410e],

[9410m] $$u'=b\,.\sin.\theta', \qquad u_{,}=b\,.\sin.\theta_{,}, \qquad u'+u_{,}=2\lambda;$$

so that, if 2λ, θ', $\theta_{,}$, are given, we may determine the values of b, u', $u_{,}$. If we suppose
[9410n] $\theta'=\theta_{,}=90^d$, the equations [9410m] will give $u'=u_{,}=b=\lambda$, which correspond to the actual form of the figure 119, where the extreme points M', N', appear to be upon the
[9410o] horizontal line $M'C'N'$, passing through the centre C' of the circular arc. We may, finally, remark that the equations [9410m] are similar to those which are given by M. Poisson, in page 114 of his *Nouvelle Théorie, &c.*

[9410p] These calculations are limited to the case where the fluid is acted upon by both surfaces in the *same* manner; but it may happen, when the tube and cylinder are of
[9410q] different substances, that the fluid may be *elevated* near one of the surfaces, and *depressed* near the other, so that there may be a point of *inflexion* of the surface, as is observed between
[9410r] two parallel planes in [10158, &c.]. In the case of two parallel planes, the surface
[9410s] may be concave in one part $NO'O$, fig. 112, page 695, and convex in the other part OM', the point of inflexion being O, which, we shall soon see, is on a level with the fluid in the vessel AVB; the point O falling in H; the fluid being elevated above this level in the
[9410t] part ON, and depressed below the level in the part OM'. For the capillary action, at any point O' of the concave surface $NO'O$, will be represented, as in [9294, 9294'], by
[9410u] $K-\frac{1}{2}H.\left(\frac{1}{b}+\frac{1}{b'}\right)$; b' and b being the greatest and the least radii of curvature corresponding to that point; and we shall evidently have, as in [9383o], $b'=\infty$; so that this force becomes
[9410v] $K-\frac{H}{2b}$, instead of $K-\frac{H}{b}$, which is used in [9355a]; and if we make the calculation as in [9355a—e], we must change b into $2b$, to conform to this alteration in the expression of the capillary action. By this means the expression of q [9351], or q' in the present
[9410w] notation, becomes $q'=\frac{H}{g}\cdot\frac{1}{2b}$; and by putting, as in [9323o], $\frac{H}{g}=\alpha^2$, we shall get
[9410x] $q'=\frac{\alpha^2}{2b}$. This value of q' expresses the elevation QO' of the fluid in the *concave* part NO of the capillary surface, above the level AVB of the fluid in the vessel; it becomes

perpendicular to these two planes; if we put $OM=z$, $NM=y$, *z will be a function of y only.* Moreover, b and b' being the greatest and the least radii of curvature of the surface of the fluid at the lowest point O [9316], *b will be infinite*, and b' the radius of curvature of the curve AOB, at the point O. Hence we shall have, in the equations of partial differentials* [9312, 9313], [9413'] Elevation of a fluid between two parallel planes. [9414]

$$\mathrm{p}=\left(\frac{dz}{dx}\right)=0, \qquad \mathrm{q}=\left(\frac{dz}{dy}\right); \qquad \mathrm{r}=0, \qquad \mathrm{s}=0, \qquad \mathrm{t}=\left(\frac{ddz}{dy^2}\right), \qquad \frac{1}{b}=0;$$ [9415]

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negative in the *convex* part OM' [9411, &c.]; and we may in both cases use the same [9410y]
expression of $q'=\frac{\alpha^2}{2b}$ [9410x]; considering the radius of curvature b to be *positive* in the *concave* part of the surface NO, and *negative* in the convex part OM'. If we suppose the radius of curvature b, corresponding to the point O, to become infinite, we shall have from [9410x] $q'=0$, or $HO=0$; so that the point O will then fall to the level AVB of the fluid in the vessel; and O will be a point of inflexion, because it is the part which separates the positive from the negative values of the radii of curvature; or, in other words, it is the part which separates the concave part NO from the convex part OM'; this agrees with what we have stated above in [9410s, &c.]. [9410z] [9411a] [9411b] [9411c]

* (4165) As z does not contain x [9413'], the values of p, r, s [9312, 9313], must vanish, and those of q, t [9312, 9313], become $\mathrm{q}=\left(\frac{dz}{dy}\right)$, $\mathrm{t}=\left(\frac{ddz}{dy^2}\right)$. Substituting these and $\frac{1}{b}=0$, also $\frac{g}{H}=\alpha$ [9328], in [9318], we get [9416]. Multiplying this by dz, and taking its integral, we obtain [9417]. At the point O, where $z=0$, we also have $\frac{dz}{dy}=0$, because the tangent at O is parallel to the horizon; substituting these in [9417], we get $-1=\text{constant}$; and by substituting this last expression in [9417], we obtain [9415a] [9415b]

$$-\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}-\alpha z^2=\frac{z}{b'}-1,$$ [9415c]

which, by changing the signs of the terms, and making a slight reduction, becomes as in [9419]. We may here remark, that the author supposes, in this article, that the angle θ' is the same for both planes, so that the fluid is either *elevated* near both planes or *depressed* near both. The case where the fluid is *elevated* near one of the planes, and depressed near the other, is treated of in [10158—10257]. [9415d]

The author simplifies the equation [9416], by taking, as in [9323m], the origin of z at the level surface of the fluid in the vase, instead of at the point O, fig. 116, page 713; or, in other words, by changing z into $z-\frac{1}{2\alpha b'}$; for by this means the differential equation [9416] changes into the following, which is the same as [10164]; [9415e]

[9415′] hence the equation [9318] becomes

[9415f]
$$\frac{\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}}=2\alpha z.$$

Multiplying this by α^2, and putting $\alpha\alpha^2=1$, as in [9323p], it becomes

[9415g]
$$\frac{\alpha^2.\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}}=2z,$$

being the same as is used by M. Poisson, in page 174 of his *Nouvelle Théorie, &c.*

[9415h] Multiplying [9415g] by $-dz$, integrating, and adding the constant quantity b, we get

[9415i]
$$\frac{\alpha^2}{\sqrt{1+\frac{dz^2}{dy^2}}}=\mathrm{b}-z^2.$$

[9415k] Putting h for the elevation of the point O above the level of the fluid in the vase, we shall have, at this point, where $\frac{dz}{dy}=0$, $\alpha^2=\mathrm{b}-h^2$ [9415i], or $\mathrm{b}=\alpha^2+h^2$; substituting this in [9415i], we get

[9415l]
$$\frac{\alpha^2}{\sqrt{1+\frac{dz^2}{dy^2}}}=\alpha^2+h^2-z^2.$$

Deducing the value of dy from this equation, we obtain

[9415m]
$$dy=\frac{(\alpha^2+h^2-z^2).dz}{\sqrt{(z^2-h^2).(h^2+2\alpha^2-z^2)}},$$

[9415n] which is of the same nature as the equation [10172], and is easily integrated by means of elliptical functions. It is rather remarkable that these functions are not mentioned by La Place throughout his whole work, which can be accounted for upon no other principle than his dislike to Le Gendre, the great promoter and improver of this calculus; since there are many parts of the Mécanique Céleste where it would have been very advantageous to have used

[9415n′] this method of integration. The process of reduction to elliptical integrals, in the present case, is easily obtained from what Le Gendre has published on the subject, and we shall here give the calculation, with all the necessary details.

[9415o] We shall assume for z^2 an expression of the form [9415p], which, by dividing the numerator and denominator by $\cos.^2\varphi$, or multiplying by $1+\mathrm{tang.}^2\varphi$, gives [9415q]; whence we easily deduce the value of $\mathrm{tang.}^2\varphi$ [9415r].

[9415p]
$$z^2=\frac{(h^2+2\alpha^2).h^2}{h^2+2\alpha^2.\cos.^2\varphi}$$

[9415q]
$$=\frac{(h^2+2\alpha^2).h^2.(1+\mathrm{tang.}^2\varphi)}{(h^2+2\alpha^2)+h^2.\mathrm{tang.}^2\varphi};$$

[9415r]
$$\mathrm{tang.}^2\varphi=\frac{(h^2+2\alpha^2).(z^2-h^2)}{h^2.(h^2+2\alpha^2-z^2)}.$$

$$\frac{\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}}-2\alpha z=\frac{1}{b'}. \qquad [9416]$$

The limits of the value of the first member of [9415l] being α^2 and 0, it follows from the second member of the same equation, that the limits of z^2 must be h^2 and α^2+h^2; so that the limits of z^2-h^2 must be 0 and α^2; therefore the value of $\text{tang.}^2\varphi$ [9415r] must be always positive, and φ a real angle. Now substituting the value of z^2 [9415p], in the first member of [9415u], we get, by a slight reduction, its second member; and by using this in [9415v], it is reduced to the form [9415w]. The square root of the product of the two expressions [9415u, w] gives [9415x], using for brevity the symbol $w=\sqrt{h^2+2\alpha^2.\cos.^2\varphi}$. [9415$s$] [9415$t$]

$$z^2-h^2=\frac{2\alpha^2h^2-2\alpha^2h^2.\cos.^2\varphi}{w^2}=\frac{2\alpha^2h^2.\sin.^2\varphi}{w^2}; \qquad [9415u]$$

$$h^2+2\alpha^2-z^2=2\alpha^2-(z^2-h^2)=2\alpha^2-\frac{2\alpha^2h^2.\sin.^2\varphi}{w^2}=\frac{2\alpha^2.(h^2+2\alpha^2.\cos.^2\varphi)-2\alpha^2h^2.\sin.^2\varphi}{w^2} \qquad [9415v]$$

$$=\frac{2\alpha^2h^2.(1-\sin.^2\varphi)+4\alpha^4.\cos.^2\varphi}{w^2}=\frac{2\alpha^2.(h^2+2\alpha^2).\cos.^2\varphi}{w^2}; \qquad [9415w]$$

$$\sqrt{(z^2-h^2).(h^2+2\alpha^2-z^2)}=\frac{2\alpha^2h.(h^2+2\alpha^2)^{\frac{1}{2}}.\sin.\varphi.\cos.\varphi}{w^2}. \qquad [9415x]$$

Substituting w [9415t] in [9415p], and taking the square root, we get

$$z=(h^2+2\alpha^2)^{\frac{1}{2}}.h.w^{-1},$$

whose differential is

$$dz=-(h^2+2\alpha^2)^{\frac{1}{2}}.hw^{-2}dw;$$

and as the differential of

$$w^2=h^2+2\alpha^2.\cos.^2\varphi \qquad [9415y]$$

[9415t] is

$$2wdw=-4\alpha^2.d\varphi.\sin.\varphi.\cos.\varphi,$$

we shall have

$$dz=(h^2+2\alpha^2)^{\frac{1}{2}}.w^{-3}.2\alpha^2h.d\varphi.\sin.\varphi.\cos.\varphi.$$

Dividing this by [9415x], we obtain

$$\frac{dz}{\sqrt{(z^2-h^2).(h^2+2\alpha^2-z^2)}}=\frac{d\varphi}{w}=\frac{d\varphi}{\sqrt{h^2+2\alpha^2.\cos.^2\varphi}}; \qquad [9415z]$$

multiplying this by $\alpha^2+h^2-z^2=\alpha^2+h^2-\frac{(h^2+2\alpha^2).h^2}{w^2}$ [9415p, t], we get the value of dy [9415m], under the following form;

$$dy=\frac{(\alpha^2+h^2).d\varphi}{(h^2+2\alpha^2.\cos.^2\varphi)^{\frac{1}{2}}}-\frac{(h^2+2\alpha^2).h^2.d\varphi}{(h^2+2\alpha^2.\cos.^2\varphi)^{\frac{3}{2}}}. \qquad [9416a]$$

To reduce the radical $\sqrt{h^2+2\alpha^2.\cos.^2\varphi}$ to the usual form of the elliptical functions $\sqrt{1-c^2.\sin.^2\varphi}$ [8910i, &c.], we may put $\cos.^2\varphi=1-\sin.^2\varphi$, and $c^2=\frac{2\alpha^2}{h^2+2\alpha^2}$; then from this value of c^2, we easily deduce [9416b]

$$1-c^2=\frac{h^2}{h^2+2\alpha^2}; \qquad 2-c^2=\frac{2.(\alpha^2+h^2)}{h^2+2\alpha^2}; \qquad h^2=\frac{2\alpha^2.(1-c^2)}{c^2}. \qquad [9416c]$$

[9416′] Multiplying this by dz, and integrating, we shall get

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[9416d] $$\alpha^2+h^2=\frac{\alpha^2.(2-c^2)}{c^2}; \qquad h^2+2\alpha^2=\frac{2\alpha^2}{c^2}.$$

Now substituting $\cos.^2\varphi=1-\sin.^2\varphi$ in [9416a], and using the values [9416b, c, d], we get

[9416e] $$dy=\frac{(2-c^2).\alpha}{c.\sqrt{2}}\cdot\frac{d\varphi}{\sqrt{1-c^2.\sin.^2\varphi}}-\frac{2(1-c^2).\alpha}{c.\sqrt{2}}\cdot\frac{d\varphi}{(1-c^2.\sin.^2\varphi)^{\frac{3}{2}}}.$$

[9416f] If we put for a moment, for brevity, $\Delta=\sqrt{1-c^2.\sin.^2\varphi}$, and $X=\frac{\sin.\varphi.\cos.\varphi}{\Delta}$, we shall have, by taking the differential of X, and dividing by $d\varphi$, then putting $\cos.^2\varphi=1-\sin.^2\varphi$,

[9416g] $$\left(\frac{dX}{d\varphi}\right)=\frac{\cos.^2\varphi-\sin.^2\varphi}{\Delta}+\frac{c^2.\sin.^2\varphi.\cos.^2\varphi}{\Delta^3}=\frac{1-2\sin.^2\varphi}{\Delta}+\frac{c^2.\sin.^2\varphi-c^2.\sin.^4\varphi}{\Delta^3};$$

adding $\frac{1-c^2}{c^2.\Delta^3}$ to both sides of this equation, and making successive reductions, we get

[9416h] $$\left(\frac{dX}{d\varphi}\right)+\frac{1-c^2}{c^2.\Delta^3}=\frac{1-2\sin.^2\varphi}{\Delta}+\frac{(1-c^4.\sin.^4\varphi)-c^2(1-c^2.\sin.^2\varphi)}{c^2.\Delta^3}$$

$$=\frac{1-2\sin.^2\varphi}{\Delta}+\frac{(1-c^2.\sin.^2\varphi).(1+c^2.\sin.^2\varphi-c^2)}{c^2.\Delta^3}$$

$$=\frac{1-2\sin.^2\varphi}{\Delta}+\frac{(1+c^2.\sin.^2\varphi-c^2)}{c^2.\Delta}=\frac{c^2.(1-2\sin.^2\varphi)+(1+c^2.\sin.^2\varphi-c^2)}{c^2.\Delta}$$

[9416i] $$=\frac{1-c^2.\sin.^2\varphi}{c^2.\Delta}=\frac{\Delta^2}{c^2.\Delta}=\frac{\Delta}{c^2}.$$

Multiplying this by $-\frac{2\alpha c.d\varphi}{\sqrt{2}}$, and transposing the first term, we obtain

[9416k] $$-\frac{2(1-c^2).\alpha.d\varphi}{c.\sqrt{2}.\Delta^3}=-\frac{2\alpha}{c.\sqrt{2}}.d\varphi.\Delta+\alpha c.\sqrt{2}.dX;$$

substituting this in [9416e], it becomes, by resubstituting the values of Δ, X [9416f],

[9416l] $$dy=\frac{(2-c^2).\alpha}{c.\sqrt{2}}\cdot\frac{d\varphi}{\sqrt{1-c^2.\sin.\varphi}}-\frac{2\alpha}{c.\sqrt{2}}.d\varphi.\sqrt{1-c^2.\sin.^2\varphi}+\alpha c.\sqrt{2}.d.\left(\frac{\sin.\varphi.\cos.\varphi}{\sqrt{1-c^2.\sin.^2\varphi}}\right).$$

Integrating this expression, and using the elliptical symbols [8910k, l], we get

[9416m] $$y=\frac{(2-c^2).\alpha}{c.\sqrt{2}}.\mathrm{F}(c,\varphi)-\frac{2\alpha}{c.\sqrt{2}}.\mathrm{E}(c,\varphi)+\frac{\alpha c.\sqrt{2}.\sin.\varphi.\cos.\varphi}{\sqrt{1-c^2.\sin.^2\varphi}};$$

no constant quantity is added, because we have $y=0$, at the point O, where $z=h$ [9415k], which corresponds to $\varphi=0$ in [9415r]; multiplying the value of y by $\frac{\sqrt{2}}{\alpha}$, and making a slight reduction, we finally obtain

[9416n] $$\frac{y\sqrt{2}}{\alpha}=\frac{(2-c^2)}{c}.\mathrm{F}(c,\varphi)-\frac{2}{c}.\mathrm{E}(c,\varphi)+\frac{2c.\sin.\varphi.\cos.\varphi}{\sqrt{1-c^2.\sin.^2\varphi}},$$

which is the same as the equation (3) of M. Poisson in page 177 of his *Nouvelle Théorie, &c.*, changing x into y, to conform to his notation.

$$-\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}-\alpha z^2=\frac{z}{b'}+\text{constant.}$$ [9417]

Substituting $\cos.^2\varphi=1-\sin.^2\varphi$ in [9415p], then dividing the numerator and denominator by $h^2+2\alpha^2$, we get, by using c^2, h^2, $h^2+2\alpha^2$ [9416b, c],

$$z^2=\frac{2\alpha^2.(1-c^2)}{c^2.(1-c^2.\sin.^2\varphi)},$$ [9416o]

which is the same as the equation (4) page 177 of M. Poisson's work. Hence we see that the values of the coordinates y, z, may be ascertained in functions of the angle φ, by means of the formulas [9416m, o]. From these expressions we may trace the form of the curve surface, and the relation of the coordinates. If we suppose, as in [9346f], that ϖ is the [9416p] acute angle formed by the lower part of the plane and a tangent to a point of the section placed at the limit of the sphere of sensible activity of the first plane, we shall have, by the usual differential formulas,

$$\sin.\varpi=\frac{dy}{\sqrt{dy^2+dz^2}}=\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}};$$ [9416q]

and if we suppose the value of z near the fixed plane to be $z=q$, we shall have, by substitution in [9415l], $\alpha^2.\sin.\varpi=\alpha^2+h^2-q^2$; hence we easily deduce the expression [9416r] of q^2 [9416s], and by substituting the value of h^2 [9416c], we get [9416t]

$$q^2=h^2+\alpha^2.(1-\sin.\varpi);$$ [9416s]

$$q^2=\frac{\alpha^2}{c^2}.(2-c^2-c^2.\sin.\varpi).$$ [9416t]

The first of these values of q^2 gives

$$q^2-h^2=\alpha^2.(1-\sin.\varpi);\qquad h^2+2\alpha^2-q^2=\alpha^2.(1+\sin.\varpi).$$ [9416t']

Now if we suppose Θ to be the value of φ, corresponding to $z=q$, we shall have, by substituting these values in [9415r], and reducing by means of $\frac{h^2+2\alpha^2}{h^2}=\frac{1}{1-c^2}$ [9416c, d], [9416u]

$$\text{tang.}^2\Theta=\frac{(h^2+2\alpha^2).\alpha^2.(1-\sin.\varpi)}{h^2.\alpha^2.(1+\sin.\varpi)}$$ [9416v]

$$=\frac{1-\sin.\varpi}{(1-c^2).(1+\sin.\varpi)}.$$ [9416w]

If we suppose that $y=a_{,}$ corresponds to $z=q$, and to $\varphi=\Theta$, near the first plane, [9416x] we shall have, by substitution in [9416n],

$$\frac{a_{,}.\sqrt{2}}{\alpha}=\frac{2-c^2}{c}.\mathrm{F}(c,\Theta)-\frac{2}{c}.\mathrm{E}(c,\Theta)+\frac{2c.\sin.\Theta.\cos.\Theta}{\sqrt{1-c^2\sin.^2\Theta}}.$$ [9416y]

If we denote by a and ϖ' the distance and angle relative to the second plane, corresponding respectively to a, ϖ, for the first plane; we shall get the *following equation,* [9416z] *which is similar to* [9416y], *and may be deduced from it, by changing* $a_{,}$ *into* $a'_{,}$, Θ *into* Θ', *and* ϖ *into* ϖ';

[9418] At the point O, $\frac{dz}{dy}=0$; therefore constant $=-1$; consequently

[9419]
$$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}+\alpha z^2=\frac{b'-z}{b'}.$$

[9417a]
$$\frac{\alpha'_{,}\sqrt{2}}{\alpha}=\frac{2-c^2}{c}\cdot\mathrm{F}(c,\Theta')-\frac{2}{c}\cdot\mathrm{E}(c,\Theta')+\frac{2c\cdot\sin.\Theta'\cdot\cos.\Theta'}{\sqrt{1-c^2\cdot\sin.^2\Theta'}}.$$

Adding these two equations, and putting $\alpha_{,}+\alpha'_{,}=2l$ for the distance of the two planes, as in [9443e, 10236], we get

[9417b]
$$\frac{2l.\sqrt{2}}{\alpha}=\frac{2-c^2}{c}\cdot\{\mathrm{F}(c,\Theta)+\mathrm{F}(c,\Theta')\}-\frac{2}{c}\cdot\{\mathrm{E}(c,\Theta)+\mathrm{E}(c,\Theta')\}$$
$$+\frac{2c\cdot\sin.\Theta\cdot\cos.\Theta}{\sqrt{1-c^2\cdot\sin.^2\Theta}}+\frac{2c\cdot\sin.\Theta'\cdot\cos.\Theta'}{\sqrt{1-c^2\cdot\sin.^2\Theta'}}.$$

[9417c] Hence it appears that, if c, ϖ, ϖ', are given, we may deduce Θ,Θ', from the value of $\tang.^2\Theta$ [9416w], and the similar value of $\tang.^2\Theta'$ [9416z]; substituting these in [9417b], we
[9417d] obtain the distance of the two planes, $2l$. If this distance be given instead of c, we may deduce, by an inverse process, the value of c from the same three equations; but this is
[9417e] a much more laborious process than the preceding one, where c is given; and it will be
[9417f] necessary to form a little table, giving the expressions of $2l$ for values of c, increasing by small differences, which is easily done by means of Le Gendre's elliptical tables; and by entering this table, we may find by inspection the value of c, corresponding to any proposed
[9417g] value of $2l$; α being considered as a given quantity. We may remark that the equations [9416t, w, y, 9417b], are equivalent to the equations (5), (6), (7), (8), in pages 177, 178,
[9417h] of M. Poisson's *Nouvelle Théorie, &c.* The equation [9416n] being subtracted from [9416y], gives, by putting $\alpha_{,}-y=\mathrm{y}$,

[9417i]
$$\frac{\mathrm{y}\sqrt{2}}{\alpha}=\frac{2-c^2}{c}\cdot\{\mathrm{F}(c,\Theta)-\mathrm{F}(c,\varphi)\}-\frac{2}{c}\cdot\{\mathrm{E}(c,\Theta)-\mathrm{E}(c,\varphi)\}$$
$$+\frac{2c\cdot\sin.\Theta\cdot\cos.\Theta}{\sqrt{1-c^2\cdot\sin.^2\Theta}}-\frac{2c\cdot\sin.\varphi\cdot\cos.\varphi}{\sqrt{1-c^2\cdot\sin.^2\varphi}};$$

y being the distance from the first plane to the point of the surface corresponding to the
[9417k] angle φ, and Θ being the value of φ at the first plane. In like manner, if y' be the distance
[9417l] from the second plane to the point of the surface corresponding to the angle φ, and Θ' be the value of φ, at the second plane, we shall have, from [9417i], by changing y into y', Θ into Θ', and considering the angles φ, Θ', as positive,

[9417m]
$$\frac{\mathrm{y}'\sqrt{2}}{\alpha}=\frac{2-c^2}{c}\cdot\{\mathrm{F}(c,\Theta')-\mathrm{F}(c,\varphi)\}-\frac{2}{c}\cdot\{\mathrm{E}(c,\Theta')-\mathrm{E}(c,\varphi)\}$$
$$+\frac{2c\cdot\sin.\Theta'\cdot\cos.\Theta'}{\sqrt{1-c^2\sin.^2\Theta'}}-\frac{2c\cdot\sin.\varphi\cdot\cos.\varphi}{\sqrt{1-c^2\cdot\sin.^2\varphi}}.$$

When the second plane is at an infinite distance from the first, the equation of the surface
[9417n] [9417i] comes under the case which is treated of in [9435], and integrated by means of logarithms in [10208]; the elliptical functions which occur in [9417i] being, in this case, reduced to common logarithms, as is seen in [10208].

Now putting

$$Z = \frac{b'-z}{b'} - \alpha z^2, \qquad [9420]$$

we shall have *

$$dy = \frac{Zdz}{\sqrt{1-Z^2}}; \qquad [9421]$$

and this is the equation of the elastic curve, as ought to be the case, since, as in the elastic curve, the force which depends upon the curvature is inversely [9422]
proportional to the radius of curvature.† At A, fig. 116, page 713, the most elevated point of the curve ANO, we have ‡ $\frac{dz}{dy} = \text{tang.}\theta'$, θ' *being, as in* [9423]

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* (4166) Transposing the term αz^2 [9419], and substituting [9420], we get

$$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}} = Z, \quad \text{or} \quad 1 = Z.\sqrt{1+\frac{dz^2}{dy^2}}; \qquad [9421a]$$

squaring and reducing, we get the value of dy [9421].

† (4167) Substituting [9326] in [9416], after changing u into y, we get $\frac{1}{R} - 2\alpha z = \frac{1}{b'}$;
and if we put for brevity $e = \frac{1}{2\alpha b'}$, it may be put successively under the forms [9423a]

$$\frac{b'}{R} = 1 + 2\alpha b' z = \frac{e+z}{e};$$

and, as the curvature at any point is inversely as the radius of curvature, it is evident that the curvature at any point N, fig. 116, page 713, will be directly as the quantity $e+z$, that [9423b]
is, directly as the absciss z, augmented by the constant quantity e, being the principle upon which the properties of the elastic curve are founded in [9422].

‡ (4168) This is similar to [9383f], changing u into y, and taking dy positive, because y and z increase together. Substituting this value of $dz = dy.\text{tang.}\theta'$ in [9421], dividing by dy, and reducing, we get $1 - Z^2 = Z^2.\text{tang.}^2\theta'$, or $Z^2.(1+\text{tang.}^2\theta') = 1$; whence [9425a]
$Z = \cos.\theta'$, as in [9425]. Substituting this in [9420], we get [9426], and then dividing [9425b]
by $-\alpha$, we obtain $z^2 + \frac{1}{\alpha b'}.z = \frac{1}{\alpha}.(1-\cos.\theta') = \frac{2}{\alpha}.\sin.^2\tfrac{1}{2}\theta'$. From this quadratic [9425c]
equation we get z [9427], and when $b' = \infty$, it becomes as in [9429]. Now from [9430] [9425d]
we have $\alpha = \frac{g}{H} = \frac{\sin.\theta'}{lq} = \frac{2\sin.\frac{1}{2}\theta'.\cos.\frac{1}{2}\theta'}{lq} = \frac{2\sin.^2\frac{1}{2}\theta'}{lq.\text{tang.}\frac{1}{2}\theta'}$; hence we get $\sqrt{2\alpha} = \frac{2\sin.\frac{1}{2}\theta'}{\sqrt{lq.\text{tang.}\frac{1}{2}\theta'}}$. [9425e]
Substituting this in [9429], it becomes as in [9431]; and if in this we put $l = 1^{\text{mi.}}$, [9425f]
$q = 6^{\text{mi.}},784$, $\theta' = 90^{d}$, or $\text{tang.}\frac{1}{2}\theta' = 1$, we obtain $z = \sqrt{6^{\text{mi.}},784} = 2^{\text{mi.}},6046$, as [9425g]
in [9433].

Substituting $\alpha = \frac{1}{\alpha^2}$ [9323p] in [9429], it becomes $z = \alpha.\sqrt{2}.\sin.\frac{1}{2}\theta'$, which is the same [9425h]

[9424] [9346], *the complement of the angle which the extreme side of the curve makes with the plane;* therefore we shall have, at this point,

[9425] $$Z = \cos.\theta' \quad [9425b],$$

which gives, to determine the extreme value of z [9420, 9425],

[9426] $$\frac{b'-z}{b'} - \alpha z^2 = \cos.\theta', \quad \text{or}$$

[9427] $$z = -\frac{1}{2\alpha b'} + \sqrt{\frac{2\sin.^2\tfrac{1}{2}\theta'}{\alpha} + \frac{1}{4\alpha^2 b'^2}}. \qquad \text{[Extreme value of } z \text{]}$$

[9428] *If the two planes are at an infinite distance from each other,* b' will be infinite,

[9429] and we shall have

When the parallel planes are at a very great distance from each other.

$$z = \frac{2\sin.\tfrac{1}{2}\theta'}{\sqrt{2\alpha}}. \qquad \left[\text{Extreme value of } z\text{, when the distance of the planes is infinite.}\right]$$

We have $\frac{g}{H} = \alpha$ [9328]; moreover, *in a capillary tube whose semi-diameter*

[9430] *is* l, *we have* $\frac{g}{H} = \frac{\sin.\theta'}{lq}$ [9379], q *being the height to which the fluid in the tube rises above the level* [9378]; therefore we have

[9431] $$z = \sqrt{ql.\tan g.\tfrac{1}{2}\theta'}. \qquad \left[\text{Extreme value of } z\text{, when the distance of the planes is infinite.}\right]$$

If we suppose θ' *to be equal to a right angle, which appears to be the case for water*

[9432] [*and alcohol*] *relative to glass,* l being a millimetre, we shall have $q = 6^{\text{mi.}},784$ [9364], which gives for the height to which the water is elevated, by a glass plane dipped vertically into a vessel filled with that fluid, $2^{\text{mi.}},6046$ [9425g]. This ought to be rather less by experiment, because the point which we take for the

[9433] origin of the curve can become sensible only by being at a distance from the sides of the tube; it must therefore be a little below the point A, fig. 116, page 713. We may observe that, *by the extreme point A, we always mean the point*

[9434] *nearest to the tube, but situated without its sphere of sensible activity; and as this point is at an insensible distance from the tube, it may be supposed to touch the tube.*

In case the two planes are at an infinite distance, the differential equation of the curve becomes *

[9425i] as is given by M. Poisson, in page 184 of his *Nouvelle Théorie, &c.*, changing z into l, and θ' into μ, to conform to his notation. When the fluid wholly moistens the plate, as is

[9425k] the case with water, we shall have $\theta' = 90^d$ [9362], and $\sqrt{2}.\sin.\tfrac{1}{2}\theta' = 1$; whence $z = \alpha$, which represents the elevation of the fluid near the sides of a plate which is perfectly moistened by it. The same is found in another way in [10281a].

* (4169) When the planes are at an infinite distance, and $b' = \infty$ [9428], the

[9435a] expression [9420] becomes $Z = 1 - \alpha z^2$; whence

$$dy=\frac{(1-\alpha z^2)\,.dz}{z.\sqrt{\alpha}.\sqrt{2-\alpha^2}}.$$ [Equation of the surface, when the planes are at an infinite distance.] [9435]

Two planes at an infinite distance from each other.

Thus in fig. 120, PQ being the line of level of the fluid, we have $PV=z$, and by making $VN=y'$, we shall have $dy'=-dy$; the differential equation [9436] of the curve ANQ, which the fluid forms near the plane AP, will therefore be

$$dy'=-\frac{(1-\alpha z^2)\,.dz}{z.\sqrt{2\alpha}.\sqrt{1-\frac{1}{2}\alpha z^2}},$$ [Equation of the surface near one single plane.] [9437]

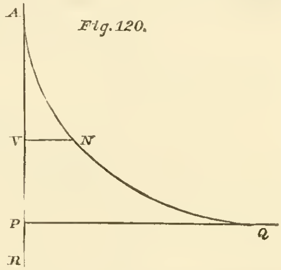

Fig. 120.

which is easily integrated [10208].

Elevation of a fluid between two very near parallel planes.

If the distance of the two planes from each other is very small, the equation

$$\frac{b'-z}{b'}-\alpha z^2=Z, \quad [9420], \quad \text{or}$$ [9438]

$$\frac{z}{b'}=1-Z-\frac{\alpha b'^2.z^2}{b'^2},$$ [9439]

gives, by using the formula* [629],

$$\frac{z}{b'}=1-Z-\alpha b'^2.(1-Z)^2+2\alpha^2.b'^4.(1-Z)^3-5\alpha^3.b'^6.(1-Z)^4+\&c.;$$ [9440]

$$\sqrt{1-Z^2}=\sqrt{2\alpha z^2-\alpha^2 z^4}=z.\sqrt{2\alpha}.\sqrt{1-\tfrac{1}{2}\alpha z^2};$$ [9435b]

hence [9421] becomes as in [9435]. Moreover, by continuing the line $MN=y$, fig. 116, page 713, to meet the vertical line AV in V, we have $VN=AP-NM$, or $y'=AP-y$ [9435c] [9413′, 9436]; and as AP is constant, while the point N varies, the differential will be $dy'=-dy$, as in [9436]. Substituting this in [9435], we get [9437], which is here left by the author without integration, but it is done in [10181a—v], by means of logarithmic [9435d] functions, and also in [10208].

* (4170) If we put $\frac{z}{b'}=u'$ in [9439], it becomes $u'=1-Z-\alpha b'^2 u'^2$, which may [9440a] be solved by means of the formula [629a, 629]; but to avoid any confusion in the symbols, we shall suppose the letters u, x, t, z, α, Z, to be accented in [629a, 629]; then, comparing the expression of u' [9440a], with the general form $x'=\varphi(t'+\alpha' z')$ [629a], we get $x'=u'$, [9440b] $\varphi=1$, $t'=1-Z$, $\alpha'=-\alpha b'^2$, $z'=u'^2$; and the expression $u'=\psi(x')=\psi\{\varphi(t'+\alpha' z')\}$ [9440c] [629a], becomes simply $u'=t'+\alpha' z'$, or $\psi\varphi=1$; moreover, u being the value of u', when $\alpha'=0$, we have $\mathrm{u}=t'=1-Z$; hence $\left(\frac{d\mathrm{u}}{dt'}\right)=1$. Substituting this in [629], we get [9440d]

$$u'=\mathrm{u}+\alpha' Z'.\frac{d\mathrm{u}}{dt'}+\frac{\alpha'^2}{1.2}.\frac{d.\left(Z'^2.\frac{d\mathrm{u}}{dt'}\right)}{dt'}+\&c.=\mathrm{u}+\alpha' Z'+\frac{\alpha'^2}{1.2}.\frac{d.Z'^2}{dt'}+\&c.$$ [9440e]

Now substituting $\alpha'=-\alpha b'^2$, $Z'=\mathrm{u}^2$ [9440c], we get

whence we deduce

[9441] $$dz = -b'dZ.\{1 - 2\alpha b'^2.(1-Z) + \&c.\};$$

consequently

[9442] $$\frac{dy}{b'} = -\frac{ZdZ.\{1-2\alpha b'^2.(1-Z)+\&c.\}}{\sqrt{1-Z^2}}.$$

[9443] Putting $Z = \cos.\theta$, we shall have *

[9444] $$\frac{dy}{b'} = d\theta.\cos.\theta.\{1 - 2\alpha b'^2.(1-\cos.\theta) + \&c.\};$$

whence, by integration,

[9445] $$\frac{y}{b'} = \sin.\theta - \alpha b'^2.(2\sin.\theta - \theta - \tfrac{1}{2}.\sin.2\theta);$$

[9440f] $$u' = u - \alpha b'^2.u^2 + \frac{\alpha^2.b'^4}{1.2}.\frac{d.u^4}{dt'} - \frac{\alpha^3.b'^6}{1.2.3}.\frac{d^2.u^6}{dt'^2} + \&c.,$$

in which we must put $u = t'$ [9440d], and we shall get

$$u' = t' - \alpha b'^2.t'^2 + \frac{\alpha^2.b'^4}{1.2}.\frac{d.t'^4}{dt'} - \&c. = t' - \alpha b'^2.t'^2 + 2\alpha^2 b'^4.t'^3 - 5\alpha^3 b'^6.t'^4 + \&c.$$

Lastly, substituting $u' = \frac{z}{b'}$ [9440a], and $t' = 1 - Z$ [9440c], it becomes as in [9440];

[9440g] multiplying this by b', and taking its differential, we obtain [9441]. Multiplying [9441] by $\frac{Z}{b'.\sqrt{1-Z^2}}$, we get the expression of $\frac{dy}{b'}$ [9421], as in [9442].

* (4171) The assumed value of Z [9443] gives $\sqrt{1-Z^2} = \sin.\theta$, whose differential

[9443a] is $-\frac{ZdZ}{\sqrt{1-Z^2}} = d\theta.\cos.\theta$; also $1 - Z = 1 - \cos.\theta$; substituting these in [9442], we get [9444], and by using $2\cos.^2\theta = 1 + \cos.2\theta$ [6] Int., it becomes

[9443b] $$\frac{dy}{b'} = d\theta.\cos.\theta - \alpha b'^2.(2d\theta.\cos.\theta - 2d\theta.\cos.^2\theta) = d\theta.\cos.\theta - \alpha b'^2.(2d\theta.\cos.\theta - d\theta - d\theta.\cos.2\theta).$$

Its integral gives [9445]; and by substituting $\frac{1}{2}\sin.2\theta = \sin.\theta.\cos.\theta$ [31] Int., it becomes

[9443b'] $$\frac{y}{b'} = \sin.\theta.\left\{1 - 2\alpha b'^2.\left(1 - \frac{\theta}{2\sin.\theta} - \tfrac{1}{2}\cos.\theta\right)\right\}.$$

Now substituting $y = l$, $\theta = \theta'$ [9446], and dividing by l, we get

[9443c] $$\frac{1}{b'} = \frac{\sin.\theta'}{l}.\left\{1 - 2\alpha b'^2.\left(1 - \frac{\theta'}{2\sin.\theta'} - \tfrac{1}{2}\cos.\theta'\right)\right\}.$$

If we neglect terms of the order α, we get very nearly $\frac{1}{b'} = \frac{\sin.\theta'}{l}$; using this and α

[9443d] [9446], we get $\alpha b'^2 = \frac{\sin.\theta'}{lq}.\left(\frac{l}{\sin.\theta'}\right)^2 = \frac{l}{q.\sin.\theta'}$; substituting this in [9443c], we obtain

[9443e] [9447]; observing that l is the distance of the lowest point of the surface from either of the planes, so that the distance of the planes is $2l$.

therefore, by supposing the extreme value of y to be l, that of θ to be θ', and using the value of $\alpha=\frac{\sin.\theta'}{lq}$ [9425c], we shall have [9446]

$$\frac{1}{b'}=\frac{\sin.\theta'}{l}\cdot\left\{1-\frac{2l}{q.\sin.\theta'}\cdot\left(1-\frac{\theta'}{2\sin.\theta'}-\tfrac{1}{2}.\cos.\theta'\right)\right\};$$ [9447]

and as l is very small relative to q, when the planes are very nigh to each other, we shall have nearly

$$\frac{1}{b'}=\frac{\sin.\theta'}{l}.$$ [9448]

In the case of θ' equal to a right angle, the fraction

$$\frac{2l}{q.\sin.\theta'}\cdot\left(1-\frac{\theta'}{2\sin.\theta'}-\tfrac{1}{2}\cos.\theta'\right)\quad[9447]$$ [9449]

becomes

$$\frac{2l}{q}\cdot(1-\tfrac{1}{4}\pi).$$ [9450]

If $l=1^{\text{mi.}}$, we shall have for water $q=6^{\text{mi.}},784$ [9432], and this fraction will become $\frac{2}{6,784}\cdot(1-\frac{1}{4}\pi)$, or $\frac{1}{15,8}$; it may therefore be neglected in comparison with unity. [9451]

The preceding value of $\frac{1}{b'}$ [9447] gives, for *the elevation* q' [9406] *of a fluid between two vertical and parallel planes, distant from each other by* $2l$, the following expression,* [9452]

Elevation between two vertical planes.

$$q'=\frac{H}{g}\cdot\frac{\sin.\theta'}{2l}\cdot\left\{1-\frac{2l}{q.\sin.\theta'}\cdot\left(1-\frac{\theta'}{2\sin.\theta'}-\tfrac{1}{2}\cos.\theta'\right)\right\},$$ [Elevation q' of a fluid between two vertical and parallel planes whose distance is $2l$.] [9453]

which is also the expression of the depression of the fluid below its level,

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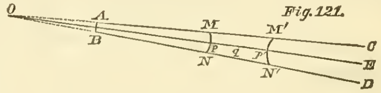
* (4172) Putting $b=\infty$ [9414], in the expression of the capillary action on a concave surface [9315b], it becomes $K-\frac{H}{2b'}$; adding this to gq', the sum must be equal to K, as is evident from the reasoning in [9355a, &c.]; hence we have $K-\frac{H}{2b'}+gq'=K$, or $q'=\frac{H}{g}\cdot\frac{1}{2b'}$, and by substituting $\frac{1}{b'}$ [9447], it becomes as in [9453]; which is nearly equal to $\frac{H}{g}\cdot\frac{\sin.\theta'}{2l}$ [9454], when l is very small. Moreover it is evident, from the reasoning in [9369, &c.], that this same expression [9453] will also denote the depression of the fluid below its level, when the surface of the fluid is concave; the effect being merely to change the signs of q, θ', q' [9353, 9424, 9452], as in [9352d, &c.], which does not alter the value of q' [9453], as is evident by inspection. [9452a] [9452b] [9452c]

between the same planes, when the interior surface of the fluid, instead of being concave, is convex; and when the distance of the planes $2l$ is very small, it
[9454] *becomes very nearly* $q' = \frac{H}{g} \cdot \frac{\sin.\theta'}{2l}$, *being inversely as that distance $2l$.*

If the two parallel planes, instead of being vertical, are inclined to the horizon, the surface of the interior fluid, and its position relative to the planes which
[9455] *include it, are very nearly the same as if the planes were vertical, as we have seen in* [9376] *relative to inclined tubes. The vertical height q' of the fluid above the level is therefore the same, whatever be the inclination of the planes.*

9. *We shall now consider a small column of fluid, contained within a conical capillary tube*, open at both ends. We shall suppose $ABCD$, fig. 121, to be the tube, and $MM'N'N$ the column of fluid; and we shall at first suppose the axis OE *of the tube to*



[9456] *be horizontal*, O being the summit of the cone, or the intersection of the lines CA, DB, if they were continued to O; we shall also suppose that the surfaces of the fluid MpN, $M'p'N'$, are concave. It is evident that, the tube being narrower at p than at p', the radius of curvature of its surface will be less in the first point, than in the second. *Representing these radii by b, b', respectively*, we shall have for the action of the fluid at p, upon an infinitely
[9457] narrow canal pp', in the direction pp', the expression $K - \frac{H}{b}$ [9254]; and
[9458] in p' this action will be $K - \frac{H}{b'}$, in the opposite direction $p'p$; and as b' is greater than b, this action will be greater in p' than in p; consequently the fluid, contained in the canal, will tend to move towards the summit O of the cone. The contrary will take place, if the surface of the fluid be convex;
[9459] for then these forces will be respectively $K + \frac{H}{b}$, and $K + \frac{H}{b'}$ [9276]; consequently the action of the fluid upon the canal will be greater at p than at
[9460] p'; therefore the fluid tends to move from p towards p'.

To determine the radii of curvature b, b', we shall put

[9461] $a =$ the line Oq, or the distance from the vertex O of the cone to the middle point p of the drop, or to the middle of the line pp';

[9461'] $\mathrm{a} =$ the half-width of the drop pq or qp';

[9462] $\varpi =$ the very small angle MOp, being the angle formed by the side of the cone and its axis;

[9463] $\theta' =$ the complement of the inclination of the extreme side of the arc pM to the side of the cone OM [9346].

Then, if we suppose the curves MpN, $M'p'N'$, to be circular, it will be easy to prove that we shall have *

$$b = \frac{(a - \mathrm{a}).\sin.\varpi}{\sin.\theta' + \sin.\varpi}, \qquad b' = \frac{(a + \mathrm{a}).\sin.\varpi}{\sin.\theta' - \sin.\varpi}, \qquad [9464]$$

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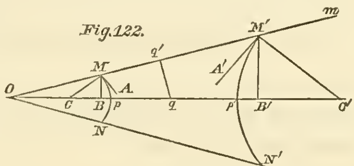
* (4173) We have changed tang.ϖ into sin.ϖ in all the formulas [9464—9482], to correct a mistake of the author, in the original work, in the values of b, b' [9464]. The correctness of this change is easily perceived by referring to fig. 122, which is similar to fig. 121, the angle at O being enlarged so as to make the lines more distinct. In this figure, C is the centre of the surface MpN; MA is perpendicular to CM, MB perpendicular to Op. Then $OMC = \theta'$, $MOC = \varpi$, $MCB = \theta' + \varpi$; and as $MC = b$, we shall have [9463a] [9463b]



$$MB = MC.\sin.MCB = b.\sin.(\theta' + \varpi); \quad CB = b.\cos.(\theta' + \varpi); \qquad [9463c]$$

$$Bp = Cp - CB = b - b.\cos.(\theta' + \varpi).$$

Now $OB = Oq - pq - Bp = a - \mathrm{a} - \{b - b.\cos.(\theta' + \varpi)\}$; multiplying it by tang.MOB, or tang.ϖ, we get $MB = (a - \mathrm{a}).\mathrm{tang}.\varpi - b.\mathrm{tang}.\varpi + b.\mathrm{tang}.\varpi.\cos.(\theta' + \varpi)$. Putting [9463d] this equal to the value of MB [9463c], we get $b = \frac{(a - \mathrm{a}).\mathrm{tang}.\varpi}{\sin.(\theta' + \varpi) - \mathrm{tang}.\varpi.\cos.(\theta' + \varpi) + \mathrm{tang}.\varpi}$. [9463e] Multiplying the numerator and denominator by cos.ϖ, and substituting in the denominator $\cos.\varpi.\sin.(\theta' + \varpi) - \sin.\varpi.\cos.(\theta' + \varpi) = \sin.\{(\theta' + \varpi) - \varpi\} = \sin.\theta'$, it becomes, as in [9464], $b = \frac{(a - \mathrm{a}).\sin.\varpi}{\sin.\theta' + \sin.\varpi}$; but differs from the original value of b [9464] by changing [9463f] tang.ϖ into sin.ϖ.

We may find in the same manner the value of $b' = C'M'$, fig. 122, corresponding to the surface $M'p'N'$, the tangent being $M'A'$, and the perpendicular $M'B'$. In this case, we have $C'M'm = \theta'$, $C'OM' = \varpi$, $M'C'B' = \theta' - \varpi$; hence $M'B' = b'.\sin.(\theta' - \varpi)$, [9463g] $C'B' = b'.\cos.(\theta' - \varpi)$, $p'B' = b' - b'.\cos.(\theta' - \varpi)$; adding this to $Op' = Oq + qp' = a + \mathrm{a}$, we get $OB' = a + \mathrm{a} + b' - b'.\cos.(\theta' - \varpi)$. Multiplying this by tang.ϖ, we get a second expression of $M'B' = (a + \mathrm{a}).\mathrm{tang}.\varpi + b'.\mathrm{tang}.\varpi - b'.\mathrm{tang}.\varpi.\cos.(\theta' - \varpi)$. From these two expressions of $M'B'$, we get $b' = \frac{(a + \mathrm{a}).\mathrm{tang}.\varpi}{\sin.(\theta' - \varpi) + \mathrm{tang}.\varpi.\cos.(\theta' - \varpi) - \mathrm{tang}.\varpi}$. Multiplying [9463h] the numerator and denominator by cos.ϖ, and substituting in the denominator

$$\sin.(\theta' - \varpi).\cos.\varpi + \cos.(\theta' - \varpi).\sin\,\varpi = \sin.\{(\theta' - \varpi) + \varpi\} = \sin.\theta',$$

we get, as in [9464], $b' = \frac{(a + \mathrm{a}).\sin.\varpi}{\sin.\theta' - \sin.\varpi}$; differing from that in the original work, in [9464], [9463i] by having in it sin.ϖ, instead of tang.ϖ.

which gives *

[9465] $$\frac{H}{b}-\frac{H}{b'}=\frac{H.\sin.\theta'}{\sin.\varpi}\cdot\left\{\frac{2\mathrm{a}}{a^2}+\frac{2\mathrm{a}^3}{a^4}+\&c.\right\}+\frac{2H}{a}+\frac{2H.\mathrm{a}^2}{a^3}+\&c.$$

[9466] If we now elevate the point A, so as to incline the axis OE to the horizon by
V. an angle V, the weight of the column pp' will be † $2g\mathrm{a}.\sin.V$, g being, as
[9467] in [9316], the force of gravity; and when the column remains suspended in

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* (4174) Dividing H by the value of b [9464], we get $\frac{H}{b}=\frac{H.\sin.\theta'}{(a-\mathrm{a}).\sin.\varpi}+\frac{H}{a-\mathrm{a}}$, which can be developed according to the powers of $\frac{\mathrm{a}}{a}$, as in [9465a]; in like manner, by dividing H by the value of b' [9464], and developing, we get [9465b], and their difference is as in [9465c];

[9465a] $$\frac{H}{b}=\frac{H.\sin.\theta'}{\sin.\varpi}\cdot\left\{\frac{1}{a}+\frac{\mathrm{a}}{a^2}+\frac{\mathrm{a}^2}{a^3}+\&c.\right\}+\frac{H}{a}+\frac{H.\mathrm{a}}{a^2}+\frac{H.\mathrm{a}^2}{a^3}+\&c.;$$

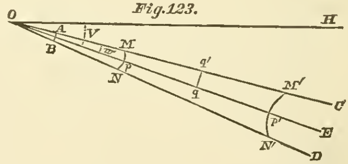
[9465b] $$\frac{H}{b'}=\frac{H.\sin.\theta'}{\sin.\varpi}\cdot\left\{\frac{1}{a}-\frac{\mathrm{a}}{a^2}+\frac{\mathrm{a}^2}{a^3}-\&c.\right\}-\frac{H}{a}+\frac{H.\mathrm{a}}{a^2}-\frac{H.\mathrm{a}^2}{a^3}+\&c.;$$

[9465c] $$\frac{H}{b}-\frac{H}{b'}=\frac{H.\sin.\theta'}{\sin.\varpi}\cdot\left\{\frac{2\mathrm{a}}{a^2}+\frac{2\mathrm{a}^3}{a^4}+\&c.\right\}+\frac{2H}{a}+\frac{2H.\mathrm{a}^2}{a^3}+\&c.$$

[9465d] This last expression agrees with that in [9465], *where, for greater accuracy, we have changed* tang.ϖ *into* sin.ϖ, *as in* [9463f, i]; and we have also changed La Place's symbol α into a, to avoid confusion, as α is used in another sense in [9328, &c.].

[9467a] † (4175) We shall suppose the axis of the tube OE, fig. 123, to be inclined to the
[9467b] horizontal plane OH, by the angle $HOE=V$, instead of being exactly horizontal, as in [9456]; and shall then take into consideration the fluid in a cylindrical canal, formed about the axis
[9467c] $pp'=2\mathrm{a}$, its base being extremely small, and represented by unity. In this case, the volume of the cylinder is represented by $2\mathrm{a}$, its gravity $2g\mathrm{a}$ [9316], and its pressure in the direction
[9467d] pp' equal to $2g\mathrm{a}.\sin.V$. This is to be added to the capillary action in p, in the same direction pp', namely $K-\frac{H}{b}$ [9457]; so that the whole column presses in that direction
[9467e] with a force which is represented by $K-\frac{H}{b}+2g\mathrm{a}.\sin.V$; and when the fluid is in equilibrium, this force must be equal, and contrary to the capillary action at p', namely
[9467f] $K-\frac{H}{b'}$ [9458]; hence $K-\frac{H}{b}+2g\mathrm{a}.\sin.V=K-\frac{H}{b'}$; consequently $2g\mathrm{a}.\sin.V=\frac{H}{b}-\frac{H}{b'}$. Substituting the value of the second member of this expression [9465], and neglecting terms of the order a^2, we get [9469].

Fig. 123.



equilibrium, by means of this action, this weight must balance the force $\frac{H}{b}-\frac{H}{b'}$, [9468]
by which it is urged towards the point O in consequence of the attraction of the fluid; we shall therefore have, by neglecting the insensible terms,

$$2ga.\sin.V=\frac{H.\sin.\theta'}{\sin.\varpi}.\frac{2a}{a^2}+\frac{2H}{a}, \quad \text{or} \quad \sin.V=\frac{H}{g}.\frac{\sin.\theta'}{\sin.\varpi}.\frac{1}{a^2}+\frac{H}{g}.\frac{1}{aa}. \qquad [9469]$$

We shall put l *for the height to which the fluid is elevated, in a cylindrical* [9470]
tube, whose interior semi-diameter or radius is $a.\sin.\varpi$, which is equal to the [9470′]
nearest distance of the side of the tube from the centre of the drop q; we shall have, by using [9360],*

$$gl=\frac{H.\sin.\theta'}{a.\sin.\varpi}, \quad \text{or} \quad l=\frac{H.\sin.\theta'}{ga.\sin.\varpi}; \qquad [9471]$$

therefore we shall have

$$\sin.V=\frac{l}{a}+\frac{l.\sin.\varpi}{a.\sin.\theta'}. \qquad [9472]$$

[Sine of the inclination V of the axis of the cone to the horizon, when in equilibrium.]

The term $\frac{l.\sin.\varpi}{a.\sin.\theta'}$ may be put under the form $\frac{l}{a}.\frac{a.\sin.\varpi}{a.\sin.\theta'}$; it will be very small, relative to the term $\frac{l}{a}$, if $a.\sin.\varpi$ [9470′] be very small relative to a [9473]
[9461′], or *if the length of the little column be much greater than the width of the cone at the point* q; *in this case, we shall have very nearly*, by neglecting [9473′]
that term in [9472],

$$\sin.V=\frac{l}{a}. \qquad [9474]$$

[Approximate value of the sine of the inclination V of the axis of the cone to the horizon.]

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* (4176) If we draw qq' perpendicular to OM', fig. 123, it will represent the radius of the tube [9470′]; which will therefore be expressed by $Oq.\sin.\varpi$, or $a.\sin.\varpi$. If [9471a]
we put this for l in [9360], we shall obtain q, the elevation of the fluid in a cylindrical tube whose radius is $a.\sin.\varpi$; and the value of q, which is the same as l, in the present notation [9470], will be $l=\frac{H.\sin.\theta'}{ga.\sin.\varpi}$; hence we obtain gl [9471], or $H=\frac{gla.\sin.\varpi}{\sin.\theta'}$. [9471b]
Substituting this in $\sin.V$ [9469], we get [9472]. We may remark that we have supposed the line qq' to be drawn perpendicular to the side of the cone OC, instead of being [9471c]
perpendicular to OE, as in the original work; this is done to avoid the use of tang.ϖ, which is changed into sin.ϖ, in [9463f]; terms of the order of the difference of these quantities [9471d]
being omitted in [9469, &c.]; so that this cannot be considered as making any essential alteration.

[9475] *As l [9471] is inversely as a, $\frac{l}{a}$ will be in the inverse ratio of* a^2; and as V is a small angle, it follows that this angle is then very nearly in the inverse*
[9475′] *ratio of the square of the distance of the middle of the drop from the summit of the cone.*

The term $\frac{l.\sin.\varpi}{\text{a}.\sin.\theta'}$ [9472] depends upon the difference in the number of degrees
[9476] contained in the arcs† MpN, $M'p'N'$; and this difference arises from the circumstance that the concavity of one of the arcs is turned towards the summit of the cone O, and the convexity of the other. The term depending upon this difference may therefore be neglected without any sensible error, when the
[9477] length of the column is much greater than its thickness, or the diameter of the cone at the point q; and then we may suppose that the two curves MpN and $M'p'N'$ are similar.

We have supposed the two surfaces of the fluid column to be spherical; but
[9477′] this supposition is not accurate, and we see in [9351, &c.], that, on account of the action of gravity, the value of $\frac{1}{b}$ will be decreased by a small term of the

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[9475a] * (4177) Dividing the value of l [9471] by a, we get $\frac{l}{a}=\frac{H.\sin.\theta'}{g.\sin.\varpi}\cdot\frac{1}{a^2}$, being inversely as a^2, as in [9475]; substituting this in [9474], we get $\sin.V=\frac{H.\sin.\theta'}{g.\sin.\varpi}\cdot\frac{1}{a^2}$; and when
[9475b] V is small, it becomes very nearly $V=\frac{H.\sin.\theta'}{g.\sin.\varpi}\cdot\frac{1}{a^2}$, as in [9475′].

† (4178) We have in fig. 122, page 751, the angle $MCB=\theta'+\varpi$ [9463b], the angle $M'C'B'=\theta'-\varpi$ [9463g]; hence $MCB-M'C'B'=2\varpi$; and as the quantity ϖ occurs
[9476a] in the term $\frac{l.\sin.\varpi}{\text{a}.\sin.\theta'}$, it would seem, at the first view, that, when ϖ is small, this term must be nearly proportional to the difference 2ϖ of the angles MCB, $M'C'B'$; but we ought to observe that, if we multiply the value l [9471] by $\frac{\sin.\varpi}{\text{a}.\sin.\theta'}$, we shall get

[9476b]
$$\frac{l.\sin.\varpi}{\text{a}.\sin.\theta'}=\frac{H}{g}\cdot\frac{1}{a}\cdot\frac{1}{\text{a}},$$

[9476c] which depends on the constant factor $\frac{H}{g}$, and on the lines $Oq=a$ [9461], $pq=\text{a}$ [9461′], without containing ϖ explicitly, which seems to be contrary to what the author supposes in [9476].

form * $\frac{1}{b}.Q.\frac{g}{H}.b^2$; Q being a coefficient independent of b. Likewise $\frac{1}{b'}$ [9478]
will be decreased by the term $\frac{1}{b'}.Q.\frac{g}{H}.b'^2$; the difference $\frac{H}{b}-\frac{H}{b'}$ will [9479]
therefore be increased by $Qg.(b'-b)$, or very nearly $\frac{2Qg.\mathrm{a}.\sin.\varpi}{\sin.\theta'}$. The
value of $\sin.V$ will therefore be increased by the term $\frac{Q.\sin.\varpi}{\sin.\theta'}$. Without [9480]
determining Q, we see that it must be a small number, and there is reason to
believe that it is less than unity, as in the expression of $\frac{1}{b}$ [9478b], where it is
only $\frac{1}{3}$, when θ' is a right angle.† The value of V will therefore be increased [9481]
from this cause by a very small angle which is much less than ϖ; so that we [9482]
may, without any sensible error, neglect this increment.

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* (4179) Neglecting terms of the order α, we get from [9351] $\frac{1}{b}=\frac{\sin.\theta'}{l}$, or
$l=b.\sin.\theta'$; substituting this in the part of [9351] containing α, and putting for brevity [9478a]
$Q=1-\frac{2}{3}.\frac{(1-\cos.^3\theta')}{\sin.^2\theta}$, we get $\frac{1}{b}=\frac{\sin.\theta'}{l}-\frac{1}{b}.Q.\alpha b^2$, as in [9478], observing that [9478b]
$\alpha=\frac{g}{H}$ [9328]. In like manner $\frac{1}{b'}=\frac{\sin.\theta'}{l'}-\frac{1}{b'}.Q.\alpha b'^2$. Hence the terms depending on [9478c]
Q will produce in $\frac{H}{b}-\frac{H}{b'}$ the quantity $H.\left\{\frac{1}{b'}.Q.\alpha b'^2-\frac{1}{b}.Q.\alpha b^2\right\}=H\alpha.Q.(b'-b)$; [9478d]
and by substituting $H\alpha=g$ [9478c], it becomes $Qg.(b'-b)$, as in [9480]. But, $\sin.\varpi$
being small in comparison with $\sin.\theta'$, we have nearly, from [9464], $b=(a-\mathrm{a}).\frac{\sin.\varpi}{\sin.\theta'}$,
$b'=(a+\mathrm{a}).\frac{\sin.\varpi}{\sin.\theta'}$, $b'-b=2\mathrm{a}.\frac{\sin.\varpi}{\sin.\theta'}$; consequently the preceding expression $Qg.(b'-b)$ [9478e]
becomes very nearly $\frac{2Qg\mathrm{a}.\sin.\varpi}{\sin.\theta'}$, as in [9480]. This increment of $\frac{H}{b}-\frac{H}{b'}$ ought to
be added to the second members of [9465, 9469]; and as the last of these expressions
represents the value of $2g\mathrm{a}.\sin.V$ [9467f], its increment will be $\frac{2Qg\mathrm{a}.\sin.\varpi}{\sin.\theta'}$; dividing [9478f]
this by $2g\mathrm{a}$, we get the increment of $\sin.V$, equal to $\frac{Q.\sin.\varpi}{\sin.\theta'}$, as in [9480]. We may [9478g]
finally observe, that, when the surface of the fluid in a tube is spherical, we have $u=b.\sin.\theta$
[9345b]; and at the side of the tube where $u=l$, $\theta=\theta'$, it becomes $l=b.\sin.\theta'$, or [9478h]
$\frac{1}{b}=\frac{\sin.\theta'}{l}$, which differs from the general value of $\frac{1}{b}$ [9478b] by the terms depending on
Q; therefore the terms depending on Q arise from the variation of the figure of the surface [9478i]
from a spherical form.

† (4180) When $\theta'=90^d$, the value of Q [9478b] evidently becomes $Q=\frac{1}{3}$, and
then the increment of $\sin.V$ [9478g] is $\frac{1}{3}\sin.\varpi$, or $\frac{1}{3}\varpi$ nearly. [9481a]

[9483] 10. We shall now consider, in the same manner, a drop of fluid between two planes, which meet each other at their border, or line of intersection, supposing this line to be in a horizontal position. The drop will assume, between these planes, a form nearly circular, and similar to a pulley. [9483′] *We shall first determine the figure it would have between two horizontal planes which are very near to each other.* [9484] The surface of the drop will be that of a solid of revolution about a vertical axis passing through its centre of gravity. Therefore, by taking this [9485] point for the origin of the vertical ordinates z, and of the horizontal ordinates u, we shall have, as in [9315, &c.],*

Figure of a drop of fluid between two parallel planes.

[9486]
$$\frac{\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}+\frac{2gz}{H}=\frac{1}{b'}-\frac{1}{b},$$

[Equation of the figure of a drop between two parallel planes.]

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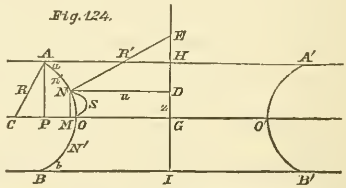
* (4181) Let $AA'O'B'BOA$, fig. 124, [9485a] be a section of the drop, by a vertical plane passing through its centre of gravity G; AOB, $A'O'B'$, the surface of the drop; AA', BB', [9485b] the planes. The coordinates of any point N of the surface being denoted by $GD=z$, $DN=u$, C is the centre of curvature of the point O of the surface; so that $CO=b$ [9488], and it is [9485c] evident that $GO=b'$ [9487]. The radius of curvature of the arc ANO, at the point [9485d] N, is R; the other radius of curvature of the arc passing through the same point N, [9485e] perpendicular to the plane of the figure, is R'. In the calculation of the equation [9315], the external surface is supposed to be concave; and to conform to the present case, we must change the signs of R', b', as in [9301a, b], and then [9315] becomes



[9485f]
$$K+\frac{H}{2}\cdot\left(\frac{1}{R'}-\frac{1}{R}\right)+gz=K+\frac{H}{2}\cdot\left(\frac{1}{b'}-\frac{1}{b}\right).$$

Rejecting K from both sides of the equation, and then multiplying by $\frac{2}{H}$, we get

[9485g]
$$\frac{1}{R'}-\frac{1}{R}+\frac{2gz}{H}=\frac{1}{b'}-\frac{1}{b}.$$

Now we have

[9485g′]
$$\frac{1}{R'}=\frac{\frac{1}{u}\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}\ [9326']; \text{ also, in } [9327e],\ \frac{1}{R}=-\frac{\frac{1}{du}\cdot d\cdot\frac{dz}{du}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}};$$

observing that the sign of this expression is negative, being different from that in [9326]. [9485h] The reason of this difference in the signs is, that, in computing $\frac{1}{R}$ [9326], the curve ON

$b' =$ the radius of the circumference formed by the section of the drop with a horizontal plane drawn through its centre of gravity; [9487]

$b =$ the radius of curvature of the section of the surface of the drop by a vertical plane passing through the centre of gravity, at a point where z is nothing. [9488]

We have given to the fractions $\frac{1}{b}, \frac{1}{b'}$, contrary signs in [9486], because the surface is concave towards the centre of gravity in a horizontal direction, and convex towards the centre of gravity in a vertical direction. [9488']

If we neglect the action of gravity g, as we have done in [9333, 9334], the equation [9486] will give, by multiplying by udu, and then integrating,*

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fig. 116, page 713, is supposed to pass through the origin of the coordinates O, with its concavity upwards, or in the direction of the positive values of z; being the same as if a curve was to pass from G to N in fig. 124, with its concavity upwards; but in the present case, the curve ONA has its *convexity* upwards; hence it is evident that the differential of $\frac{dz}{du}$, on which the curvature depends, must have a different sign in fig. 124 from that in fig. 116. Substituting the values [9485g'], in [9485g], and reducing the two first terms to the same denominator, we get the differential equation of the surface [9486], being under the same form as it is given by M. Poisson, in page 202, &c. of his *Nouvelle Théorie, &c.* When the point N falls below the plane OGO', as at N', z will become negative, and equal to $-z_{,}$, $z_{,}$ being considered as positive, and the equation [9485g] becomes [9485i] [9485k] [9485l]

$$\frac{1}{R'}-\frac{1}{R}-\frac{2gz_{,}}{H}=\frac{1}{b'}-\frac{1}{b}; \qquad [9485m]$$

and as the *signs* of R', R, remain the same at N', N, the arc AOB being supposed concave to C, it follows that the radical $\sqrt{1+\frac{dz^2}{du^2}}$, in the values of $\frac{1}{R'}, \frac{1}{R}$ [9485g], must also change its sign, when we change z into $-z_{,}$; hence it will follow that the only difference in the equation [9485g, or 9486], when applied to the lower part of the curve at N', will be to change the sign of the term $+\frac{2gz}{H}$ [9486], so as to make it $-\frac{2gz}{H}$, and then read $z_{,}$ for z. [9485n] [9485o] [9485p]

* (4182) Neglecting the term depending on $\frac{g}{H}$ in [9486], multiplying the rest of the expression by udu, and then integrating, we obtain the equation [9489], the process being similar to that in [9329]. Its first member may be put under the form [9489a]

$$\frac{u}{\sqrt{\frac{du^2}{dz^2}+1}}; \qquad [9489b]$$

[9489]
$$\frac{u\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\text{constant}-\frac{u^2}{2b}+\frac{u^2}{2b'}.$$

To determine the constant quantity, we shall observe that, when $z=0$, we shall have $\frac{dz}{du}=\infty$, and $u=b'$ [9487]; hence we deduce

[9490]
$$\text{constant}=\frac{b'^2}{2b}+\frac{b'^2}{2b'};$$

consequently

[9491]
$$\frac{u\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\frac{b'^2-u^2}{2b}+\frac{b'^2+u^2}{2b'}.$$

We shall now put

[9492]
$$U=\frac{b'^2-u^2}{2b}+\frac{b'^2+u^2}{2b'};$$

and then we shall have *

[9493]
$$dz=\frac{U.du}{\sqrt{u^2-U^2}},\quad\text{or}\quad\frac{dz}{du}=\frac{U}{\sqrt{u^2-U^2}}.$$

[9493′] The integral of this differential equation depends on the rectification of the conic sections; and, by taking the integral, we get z in terms of u. We shall put

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and at the point O, fig. 124, page 756, where $u=b'$, and $\frac{du}{dz}=0$, it becomes equal to

[9489c] b'; and then the equation [9489] changes into $b'=\text{constant}-\frac{b'^2}{2b}+\frac{b'^2}{2b'}$, which gives the constant quantity as in [9490]; substituting this in [9489], we get [9491].

* (4183) Substituting [9492] in [9491], we get

[9493a]
$$\frac{u\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=U;$$

squaring, and multiplying by du^2+dz^2, we easily obtain dz [9493], which is similar to that in [9393], and like it can be integrated by means of elliptical functions, or by the rectification of the conic sections [9392c].

$2h=$ the distance of the two planes between which the drop is included, so that h is the extreme value of z; [9494]

$f=$ the extreme value of the horizontal radius u; then the integral of the equation [9493] will give h, in terms of f, b, b'; [9495]

$\theta'=$ the complement of the angle which the extreme side of the vertical section of the surface makes with the horizontal plane; this notation being similar to that in [9346]; then we shall have, at this point,* [9496]

$$\frac{dz}{du}=\frac{\cos.\theta'}{\sin.\theta'}; \qquad [9496']$$

therefore

$$\frac{\cos.\theta'}{\sin.\theta'}=\frac{U}{\sqrt{u^2-U^2}}, \quad \text{or} \quad \frac{dz}{du}=\frac{\cos.\theta'}{\sin.\theta'}=\frac{\frac{U}{u}}{\sqrt{1-\frac{U^2}{u^2}}}. \qquad [9497]$$

Substituting f for u, in the second member of the equation [9492], we obtain, as in [9496c], [9497']

$$\cos.\theta'=\frac{b'^2}{2fb}-\frac{f}{2b}+\frac{f}{2b'}+\frac{b'^2}{2fb'}. \qquad [9498]$$

If we substitute in the expression of h in terms of f, b, b' [9495], the value of f deduced from the equation [9498], we shall have an equation between h, b, and b', from which we may deduce b in terms of h and b'.

If we suppose b' to be considerably greater than h, or, in other words, if the thickness of the drop be very small in comparison with its width, as is supposed [9499]

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* (4184) This is similar to [9383f], changing the sign of $L'K'$ from negative to positive, to conform to fig. 124, page 756, and observing that, in this case, θ' represents the angle PAa, formed at A by the line AP and the tangent Aa of the arc at A. Substituting this in the second equation [9493], we get [9497]; and from the second form of this equation we easily deduce $\cos.\theta'=\frac{U}{u}$, $\sin.\theta'=\sqrt{1-\frac{U^2}{u^2}}$. This value of $\cos.\theta'$ gives $U=u.\cos.\theta'$, which is to be substituted in the first member of [9492]; then, putting $u=f$ [9497'], and dividing by f, we get [9498]. We ought to remark, that the author supposes, throughout this calculation, that the two planes are of the same nature, or have the same action on the fluid; or, in other words, that the extreme angle of inclination of the fluid to the plane θ' is the same for both planes; it is, however, very easy to modify his calculation so as to take up the case where these angles differ from each other, as we shall show hereafter, in [9514e, &c.]. [9496a] [9496b] [9496c] [9496d] [9496e]

in the preceding article [9473'], we can determine z by a converging process. For this purpose we shall put

[9500] $$\frac{1}{b}-\frac{1}{b'}=\frac{1}{b''},$$

[9501] $$u=b'+u',$$

[9502] u' being very small in comparison with b'; and we shall then have*

[9503] $$U=\frac{b'}{b''}\,(b''-u')-\frac{u'^2}{2b''}.$$

We shall now put

[9504] $$u'=u''-\frac{u''^2}{2b'},$$

[9505] and we shall have, by neglecting quantities of the order† $\frac{u'^2}{b'}$,

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* (4185) Substituting the value of u [9501] in [9492], we obtain successively, by using [9500], the same value of U as in [9503];

[9503a] $$U=-\frac{(2b'u'+u'^2)}{2b}+\frac{2b'^2+(2bu'+u'^2)}{2b'}=b'-(2b'u'+u'^2)\cdot\tfrac{1}{2}\left(\frac{1}{b}-\frac{1}{b'}\right)=b'-(2b'u'+u'^2)\cdot\frac{1}{2b''}$$

[9503b] $$=\frac{b'}{b''}\cdot(b''-u')-\frac{u'^2}{2b''}.$$

† (4186) Substituting u' [9504] in [9503], we get successively

[9505a] $$U=\frac{b'}{b''}\cdot\left(b''-u''+\frac{u''^2}{2b'}\right)-\frac{1}{2b''}\cdot\left(u''^2-\frac{2u''^3}{2b'}+\frac{u''^4}{4b'^2}\right)=\frac{b'}{b''}\cdot(b''-u'')+\frac{1}{2b''}\cdot\left(\frac{2u''^3}{2b'}-\frac{u''^4}{4b'^2}\right).$$

Now as b' is much greater than b, the quantity $\frac{1}{b''}$ [9500] must be of the same order as $\frac{1}{b}$;

[9505b] that is, b, b'', must be of the same order; consequently b'' *must be small in comparison with* b'. Again, it follows from [9504], that u'' is of the same order as u' or b [9501, &c.].

[9505c] Hence the term $\frac{1}{2b''}\cdot\frac{2u''^3}{2b'}$ is of the order $\frac{1}{2b}\cdot\frac{2b^3}{2b'}$, or $\frac{b^2}{2b}$, which is to the first term of U [9503], namely $\frac{b'}{b''}\cdot b''$ or b', as $\frac{b^2}{2b'^2}$ to 1, and, on account of the smallness of $\frac{b}{b'}$, it

[9505d] may be neglected. Moreover the last term of [9505a] is of the order $\frac{u''}{b'}$, in comparison with that just considered, and it must therefore be smaller, and, like it, may be neglected; and then the last value of U [9505a] will become as in [9506]. We may observe that, when

[9505e] $z=0$, u becomes equal to b' [9487], and u' [9501] *becomes* $u'=0$; *moreover, at this point, the value of* u'' *deduced from* [9504], *is* $u''=0$, *and then we find the assumed*

[9505f] *value of* θ, *at the same point, to be* $\theta=0$, as is evident from the equation [9508']; these values are used in [9510].

$$U=\frac{b'}{b''}\cdot(b''-u''),$$ [9506]

which gives *

$$u^2-U^2=\frac{b'^2}{b''^2}\cdot\left\{2b''\cdot\left(1+\frac{b''}{b'}\right)\cdot u''-u''^2\right\}.$$ [9507]

We shall now put

$$B=b''\cdot\left(1+\frac{b''}{b'}\right),$$ [9508]

$$u''=B\cdot(1-\cos.\theta),$$ [9508′]

and we shall have very nearly †

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* (4187) Substituting the values of u, U [9501, 9506], in u^2-U^2, we get [9507a], or, by reduction, [9507b]; and by using the value of u' [9504], it becomes as in [9507c]. Neglecting the terms containing u''^3, u''^4, on account of their smallness, we obtain the first form of [9507d], which is easily reduced to the second form, as in [9507];

$$u^2-U^2=(b'^2+2b'u'+u'^2)-\frac{b'^2}{b''^2}\cdot(b''^2-2b''u''+u''^2)$$ [9507a]

$$=2b'u'+u'^2+\frac{2b'^2}{b''}\cdot u''-\frac{b'^2}{b''^2}\,u''^2$$ [9507b]

$$=(2b'u''-u''^2)+\left(u''^2-\frac{u''^3}{b'}+\frac{u''^4}{4b'^2}\right)+\frac{2b'^2}{b''}\cdot u''-\frac{b'^2}{b''^2}\cdot u''^2$$ [9507c]

$$=2b'u''+\frac{2b'^2}{b''}\cdot u''-\frac{b'^2}{b''^2}\cdot u''^2=\frac{b'^2}{b''^2}\cdot\left\{2b''\cdot\left(\frac{b''}{b'}+1\right)\cdot u''-u''^2\right\}.$$ [9507d]

† (4188) Substituting the expressions [9508, 9508′], in the equation [9507], it becomes, by successive reductions,

$$u^2-U^2=\frac{b'^2}{b''^2}\cdot\{2B^2\cdot(1-\cos.\theta)-B^2\cdot(1-\cos.\theta)^2\}=\frac{b'^2}{b''^2}\cdot B^2\cdot\{1-\cos.^2\theta\}=\frac{b'^2}{b''^2}\cdot B^2\cdot\sin.^2\theta;$$ [9509a]

whence $\sqrt{u^2-U^2}=\frac{b'}{b''}\cdot B\cdot\sin.\theta$. From [9501, 9504], we obtain $u=b'+u''-\frac{u''^2}{2b'}$, [9509b]
whose differential is $du=du''\cdot\left(1-\frac{u''}{b'}\right)$; multiplying this by U [9506], we get

$$Udu=du''\cdot\frac{b'}{b''}\cdot\left(1-\frac{u''}{b'}\right)\cdot(b''-u'').$$ [9509c]

Substituting $du''=B\cdot d\theta\cdot\sin.\theta$ [9508′], and dividing by $\sqrt{u^2-U^2}$, we get the expression of dz [9493], under the following form;

$$dz=\frac{Udu}{\sqrt{u^2-U^2}}=d\theta\cdot\left(1-\frac{u''}{b'}\right)\cdot(b''-u'')=b''d\theta\cdot\left(1-\frac{u''}{b'}\right)\cdot\left(1-\frac{u''}{b''}\right)=b''d\theta\cdot\left\{1-u''\cdot\left(\frac{1}{b'}+\frac{1}{b''}\right)+\frac{u''^2}{b'b''}\right\}.$$ [9509d]

Now the value of u'' [9508′] gives

$$u''^2=B^2\cdot(1-2\cos.\theta+\cos.^2\theta)=B^2\cdot(\tfrac{3}{2}-2\cos.\theta+\tfrac{1}{2}\cos.2\theta)\ \ [6],\ \text{Int.};$$ [9509e]

[9509] $$dz = b''d\theta . \left\{ \cos.\theta - \frac{b''}{2b'} + \frac{b''}{2b'} . \cos.2\theta \right\}.$$

[9510] Integrating this expression, and observing that z vanishes at the same point as u'', and therefore also with θ [9505f], we get

[9511] $$z = b''. \left\{ \sin.\theta - \frac{b''}{2b'} . \theta + \frac{b''}{4b'} . \sin.2\theta \right\}. \qquad (a)$$

[9512] Putting h for the extreme value of z, and θ'' for the corresponding value of θ, we shall have

θ''.

[9513] $$h = b''.\sin.\theta''. \left\{ 1 - \frac{b''}{2b'} . \frac{\theta''}{\sin.\theta''} + \frac{b''}{2b'} . \cos.\theta'' \right\};$$

whence we deduce very nearly *

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substituting these in [9509d], we obtain

[9509f] $$dz = b''d\theta . \left\{ 1 - B.(1 - \cos.\theta) . \left(\frac{1}{b'} + \frac{1}{b''} \right) + \frac{B^2}{b'b''} . (\tfrac{3}{2} - 2\cos.\theta + \tfrac{1}{2}\cos.2\theta) \right\}$$

[9509g] $$= b''d\theta . \left\{ 1 - \frac{B}{b'} - \frac{B}{b''} + \frac{3B^2}{2b'b''} + \left(\frac{B}{b'} + \frac{B}{b''} - \frac{2B^2}{b'b''} \right) . \cos.\theta + \frac{B^2}{2b'b''} . \cos.2\theta \right\}.$$

If we neglect terms of the order b''^3, in this value of dz, we may neglect those of the order b''^2, in the terms between the braces in [9509g]; and in this case, the value of B [9508]

[9509h] will give $\frac{B}{b''} = 1 + \frac{b''}{b'}$, $\frac{B}{b'} = \frac{b''}{b'}$, $\frac{B^2}{b'b''} = \frac{b''}{b'}$. Substituting these in [9509g], and neglecting

[9509i] the terms which destroy each other, we get $dz = b''d\theta . \left\{ - \frac{b''}{2b'} + \cos.\theta + \frac{b''}{2b'} . \cos.2\theta \right\}$, as in

[9509k] [9509], whose integral is given in [9511]; and by substituting $\sin.2\theta = 2\sin.\theta.\cos.\theta$ [31], Int., and then changing θ into θ'', and z into h, we get [9513], corresponding to the extreme point of the curve.

[9514a] * (4189) As $\frac{b''}{b'}$ is small [9505b], the expression [9513] gives very nearly $h = b''.\sin.\theta''$,

[9514b] and the value of z [9511] will be nearly $z = b''.\sin.\theta$, corresponding to a circular arc whose radius is b'', and sine z; therefore the surface AOB, fig. 124, page 756, must be

[9514c] nearly circular. The value of h [9514a] gives $b'' = \frac{h}{\sin.\theta''}$; substituting this, in the terms between the braces in [9513], we get very nearly

[9514d] $$h = b''.\sin.\theta''. \left\{ 1 - \frac{h}{2b'} . \frac{\theta''}{\sin.^2\theta''} + \frac{h.\cos.\theta''}{2b'.\sin.\theta''} \right\};$$

dividing this by hb'', we obtain [9514].

[9514e] All the calculations which have been made in this article are founded upon the supposition that the two planes are of the same materials, and that the angles θ'

[9514f] corresponding to them are equal to each other. We shall now point out briefly the modifications which must be made when these angles are different. We shall still refer to

$$\frac{1}{b''}=\frac{\sin.\theta''}{h}.\left\{1-\frac{h}{2b'}.\frac{\theta''}{\sin.^2\theta''}+\frac{h.\cos.\theta''}{2b'.\sin.\theta''}\right\}. \qquad [9514]$$

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fig. 124, page 756, supposing the horizontal line OGO' to be drawn through the points O, O', where the tangents to the curves AOB, $A'O'B'$, are vertical; and taking, as before, [9514g] the middle point G as the origin of the coordinates z, u; the positive values of z being taken above the line OGO', and the negative values below that line. Then, instead of the notation in [9494—9496], we shall suppose that the symbols h, f, θ', correspond to the upper plane [9514h] AHA', and that the same letters, with a mark below them, $h_{,}, f_{,}, \theta'_{,}$, represent respectively the similar quantities for the lower plane BIB'; $h_{,}$ being considered as a positive quantity; so that $HI=h+h_{,}$ represents the distance of the two planes; and we have [9514i] $GO=GO'=b'$, $HA=f$, $IB=f_{,}$, angle $HAa=90^d-\theta'$, angle $IBb=90^d-\theta'_{,}$; [9514k] lastly we shall suppose, as in [9488], that b is the radius of curvature of the curve AOB at [9514l] O, or of the curve $A'O'B'$ at O'. Then the whole calculation in this article, from [9486] to [9525], may be considered as appertaining to the part of the drop which is situated above [9514m] the plane OGO'; and the calculation relative to the lower part of the drop is perfectly similar. For we have merely to repeat the calculation, altering the sign of the term $\frac{2gz}{H}$ in [9514n] [9486], and then writing $z_{,}$ for z, as in [9485p], $z_{,}$ being considered as positive [9485l]; changing also, as above, h into $h_{,}$, f into $f_{,}$, θ' into $\theta'_{,}$, without altering the symbols b, b', b'' [9514o] [9514l, k, 9500], or B, θ [9508, 9508']. Then we shall get, as in [9511],

$$z_{,}=b''.\left\{\sin.\theta-\frac{b''}{2b'}.\theta+\frac{b''}{4b'}.\sin.2\theta\right\}. \qquad [9514p]$$

Taking, in like manner as in [9512], $h_{,}$ for the extreme value of $z_{,}$, and $\theta''_{,}$ for the extreme value of θ, we shall find, as in [9513],

$$h_{,}=b''.\sin.\theta''_{,}.\left\{1-\frac{b''}{2b'}.\frac{\theta''_{,}}{\sin.\theta''_{,}}+\frac{b''}{2b'}.\cos.2\theta''_{,}\right\}. \qquad [9514q]$$

The same changes may be made in the symbols in the formulas [9516—9524]; but, for the sake of brevity, we shall not repeat these formulas with their additional accents. If b or b'' [9514r] be so small, in comparison with b', that the terms of the order $\frac{b''}{b'}$ may be neglected, we shall have, from [9513, 9514q],

$$h=b''.\sin.\theta'', \qquad h_{,}=b''.\sin.\theta''_{,}; \qquad [9514s]$$

and, by continuing to neglect quantities of the same order $\frac{b''}{b'}$, we may also put $b''=b$ [9514t] [9500], $\theta''=\theta'$ [9521], and in like manner $\theta''_{,}=\theta'_{,}$; hence [9514s] become

$$h=b.\sin.\theta', \qquad h_{,}=b.\sin.\theta'_{,}, \qquad [9514u]$$

whose sum gives the thickness of the drop $h+h_{,}=b.\{\sin.\theta'+\sin.\theta'_{,}\}$; and from this we [9514v] get $b=\frac{h+h_{,}}{\sin.\theta'+\sin.\theta'_{,}}$, which is similar to the expression given by M. Poisson in page 206 [9514w] of his *Nouvelle Théorie, &c.*, changing $h+h_{,}$ into k, &c., so as to conform to his notation. [9514x]

We may determine θ'' by means of the angle θ', the complement of the inclination
[9515] of the extreme sides of the curve to the two planes [9496]. At the extreme points of the curve, we have, as in [9497],

[9516] $$\frac{dz}{du}=\frac{\cos.\theta'}{\sin.\theta'}=\frac{\frac{U}{u}}{\sqrt{1-\frac{U^2}{u^2}}};$$

[9517] $\frac{U}{u}$ being, in this case, its extreme value, which is therefore $\frac{U}{u}=\cos.\theta'$, as in [9496b]. Then we have, at these extreme points, as in [9508', 9512, 9506, 9501, 9504],

[9518] $$u''=B.(1-\cos.\theta''),$$

[9519] $$U=\frac{b'}{b''}.(b''-u'')=b'-\frac{b'u''}{b''},$$

[9520] $$u=b'+u'=b'+u''-\frac{u''^2}{2b'}.$$

Hence we easily deduce *

[9521] $$\theta''=\theta'-\frac{b''}{b'}.\sin.\theta';$$

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* (4190) Dividing the value of U [9519] by that of u [9520], and substituting for $\frac{U}{u}$ its value $\cos.\theta'$ [9517], we get [9521a]; if we neglect the square of $\frac{u''}{b'}$, on account of its smallness, it will become successively, as in the last form of [9521a],

[9521a] $$\cos.\theta'=\frac{b'-\frac{b'u''}{b''}}{b'+u''-\frac{u''^2}{2b'}}=\frac{1-\frac{u''}{b''}}{1+\frac{u''}{b'}-\frac{u''^2}{2b'^2}}=\left(1-\frac{u''}{b''}\right).\left(1-\frac{u''}{b'}\right)=1-\frac{u''}{b''}-\frac{u''}{b'}.\left(1-\frac{u''}{b''}\right).$$

Now from [9518, 9508], we have

[9521b] $$u''=b''.\left(1+\frac{b''}{b'}\right).(1-\cos.\theta''),\quad\text{or}\quad \frac{u''}{b''}=(1-\cos.\theta'')+\frac{b''}{b'}.(1-\cos.\theta'');$$

whence

[9521b'] $$1-\frac{u''}{b''}=\cos.\theta''-\frac{b''}{b'}.(1-\cos.\theta'');$$

substituting this in [9521a], and neglecting the square of $\frac{b''}{b'}$, we get

[9521c] $$\cos.\theta'=\cos.\theta''-\frac{b''}{b'}.(1-\cos.\theta'')-\frac{u''}{b'}.\cos.\theta'';$$

and in this last term, we may put simply $b''.(1-\cos.\theta'')$ for u'' [9521b], and it will become, by successive reductions,

therefore we shall have * [9521']

$$\cos.\theta' = \cos.\theta'' - \frac{b''}{b'}.(1-\cos.\theta'') - \frac{b''}{b'}.(1-\cos.\theta'').\cos.\theta'' = \cos.\theta'' - \frac{b''}{b'}.(1-\cos.^2\theta'')$$ [9521d]
$$= \cos.\theta'' - \frac{b''}{b'}.\sin.^2\theta''.$$

Now if we put $\theta' = \theta'' + \delta$, we shall have, as in [61], Int., $\cos.\theta' = \cos.\theta'' - \delta.\sin.\theta''$; Putting this equal to the last expression in [9521d], neglecting the terms $\cos.\theta''$, which [9521e] mutually destroy each other, and then dividing by $-\sin.\theta''$, we get $\delta = \frac{b''}{b'}.\sin.\theta''$; consequently

$$\theta' = \theta'' + \delta = \theta'' + \frac{b''}{b'}.\sin.\theta''; \quad \text{whence} \quad \theta'' = \theta' - \frac{b''}{b'}.\sin.\theta'', \quad \text{or} \quad \theta'' = \theta' - \frac{b''}{b'}.\sin.\theta'$$ [9521f]

very nearly, as in [9521].

* (4191) Taking the sine of the value of θ'' [9521], we get

$$\sin.\theta'' = \sin.\theta' - \frac{b''}{b'}.\sin.\theta'.\cos.\theta'$$ [9522a]

nearly; and as we have very nearly $b'' = \frac{h}{\sin.\theta''}$ [9514c], or $b'' = \frac{h}{\sin.\theta'}$ nearly, the [9522a'] preceding expression becomes very nearly, by successive operations,

$$\sin.\theta'' = \sin.\theta'.\left\{1 - \frac{b''}{b'}.\cos.\theta'\right\} = \sin.\theta'.\left\{1 - \frac{h.\cos.\theta'}{b'.\sin.\theta'}\right\}.$$ [9522b]

Dividing this by h, we get the factor, without the braces, in the second member of [9514]; substituting this, and changing θ'' into θ', in the small terms of [9514] depending on h between the braces, it becomes

$$\frac{1}{b''} = \frac{\sin.\theta'}{h}.\left\{1 - \frac{h.\cos.\theta'}{b'.\sin.\theta'}\right\}.\left\{1 - \frac{h}{2b'}.\frac{\theta'}{\sin.^2\theta'} + \frac{h.\cos.\theta'}{2b'.\sin.\theta'}\right\}.$$ [9522c]

Multiplying together the two factors of the second member, neglecting terms of the order h^2, we get [9522], or, as it may be written,

$$\frac{1}{b''} = \frac{\sin.\theta'}{h} - \frac{1}{b'}.\left(\frac{\theta'}{2\sin.\theta'} + \tfrac{1}{2}\cos.\theta'\right) = \frac{\sin.\theta'}{h} - \frac{Q}{b'} \quad [9524],$$ [9522d]

agreeing with [9523].

In all these calculations, b' is supposed to be much larger than h [9499]; and if we suppose it to be so large that we may neglect terms of the order $\frac{b''}{b'}$ in [9511], it will become $z = b''.\sin.\theta$; moreover [9508] will give $B = b''$, and from [9508'], we have [9522e] $b'' - u'' = b''.\cos.\theta$; and then, from the sum of the squares of these two equations, we get $z^2 + (b''-u'')^2 = b''^2$, which is the equation of a circle whose radius is b''; therefore the arc [9522f] AOB, fig. 124, page 756, will be nearly circular when the height of the drop is extremely small relative to its width; agreeing with M. Poisson, in page 205 of his *Nouvelle Théorie, &c.*

[9522] $$\frac{1}{b''}=\frac{\sin.\theta'}{h}\cdot\left\{1-\frac{h}{2b'}\cdot\frac{\theta'}{\sin.^2\theta'}-\frac{h.\cos.\theta'}{2b'.\sin.\theta'}\right\},$$

or

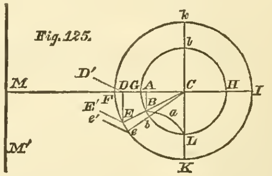
[9523] $$\frac{1}{b''}=\frac{\sin.\theta'}{h}-\frac{Q}{b'};$$

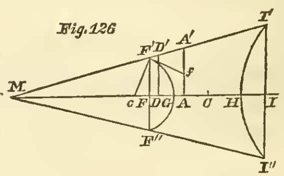
putting

[9524] $$Q=\frac{\theta'}{2\sin.\theta'}+\tfrac{1}{2}\cos.\theta'.$$

[9525] When θ' is equal to a right angle, or $\frac{1}{2}\pi$, Q will become equal to $\frac{1}{4}\pi$.

We shall now consider a drop of fluid, suspended in equilibrium between two
[9525′] *planes, which touch each other in a horizontal line at their two edges; and we shall put* *





* (4192) To illustrate this description, we have given
[9526a] the adjoined figures 125, 126. The first is on the *intermediate plane* passing through the centre of the drop C; the second figure is drawn through the same centre,
[9526b] perpendicular to the *intermediate plane*, and this we shall call the *second plane*. The corresponding points in these two figures are marked with the same letters. MM' is the intersection of the two glass planes, CM the line drawn perpendicular to that intersection. In fig. 125, the
[9526c] curve $GLHl$ represents the curve formed by the drop, where it is intersected by the intermediate plane; and the curve $KIkF$ represents the curve formed from that of the outer surface of the drop, where it intersects the glass plane by projecting it perpendicularly upon the intermediate plane. CLK is drawn perpendicular to
[9526d] CM. Then CM is the axis of x, CK the axis of y, the axis of z being perpendicular to the intermediate plane; and if through any point B of the curve $GLHl$, we draw BA
[9526e] perpendicular to CM, we shall have $CA=x$, $CM=a$, $MA=a-x$, $AB=y$, also $HG=2a$. These same values correspond to fig. 126, in which F', D', A', I', &c., represent respectively the points whose projections in both figures are F, D, A, I, &c. Moreover
[9526f] $F'GF''$, $I'HI''$, represent the sections of the surface by the *second plane*; these sections are almost circular, and the radii nearly equal to b'', when the planes are but little inclined to
[9526g] each other, as is shown in [9514b]; and if we draw $F'f$ tangent to the curve at F', also $F'c$ perpendicular to $F'f$, to meet MC in c, we shall have nearly $F'c=b''$ [9514b], also the
[9526h] angle $A'F'f=90^d-\theta'$ [9496, &c.]; and as $cF'f=90^d$, we shall have $MF'c=\theta'$,
[9526i] also $F'Mc=F''Mc=\varpi$ [9526]. Hence $F'cF=F'Mc+MF'c=\theta'+\varpi$. Lastly, h, as in [9494], represents the distance of the point E from the glass plane, measured on a line EE' drawn perpendicular to the intermediate plane, and this line is evidently equal and
[9526k] parallel to DD', so that we have $h=DD'=EE'$.

$2\varpi =$ the angle which these two planes form with each other. Then, having drawn an *intermediate plane*, which divides this angle into two equal parts, we shall put [9526]

Figure of a drop of fluid between two inclined planes intersecting each other in a horizontal line.

$V =$ the inclination of the *intermediate plane* to the horizon. [9527]

The section of the surface of the drop by the intermediate plane, will be very nearly a circle, supposing, as we have just done, that the width of the drop is considerable in comparison with its thickness. We shall also suppose that, from the middle of this drop, through any part of the section, a plane is drawn, perpendicular to the intermediate plane; then the section of the fluid surface by this plane will be very nearly expressed by the equation [9511]. In defining the curve which is formed by the section of the surface of the drop by the intermediate plane, we shall use the rectangular coordinates x, y, whose origin is in the centre of this curve, or centre of the drop, putting, [9527'] [9528] [9528']

For *the axis of* x, the line drawn from the centre of the drop perpendicular to the line of intersection of the two planes; [9529]

For *the axis of* y, the line drawn from the centre of the drop parallel to the line of intersection of the two planes; [9530]

$a =$ the distance from the centre of the drop to the line of intersection of the two planes; [9531]

$2\mathrm{a} =$ the width of the drop, measured in the direction of the line a, or in a direction perpendicular to the line of intersection of the two planes; [9532]

$a - x =$ the distance from the line of intersection of the two planes to the point of the curve whose coordinates are x, y. [9533]

Then it is evident that we shall have very nearly*

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* (4193) The glass planes being nearly parallel, the curves $GLHl$, $FKIk$, fig. 125, will be nearly circular, having the common centre C. Through this centre C suppose a plane to be drawn perpendicular to the intermediate plane, and cutting it in the line CBE; this plane will cut the extreme part of the drop in the point E', situated perpendicularly over the point E, so that the coordinate z will be represented by $EE' = DD' = h$ [9526k], or [9494]. Moreover $CB (= \mathrm{a})$ nearly [9532], and in the similar triangles CAB, CDE, we shall have $CB (= \mathrm{a}) : CA (= x) :: BE (= FG) : AD = FG \times \frac{x}{\mathrm{a}}$. Now the arc $F'GF''$, fig. 126, being nearly circular, we have [9534a] [9534b] [9534c]

$$FG = Gc - Fc = Gc - F'c.\cos.F'cF = b'' - b''.\cos.(\theta' + \varpi) \qquad [9534d]$$

nearly [9526i]; hence $AD = \frac{x}{\mathrm{a}}.b''.\{1 - \cos.(\theta' + \varpi)\}$; if in this we neglect ϖ, on account of its smallness, and substitute for $1 - \cos.\theta'$ its value $2\sin.^2\tfrac{1}{2}\theta'$ [1], Int., we shall get $AD = \frac{x}{\mathrm{a}}.b''.2\sin.^2\tfrac{1}{2}\theta'$; but from [9523], we have nearly $\frac{1}{b''} = \frac{\sin.\theta'}{h} = \frac{2\sin.\frac{1}{2}\theta'.\cos.\frac{1}{2}\theta'}{h}$ [9534e] [9534f]

[9534] $$h=\left(a-x-\frac{hx}{\mathrm{a}}.\mathrm{tang}.\tfrac{1}{2}\theta'\right).\mathrm{tang}.\varpi,\quad \text{or}\quad h=\frac{(a-x).\mathrm{tang}.\varpi}{1+\frac{x}{\mathrm{a}}.\mathrm{tang}.\frac{1}{2}\theta'.\mathrm{tang}.\varpi},$$

which gives*

[9535] $$\frac{1}{h}=\frac{1+\frac{x}{\mathrm{a}}.\mathrm{tang}.\frac{1}{2}\theta'.\mathrm{tang}.\varpi}{(a-x).\mathrm{tang}.\varpi}=\frac{1}{a.\mathrm{tang}.\varpi}+\frac{x}{a^2.\mathrm{tang}.\varpi}.\left(1+\frac{a}{\mathrm{a}}.\mathrm{tang}.\tfrac{1}{2}\theta'.\mathrm{tang}.\varpi\right)+\&c.$$

[9534g] [31], Int.; hence we get $b''.2\sin.^2\frac{1}{2}\theta'=h.\frac{\sin.\frac{1}{2}\theta'}{\cos.\frac{1}{2}\theta'}=h.\mathrm{tang}.\frac{1}{2}\theta'$, and by substituting it, in the preceding value of AD, we get $AD=\frac{hx}{\mathrm{a}}.\mathrm{tang}.\frac{1}{2}\theta'$. Subtracting this from $MA=a-x$
[9534h] [9526e], we get $MD=a-x-\frac{hx}{\mathrm{a}}.\mathrm{tang}.\frac{1}{2}\theta'$, and by multiplying it by tang.$D'MD$, or tang.ϖ [9526], we obtain for DD' or h [9534c] the expression

[9534i] $$h=\left(a-x-\frac{hx}{\mathrm{a}}.\mathrm{tang}.\tfrac{1}{2}\theta'\right).\mathrm{tang}.\varpi.$$

[9534k] The value given by the author is $h=\left(a-x-\frac{hx}{\mathrm{a}}\right).\mathrm{tang}.\varpi$, the factor $\mathrm{tang}.\frac{1}{2}\theta'$ being
[9534l] omitted by mistake. To remedy this defect, we have altered the original work, by changing
[9534m] a into $\frac{\mathrm{a}}{\mathrm{tang}.\frac{1}{2}\theta'}$, in the formulas [9534, 9535, 9540, 9542, 9547], where a occurs. This correction is of no importance with fluids which, like water, alcohol, &c., completely moisten
[9534n] the planes, and make $\theta'=90^d$, because in this case $\mathrm{tang}.\frac{1}{2}\theta'=1$. To preserve a symmetry with the formulas for a cone [9469, 9472, &c.], we may suppose, in like manner as in
[9534o] [9471c, d], that the lines EE', DD', &c., fig. 125, page 766, are drawn perpendicular to the upper plane, instead of being perpendicular to the intermediate plane. In this case, we must multiply MD [9534h] by sin.ϖ instead of tang.ϖ [9534i], to obtain h [9534i], which will become

[9534p] $$h=\left(a-x-\frac{hx}{\mathrm{a}}.\mathrm{tang}.\tfrac{1}{2}\theta'\right).\sin.\varpi,\quad \text{or}\quad h=\frac{(a-x).\sin.\varpi}{1+\frac{x}{\mathrm{a}}.\mathrm{tang}.\frac{1}{2}\theta'.\sin.\varpi},$$

[9534q] instead of the value in [9534]. The effect of this modification will be to change tang.ϖ into sin.ϖ, in all the formulas where it occurs from [9534] to [9549], as well as in the corresponding notes.

[9535a] * (4194) The expression of h [9534] gives $\frac{1}{h}$, as in the first formula in [9535], or, as it may be written,

[9535b] $$\frac{1}{h}=\frac{1}{a.\mathrm{tang}.\varpi}.\left\{\frac{1+\frac{x}{\mathrm{a}}.\mathrm{tang}.\frac{1}{2}\theta'.\mathrm{tang}.\varpi}{1-\frac{x}{a}}\right\}.$$

Developing this, according to the powers of x, and neglecting x^2, &c., it becomes of the same form as the second expression in [9535].

We shall suppose a canal to be so situated in the intermediate plane, that one of its extremities is at the point of the section whose coordinates are x, y, the [9536] other at the point of the section through which the axis of y passes; the equilibrium of the fluid in this canal will give the equation *

$$K-\frac{H}{2b''}+gx.\sin.V=K-\frac{H}{2b''_{,}}, \qquad [9537]$$

placing one accent below the letters relative to this last point. But we have, by [9537′] what precedes † [9523],

$$\frac{1}{b''}=\frac{\sin.\theta'}{h}-\frac{Q}{b'}, \qquad \frac{1}{b''_{,}}=\frac{\sin.\theta'}{h_{,}}-\frac{Q}{b'_{,}}, \qquad [9538]$$

b' being, in this case, the radius of curvature of the curve formed by the intersection of the intermediate plane with the surface of the drop. Moreover,‡ [9539]

$$\frac{1}{h}=\frac{1}{a.\text{tang.}\varpi}+\frac{x}{a^2.\text{tang.}\varpi}.\left(1+\frac{a}{\text{a}}.\text{tang.}\tfrac{1}{2}\theta'.\text{tang.}\varpi\right), \qquad [9540]$$

$$\frac{1}{h_{,}}=\frac{1}{a.\text{tang.}\varpi}; \qquad [9541]$$

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* (4195) We shall suppose, in fig. 125, page 766, that BaL is an extremely narrow [9537a] canal, of uniform diameter, passing along the *intermediate plane*, and terminated at the points B, L; and that the radii of curvature at B are b' for the arc GBL, and b for the arc passing through B, perpendicular to the plane of the figure; also $b'_{,}$, $b_{,}$, for the radii of the [9537b] similar arcs at the point L. Then the capillary action at B upon the canal BaL, is as in [9485f], equal to $K+\frac{H}{2}.\left(\frac{1}{b'}-\frac{1}{b}\right)$; and at L, this action is $K+\frac{H}{2}.\left(\frac{1}{b'_{,}}-\frac{1}{b_{,}}\right)$. Now [9537c] the point B is elevated above L by the quantity $CA\times\sin.V$ [9527], or $x.\sin.V$. [9537d] Multiplying this by g, we get the pressure of the column of fluid, to be added to the action at B, to obtain that at L, when in equilibrium, as in [9485e, &c.]. Hence we get $K+\frac{H}{2}.\left(\frac{1}{b'}-\frac{1}{b}\right)+gx.\sin.V=K+\frac{H}{2}.\left(\frac{1}{b'_{,}}-\frac{1}{b_{,}}\right)$. Substituting $\frac{1}{b}-\frac{1}{b'}=\frac{1}{b''}$ [9500], and [9537e] the similar expression $\frac{1}{b_{,}}-\frac{1}{b'_{,}}=\frac{1}{b''_{,}}$, it becomes as in [9537]. [9537f]

† (4196) The first equation [9538] corresponding to the point E, fig. 125, page 766, is the same as [9523], which is found by a calculation similar to that in [9486—9523], [9538a] using, instead of f, h [9495, 9494], the values CE, EE' [9526i—k]. When the point E falls in K, the quantity h changes into $h_{,}$ [9537′], and the value of $\frac{1}{b''}$ [9538] changes into [9538b] that of $\frac{1}{b''_{,}}$ [9538].

‡ (4197) The equation [9540] is the same as [9535], and at the point L, fig. 125, page 766, where $x=0$, and $h=h_{,}$ [9537′], it becomes as in [9541]. [9540a]

therefore we shall have *

[9542] $$-\frac{Hx.\sin.\theta'}{2a^2.\text{tang.}\varpi}.\left(1+\frac{a}{\mathrm{a}}.\text{tang.}\tfrac{1}{2}\theta'.\text{tang.}\varpi\right)+\tfrac{1}{2}QH.\left(\frac{1}{b'}-\frac{1}{b'_{\prime}}\right)+gx.\sin.V=0.$$

As the section differs but little from a circle, b' is very nearly equal to the half width of the drop a; b'' is, by what has been said [9522, 9514c], very nearly

[9543] equal to $\frac{h}{\sin.\theta'}$, and h is half the thickness of the drop; b' is therefore very large in comparison with b''; consequently $\frac{1}{b'}$ is very small in comparison with

[9544] $\frac{1}{b''}$; therefore the difference $\frac{1}{b'}-\frac{1}{b'_{\prime}}$ may be neglected in comparison with $\frac{1}{b''}-\frac{1}{b''_{\prime}}$. This may be done with more propriety, since, $b'_{\prime}$ being a mean

[9545] between the extreme values of b', the greatest value of the difference $\frac{1}{b'}-\frac{1}{b'}$ is only about half of the difference of the extreme values of $\frac{1}{b'}$. Moreover, the figure of the drop being very nearly circular, as is found by experiment, the

[9546] difference $\frac{1}{b'}-\frac{1}{b'_{\prime}}$ is nearly insensible. We may also, in the preceding

[9547] equation, neglect the fraction $\frac{a}{\mathrm{a}}.\text{tang.}\tfrac{1}{2}\theta'.\text{tang.}\varpi$, in comparison with unity; because, $2a.\text{tang.}\varpi$ being the thickness of the drop whose width is 2a, this

[9548] fraction does not exceed the ratio of the thickness of the drop to its width, which ratio, by hypothesis, is very small. This being premised, the preceding equation will give †

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* (4198) Transposing all the terms of [9537] to the first member, we get

[9542a] $$\tfrac{1}{2}H.\left\{\frac{1}{b''_{\prime}}-\frac{1}{b''}\right\}+gx.\sin.V=0;$$

and by substituting the values [9538], it becomes

[9542b] $$\tfrac{1}{2}H.\sin.\theta'.\left\{\frac{1}{h_{\prime}}-\frac{1}{h}\right\}+\tfrac{1}{2}QH.\left\{\frac{1}{b'}-\frac{1}{b'_{\prime}}\right\}+gx.\sin.V=0.$$

Now if we subtract [9540] from [9541], we get

[9542c] $$\frac{1}{h_{\prime}}-\frac{1}{h}=-\frac{x}{a^2.\text{tang.}\varpi}.\left\{1+\frac{a}{\mathrm{a}}.\text{tang.}\tfrac{1}{2}\theta'.\text{tang.}\varpi\right\};$$

and by substituting it in [9542b], we obtain [9542].

† (4199) Neglecting the second and third terms of [9542], on account of their smallness

[9549a] [9546, &c.], it becomes $-\frac{Hx.\sin.\theta'}{2a^2.\text{tang.}\varpi}+gx.\sin.V=0$; dividing this by gx, and

$$\sin.V = \frac{H.\sin.\theta'}{2a^2.g.\tang.\varpi}. \qquad [9549]$$

Thus the angle V is very nearly in the inverse ratio of the square of a, as in a drop suspended in equilibrium in a cone [9475b]. Comparing this expression of $\sin.V$ with that of the preceding article, we see that, the angle formed by the two planes [9526] being supposed equal to the angle formed by the axis of the cone and its side [9462], the sine of the angle V relative to the intermediate plane, is equal to the sine of the angle relative to the axis of the cone.* We ought not, however, to forget, in comparing the results of this analysis with experiments, that the expressions of $\sin.V$ are only approximate values. [9550] [9550'] [9551]

11. *The preceding analysis furnishes an explanation and the measure of a singular phenomenon discovered by experiment; namely, that, whether the fluid be elevated or depressed between two vertical and parallel planes, dipped into a fluid at their lower extremities, the planes will tend towards each other;* so that, if two small glass vases, in the form of a parallelopiped, floating upon water or mercury, happen to approach near to each other, they will immediately unite together. To prove that this must take place, we shall consider the two planes MB [9552] [9552']

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transposing the first term, we get [9549]. Poisson, in computing this problem in his *Nouvelle Théorie, &c.*, uses the same fundamental equation as in [9318], carrying on the approximation so as to include some terms which are neglected by La Place, as being within the limits of the errors of the observations. These terms are, however, omitted by M. Poisson, in his final expression of $\sin.V$, which is nearly the same as that given in [9549], as may be seen by changing, in his formula (10), page 258 of that work, the symbols $a^2, c, i, \theta, \varpi$, into $\frac{H}{g}, a, \varpi, V, \theta'+90^d$, respectively, in order to conform to the present notation; putting also, as in [9534q], $\sin.\varpi$ for $\tang.\varpi$, in [9534, &c.]; this last change being within the limits of the degree of accuracy aimed at by La Place, throughout his whole calculation; so that, instead of [9549], we may put very nearly $\sin.V = \frac{H.\sin.\theta'}{2a^2.g.\sin.\varpi}$. [9549b] [9549c] [9549d] [9549e] [9549f]

* (4200) If we suppose that the angle formed by the axis of the cone and its sides is 2ϖ, instead of ϖ [9462], we shall find that the expression [9475a] will become $\frac{l}{a} = \frac{H.\sin.\theta'}{2a^2.g.\sin.\varpi}$ nearly, because, on account of the smallness of ϖ, we have $\sin.2\varpi = 2\sin.\varpi$ nearly. This being equal to the expression [9549f], proves the correctness of the remarks in [9551]. [9550a] [9550b]

[9553] and NR, fig. 127, and shall suppose, in the first place, that the fluid is elevated between them. The infinitely small external part of the plane NR, situated at R, below the level VP, will be pressed by a force which may be thus estimated: Imagine a canal VSR, of which the branch VS is vertical, and the branch SR horizontal; then the force acting upon the fluid in the canal VS is equal to $g.VS$,

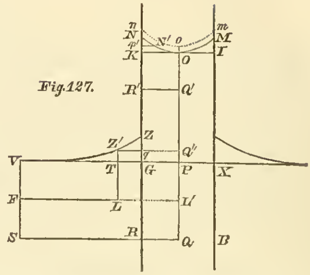

Fig. 127.

[9554] augmented by the force acting in V, either by the action of the fluid upon the canal, or by the pressure of the atmosphere. The first of these two forces is represented by K [9259], and we shall suppose
[9554'] that the *pressure of the atmosphere is P*; then the whole force of the column
[9555] VS will be $P+K+g.VS$. The action, with which the fluid in the canal RS is urged, is composed of two parts. *First*, the action of the fluid upon the canal, which is equal to K [9259]. *Second*, the action of the plane upon the same canal; but this action is destroyed by the attraction of the fluid upon the
[9556] plane, and there cannot arise from this source any tendency in the plane to move. For, by considering only these reciprocal attractions, the fluid and the plane will be at rest, the action being equal and contrary to the reaction; these attractions can produce only an adhesion of the plane to the fluid, which need not be noticed. Hence it follows that the fluid presses the point R with a
[9557] force equal to $P+K+g.VS-K$, or simply $P+g.VS$.

We shall now determine the corresponding internal pressure. For this purpose, we shall suppose the canal OQR to have a vertical branch OQ, terminated at the surface O, and a horizontal branch QR. The force with
[9558] which the fluid is urged in the branch OQ, is equal to $g.OQ$, increased by *the pressure of the atmosphere P*, also by the force with which the fluid acts
[9558'] upon the column OQ; and this is, by what has been said,* equal to $K-\frac{H}{2b}$,
[9559] *b being the radius of curvature at O.* Therefore the force with which the

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[9558a] * (4201) This force is represented in [9315b] by $K-\frac{H}{2b}-\frac{H}{2b'}$; and as $b'=\infty$,
[9558b] for reasons similar to that of putting b infinite in [9414], it becomes $K-\frac{H}{2b}$, as in [9452a].
[9558c] In the original work, it is erroneously printed $K-\frac{H}{b}$; to correct this mistake we have changed b into $2b$, in all the formulas [9559—9578], b being the radius of curvature at O.

fluid OQ is urged, is $P+K-\frac{H}{2b}+g.OQ$. Now, by what has been said,* [9559′]

$$\frac{H}{2b}=g.OP. \qquad [9560]$$

Therefore the force of the canal OQ is $P+K+g.PQ$. The force of the [9561] canal QR is equal to K [9259]; the point R will therefore be pressed on the inner surface by the difference of these forces, which is $P+g.PQ$, or [9562] $P+g.VS$. Thus the plane is pressed with equal forces [9557, 9562] on the outer and inner surfaces, and it will therefore be in equilibrium in virtue of these pressures.

The exterior fluid rises as high as Z, forming a curve $VZ'Z$; and between [9563] the planes, it rises as high as N, forming the curve $ON'N$. The parts of the plane extremely near to Z and N, and similarly placed at a distance from Z and N equal to or less than the sensible sphere of activity of the plane, are equally pressed within and without;† because the surfaces of the fluid, comprised [9564] within this sphere, and near the points Z, N, are very nearly of the same form. Besides, the extremely small difference, which may exist between the internal and external pressures of the fluid, being limited to an insensible extent, we may neglect it, *and notice only the pressure exerted by the fluid at the points* [9565] *where the action of the plane upon the surface ceases to be sensible.* Therefore let Z' be one of the points of this surface, and suppose the horizontal canal $Z'q$ to be formed. The force in Z' will be ‡ $P+K-\frac{H}{2R}$, R being the radius [9566]

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* (4202) The expression [9452b] gives $\frac{H}{2b'}=g.q'$; and in like manner, in the [9559a] present case, we get, as in [9560], $\frac{H}{2b}=g.OP$, by changing b' into b, and q' into OP, to conform to the present notation. Substituting this in [9559′], it becomes

$$P+K+g.(OQ-OP)=P+K+g.PQ, \qquad [9559b]$$

as in [9561].

† (4203) If the angle θ' [9346] is the same on both sides of the plane, as the author supposes in this article, the part of the pressure here spoken of will be the same on both [9565a] sides of that plane; but if there is a difference in the value of this angle, on account of the plane being more or less wet on the one side than on the other, or being made of a different [9565b] substance on opposite sides, the pressures may be *unequal*, as we shall see when treating of this subject in the second supplement [9983w, &c.].

‡ (4204) The capillary action, which in [9558′] is called $K-\frac{H}{2b}$, here becomes

[9567] of curvature in Z'. If we put $TZ'=x$, the equilibrium of the fluid in the canal $Z'LFV$ will give, by using [9416],*

[9568] $$\frac{H}{2R}=gx.$$

The point V being placed at a distance from the plane, such that the radius of
[9568'] curvature of the surface in V may be considered as infinite, the external pressure in q will be

[9569] $$P+K-gx.$$

The corresponding internal pressure will be †

[9570] $$P+K-\frac{H}{2b}+g.(OP-x),$$

[9571] or $P+K-gx$; *therefore the internal and external pressures are equal through the whole extent of* ZG.

We shall now consider the pressure above the point Z. *The external pressure*
[9572] *is reduced to* P [9554']. The internal pressure upon a point R' is determined by considering a canal $OQ'R'$, $Q'R'$ being horizontal. The pressure of the

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[9566a] $K-\frac{H}{2R}$, because the radius of curvature b changes into R at the point Z'. Adding to this the atmospherical pressure P [9558], and we get the whole external action at q, equal to
[9566b] $P+K-\frac{H}{2R}$, as in [9566].

[9568a] * (4205) Substituting $\frac{1}{R}$ [9326] in [9416], we get $\frac{1}{R}-2\alpha z=\frac{1}{b'}$; but at the point V, we have $b'=\infty$, or $\frac{1}{b'}=0$ [9558a, 9568'], also $\alpha=\frac{g}{H}$ [9328]; hence $\frac{1}{R}-2.\frac{g}{H}.z=0$,
[9568b] or $\frac{H}{2R}=gz$; and by changing z [9309] into x [9567], to conform to the present notation, it becomes as in [9568]. Substituting this in [9566], it becomes as in [9569].

[9570a] † (4206) This is found in precisely the same manner as in [9559'], supposing the line $Z'q$ to be continued till it meets OQ in Q''. For then the action at q, on the canal $Q''q$, is found by adding together the atmospherical pressure P [9554']; the gravity of the column
[9570b] OQ'', which is equal to $g.OQ''$; and the capillary action at O, which is equal to $K-\frac{H}{2b}$ [9558']. The sum is $P+K-\frac{H}{2b}+g.OQ''$; and as

[9570c] $$OQ''=OP-PQ''=OP-TZ'=OP-x \quad [9567],$$

it becomes as in [9570]; and by using $\frac{H}{2b}=g.OP$ [9560], it becomes $P+K-gx$, as in [9571].

column OQ' is* $P+K-\frac{H}{2b}+g.OQ'$, or $P+K-g.OP+g.OQ'$, or, [9573]
lastly, $P+K-g.PQ'$. The contrary pressure of the canal $R'Q'$ is K [9259], [9574]
so that the point R' is pressed at its inner surface by the force $P-g.PQ'$; *therefore the plane is pressed inwards, at that point, by the force*† $g.PQ'$. [9575]

In the part $NN'O$, the pressure at N' is‡ $P+K-\frac{H}{2b'}$, *b' being the radius of curvature in N'*; and by supposing the horizontal canal $N'p'$ to be [9576]
formed, the interior pressure in p' will be $P-\frac{H}{2b'}$. Now putting x' for the [9577]
height of the point N' above IK, we shall have, from [9416],§

$$\frac{H}{2b'}=\frac{H}{2b}+gx'=g.Gp'; \qquad [9578]$$

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* (4207) This is the same as [9570*b*], changing OQ'' into OQ', and then substituting $\frac{H}{2b}=g.OP$ [9560]; for by this means it becomes $P+K-g.OP+g.OQ'$, as in [9573]. [9573*a*]

† (4208) The difference between the internal pressure $P-g.PQ'$ [9575], and the external pressure P [9572], is $g.PQ'$, as in [9575], in the direction $R'Q'$. [9575*a*]

‡ (4209) This is similar to the force at Z' [9566], changing the radius R, relative to the point Z', into b' [9576], corresponding to the point N'. The force of the canal $N'p'$ at p' is K [9259], and by subtracting it from the preceding value [9576], we get the pressure in p', in the direction $N'p'$ equal to $P-\frac{H}{2b'}$, as in [9577]. [9576*a*] [9576*b*]

§ (4210) The equation [9416] is, in [9568*a*], put under the form $\frac{1}{R}-2\alpha z=\frac{1}{b'}$; [9577*a*]
R, b', being the radii of curvature corresponding, in the present case, to the points N', O, respectively [9326, 9413']; and to conform to the notation here used, we must change them into b', b [9576, 9559]; also z [9309] must be changed into x' [9577], and then the preceding equation will become $\frac{1}{b'}-2\alpha x'=\frac{1}{b}$. Substituting $\alpha=\frac{g}{H}$ [9568*b*], multiplying [9577*b*]
by $\frac{1}{2}H$, and transposing the second term, we get $\frac{H}{2b'}=\frac{H}{2b}+gx'$, as in [9578]. [9577*c*]
Substituting the value of $\frac{H}{2b}$ [9560], it becomes $\frac{H}{2b'}=g.(OP+x')=g.Gp'$, as in the second form of [9578]. Substituting this in the function $P-\frac{H}{2b'}$ [9577], denoting the internal pressure at p', it becomes $P-g.Gp'$; taking the difference between this and the [9577*d*]
external pressure at p', namely P [9572], we obtain the whole inward pressure $g.Gp'$ [9577*e*]
[9579], in the direction $p'N'$.

[9579] *therefore the plane at the point p' will be pressed inwards, in the direction $p'N'$, by the force $g.Gp'$.*

Hence we easily perceive that the force which presses the plane NR inwards,
[9580] *is equal to the pressure of a column of fluid whose height is $\frac{1}{2}.(GN+GZ)$, and whose base is the part of the plane included between Z and* N.*

[9580a] * (4211) From what has been proved in [9575, 9579], it follows that, at any part of the line ZN, as at R', the pressure will be as $g.GR'$; and by putting $GR'=w$, it becomes $g.w$; therefore the whole pressure on the part dw of w, will be $gwdw$, whose integral,
[9580b] supposing it to commence at the point Z, will be $\frac{1}{2}g.(w^2-GZ^2)$; and the whole pressure upon ZN is found by putting $w=GN$; hence it becomes

[9580c] $$\tfrac{1}{2}g.(GN^2-GZ^2)=\tfrac{1}{2}g.(GN+GZ).(GN-GZ)=\tfrac{1}{2}g.(GN+GZ).NZ,$$

[9580d] which is evidently equal to the pressure of a column whose base is NZ, and height $\frac{1}{2}(GN+GZ)$, as in [9580].

In all these calculations, it is supposed, as in [9434, &c.], that the point N, which is taken
[9580e] as the extreme point of elevation of the fluid near the plane, is not in fact the actual point of contact with the plane, or with the upper surface of the fluid, but is distant from it by an
[9580f] insensible quantity of the order λ, corresponding to the radius of the sphere of activity of the corpuscular attraction, or to the distance between the upper surface *nom* of the fluid, and
[9580g] the assumed surface NOM; the fluid varying in density, from the upper surface *nom*, where the density is very small or nothing, to the lower surface NOM, where the density
[9580h] is the same as that of the internal fluid mass, which is represented by unity. We may also remark, relative to the actual measures of the elevation of the point N above the horizontal
[9580i] level of the fluid in the vase, which we shall represent by $GN=k$, that it is a matter of perfect indifference whether we consider that point as being situated in the surface *nom*, or
[9580k] in the surface NOM, the interval between these surfaces being wholly insensible to our senses, on account of its smallness, so that the one may be used for the other without any
[9580l] appreciable error. Similar remarks may be made relative to the surface of the fluid on the outside of the plane at Z, supposing its elevation above the level of the fluid in the vase to
[9580m] be represented by $GZ=k_{,}$.

The values of k, $k_{,}$ [9580i, m], being substituted in the difference of the pressures on the opposite sides of the plane [9580c], it becomes

[9580n] $$\tfrac{1}{2}g.(k^2-k_{,}^2)=\text{the difference of the pressures computed in [9580].}$$

We shall see, in the second supplement to this book [9983w, &c.], that this expression of the
[9580o] pressure requires some modification when the planes are of a different nature, or, in other words, when the angle ϖ [9346′] corresponding to the inner side of the plane at N, differs from the similar angle $\varpi_{,}$ relative to the outer side of the plane at Z; the correction of this
[9580p] pressure being $\frac{1}{2}g.\alpha^2.(\sin.\varpi-\sin.\varpi_{,})$; so that we shall have, as in [9983w],

[9580q] $$\tfrac{1}{2}g.(k^2-k_{,}^2)+\tfrac{1}{2}g\alpha^2.(\sin.\varpi-\sin.\varpi_{,})=\left\{\begin{array}{c}\text{The corrected differences of the pressures}\\ \text{on opposite sides of the plane } GN.\end{array}\right\}$$

This expression is the same as that given by M. Poisson, in page 172 of his *Nouvelle Théorie, &c.*, and it is reduced to a much more simple form in [9984f].

A similar result holds good for the plane MB; thus we have the force with [9581]
which the two planes tend towards each other, and we see that this force increases
*in the inverse ratio of their distance from each other.** [9582]

Pressure on two planes dipped in a fluid.

In a vacuum, the two planes will also tend towards each other; the adhesion [9583]
of the plane to the fluid produces then the same effect as the pressure of the atmosphere.

We may prove in the same manner, that when the fluid is depressed between [9584]
two planes, the pressure which each plane suffers in an inward direction, is equal to the pressure of a column of the fluid whose height is the half sum of the
depressions below the level, at the points of contact of the internal and external [9585]
surfaces of the fluid with the plane, and whose base is the part of the plane [9586]
included between the two horizontal lines drawn through those points.†

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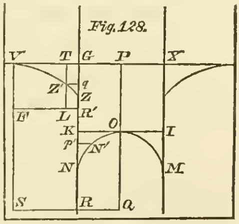
* (4212) When the planes are very near to each other, we shall have GZ^2 very small in comparison with GN^2; and then the whole pressure [9580c], $\frac{1}{2}g.(GN^2-GZ^2)$, is [9582a]
nearly equal to $\frac{1}{2}g.GN^2$; and as GN is nearly in an inverse ratio to GX [9454], the [9582b]
whole pressure will be very nearly in the *inverse ratio of the square of* GX. Moreover, if we divide the whole pressure $\frac{1}{2}g.GN^2$ [9582b], by the height GN, we shall get $\frac{1}{2}g.GN$, [9582c]
for the *mean pressure* on any given point of the column ZN, which will therefore be *inversely as the distance of the planes* GX.

† (4213) The annexed figure 128 is similar to fig. 127, but is adapted to a convex surface NOM, depressed below [9586a]
the level VX of the fluid in the vessel; in this case, the demonstration is nearly the same as in [9552—9578], merely [9586b]
changing the signs of the terms. Thus upon the principles mentioned in [9276, 9294'], we must change the sign of b, in [9559'], and by this means it becomes



$$P+K+\frac{H}{2b}+g.OQ.$$ [9586c]

Also, in [9560], we must change the signs of b, OP; and we get, as in that formula, $\frac{H}{2b}=g.OP$. Substituting this in the preceding expression, it [9586d]
becomes $P+K+g.(OP+OQ)=P+K+g.PQ$, as in [9561]; and the rest of the [9586e]
calculation [9561, 9562] is not altered; so that we find, as in [9562], that the opposite actions at
R, and of course upon any point of the plane NR below N, mutually balance each other. The [9586f]
same takes place on the part GZ, the pressures of the atmosphere on opposite sides mutually destroying each other. It now remains to examine the action upon the part NZ. To determine
this, we shall draw through any point R', a narrow horizontal canal $R'F$, which is bent upwards [9586g]
in FV to meet the level surface of the water in V. Then the pressure at V is $P+K$; and at
R' this will produce a force similar to that computed in [9557], namely, $P+K+g.VF-K$, [9586h]

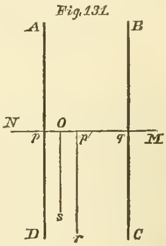
[9587] 12. *It now remains, in order to complete this theory of capillary attraction, to inquire into what produces the concavity or convexity of the fluid contained in the tube or between two planes.* The principal cause is the reciprocal attraction of the tube and fluid, compared with the action of the fluid upon its own particles. *We shall here suppose that these attractions follow the same law, with respect to*

[9587'] *the distance, both for the particles of the tube and for those of the fluid, and that they differ only by their intensities at the same distance;* moreover we shall put

[9588] $\rho =$ the intensity of the action of the particles of the tube upon those of the fluid;

[9588'] $\rho' =$ the intensity of the particles of the fluid upon each other.

This being premised, we shall suppose, in fig. 131, that the lower part of the vertical tube $ABCD$ is dipped into the fluid contained in the vessel, and that MN is the line of the level of the fluid in the vessel. We shall also suppose that the whole surface of the fluid in the tube is plane, and at the same level. The point O of this surface



[9589] included within the sphere of the sensible activity of the tube, will be attracted both by the tube and by the fluid contained within it. The attraction upon this point, produced by *the part of the tube below MN*, may be resolved into two forces, namely,

[9590] A *vertical* force, in the direction of gravity, and represented by ρx;

[9591] A *horizontal* force, in the direction Op, and represented by ρy.

The attraction of the *upper part of the tube* upon the same point O, may, in like manner, be resolved into the following forces;

[9592] A *vertical* force, in the direction of gravity, and represented by $-\rho x$;

[9593] A *horizontal* force, in the direction Op, and represented by ρy.

[9594] We have prefixed the sign — to this last vertical force [9592], because its direction is opposite to that of the force [9590], produced by the attraction of

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or simply $P+g.VF$, pressing in the direction FR'; this is partly balanced by the pressure of the atmosphere P, which acts between the planes, and reduces it to $g.VF$, or $g.GR'$, in the direction FR'; this is of the same form as the pressure mentioned in [9580a]; and by following the same method as in [9580a—c], we obtain the whole pressure on ZN, namely,

[9586i] $$\tfrac{1}{2}g.(GN+GZ).(GN-GZ)=\tfrac{1}{2}g.(GN+GZ).NZ,$$

[9586k] which agrees with what is said in [9585, 9586]. This is to be corrected for the terms depending on $\frac{1}{2}g\alpha^2.(\sin.\varpi-\sin.\varpi_{,})$, as in [9580q, &c.].

the lower part of the tube. To determine the action of the fluid upon the point O, we shall take Op' equal to* Op; then it is evident that the attraction of the part $pp'rD$ of the fluid will be

A *vertical* force, in the direction of gravity, which we shall represent by $\rho' z$. [9595]

The attraction of the part $rp'qC$ of the fluid differs from the attraction of the lower part of the tube by its intensity only;† it may therefore be resolved into the two following forces;

A *vertical* force, in the direction of gravity, and represented by $\rho' x$; [9596]

A *horizontal* force, considered as in the direction Op, and represented by $-\rho' y$. [9597]

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* (4214) In taking $Op' = Op$, the author supposes the fluid to be homogeneous, and neglects the consideration of the change of density, &c., near the surface pOp' of the fluid, [9595a] and near the side ApD of the tube. If we notice this circumstance, we may suppose that Op' is taken of such a magnitude, in comparison with Op, that the horizontal action of the [9595b] fluid $OpDs$ upon the point O may just balance the horizontal action of the fluid $sOp'r$ upon this same point, and by this means the whole action of the fluid $pp'rD$ is reduced to one vertical force $\rho z'$ [9595]. [9595c]

† (4215) The author, in supposing that the action of the tube and that of the fluid differ only by their intensities, neglects wholly the consideration of the variations of the density, as [9596a] well as the corresponding variations in the action of the fluid near its surface and near the side of the tube; and upon this principle he deduces the vertical force of the fluid [9596] [9596b] from that of the tube [9590], by supposing them to be in the ratio of the intensities ρ, ρ' [9588, 9588']. In a similar way he obtains the horizontal force [9597] from that in [9591]. [9596c] The forces $\rho' x$, $-\rho' y$ [9596, 9597], will be changed when we suppose the density to be variable; and then we shall represent them by

A *vertical* force, in the direction of gravity, and represented by $\rho'_{,} x$; [9596d]

A *horizontal* force, in the direction Op, and represented by $-\rho'_2 y$; [9596e]

$\rho'_{,}$, ρ'_2, being functions depending on the nature of the fluid, on that of the tube, on the laws of their corpuscular attraction, and on the action of heat [9173e]. These are to be used instead of $\rho' x$, $-\rho' y$, in [9598—9600]; and we shall obtain, instead of the whole [9596f] resulting forces [9599, 9600], the following expressions;

The *whole vertical* force $= \rho'_{,} x + \rho' z$; [9596g]

The *whole horizontal* force $= (2\rho - \rho'_2) \cdot y$. [9596h]

To make the horizontal force vanish, we must have $\rho'_2 = 2\rho$, instead of $\rho' = 2\rho$, which is given by the author in [9601]; and as ρ'_2 differs from ρ', it is incorrect to state, as the [9596i] author has done in [9601, &c.], that, when $\rho' = 2\rho$, or the intensity of the action of the [9596k] tube is half of that of the fluid, the surface will be horizontal; for this is true only when the density of the fluid is considered as uniform, or $\rho' = \rho'_2$. [9596l]

Hence the point O will be acted upon by the following forces, namely,

[9598] The *vertical* forces ρx, $-\rho x$, $\rho' z$, $\rho' x$ [9590, 9592, 9595, 9596];

[9598′] The *horizontal* forces ρy, ρy, $-\rho' y$ [9591, 9593, 9597].

Connecting together the expressions [9598], we get the whole vertical force [9599]; and in like manner the sum of these in [9598′] gives the horizontal force [9600], namely,

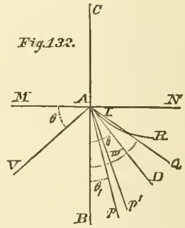
[9599] The *whole vertical* force $= \rho' x + \rho' z$;

[9600] The *whole horizontal* force $= (2\rho - \rho') \cdot y$.

[9601] *This horizontal force will vanish* if $\rho' = 2\rho$, or if *the intensity of the attractive force of the matter of the tube is the half of that of the fluid* [of uniform density] [9596*l*]. Then the point O will be subjected only to the vertical force; and as
[9601′] this is perpendicular to the surface of the fluid, it will maintain it in equilibrium.

We shall now consider a vertical plane CAB, fig. 132, *whose lower extremity is dipped into the fluid contained in the vessel MNB*, supposing AR to be the section of the surface of this fluid, by a vertical plane drawn perpendicular to the first plane CAB, and the line AD to be a tangent to the curve AR at the point A; putting
[9601″] $\theta =$ angle BAD [θ being the angle which is called ϖ in 9346′, &c.]. Then the lower part AB of the plane CAB exerts upon a particle of the fluid which touches it at A, a force which can be resolved into the two following forces, namely,*

Fig. 132.



[9602] A *vertical* force, in the direction AB, and represented by ρK;

[9603] A *horizontal* force, in the direction AM, perpendicular to AB, and represented also by ρK.

In like manner the upper part AC of the plane exerts upon the same particle of the fluid at A, a force which can be resolved into the two following forces, namely,

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* (4216) On account of the excessive smallness of the radius of the sphere of activity
[9602a] of the corpuscular action, we may suppose the lower part of the plane to be a prism, formed by planes drawn through AM, AB, perpendicular to the plane of the figure, and extending infinitely above and below it. Then, as the angle MAB is a right angle, it is evident that
[9602b] the action of this prism in the direction AM is the same as that in the direction AB, both
[9602c] being represented by ρK, as in [9602, 9603]: changing ρ [9588] into ρ' [9588′], we get,
[9602d] *upon the supposition that the fluid is of the uniform density unity*, $\rho' K$ for the vertical action of the fluid in the rectangular space BAN, which agrees with [9605′].

A *vertical* force, in the direction AB, and represented by $-\rho K$; [9604]

A *horizontal* force, in the direction AM, perpendicular to AB, and represented by ρK. [9605]

The particle A will also be attracted by the fluid BAD, and it is evident that as $\rho' K$ represents the vertical action of the fluid [of uniform density], if the angle BAD be a right angle [9602*d*], we shall have, when this angle is θ, a force which may be resolved into the two following forces, namely,* [9605′]

A *vertical* force, in the direction AB, and represented by $\rho' K . \sin.\theta$; [9606]

A *horizontal* force, in the direction AN, perpendicular to AB, and represented by $\rho' K . (1 - \cos.\theta)$. [9607]

For $\rho' K . d\theta_{,} . \cos.\theta_{,}$, and $\rho' K . d\theta_{,} . \sin.\theta_{,}$, will be the elementary attractions of the infinitely small part pAp', in which $d\theta_{,} =$ the angle pAp', $\theta_{,} =$ angle BAp; and by integrating them from $\theta_{,} = 0$, to $\theta_{,} = \theta = BAD$, we shall obtain the preceding expressions [9606, 9607]. [9607′]

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* (4217) We shall suppose the lines Ap, Ap', to be drawn infinitely near to each other, and shall put the angle $BAp = \theta_{,}$, the angle $BAD = \theta$, and the angle $pAp' = d\theta_{,}$; observing that, to avoid confusion, we have inserted the symbol $\theta_{,}$ in [9607′], instead of θ, which is used by the author. We shall also suppose that planes are drawn through the lines Ap, Ap', perpendicular to the plane of the figure, and continued indefinitely above and below it, so as to include a portion of the fluid, corresponding to the angle $d\theta_{,}$, whose action on the point A, in the direction of Ap or Ap' is $M . d\theta_{,}$, M being a quantity depending on the nature of the fluid; and if we suppose, with the author, that the fluid is of uniform density, this quantity M will evidently be wholly independent of $\theta_{,}$. Multiplying this force by $\cos.BAp$, or $\cos.\theta_{,}$, we get $Md\theta_{,} . \cos.\theta_{,}$, for the vertical action of this wedge; and its integral $M . \sin.\theta_{,}$, represents the action of the whole wedge BAp, in a vertical direction. In like manner, the force $Md\theta_{,}$ [9606*c*], resolved in a horizontal direction, is $Md\theta_{,} . \sin.\theta_{,}$; its integral, supposing it to commence with $\theta_{,}$, is $M . (1 - \cos.\theta_{,})$, and this represents the whole horizontal action of the wedge BAp. When $\theta_{,}$ becomes equal to 90^d, both these forces, $M . \sin.\theta_{,}$, $M . (1 - \cos.\theta_{,})$, become equal to M, and as they are represented by $\rho' K$ in [9602*c*, *d*, &c.], we shall have $M = \rho' K$; substituting this, in the preceding expressions of the horizontal and vertical forces, they become respectively $\rho' K . \sin.\theta_{,}$, $\rho' K . (1 - \cos.\theta_{,})$; and when $\theta_{,}$ becomes θ [9606*a*], they change into $\rho' K . \sin.\theta$, and $\rho' K . (1 - \cos.\theta)$, as in [9606, 9607], which correspond to a fluid of uniform density. When the fluid varies in density near its surface, and near the side of the plane, we may suppose the vertical force [9606] to change into $\rho'_3 . K . \sin.\theta$, and the horizontal force [9607] to change into $\rho'_4 . K . (1 - \cos.\theta)$; ρ'_3, ρ'_4, being functions of ρ', θ, depending on the nature of the fluid and plane. The same change of ρ' into ρ'_3, must be made in the vertical forces [9610, 9614]; and ρ' must be changed into ρ'_4, in the horizontal forces [9611, 9615].

[9606*a*] [9606*b*] [9606*c*] [9606*d*] [9606*e*] [9606*f*] [9606*g*] [9606*h*] [9606*h*′] [9606*i*] [9606*k*] [9606*l*] [9606*m*]

The part of the fluid [of uniform density] intercepted between the tangent AD and the curve AR, will act upon the particle A with a force which we
[9608] shall denote by Q, whose direction we shall suppose to be AQ. Now if we put the angle $QAB = \varpi$, we shall have for the action of the fluid DAR, the two following forces;

[9609] A *vertical* force, in the direction AB, and represented by $Q.\cos.\varpi$;

[9609′] A *horizontal* force, in a direction AN, perpendicular to AB, and represented by $Q.\cos.\varpi$.

Then the particle at A will be acted upon by the following forces, namely,

[9610] The *vertical* forces ρK, $-\rho K$, $\rho' K.\sin.\theta$, $Q.\cos.\varpi$ [9602, 9604, 9606, 9609].

[9611] The *horizontal* forces ρK, ρK, $-\rho' K.(1-\cos.\theta)$, $-Q.\sin.\varpi$ [9603, 9605,] [9607, 9609′].

The sign $-$ is prefixed to the two last, because they act from A towards N, or
[9612] in a contrary direction to the two first horizontal forces, which act from N towards A.

[9613] Combining together all these forces, we obtain a single force AV, which must be perpendicular to AD; and we shall put this resultant equal to R. If we resolve it into two others, the one vertical, the other horizontal, we shall have,*

[9614] $$R.\sin.\theta = \rho' K.\sin.\theta + Q.\cos.\varpi;$$

[9615] $$R.\cos.\theta = 2\rho K - \rho' K + \rho' K.\cos.\theta - Q.\sin.\varpi.$$

[When the density of the fluid is not uniform, ρ' must be changed into ρ'_3 in [9614], or into ρ'_4 in [9615], as in [9606m].]

Hence we deduce [for a fluid of uniform density] †

[9616] $$Q.\cos.(\varpi - \theta) = (2\rho - \rho').K.\sin.\theta;$$

[Change ρ' into ρ'_6 [9615g] for a variable density.]

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* (4218) We have the angle $VAB = 90^d - \theta$, and the force in the direction AV is
[9614a] represented by R. We may resolve this force into a *vertical* force $R.\sin.\theta$, in the direction AB, and a horizontal force $R.\cos.\theta$ in the direction AM. Putting the first of these forces equal to the sum of those in [9610], we get [9614]; putting also the second of these forces equal to the sum of those in [9611], we get [9615].

† (4219) Multiplying [9614] by $\cos.\theta$, and [9615] by $-\sin.\theta$, then taking the sum of the products, we get

[9615a] $$0 = -2\rho K.\sin.\theta + \rho' K.(\sin.\theta.\cos.\theta - \sin.\theta.\cos.\theta + \sin.\theta) + Q.(\cos.\varpi.\cos.\theta + \sin.\varpi.\sin.\theta).$$

Rejecting the terms which destroy each other, and putting for the coefficient of Q its value
[9615b] $\cos.(\varpi-\theta)$, [24], Int., it becomes $0 = -2\rho K.\sin.\theta + \rho' K.\sin.\theta + Q.\cos.(\varpi - \theta) = 0$, which is easily reduced to the form [9616], corresponding to a fluid of uniform density.

$Q.\cos.(\varpi - \theta)$ and $\sin.\theta$ being positive when the curve is concave, we see that $2\rho - \rho'$ must be positive, and ρ will then exceed $\frac{1}{2}\rho'$, [supposing the density of the fluid to be uniform, 9618a, &c.]. [9617]

If the factor $2\rho - \rho'$ vanishes, the surface of the fluid [of uniform density] will be horizontal, as we have seen in [9601, 9601']. This value of $2\rho - \rho'$ satisfies the preceding equation, because Q is then equal to nothing.* [9618]

The curves AR relative to the different fluids, which are successively used to fill the same tube, differ from each other. To prove this, we shall consider a point I, placed in all these curves at the same distance from the tube, and within its sphere of sensible activity; the action of the tube upon this point will be the same and horizontal. If all these curves coincide, the action of the fluids upon the point I will have the same direction, but will vary from one [9619] [9620]

When the fluid varies in density, we must change, as in [9606m], ρ' into ρ'_3 in [9614], and ρ' into ρ'_4 in [9615]; by this means the quantity depending on ρ', in [9615a], becomes equal to [9615c]

$$K.(\rho'_3.\sin.\theta.\cos.\theta - \rho'_4.\sin.\theta.\cos.\theta + \rho'_4.\sin.\theta); \qquad [9615d]$$

and if we put $\rho'_5 = \rho'_3.\cos.\theta - \rho'_4.\cos.\theta + \rho'_4$, it becomes $\rho'_5.K.\sin.\theta$; and then the equation [9615b] becomes $0 = -2\rho K.\sin.\theta + \rho'_5.K.\sin.\theta + Q.\cos.(\varpi - \theta)$; whence [9615$e$]

$$Q.\cos.(\varpi - \theta) = (2\rho - \rho'_5).K.\sin.\theta, \qquad [9615f]$$

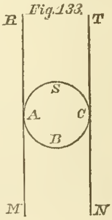
which is to be used instead of [9616]; so that, to make the equation [9616] correspond to a fluid of variable density near its surface and near the side of the tube, we must change ρ' into ρ'_5, ρ'_5 being a function depending on the nature of the fluid, on that of the tube, on the law of the capillary attraction, and on that of the repulsion of heat; and it will follow from [9615f], that $2\rho - \rho'_5$ must generally be of the same order as Q; observing that Q must be much smaller than $2\rho K$, because Q [9608] depends on the action of the very small meniscus RAD, fig. 132, whereas ρK or $\rho'_5 K$ depends on the action of the matter in the whole quadrantal space MAB or NAB. This change of ρ' into ρ'_5 in the equation [9616] obviates the objection made to it by Dr. Young; but, at the same time, it takes away its important character in determining the relation between the quantities ρ, ρ', and the angle θ, which will hereafter be illustrated in a different manner. [9615g] [9615h] [9615i] [9615k]

* (4220) In this case, the curve AR coincides with the horizontal line AN, so that the attracting fluid, included in the space NAR vanishes; consequently Q [9608] becomes nothing; and as we then have $\varpi = \theta = 90^d$, the equation [9616] becomes $0 = (2\rho - \rho').K$, or $\rho' = 2\rho$, for a fluid of uniform density, as in [9601]. When the fluid varies in density near the surface and near the side of the tube, the preceding equation becomes $0 = (2\rho - \rho'_5).K$ [9615f]; but when $\theta = 90^d$, we have $\rho'_5 = \rho'_4$ [9615e]; hence $0 = (2\rho - \rho'_4).K = 0$, or $\rho'_4 = 2\rho$, which is similar to [9596i]. [9618a] [9618b] [9618c]

fluid to another, in proportion to the respective intensity of the action of these fluids. This action, being combined with the horizontal action of the tube, will
[9621] therefore produce a resultant whose direction will differ in the different fluids. This resultant ought to be, by the condition of the equilibrium, perpendicular to the surface; therefore the inclination of the planes of that surface will not be
[9622] the same for the different fluids, which is contrary to the hypothesis. Thus the curves AR differ according to the ratio of the respective intensities ρ and ρ'. Their extreme sides, at the limit of the sphere of sensible activity of the tube, have different inclinations relative to the sides of the tubes. This inclination
[9623] determines, as we have seen, the magnitude of the segment of the spherical surface, which the surface of the fluid included in a very narrow tube assumes,
[9624] beyond the sphere of sensible activity of the tube; and this magnitude determines the radius of that surface, the inverse ratio of which determines the ascent of the fluid in the tube [9368, &c.].

[9625] In proportion as the ratio ρ to ρ' increases, the curve AR becomes more and more concave; *and when ρ is equal to ρ', the surface of the fluid in the tube is a concave hemisphere.* To prove this, we shall suppose, in fig. 133, that the tube is of the same matter as the fluid, and that its surface ABC is a hemisphere; we shall also suppose that the whole
[9626] spherical surface $ABCS$ is completed, and that the fluid fills the
[9627] upper part of the tube* $RASC$. *Neglecting the force of gravity*, as we may do in very narrow tubes, it is evident that, by reason of the homogeneousness of the matter of the tube and of the fluid, all the points of the concave surface ABC will be affected, in consequence of the attractions of the tube and of the
[9627′] fluid, by equal forces perpendicular to the surface, which suffices

Fig. 133.



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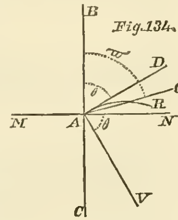
* (4221) The tube being filled with a fluid of uniform density, having the same intensity
[9626a] of action as that of the tube, and the effect of gravity being insensible [9627], it is evident that, in noticing the conditions of the equilibrium of the spherical surface $ABCS$, we
[9626b] may neglect the consideration of the action of the fluid contained within that surface, because this action is perpendicular to that surface. Then the equilibrium of the surface $ABCS$
[9626c] will be sustained by the combined action of the tube, of the upper part $RASCT$, and of the lower part $MABCN$, of the fluid; and it is in this way that the author considers the
[9626d] subject in [9625—9629], and finally demonstrates that, in this case of $\rho=\rho'$, with a fluid of
[9626e] uniform density, the surface ABC in a capillary tube is a concave hemisphere, making the
[9626f] angle $\varpi=0$; ϖ [9601″] being the angle formed by the line AM and the tangent of the surface drawn through A in a downward direction.

A little consideration will make it evident, that the same result must very nearly obtain,

for the equilibrium of the fluid. Now if we take away the upper fluid $RASC$, there can be produced only insensible changes in the magnitudes of the forces which act upon the different points of the surface ABC, and in the directions of these forces. For, AR being a tangent to the spherical surface, it is evident [9628]
that the action of the part RAS of the fluid upon the point A, is incomparably less than the action of the tube upon that point, since the attraction becomes insensible at any sensible distance;* therefore the equilibrium of the lower fluid $ABCNM$, will not be affected by the suppression of the upper part $RASC$; *whence it follows that the surface of the fluid is a concave hemisphere* [9629]
when ρ *is equal to* ρ'.

The surface of the fluid is a concave hemisphere when ρ, ρ', are equal.

If the intensity of the attraction of the tube upon the fluid exceeds that of the attraction of the fluid towards its own particles, it appears to me probable that, [9630]
in that case, the fluid, by attaching itself to the tube, forms an interior tube, which alone produces the elevation of the fluid in the tube, whose surface thus becomes concave and hemispherical [9629]. [9631]
I suspect this to be the case with water [*alcohol*] *or oil in a glass tube.*



We shall now consider the case, in which the surface of the fluid, instead of being concave, is convex; and we shall suppose BAC, fig. 134, to be a vertical plane, whose lower extremity is dipped into a fluid contained in the vessel MNC; AR a section of the surface of the fluid, by a [9632]

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when we notice the change of density of the fluid, arising from the corpuscular action. For [9626g]
when $\rho = \rho'$, the action of the tube upon the fluid is the same as the action of the fluid [9626h]
upon its own particles; therefore the fluid will not vary in density near the sides AM, CN, of the tube; so that the particles of the fluid near the points A, C, must be very nearly in the same state as if the density of the whole fluid was equal to unity. For the action of the part [9626i]
of the surface of variable density near B, where this variation of density is the greatest, is [9626k]
insensible at A, on account of its distance; and the same is very nearly true for the other parts of the surface BA, until we approach quite near to A; so that we may suppose the [9626l]
fluid on the surface between B and A, to have the same uniform density unity without altering sensibly the action near A, or changing the corresponding angle of inclination, [9626m]
which must therefore be very nearly represented, as in [9626f], by $\varpi = 0$, if $\rho = \rho'$, even when we take into consideration the change of density at the surface of the fluid from the [9626n]
corpuscular action, &c.

* (4222) Observing that the angle of contact between the curve AS and the tangent AR is infinitely small. [9628a]

plane perpendicular to the first plane; AD a tangent to the curve AR; and we
[9633] shall put the angle $BAD=\theta$. Then the *vertical* attraction of the fluid DAN [of uniform density] upon the point A, will be, by what has been said,*
[9634] $-\rho'K.(1-\sin.\theta)$, *downwards*, or in the direction AC; and the *horizontal*
[9635] attraction will be $\rho'K.\cos.\theta$, from A towards N. To obtain the attraction of the fluid RAN, we must subtract from the preceding attractions those of the
[9636] segment DAR. We shall put Q for the action of this segment DAR upon the point A, and AQ for its direction, and we shall also put the angle
[9637] $BAQ=\varpi$. Then the *vertical* attraction of the segment DAR is $-Q.\cos.\varpi$, in the direction AC; and its *horizontal* attraction, in the direction AN, is
[9637′] $Q.\sin.\varpi$. Hence the vertical attraction of RAN becomes†

[9638] $$Q.\cos.\varpi-\rho'K.(1-\sin.\theta), \qquad \text{[Vertical force in the direction } AC.]$$

and its horizontal attraction will be

[9639] $$-Q.\sin.\varpi+\rho'K.\cos.\theta. \qquad \text{[Horizontal force in the direction } AN.]$$

[9640] The *vertical* attraction of the fluid NAC is $\rho'K$, and its *horizontal* attraction is the same.‡ Lastly, the *vertical* attraction of the plane BAC is nothing, and

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[9634a] * (4223) It is shown in [9606, 9607], that the action of the fluid, of uniform density, contained between the planes BA, DA, fig. 132, are $\rho'K.\sin.BAD$, in the direction AB, and $\rho'K.(1-\cos.BAD)$, in the direction AN. In like manner, in fig. 134, the action of
[9634b] the fluid, corresponding to the space DAN, is $\rho'K.\sin.DAN$, in the direction AN, and $\rho'K.(1-\cos.DAN)$, in the direction AB; and by putting, as in [9633], $DAN=90^d-\theta$,
[9634c] these forces become respectively $\rho'K.\cos.\theta$, in the horizontal direction AN, as in [9635], and
[9634d] $\rho'K.(1-\sin.\theta)$, in the direction AB, or $-\rho'K.(1-\sin.\theta)$, in the direction AC, as in [9634]; the vertical action being always reckoned downwards, or in the direction AC. Moreover
[9634e] the force Q, in the direction AQ, produces the force $-Q.\cos.\varpi$, in the direction AC, and the force $Q.\sin.\varpi$, in the direction AN, as in [9637]. If we wish to notice the change of density of the fluid, we must, in like manner as in [9606*l*, *m*], change the forces
[9634f] $-\rho'K.(1-\sin.\theta)$, $\rho'K.\sin.\theta$, into $-\rho'_6K.(1-\sin.\theta)$, $\rho'_7K.\sin.\theta$, respectively, ρ'_6, ρ'_7, being functions similar to ρ'_3, ρ'_4.

[9637a] † (4224) Subtracting the vertical action of the part DAR [9637] from that of DAN [9634], we obtain the action of the part RAN, as in [9638]. In like manner, by subtracting
[9637b] the horizontal action $Q.\sin.\varpi$ [9637′], corresponding to the part DAR, from that of DAN [9635], we obtain that corresponding to RAN, as in [9639].

‡ (4225) This is proved in [9606*h*]; it may also be deduced from [9638, 9639]; for,
[9638a] by putting $\varpi=0$, $\theta=0$, $Q=0$, the space NAR becomes NAB, and the expressions
[9638b] [9638, 9639] become respectively $-\rho'K$, $\rho'K$, representing the whole action of NAB in

its *horizontal* attraction is* $-2\rho K$; therefore the fluid will be acted upon by the vertical force [9641]

$$Q.\cos.\varpi - \rho' K.(1-\sin.\theta) + \rho' K, \quad \text{or} \quad Q.\cos.\varpi + \rho' K.\sin.\theta, \qquad \left[\begin{smallmatrix}\text{Vertical force in the}\\ \text{direction } AC.\end{smallmatrix}\right] \qquad [9642]$$

and by the horizontal force

$$-Q.\sin.\varpi + \rho' K.\cos.\theta + \rho' K - 2\rho K. \qquad \left[\begin{smallmatrix}\text{Horizontal force in}\\ \text{the direction } AN.\end{smallmatrix}\right] \qquad [9643]$$

We shall suppose $AV = R$ to be the resultant of these forces, which we shall suppose to be perpendicular to AD; then we shall have† [9644]

$$R.\sin.\theta = Q.\cos.\varpi + \rho' K.\sin.\theta, \qquad [9645]$$

$$R.\cos.\theta = -Q.\sin.\varpi + \rho' K.\cos.\theta + (\rho' - 2\rho).K; \qquad [9646]$$

whence we deduce [for fluids of a uniform density]

$$Q.\cos.(\varpi - \theta) = (\rho' - 2\rho).K.\sin.\theta; \qquad \left[\begin{smallmatrix}\text{If we notice the variation of density,}\\ \text{we must change } \rho' \text{ into } \rho'_8.\end{smallmatrix}\right] \qquad [9647]$$

$\sin.\theta$, Q, and $\cos.(\varpi - \theta)$, being positive, when the curve AR is convex; the

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a vertical or horizontal direction respectively. That of NAC will evidently be the same, except in the sign of the vertical force $-\rho' K$, which must evidently be changed into $+\rho' K$, as in [9640]. [9638c]

* (4226) The vertical forces of the two parts of the plane AB, AC [9602, 9604], represented by ρK, $-\rho K$, mutually destroy each other. The horizontal forces of each of these parts is also equal to ρK [9603, 9605], in the direction AM; or $-\rho K$, in the direction AN, in which the horizontal force is reckoned in [9634c, 9639]. Connecting together the vertical forces of the fluid and plane [9638, 9640], it becomes as in [9642]; also the horizontal forces [9639, 9640, 9641], being connected together, give the whole force [9643]. [9639a] [9639b] [9639c] [9639d]

† (4227) The force R, in the direction AV, produces the vertical force $R.\cos.CAV$, and the horizontal force $R.\sin.CAV$; and as $BAD = \theta$, $DAV = 90^d$, $CAV = 90^d - \theta$, these forces become $R.\sin.\theta$, and $R.\cos.\theta$; so that, by putting them respectively equal to the forces [9642, 9643], we obtain the equations [9645, 9646]. Now multiplying [9645] by $\cos.\theta$, also [9646] by $-\sin.\theta$, and adding together the products, we get, by neglecting the terms which destroy each other, $0 = Q.(\cos.\varpi.\cos.\theta + \sin.\varpi.\sin.\theta) - (\rho' - 2\rho).K.\sin.\theta$; and as the coefficient of Q is $\cos.(\varpi - \theta)$, it is easily reduced to the form [9647], which corresponds to a fluid of uniform density. The case of a variable density may be considered as in [9606*l*, *m*, 9615*d*, &c., 9618*c*]; and the final result will be, that we must change, in [9647], ρ' into a function of ρ', ρ, θ, which we shall denote by ρ'_8; so that [9647] will become $Q.\cos.(\varpi - \theta) = (2\rho - \rho'_8).K.\sin.\theta$; observing, as in [9615*h*—*k*], that $2\rho K$, $\rho'_8 K$, are much greater than Q, and $(2\rho - \rho'_8).K$ is generally of the same order as Q. [9645a] [9645b] [9645c] [9645d] [9645e] [9645f] [9645g]

[9648] factor $\rho' - 2\rho$ *will be positive, or the intensity* ρ *will be less than* $\frac{1}{2}\rho'$ [the density of the fluid being supposed uniform].

If we compare this result with the preceding [9617], *we shall see that the*
[9649] *surface of the fluid in a tube will be concave or convex, according as* ρ *is greater or less than* $\frac{1}{2}\rho'$ [*the fluid being supposed of a uniform density*].

[9650] The tube being capillary, the surface will approach so much the more towards the form of a convex hemisphere, as ρ shall be diminished; and if ρ *be nothing, or insensible, this surface will be a convex hemisphere.* To prove this, we shall suppose, that this surface ASC, fig. 133, page 784, is a hemisphere, and we
[9651] shall continue it below A, so as to complete the whole sphere $ASCB$. Then, if we suppose the fluid $ABCNM$ to be suppressed, and neglect the force of gravity, it is evident that all the points of the surface ASC, will be urged by equal forces perpendicular to that surface; therefore the fluid will be in
[9652] equilibrium. If we now replace the suppressed fluid, it is evident that, AM
[9652'] being a tangent to the sphere, the action of the fluid MAB upon the point A will be incomparably less than the action of the sphere upon this point; we may therefore neglect it, and with much greater reason we may neglect the
[9653] action of the same fluid upon the other points of the surface ASC; *therefore*
[9654] *the equilibrium takes place when the convex surface of the fluid is hemispherical.** Between the limits $\rho = 0$ and $\rho = \frac{1}{2}\rho'$, the surface becomes less and less
[9655] convex [9649]. It is horizontal when $\rho = \frac{1}{2}\rho'$ [9618]; when it exceeds $\frac{1}{2}\rho'$,

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* (4228) The surface will be very nearly that of a convex hemisphere, if the action of
[9654a] the tube on the fluid is nothing, or $\rho = 0$, even when we notice the change of density of the fluid near its surface and near the sides of the tube. For as the tube has no action on the
[9654b] fluid, the part of the fluid which is situated about the point A, fig. 133, page 784, must by very nearly in a similar situation to that which is in contact with the atmosphere at the surface
[9654c] ASC; because the atmosphere has been found, by observation, to have no sensible effect
[9654d] on the capillary phenomena; therefore we may suppose the thin stratum of variable density at A and C to be very nearly the same as at S; and we may suppose that this
[9654e] stratum is continued round the lower part of the spherical surface ABC, without producing any sensible effect on the fluid in the upper surface ASC, on account of its great distance, in comparison with the radius of the sphere of activity of the corpuscular attraction, as is
[9654f] supposed in [9652']. Now with a spherical mass $ABCS$, whose exterior part is formed of concentrical strata, varying in density from one stratum to another, the equilibrium
[9654g] will hold good as in the case of uniform density, as is evident from the process of reasoning which is used in [9650—9654] for a sphere of uniform density. We shall also prove by
[9654h] another method, in [9980x], that, when $\rho = 0$, the surface of the fluid is that of a concave hemisphere.

the surface becomes more and more concave [9617]; and lastly it is that of a concave hemisphere when $\rho = \rho'$ [9625].* [9655']

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* (4229) We have seen that, in all the calculations of this article [9587—9655], the author supposes the whole mass of the fluid to be of the same uniform density, and neglects [9655a]
wholly the consideration of the change of density near the surface of the fluid and near the sides of the tube. The subject is resumed by him, in a different manner, but upon the same [9655b]
principles, in his second supplement; and he finally obtains the formula $\rho = \rho' . \cos.^2 \frac{1}{2}\varpi$ [9935], [9655c]
for the relation between the intensities ρ and ρ'; supposing $\varpi =$ the angle BAD, fig. 132, page 780, to represent the angle contained between the vertical line AB, and the tangent AD to the curve AR, at the point A. When $\varpi = 0$, this formula gives $\rho = \rho'$, as in [9655d]
[9625]; when $\varpi < 90^d$, it becomes $\rho > \frac{1}{2}\rho'$, or $2\rho - \rho'$ positive, as in [9617]; when [9655e]
$\varpi = 90^d$, it becomes $\rho = \frac{1}{2}\rho'$, or $2\rho - \rho' = 0$, as in [9618]; when $\varpi > 90^d$, it becomes $\rho < \frac{1}{2}\rho'$, as in [9648]; lastly, when $\varpi = 180^\circ$, it becomes $\rho = 0$, as in [9650]. [9655f]

These results require some modification when we notice the change of density in the fluid. For we have seen, in [9654a—g], that, when $\rho = 0$, the surface of the fluid in a capillary [9655g]
tube is *a convex hemisphere*, and the angle $\varpi = 180^d$. This is the situation of water in a capillary tube whose internal surface has been well covered with an oily substance. We have also found, in [9626n], that, when $\rho = \rho'$, the surface of the fluid in a capillary tube will be very [9655h]
nearly that of a *concave hemisphere* corresponding to $\varpi = 0$. Hence it appears that, while [9655i]
ρ varies from $\rho = 0$ to $\rho = \rho'$, the angle ϖ varies from 180^d to 0^d. The mean between these extreme values would give $\rho = \frac{1}{2}\rho'$, corresponding to $\varpi = 90^d$, or to a horizontal [9655k]
surface; which is found to be true in [9601], when the fluid is supposed to be of uniform density; but we cannot infer that, when $\varpi = 90^d$, we shall have $\rho = \frac{1}{2}\rho'$, if we notice the [9655l]
change of density of the fluid as we have already observed in [9596l]; though it may be considered as an approximation towards the true value of ρ, which cannot be accurately [9655m]
determined till the law of the corpuscular attraction is known. Finally, when ρ exceeds ρ', [9655n]
the fluid attaches itself to the side of the tube, as in [9630, 9631], and forms an interior tube, of the same matter as the fluid itself, and which will correspond to the case of $\rho = \rho'$ [9655o]
[9655h]; then the fluid in a capillary tube will have very nearly, for its surface, that of a concave hemisphere [9655i].

SECTION II.

COMPARISON OF THE PRECEDING THEORY WITH OBSERVATION.

13. We have seen, [in 9379, 9409—9411, 9454], that, by the theory, a fluid
[9656] is elevated or depressed in capillary tubes, of the same kind, in the inverse ratio of their diameters; that, between two vertical and parallel planes, which are very near to each other, the fluid is elevated or depressed in the inverse ratio of their distance; lastly, that the elevation or depression of the fluid between these
[9657] planes, is the same as in a tube whose interior semi-diameter is equal to the distance of the planes [9410]. All these phenomena have been observed for a long time by philosophers, as may be seen by the following passage in Newton's Optics, question 31, vol. 4, page 253, of Horsley's edition of Newton's works.

"If two plane polished plates of glass (suppose two pieces of a polished looking-glass) be laid together so that their sides be parallel, and at a very small distance from one another, and then their lower edges be dipped into water, the water will rise up between them. And the less the distance of the glass is, the greater will be the height to which the water will rise. If the
[9658] distance be about the hundredth part of an inch, the water will rise to the height of about an inch; and if the distance be greater or less in any proportion, the height will be reciprocally proportional to the distance very nearly. For the attractive force of the glasses is the same, whether the distance between
[9659] them be greater or less; and the weight of the water drawn up is the same, if the height of it be reciprocally proportional to the distance of the glasses. And in like manner, water ascends between two marbles, polished plane, when their polished sides are parallel, and at a very small distance from one another. And if slender pipes of glass be dipped at one end into stagnating water, the
[9660] water will rise up within the pipe; and the height to which it rises will be reciprocally proportional to the diameter of the cavity of the pipe, and will equal the height to which it arises between two planes of glass, if the semi-diameter of the cavity of the pipe be equal to the distance between the

planes, or thereabouts. And these experiments succeed after the same manner [9661] *in vacuo* as in the open air (as hath been tried before the Royal Society), and therefore are not influenced by the weight or pressure of the atmosphere."

Messrs. Haüy and Tremery have been willing to make, at my request, some experiments of the same kind. In a glass tube whose internal diameter is two millimetres, they have found the elevation of the water, above the level, to be [9662] $6^{mi.},75$, and that of oil of orange, $3^{mi.},4$.

In a second glass tube, $\frac{4}{3}$ of a millimetre in diameter, the elevation of the [9663] water was 10 millimetres, and that of oil of orange 5 millimetres.

In a third glass tube, $\frac{3}{4}$ of a millimetre in diameter, the elevation of the water [9664] was $18^{mi.},5$, and that of oil of orange, 9 millimetres.

If the elevation of any fluid is in the inverse ratio of the diameter of the [9665] tube [9379], the product of this elevation by the corresponding diameter of the tube, must be the same for all the tubes; and this product reduced to square millimetres, and divided by one millimetre, will give the ascent of the fluid in a [9666] tube whose diameter is one millimetre. Thus, by multiplying each of the preceding elevations by the corresponding diameter of the tube, we shall have the three following results for the ascension in a tube whose diameter is [9666′] 1 millimetre;*

	Water.	Oil of Orange.		
First tube,	$13^{mi.},50$;	$6^{mi.},8$.	Elevations reduced to that of a tube whose diameter is $1^{mi.}$, being the same as the values of $\frac{2}{\alpha}$, or $2a^2$.	[9667]
Second tube,	$13^{mi.},333$;	$6^{mi.},667$.		[9668]
Third tube,	$13^{mi.},875$;	$6^{mi.},75$.		[9669]

The near agreement of these results, both as it respects water and oil of oranges, proves the accuracy of the law of ascension of these fluids, in the [9669′] inverse ratio of the diameter of the tube. The mean of the results in [9667, 9668, 9669], gives for the elevation of these fluids, the following quantities;

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* (4230) The quantities [9667] are deduced from the numbers (9662), thus, [9667a] $2\times 6^{mi.},75=13^{mi.},50$, $2\times 3^{mi.},4=6^{mi.},8$. In like manner [9668, 9669], are deduced respectively from the numbers [9663, 9664], respectively. The results of a much more accurate [9667b] set of experiments on the ascent of water, alcohol, and oil of turpentine, by Gay-Lussac, who took great care in previously wetting the tubes, are given in [10304—10348]. The expressions [9670, 9670′], represent, as in [9372y], the values of $\frac{2}{\alpha}$, corresponding to water [9667c] and alcohol, as deduced from these experiments of Haüy, who was not careful in wetting the tubes. The results of Gay-Lussac's experiments [10308, 9702k], are nearly twice as great as those in [9670, 9670′].

[9670] Elevation of water, $13^{mi.},569$, in a tube whose diameter is $1^{mi.}$ $\left[\text{or } \frac{2}{\alpha}=2\alpha^2=13^{mi.},569\right]$;

[9670′] Elevation of oil of oranges, $6^{mi.},7389$, in a tube whose diameter is $1^{mi.}$ $\left[\text{or } \frac{2}{\alpha}=2\alpha^2=6^{mi.},7389\right]$.

The two first tubes of which we have spoken, whose diameters are
[9671] 2 millimetres, and $\frac{4}{3}$ of a millimetre, were made use of to determine the depression of mercury below the level. For this purpose they were placed in a basin of mercury, at a depth which was accurately measured. Then, a very smooth plane, which prevented the fluid from running out, having been slid under the bottom of the tubes, they were taken out from the basin, and the heights of the columns of mercury above the plane were measured. The differences between these heights and the lengths of the immersed part of the tube, give the depressions of the mercury below its level. In this way was found $3^{mi.}\frac{2}{3}$ for
[9672] the depression, in the tube whose diameter is 2 millimetres, and $5^{mi.},5$ for the depression in the tube whose diameter is $\frac{4}{3}$ of a millimetre. Each of these
[9673] experiments gives $7^{mi.},333$ for the depression of the mercury in a tube whose diameter is a millimetre;* therefore we see, in this case, the accuracy of the law of the depression of the fluids, in the inverse ratio of the diameter of the tubes.

Messrs. Haüy and Tremery have also observed the elevation of water between
[9674] two vertical and parallel plates of glass, distant from each other by 1 millimetre. They have found it to be $6^{mi.},5$, which differs but very little from the elevation
[9674′] of water in a tube whose radius is a millimetre; for this last elevation ought to
[9675] be, by the preceding experiments, equal to the half of $13^{mi.},569$, or $6^{mi.},784$.† Thus the result of the theory by which water ought to ascend between these
[9676] planes as much as it would in a tube whose radius is equal to that distance [9410], is conformable to this experiment. We have seen, in the passage quoted from Newton's Optics [9658], that, when the two planes of glass are at the distance of a hundredth part of an English inch, the corresponding
[9677] elevation of the water is one inch. The product of these two quantities
[9678] is $\frac{1^{in.}}{100}$. An English inch is $25^{mi.},3918$; hence $\frac{(1^{in.})^2}{100}=\frac{(25^{mi.},3918)^2}{100}$, or
[9679] $6^{mi.},4474$. Dividing it by $1^{mi.}$, we obtain $6^{mi.},4474$, for the ascent of the water

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[9673a] * (4231) Thus $3\frac{2}{3}\times 2=7^{mi.},333$, and $5,5\times\frac{4}{3}=7^{mi.},333$, as in [9673].

† (4232) From [9670], the elevation of water in a tube whose *diameter* is one millimetre,
[9675a] is $13^{mi.},569$; and if its *radius* be $1^{mi.}$, or diameter $2^{mi.}$, the elevation will be, from [9361], $\frac{13^{mi.},569}{2}=6^{mi.},784$, as in [9675].

between two parallel planes, distant one millimetre from each other, which differs but little from the preceding result [9674'].

We have seen, in [9410], that, if we introduce into a cylindric tube a cylinder having the same axis as the tube, the elevation of the water, in the circular space [9680]
included between the interior surface of the tube and the surface of the cylinder, is equal to the elevation of the water in a tube which has for its radius the width of this circular space. One of the limits of this general case is the [9681]
particular case, where the radii of the tube and cylinder are both infinite; this corresponds to the case of two parallel planes which are very near to each other. We have just seen, in [9674—9677], that the general result at this limit is verified by experiment. The other limit is that where the radii of the tube and cylinder are very small. To verify, in this case, the result of analysis, [9682]
M. Haüy took a glass tube, of uniform caliber, whose interior diameter was five millimetres, and placed within it a glass cylinder, whose diameter was three millimetres, with all the necessary precautions to make the axis of the tube coincide with that of the cylinder. Then, dipping into the water the tube [9683]
and cylinder thus prepared, he observed the elevation of the fluid within the circular space to be very nearly seven millimetres, but rather below it. The width of the circular space being in this case one millimetre, the water ought, by the theory,* to rise as it does between two parallel planes at the distance of one millimetre; therefore this elevation ought to be $6^{mi\cdot},784$ [9675], which [9684]
agrees perfectly with experiment. Thus the general result of the theory upon the elevation of water between the circular space included between a tube and an included cylinder, is verified at both its limits.

The result of any experiment will vary a little with the temperature, and we may consider the preceding observations as being made when the temperature [9685]
was at ten degrees of the centigrade thermometer. All these experiments require particular attention relative to the caliber of the tube and the exact measure of its diameter; care must also be taken to prevent the surface from being either too dry or too wet. In measuring the elevation of the fluid, we [9686]
must keep the tube dipped into the fluid; for, by taking it out, the drop which is formed at the bottom of the tube will rise the fluid in the tube [9717, &c.]. We must also measure these elevations from the level of the fluid in the vessel to the lowest point of its surface in the tube, if the fluid be elevated; or to its [9687]
highest point, if the fluid be depressed.

14. One of the most interesting of the capillary phenomena, and the most

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* (4233) This appears by comparing formula (9409) with the remarks in (9412). [9683a]

proper to verify the preceding theory, is that of the suspension of a drop of fluid in a conical capillary tube, or between two planes which form a very small angle
[9688] with each other. We have given the analysis of it in [9472, 9549], and we shall now proceed to compare it with observation. Hauksbee very carefully made an experiment of this kind with a drop of oil of oranges suspended between two glass planes; his account is nearly as follows:

[9689] "I took two rectangular glass planes, each twenty inches long and three inches wide; that which I used for the lower plane had its surface parallel to the horizon. These planes touched at two of their narrow borders, and the axis from which the distances of the drop were counted, was on the opposite narrow
[9689′] border of the lower plane. The* planes being very clean, they were rubbed with a linen cloth dipped in oil of oranges: then, a drop or two of the same oil being let fall upon the lower plane, near the axis, the other plane was laid upon it; and as soon as it touched the oil, the drop spread itself considerably between both their surfaces. Then, the upper plane being raised a little at the same end by a screw, the oil immediately attracted itself into a body, forming a globule contiguous to both surfaces, and began to move towards the touching ends. When it was 2 inches from the axis, an elevation of 15 minutes [sexagesimal] at the touching ends stopped its progress, and it remained there without motion any way. The planes being let fall again, the drop moved forward till it was four inches from the axis; then an elevation of $25^{m.}$ was required to give it a fixed station. At 6 inches, it required an angle of $35^{m.}$; at 8, of $45^{m.}$; at 10, a
[9690] degree. At 12 inches from the axis, the elevation was $1^{d.}\,45^{m.}$, and so on, at the several stations, as they stand in the annexed table. This, after numerous trials, I take to be the most correct, though the others succeeded very nearly the same. It is to be observed, that when the globule, or drop, had arrived to near 17 inches on the planes from their axis, it would become of an oval form; and as it ascended higher, so would its figure become more and more oblong; and unless the drop was small, on such an elevation of planes as was required at
[9690′] such a progress of the drop, it would be parted, some of it descending, and the rest of it running up to the top at once; but on a drop that separated there, I found the remaining part of it, at 18 inches, would bear an angle of elevation equal to $22^{d.}$ to balance its weight. Higher than that I could not observe.
[9691] The planes were separated at their axis about $\frac{1}{16}$ of an inch. I found but

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* (4234) The rest of this paragraph [9689—9691′] is copied from the account given by
[9689a] Hauksbee in the Philosophical Transactions for 1712, and differs a little from that given by La Place.

little difference between small and large drops of oil, in regard to the experiment. The angles were measured by a quadrant marked on paper, of near 20 inches [9691'] radius, divided into degrees and quarters.

Distance from the axis in inches.	Angles of elevation in sexagesimal degrees and minutes.
2in.	15m.
4	25m.
6	35m.
8	45m.
10	1d. 0m.
12	1d. 45m. [9692]
14	2d. 45m.
15	4d.
16	6d.
17	10d.
18	22d.

Haulksbee does not say that the distances were counted from the middle of the drop; but there is reason to believe that this was the case, from a passage in Newton's Optics which we shall soon quote; we shall suppose this [9693] to be the case in the following calculations, since any inaccuracy in the supposition will have but very little influence upon the result.

We shall put

V = the inclination to the horizon of an *intermediate plane*, having a [9694] common intersection with the two planes which were used in the preceding experiments, and bisecting the angle which they form with each other;

h = the height to which the fluid would rise between two vertical and [9695] parallel planes, whose distance is equal to that of the two planes which were used in the preceding experiments, *at the distance b from the common intersection;* [9696]

a = the distance from the middle of the drop to the line of intersection of [9697] the two planes;

then we shall have very nearly, from [9549, 9454, &c.],*

$$\sin.V = \frac{h}{b}\cdot\frac{b^2}{a^2}. \qquad [9698]$$

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* (4235) The value of h [9695] is given in [9454] under the form $h = \frac{H.\sin.\theta'}{g.2l}$; [9698a] $2l$ being the distance of the two planes [9452]. Now by [9526], the inclination of the two

[9699] In the preceding experiment, the planes were distant from each other $\frac{1}{16}$ of an English inch, at 20 inches distance from their line of intersection [9691, 9689], or at their axis situated at the extremity of these planes. Therefore their distance
[9699′] from each other was only $\frac{1}{32}$ of an inch, at the distance of 10 inches; and we
[9700] shall suppose b [9696] *to be equal to ten English inches.* As a half millimetre of distance between two vertical and parallel planes, corresponds, by the
[9701] preceding article, to an elevation of the oil of orange of $6^{\text{mi.}},7389$,* we shall have

[9702] $$(0{,}5\times 6{,}7389) \text{ square millimetres} = (\tfrac{1}{32}\times h) \text{ square inches},$$

[9698b] planes is 2ϖ; therefore the distance of these planes from each other, at a point whose distance from their common section is b, will be nearly $2l=2b.\text{tang.}\varpi$, on account of the smallness
[9698c] of the angle ϖ. Substituting this in the preceding value of h, it becomes $h=\frac{H.\sin.\theta'}{g.2b.\text{tang.}\varpi}$.
[9698d] Multiplying this by $\frac{b}{a^2}$, we get $\frac{bh}{a^2}=\frac{H.\sin.\theta'}{2a^2.g.\text{tang.}\varpi}$; hence, by [9549], $\frac{bh}{a^2}=\sin.V$, or, as it may be written, $\sin.V=\frac{h}{b}.\frac{b^2}{a^2}$ [9698]. In this demonstration, we might have used
[9698e] $\sin.\varpi$ instead of $\text{tang.}\varpi$, by measuring the half interval of the planes, from the centre of the drop to either of the planes which are used in the experiment; then the formula [9549*f*] would agree with [9698].

[9702a] * (4236) The elevation of any fluid in a *tube* whose diameter is $1^{\text{mi.}}$, or radius $0^{\text{mi.}},5$, is the same as between two parallel planes whose distance is $0^{\text{mi.}},5$ [9110]. Now by [9670′],
[9702b] the elevation of oil of oranges in a tube of the diameter $1^{\text{mi.}}$, is $6^{\text{mi.}},7389$; therefore, by the
[9702c] principle used in [9665, &c.], we must have $0^{\text{mi.}},5\times 6^{\text{mi.}},7389$, for the constant quantity
[9702d] representing the product of the distance of two parallel planes, by the elevation h of the oil of oranges between the two planes. To reduce the constant quantity [9702*c*] from square millimetres to square inches of English measure, we must divide it by the square of
[9702e] $25^{\text{mi.}},3918$, and then the constant quantity becomes as in the first member of [9704], which is to be put equal to the product of the distance of the planes, $\frac{1}{32}$ of an English inch
[9702f] [9699′], by the corresponding elevation of the fluid h; hence we get the equation [9704], from which we deduce the value of h [9705]. Substituting this value of h, and that of
[9702g] $b=10^{\text{in.}}$ [9700], in [9698], we get [9706].

[9702h] The quantity $6^{\text{mi.}},7389$ [9702*c*], deduced from the experiments of Haüy, is much too small, from his not having well moistened the tubes. In an experiment made by Gay-Lussac, the temperature was $15^{\circ},5$ of the centigrade thermometer, the diameter of the tube
[9702i] $1^{\text{mi.}},296=2l$ [9343], and the elevation of the fluid $q=10^{\text{mi.}},4$ [9353]. From these we
[9702k] get $q_{,}=q+\frac{1}{3}l=10^{\text{mi.}},616$ [9372*x*], and $\frac{2}{\alpha}=2\alpha^2=2lq_{,}=13^{\text{mi.}},758$ [9372*y*], instead of $6^{\text{mi.}},7389$ found by Haüy [9670′]; and if we use this result in [9706], we shall have

h being estimated in English inches. We have seen, in the preceding article, that the English inch contains $25^{mi.},3918$ [9678]; therefore we shall have [9703]

$$\frac{0,5\times 6,7389}{(25,3918)^2}=\frac{1}{32}.h, \qquad \text{[Expressed in square inches.]} \qquad [9704]$$

which gives

$$h=\frac{16\times 6,7389}{(25,3918)^2}. \qquad \text{[In linear inches.]} \qquad [9705]$$

Hence the formula [9698] becomes

$$\sin.V=\frac{16\times 6,7389}{10\times(25,3918)^2}.\frac{100}{a^2}=\frac{1,6723}{a^2}, \qquad [9706]$$

a being estimated in English inches.

The angle formed by the two glass planes, in the experiment, having for its sine $\frac{1^{in.}}{16\times 20^{in.}}$, this angle is $10^{m.}\ 44^{s.}$. The lower plane having been placed [9707] horizontally at the commencement of the experiment, it is evident that, to obtain the inclination V of the intermediate plane, we must decrease by $5^{m.}\ 22^{s.}$ all the inclinations of Hauksbee's table. We must then subtract all the [9708] numbers of that table from $20^{in.}$, to obtain the successive values of a. This being premised, we shall have the following table:

$$\sin.V=\frac{16\times 13,758}{10\times(25,3918)^2}.\frac{100}{a^2}=\frac{3,4142}{a^2}. \qquad [9702l]$$

This formula makes the sines of V more than double of those which are deduced from the third column of the table [9709] from the formula [9706], the ratio of their sines being as [9702m] 1,6723 to 3,4142, or as 1 to 2,041; and as the computed angles in the table are generally too large, it must follow that the results of Hauksbee's experiments must differ very much [9702n] from those derived from Gay-Lussac's experiment in [9702l]. These differences arise chiefly from the difficulty of making accurate observations of this kind, and in part from the terms [9702o] neglected in the expression of sin. V [9549, &c.]; and we may add, that we have supposed, in the theory, that the drop is nearly circular, whereas by observation it is found to be oblong when a is small. We may finally observe that, instead of [9702l], Poisson uses, in page 260 of his work, $\sin.V=\frac{3,4117}{a^2}$, as the result of Gay-Lussac's experiment; the difference between [9702p] these formulas arises chiefly from his deducing the value of the numerator 3,4117 from the [9702q] formula [9372l], instead of using that in [9372v]; but the difference is of no importance in comparison with the much greater errors of the observation.

Distances a, in inches, from the middle of the drop to the intersection of the planes.	Observed values of V, in sexagesimals.	Values of V, calculated by the formula [9706].	Difference between the calculated and the observed angles, expressed in aliquot parts of the observed values.
$18^{in.}$	$9^{m.}\ 38^{s.}$	$17^{m.}\ 44^{s.}$	$\frac{1}{1,2}$.
16	19 38	22 27	$\frac{1}{7}$.
14	29 38	29 20	$\frac{1}{99}$.
12	39 38	39 55	$\frac{1}{140}$.
[9709] 10	54 38	57 29	$\frac{1}{19}$.
8	$1^{d.}$ 39 38	$1^{d.}$ 29 50	$\frac{1}{10}$.
6	2 39 38	2 39 45	$\frac{1}{1368}$.
5	3 54 38	3 50 8	$\frac{1}{52}$.
4	5 54 38	5 59 58	$\frac{1}{67}$.
3	9 54 38	10 42 31	$\frac{1}{12}$.
2	21 54 38	24 42 49	$\frac{1}{7,8}$.

The calculated values of V agree with the observed values, as well as could be expected in a formula which is only approximative, and in observations in
[9710] which the fractions of a quarter of a degree were found by mere estimation. Towards the limits of the least and greatest distances of the drop from the line of intersection of the planes, the difference is the greatest, and it is evident, from § 10, that this ought to be the case; because, in the greatest distances, the drop has not sufficient width in comparison with its thickness; and in the least
[9711] distance, its width bears too great a ratio to its distance from the line of intersection.

It is this experiment of Hauksbee which Newton refers to in his Optics, question 31. "If two plane polished plates of glass, 3 or 4 inches broad, and 20 or 25 long, be laid, one of them parallel to the horizon, the other upon the first,
[9712] so as at one of their ends to touch one another, and contain an angle of about 10 or 15 minutes; and the same be first moistened on their inward sides with a clean cloth dipped into oil of oranges or spirit of turpentine; and a drop or two of the oil or spirit be let fall upon the lower glass at the other end; so soon as the upper glass is laid down upon the lower, so as to touch it at one end, as above, and to touch the drop at the other end, making with the lower glass an angle of about 10 or 15 minutes; the drop will begin to move towards the concourse of the glasses, and will continue to move with an accelerated motion till it arrives at that concourse of the glasses. For the two glasses attract the
[9713] drop, and make it run that way towards which the attractions incline. And if, when the drop is in motion, you lift up that end of the glasses where they meet,

and towards which the drop moves, the drop will ascend between the glasses, and therefore is attracted. And as you lift up the glasses more and more, the drop will ascend slower and slower; and at length rest, being then carried downward by its weight, as much as upwards by the attraction. And by this means you may know the force by which the drop is attracted at all distances from the concourse of the glasses."

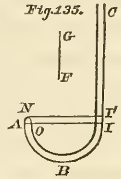
"Now by some experiments of this kind (made by the late Mr. Hauksbee), it has been found that the attraction is almost reciprocally in a duplicate proportion of the distance of the middle of the drop from the concourse [9714]
of the glasses, viz. reciprocally in a simple proportion, by reason of the spreading of a drop, and its touching each glass in a larger surface; and again reciprocally in a single proportion, by reason of the attractions growing stronger within the same quantity of attracting surface. The attraction, therefore, within the same quantity of attracting surface, is reciprocally as the distance between the glasses. And therefore, where the distance is exceeding small, the attraction must be exceeding great."

The explanation which Newton gives of the capillary phenomena, in this [9715]
extract and in that we have before given, is very proper to show the advantages of the mathematical and precise theory explained in the first section.

15. We have seen that the water rises in a capillary tube by the effect of the [9716]
concavity of its interior surface. The effect of the convexity of the surfaces becomes sensible in the following experiments:

If we dip a capillary tube into water to a small depth, and close the lower part of the tube with the finger, then draw it from the water, we shall find that, by taking away the finger, the fluid will descend in the tube, and form a drop of [9717]
water at its lower extremity. But when it has ceased to descend, the height of the column will still remain greater than the elevation of the water in the tube above the level, when it was dipped into that fluid. This excess arises from the action of the drop of water upon the column; for it is evident that, in [9718]
this experiment, the concavity of the interior surface of the column, and the convexity of the external surface, or that of the drop itself, contribute to raise the water in the tube.

Fig. 135.

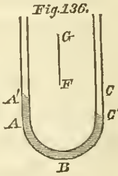


ABC, fig. 135, is a curved capillary tube, whose branches are of unequal lengths. By dipping it vertically into the water, so that its shortest branch AB may be wholly immersed, the water will rise in the branch BC above the level to a height which we shall represent by FG. Then drawing the tube from the water, there will be formed at the extremity A, a drop ANO; and when the fluid is [9719]

stationary in the tube, we shall find that, by drawing through the summit N of
[9719′] the drop, the horizontal line NI', the height $I'C$ of the water, in the longest branch, will exceed FG. If with the finger we wipe away successively the drops which are formed in A, this height will gradually decrease; and when we have rendered the surface of the water at this point plane and horizontal, the elevation of the water in the branch BC, above the horizontal line AI, will be
[9720] equal to FG. Lastly, if we successively apply drops of water at the extremity A, the surface of the water at this extremity will again become convex, and
[9721] the fluid will rise more and more in the branch BC, so that the preceding phenomena will be produced again in an inverted order. The excess of the height of the column in the branch BC, above the height FG, appears in these experiments to correspond to the convexity of the surface ANO; we must, to ascertain the exact correspondence, measure the width and the chord of that
[9721′] surface. But the great difficulty in taking these measures has prevented its being done.

The effect of a greater or less convexity in the surface, is also sensible in the following experiment: ABC, fig. 136, is a capillary siphon which contains a column of mercury ABC. By inclining the tube on the side A, the mercury moves to the height A' in
[9722] the branch AB, and to the point C' in the branch BC. By raising up the tube slowly, the mercury of the branch AB will return towards A, whilst that of the branch BC will return towards C. Then we find that the surface of the mercury in the branch AB is less convex than that of the mercury in the branch BC; and
[9723] if, through the summit of the first of these surfaces, we suppose a horizontal plane to be drawn, the summit of the second surface will be below this plane. This difference in the convexity of the two surfaces, arises from the friction of the mercury against the sides of the tube; the parts of the surface in the branch AB, which return towards A, and which touch the tube, are retarded a little by the friction, whilst the parts in the middle of this surface, do not experience
[9724] the same obstacle; whence it follows that the surface must be less convex; on the other hand, the friction must produce a contrary effect upon the surface of the mercury in the branch BC. Now as soon as the first of these surfaces is less convex than the second, it will follow that the mercury will suffer, by its action upon its own particles, a less pressure in the branch BA than in the branch BC, and therefore its height in the first of these two
[9725] branches must exceed a little its height in the second, which is conformable to experiment; a similar effect is observed in a barometer, when it is rising or falling.

Fig. 136.



Capillary siphons also furnish some phenomena which are a consequence of the theory. They may be reduced to this general phenomenon deduced from experiment: If we dip into a vessel of water any siphon ABC, fig. 137, whose two branches are of equal or unequal widths, and then [9726]
draw it forth, the water will not run out from the longest branch BC, if the difference of the two branches of the siphon be less than the height FG, to which the fluid would rise in a tube of the same width as the branch AB. To prove that this result is a consequence of the theory, we shall suppose that the fluid, whilst running from the branch C, has assumed the position $oiaBC$, the point a being very near the end A. Let q be the height of B [9727]
above the surface aio; the pressure which the fluid suffers at i, the middle point of the surface aio, will be equal, *first*, to the pressure of the atmosphere, which we shall denote by P; *second*, to the action [9728]
of the fluid upon its own particles, which is equal to $K-g.FG$,* g being the force of gravity; *third*, to the pressure of the column q, taken with the sign $-$, or to $-gq$. Thus an infinitely narrow canal, passing from i through the axis of the siphon, will be pressed upwards by the force [9729] [9730]

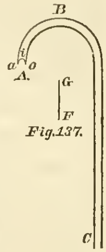


Fig. 137.

$$P+K-g.FG-gq. \qquad [9731]$$

q' being the height of the point B above the point C, the fluid at the point C will likewise be pressed upwards, by the force $P+K-gq'$, if the surface of the fluid be plane in C, or by a greater force if that surface be convex [9276]; [9732]
and the one or the other of these two cases must take place, when the fluid runs from C, or has a tendency to run from it. In this hypothesis, this second force must be less than the preceding; their difference [9733]

$$g.(q'-q-FG) \qquad [9734]$$

must therefore be a positive quantity; consequently *the excess $q'-q$ of the longer over the shorter branch, must be equal to or exceed FG, which is found to be the case by experiment.* [9735]

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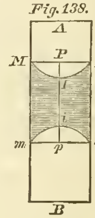
* (4237) This action is, by [9258], equal to $K-\frac{H}{b}$, but by [9354], $\frac{H}{b}=g.q$, q being, [9728a]
in this formula, the quantity denoted by FG, so that the capillary action at i is $K-g.FG$ [9728b]
pressing *upwards;* from this we must subtract the gravity of the column $iB=g.q$ [9727], and add the pressure of the atmosphere P, and it gives the whole action upwards as in [9731].

In the supplement to this theory [10005—10023, &c.], several additional cases are mentioned of the effects of the capillary action depending on the drop which forms at the [9728c]
extremity of a tube as in [9717, &c.], and in other similar phenomena.

In general, if we compare with this theory the different phenomena which
[9736] have been carefully noticed by philosophers, we shall find that they appear like corollaries deduced from the theory.

16. *It now remains to give the experiments which have been made to determine the concavity or convexity of the surfaces of the fluids in capillary tubes.* Philosophers having heretofore considered the curvature of the surfaces only as
[9737] a secondary effect, and not as the principal cause of capillary attraction, they have taken but little pains to determine the curvature. Messrs. Haüy and Tremery have endeavored, at my request, to determine that of the surface
[9738] of water. They introduced into a tube AB, fig. 138, whose interior diameter is two millimetres, a column of water $MmnN$; and, after having closed the tube at both ends, they held it vertically, and then carefully measured the two lengths Mm and Ii; I and i being the nearest points of the two surfaces MIN, min. The difference $Mm - Ii$, is equal to the sum of the two lines IP, ip; and they found this sum equal to $\frac{41}{48} \cdot MN$. According to the analysis in [9350, &c.], this sum must be equal
[9739] to MN, if the angle which we have denoted by θ', in [9346], be a right angle, or if the surface of the water be a tangent to the sides of the tube.* But we
[9740] must observe, that, if we suppose them to be tangents, we cannot accurately observe the points of contact. That which has been taken for the point M, is a point where the surface of the water begins sensibly to quit the sides of the tube;
[9740′] and it is easy to prove that, to make $IP + ip = \frac{41}{48} \cdot MN$, it is only necessary to
[9741] take, instead of M, m, the points which are† $0^{mi.},0226$ distant from the tube, which is not an improbable error. The preceding experiment seems, therefore,
[9742] to indicate that the angle θ' is a right angle, for water in a glass vessel. A similar experiment made with oil of oranges produces the same result. *Thus*

Fig. 138.



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* (4238) If we neglect the small quantity α in [9350], we shall have $z = l . \text{tang.} \frac{1}{2}\theta'$; and
[9739a] when $\theta' = 90^{d.}$, it becomes $z = l$, or $IP = MP$, fig. 138. In like manner we have $ip = pn = PN$; hence $IP + ip = MP + PN = MN$, as in [9739].

[9741a] † (4239) Having $IP = ip$, $MN = 2MP$, the equation [9740′] gives $IP = \frac{41}{48} . MP$. Now the surface MIN being supposed spherical, with the radius $1^{mi.}$, and arc $MI = \theta'$,
[9741b] we have $IP = (1 - \cos.\theta') = 2 \sin.^2 \frac{1}{2}\theta'$, $MP = \sin.\theta' = 2 \sin.\frac{1}{2}\theta' . \cos.\frac{1}{2}\theta'$ [31, 1], Int. Substituting these in the last expression of IP [9741a], and dividing by $2\sin.\frac{1}{2}\theta' . \cos.\frac{1}{2}\theta'$, we
[9741c] get $\text{tang.}\frac{1}{2}\theta' = \frac{41}{48} = \text{tang.} 40^{d.} 30^{m.}$; hence $\frac{1}{2}\theta' = 40^{d.} 30^{m.}$, or $\theta' = 81^{d.}$; consequently $MP = \sin. 81^{d.} = 0,9877 = 1 - 0.0123$; which is less than the radius by $0^{mi.},0123$, instead of $0^{mi.},0226$ [9741].

we have reason to believe that the surfaces of water, oil, and generally of the fluids which moisten glass, are very nearly hemispherical in capillary tubes. [9743]

Surfaces of water, oil, &c. in glass tubes are concave and hemi-spherical.

Determining, in the same manner, the convex surface of mercury in a very narrow glass tube, we have found that it is very nearly a hemisphere. If we compare this result with that which we have given in [9673], upon the depression of mercury below its level, in very narrow glass tubes, we may correct the effect [9744] of capillary action in the heights of the barometer. This effect is nothing in barometers with branches of equal diameters; but in a barometer formed by a tube dipped into a large cistern, the capillary effect becomes so much the more sensible as the diameter of the tube is decreased. The barometrical height, [9745] counted from the summit of the column, is always less than that which depends upon the pressure of the atmosphere; thus we see how inaccurate the method of those observers is, who measure the height of the barometer from the level [9746] to the points where the upper surface of the column touches the tube. To reduce the heights of the barometer to those which depend upon the pressure of the atmosphere, and thus to render different barometers comparable with each other, we must correct these heights for the capillary effect; and we can do [9747] this by the approximate integration of the differential equation [9324]. Integrating this equation, we get*

$$\frac{H}{b}=\frac{H}{u}\cdot\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}-\frac{2g}{u^2}\cdot\int zudu\,; \qquad [9748]$$

the ordinate z being counted downwards, from the summit of the surface of [9749] the column. The quantity $\frac{H}{gb}$ *is the capillary effect,*† *or what we must add to* [9750]

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* (4240) This integral is given in [9329]; and by substituting α [9328], and supposing the integrals to commence with $u=0$, we get

$$\frac{u\cdot\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}-\frac{2g}{H}\cdot\int zudu=\frac{u^2}{b}\,; \qquad [9748a]$$

multiplying this by $\frac{H}{u^2}$, we obtain $\frac{H}{b}$ [9748].

† (4241) The effect of the capillary action is to alter the level by the quantity q [9353], and this is equal to $\frac{H}{gb}$, or $\frac{1}{\alpha b}$ [9354], which is to be added to the height of the barometer, [9750a] to obtain its height corrected for the capillary action, as in [9750].

the height of the barometer, to obtain the height depending upon the pressure of the atmosphere. Now we have, by what has been said,*

[9751] $$\frac{2H.\sin.\theta'}{1^{mi.}}=g.7^{mi.}333.$$

Let l be the semi-diameter of the tube, estimated in millimetres. At the point where $u=l$, we have

[9752] $$\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=\sin.\theta' \quad [9389];$$

the value of $\frac{H}{gb}$ [9748, 9750] will therefore be †

[9753] $$\frac{H}{gb}=\frac{1^{mi.}\times 7^{mi.},333}{2l}-\frac{2}{l^2}.\int_0^l zdu.$$ [Capillary effect in a barometer.]

To obtain this integral, we must find z in terms of u. We may determine
[9754] it by experiment, observing that $2\pi.\int_0^l zdu$ is the volume included between the surface of the mercury at the upper extremity of the column,‡ the surface of

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[9751a] * (4242) If we multiply the first expression of q [9360] by g, we get $\frac{H.\sin.\theta'}{l}=gq$, where l represents the *radius* of the tube [9343], and q the corresponding elevation or
[9751b] depression [9353]. Now by [9673], we find, for mercury, that when $l=0^{mi.},5$, $q=7^{mi.},333$; substituting these in [9751a], we get [9751].

[9753a] † (4243) Substituting $\sin.\theta'$ [9752], in [9748], we get $\frac{H}{b}=\frac{H}{u}.\sin.\theta'-\frac{2g}{u^2}.\int zdu.$
[9753b] Dividing this by g, and putting $u=l$ [9752], it becomes $\frac{H}{gb}=\frac{H.\sin.\theta'}{gl}-\frac{2}{l^2}.\int zdu$; but from [9751], we obtain $\frac{H.\sin.\theta'}{gl}=\frac{1^{mi.}\times 7^{mi.},333}{2l}$; and by substituting it in the preceding value
[9753c] of the capillary action upon a barometer $\frac{H}{gb}$ [9753b, 9750], it becomes as in [9753].

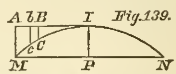

Fig.139.

[9754a] ‡ (4244) Let MIN, fig. 139, be the surface of the mercury, meeting the sides of the tube in M, N; I its vertex; $IBbA$, the horizontal tangent drawn from the point I. Upon this tangent let fall, from the points C, c, M, of the curve IM, the perpendiculars CB, cb, MA. Then,
[9754b] if we put $IB=u$, $Bb=du$, $BC=z$, we shall have the space $BbcC=zdu$; the
[9754c] volume formed by the revolution of this space about the axis IP will be equal to the quantity zdu, multiplied by the arc $2\pi u$, which it describes; so that the volume will be $2\pi.zdu$,
[9754d] whose integral $2\pi.\int_0^l zdu$ evidently represents the whole volume described by the space $IAMCI$, in revolving about the axis IP. This agrees with [9754].

The subject of the capillary action in a barometrical tube is again resumed by the

the tube, and a horizontal plane drawn through the top of the column. This volume may be accurately measured by the weight of the mercury necessary to fill it. We may therefore form, either by analysis or by experiment, a table of the correction for the capillary effect in a barometer, relative to the different [9755]
diameters of the tube [as in 10443z]. In this calculation, it is supposed that the tubes are of the same nature; but there may be a slight difference in them; [9756]
moreover the action of the tubes upon the mercury must be very small, to render the surface of the fluid in very narrow tubes nearly hemispherical [9650]. This difference cannot, therefore, have any sensible influence upon the heights of the barometer.

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author in [10454, &c.]. It is also treated of in the notes [10442a—10444a, 10456a—k]; [9754e]
and a table [10443z] is given, showing the depressions of the mercury for tubes of various [9754f]
diameters from $21^{mi.}$ to $2^{mi.}$. This table is computed upon the supposition that the angle of contact of the mercury with the upper side of the tube is $48° = 43^{d.}\,12^{m.}$; and the numbers [9754g]
would vary if this angle were increased or decreased by its friction against the sides of the tube in its ascent or descent, or from any other cause. It was observed many years since by M. Casbois, professor of medicine at Metz, that, by boiling the mercury in a barometrical [9754h]
tube, the convexity of its surface will be gradually diminished, and that, by continuing the boiling a sufficient length of time, the surface will become plane, and finally concave; and he suggested that this process might be used in obtaining a barometer with a plane surface. This experiment was afterwards confirmed by La Place and Lavoisier, who succeeded in constructing [9754i]
a barometer with a plane surface; and they adopted the opinion of M. Casbois, that this change in the convexity of the surface was produced by the expulsion of the moisture from the [9754k]
mercury by the continued process of boiling. But M. Dulong has lately given a much more [9754l]
satisfactory explanation of this phenomenon, by observing that, in the operation of boiling, the mercury in contact with the air becomes oxydized, and that this part, by adhering to the sides of the tube, or by mixing with the other parts of the fluid, produces the change in the capillary [9754m]
action which had been discovered by M. Casbois. The correctness of this explanation has been verified in several ways, namely, by viewing with a microscope the sides of the tube [9754n]
where the particles of the oxyde were visible and of a reddish hue; by agitating the barometer in an acid which decomposes the oxyde, since it was found that the surface of the mercury then resumed its convex form; finally by boiling the mercury in an atmosphere of hydrogen gas, which does not oxydize the mercury; for it was then found that no change whatever was [9754o]
produced in the surface of the mercury, however long the boiling was continued.

SUPPLEMENT

TO THE

THEORY OF CAPILLARY ATTRACTION.

The objects of this supplement are, to complete the theory which I have given of the capillary phenomena; to extend its application; to confirm its results by a comparison with experiment; and to present, in a new point of view, the effects of the capillary action, so as to render more evident the identity of the attractive forces, upon which this action depends, with those which produce the affinities of bodies.

ON THE FUNDAMENTAL EQUATION OF THE THEORY OF CAPILLARY ACTION.

The equation of partial differentials, in this theory of capillary attraction, [9318, or 9324], is deduced from the principle of the equilibrium of canals.
[9757] This principle consists in the supposition, that a *homogeneous fluid mass*, when acted upon by attractive forces, will be in equilibrium, if the equilibrium takes place in any canal whatever, whose extremities are situated at the surface of the fluid. We may prove it easily in this manner: Suppose, in the interior of the fluid, a re-entering canal, of uniform width, but infinitely small. If, from an attracting point,* taken as a centre, with any radius, we describe a spherical surface cutting this canal; and upon the same centre, with a radius which differs but infinitely
[9758] little from the first, we describe a second surface; each of these surfaces will cut the canal at least in two points, and will intercept at least two infinitely

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* (4245) This attraction may arise from any particle of the fluid, or from any other body;
[9757a] thus the earth's centre, or rather the point to which bodies upon the earth's surface tend, by means of the force of gravity, may be considered as one of the points here spoken of.

small portions of this canal. It is evident that the two columns of the fluid included in these portions, will be acted upon by equal attractive forces, and as [9758'] they have the same height in the direction of these forces, they will be in equilibrium with each other.* Hence we see that the whole canal will be in equilibrium by the action of the attracting body; therefore it follows that the [9759] equilibrium will hold good, whatever be the number of these points. Suppose

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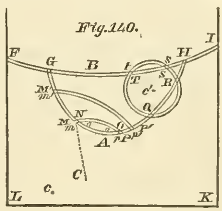
* (4246) To illustrate what is here said, let $FLKI$, fig. 140, be the tube or vessel containing the fluid of uniform density [9757], whose surface is [9759a] $FGBHI$. The proposed canal is represented by $GAHB$, whose interior part is GAH, and the part which is bent upon the surface of the fluid is GBH. Then, if C be the attracting point; $MNOP$, $mnop$, the spherical surfaces drawn about C [9759b] as a centre, cutting off the parts of the canal in the places represented by $MNnm$, $OopP$; the action of the point C upon the matter contained in these two parts of the canal will be equal, and will balance each other. For if we suppose the [9759c]



uniform area of the *base* of the canal, measured perpendicularly to its sides, to be A, and that the line CM forms the angle θ with that base, the area corresponding to the spherical surface MN, or mn, will evidently be $\frac{A}{\sin.\theta}$; and this, being multiplied by the difference of the radii [9759d] Cm, CM, represented by dr, will be the volume contained in the part corresponding to $MNnm$, which will therefore be $\frac{Adr}{\sin.\theta}$. Now the force at C, which we shall represent by [9759e] F, draws in the direction MC, and when resolved in the direction Mm, it will evidently be $F.\sin.\theta$; multiplying this by the volume $\frac{Adr}{\sin.\theta}$, we obtain the whole action upon the particles $MNnm$, in the direction of the canal Mm, equal to $\frac{Adr}{\sin.\theta}\times F.\sin.\theta = AFdr$; and as the [9759f] quantities A, F, dr, are the same at the point P as at M, the action upon the particles $OopP$ will in like manner be represented by $AFdr$, the variable angle θ having vanished [9759g] from both these expressions; hence the action at these two points must be equal, and opposite to each other, and they must therefore balance each other. The same may be proved in [9759h] other parts of the canal; so that the action of the force C upon the canal produces an equilibrium. In like manner we may prove, if the attracting point be at c', and the spherical surfaces cut the canal at the four points near Q, R, S, T, that the forces acting on the canal [9759i] near Q, R, will balance each other, and in like manner those near S, T; whence we may conclude that the canal will be in equilibrium by means of the force c'. Lastly, if we suppose [9759k] the force to be that of gravity, acting at c, we may draw about c, as a centre, the arcs $M'P'$, $m'p'$, and then proceed as we have done with the force at C, and thus prove that the [9759l] action of gravity alone produces also an equilibrium in the canal.

now that a portion of this canal is situated in the surface of the fluid, and that it is there bent down in the direction of that surface; the equilibrium in the canal will still continue. Therefore, if we suppose that the equilibrium exists separately in the interior part of the canal, it will also take place separately in
[9760] the portion which is situated at the surface. This last equilibrium can be maintained in only two ways; either because, at each point of the canal, the force by which the fluid is urged, is perpendicular to its sides; or because that,
[9760′] while the fluid is pressing in one direction at one end of the canal, this pressure is destroyed by a contrary one at the other end. But in this last case, there will not be an equilibrium in the part of the canal which is situated upon the surface,
[9760″] if the two ends of this canal are terminated in that part of the surface of the fluid which presses in the same direction. Therefore, upon the principle that there is generally an equilibrium in an internal canal whose extremities are at
[9761] the surface of the fluid, if we suppose a re-entering canal to be formed, of which a portion is situated upon the surface of the fluid, the resultant of the forces which act upon the fluid in this portion, must be perpendicular to the sides of
[9762] the canal. Now this cannot take place, for every direction of the canal, unless this resultant is perpendicular to the surface of the fluid; for, by reducing it to two forces, the one perpendicular, and the other parallel to the surface, this last force will not be destroyed by the sides of a canal which are situated in the direction of that force. The equilibrium in any internal canal is, therefore,
[9763] necessarily connected with the condition of the perpendicularity of the force to the surface; and this condition, if it be satisfied, assures the equilibrium of the whole fluid mass as we have seen in [138^{iv}]. The equations deduced either from the equilibrium of the canals, or from the perpendicularity of the force to
[9764] the surface, must therefore be identical, or, at least, the one must be the differential of the other; and it is evident that the second is the differential of the first. For the equation given by the equilibrium of the canals [9318, 9324] contains only differentials of the second order; instead of which the tangential force, at a capillary surface [9308, &c.], arises from two causes, namely, the
[9765] action of gravity, resolved in a direction parallel to that surface, and the attraction of the mass corresponding to the difference between the whole mass of the fluid and that of the osculatory ellipsoid; and it is evident that this last
[9765′] action depends upon differentials of the third order [9308]; therefore the equation resulting from the condition that the tangential force is nothing, or, in other words, that the force is perpendicular to the surface, must contain
[9766] differentials of the third order [9810]; *consequently it must be the differential of the equation given by the equilibrium of the canals* [9810h]. But it is interesting

to prove this *à posteriori*. It is what we shall now do, and the result will be a confirmation of the fundamental equation of this theory [9318], and will also be a simple method of obtaining that equation.

We shall take for the origin of the rectangular coordinates x, y, z, any point of the surface of the fluid, which we shall denote by O; and we shall define the axes of x, y, z, in the following manner: [9767]

For the axis of z, we shall take the line drawn through O, perpendicular to the surface of the fluid; [9768]

For the axes of x, y, we shall take two rectangular lines, drawn through the point O, perpendicular to the axis of z; [9768′]

Then the value of z, considered as a function of x, y, will represent the equation of the surface of the fluid, and this value can be developed in a series ascending according to the powers and products of x, y, and of the following form:* [9768″]

$$\begin{aligned} z = {} & Ax^2 + \lambda xy + By^2 \\ & + Cx^3 + Dx^2y + Exy^2 + Fy^3 \\ & + \&c. \end{aligned}$$ [Equation of the surface of the fluid.] [9769]

The three first terms of this expression of z, namely, $Ax^2 + \lambda xy + By^2$, correspond to the ellipsoid which touches the surface, or, to speak more strictly, it [9770]

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* (4247) If we put $z_{,}$ for the terms in the second member of [9769], it is evident that the general value of z, developed as in [610, 611], in a series ascending according to the powers and products of x, y, will be fully expressed by $z = a + bx + cy + z_{,}$. But the supposition that the ordinates x, y, z, commence together at the point O, will give, at that point, $x=0$, $y=0$, $z=0$; consequently $z_{,}=0$ [9769*a*, 9769], and then [9769*b*] becomes $0=a$; hence the general value of z becomes $z = bx + cy + z_{,}$. Now the differentials of $z, z_{,}$, give [9769a] [9769b] [9769c]

$$\left(\frac{dz}{dx}\right) = b + \left(\frac{dz_{,}}{dx}\right); \qquad \left(\frac{dz}{dy}\right) = c + \left(\frac{dz_{,}}{dy}\right);$$
$$\left(\frac{dz_{,}}{dx}\right) = 2Ax + \lambda y + \&c.; \qquad \left(\frac{dz_{,}}{dy}\right) = \lambda y + 2By + \&c.;$$ [9769d]

and at the point O, where $x=0$, $y=0$, these partial differentials of $z_{,}$ vanish; therefore we have, at that point, $\left(\frac{dz}{dx}\right) = b$, $\left(\frac{dz}{dy}\right) = c$; but as the tangent of the surface at the point O, is taken for the plane of x, y, we must evidently have, at that point, $\left(\frac{dz}{dx}\right) = 0$, $\left(\frac{dz}{dy}\right) = 0$; hence we get $0=b$, $0=c$; substituting these in the value of z [9769*c*], we finally obtain $z = z_{,}$, as in [9769]. [9769e] [9769f]

corresponds to the osculatory paraboloid. Now the attraction of this paraboloid upon the point O, is evidently in the direction of the axis of z, since the solid
[9771] is symmetrical on the opposite sides of this axis; therefore the tangential force at the point O, arising from the action of the whole mass, must depend wholly upon the attraction of the solid, whose surface is defined by the following equation:*

[9772]
$$z = Cx^3 + Dx^2y + Exy^2 + Fy^3 + \&c.$$

This solid is the same as the difference between the whole mass and the osculatory paraboloid. To determine the tangential force, depending upon this solid, upon the point O, we shall put f for the distance of one of the elements
[9773] of the solid from that point; also θ for the angle which this right line makes with the axis of x. The attractions upon the point O being sensible only in a very small space, *we may here consider the three right lines x, y, f, as being all in the plane which is a tangent to the surface in the point O, and we may neglect*
[9774] *the powers and products of x and y, superior to the third order.* Thus we shall have, for an element or differential of this solid,†

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* (4248) If the body attracting the point O is *symmetrical* about the axis z, that is, if the points are so situated that the value of z remains unaltered when x, y, are changed into
[9772a] $-x$, $-y$, respectively, it will evidently produce no tangential force, because the particles
[9772b] similarly situated on opposite sides of the axis z, act with equal forces in opposite tangential directions, and thus mutually balance each other. Now if we change x into $-x$, and y
[9772c] into $-y$, in [9769], the powers and products of x, y, of the even dimensions 2, 4, 6, &c., will remain unaltered, and will therefore be symmetrical, and may be neglected in computing the tangential force; and it will be only necessary to retain the uneven dimensions 3, 5, 7, &c.
[9772d] [9772]. Indeed, we may neglect the 5th, 7th, &c., dimensions, since, by a calculation similar
[9772e] to that in [9783, &c.], they will produce terms depending on $ff^6df.\varphi(f)$, $ff^8df.\varphi(f)$, &c., which must be incomparably smaller than the terms depending on $ff^4df.\varphi(f)$ [9780, &c.], depending on the *third* dimension of x, y, on account of the extreme smallness of the limit
[9772f] of f at which the attraction is sensible; so that we may put $z = Cx^3 + Dx^2y + Exy^2 + Fy^3$, as in [9774, &c.], in computing the tangential attraction.

† (4249) Let AOD, fig. 141, be the tangent plane; OA, OD, the axes of x, y, respectively;
[9775a] $AOB = \theta$, $BOC = d\theta$, $OB = f$, $Bb = df$, $BC = fd\theta$,
[9775a′] and the area $BCcb = fdf.d\theta$, representing the base of the attracting particle, whose height is z [9772]. The product of
[9775b] this height by the base, gives the whole mass as in [9775]; multiplying this by $\varphi(f)$, we get its attraction, in the direction

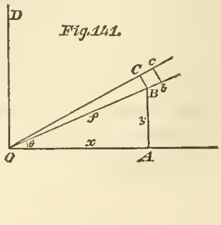

Fig. 141.

$$fdf.d\theta.\{Cx^3+Dx^2y+Exy^2+Fy^3\}.$$ [9775]

If we denote the law of attraction by $\varphi(f)$, the attraction of this element upon the point O, resolved in a direction parallel to the axis of x, will be [9776]

$$fdf.\varphi(f).d\theta.\cos.\theta.\{Cx^3+Dx^2y+Exy^2+Fy^3\};$$ [9777]

and parallel to the axis of y, it will be

$$fdf.\varphi(f).d\theta.\sin.\theta.\{Cx^3+Dx^2y+Exy^2+Fy^3\}.$$ [9778]

Moreover we shall have *

$$x=f.\cos.\theta, \qquad y=f.\sin.\theta;$$ [9779]

the tangential force of the point O, depending upon the attraction of the fluid mass, parallel to the axis of x, will be

$$\int\int f^4df.\varphi(f).d\theta.\{C.\cos.^4\theta+D.\cos.^3\theta.\sin.\theta+E.\cos.^2\theta.\sin.^2\theta+F.\cos.\theta.\sin.^3\theta\},$$ [9780]

and parallel to the axis of y, it will be

$$\int\int f^4df.\varphi(f).d\theta.\{C.\cos.^3\theta.\sin.\theta+D.\cos.^2\theta.\sin.^2\theta+E.\cos.\theta.\sin.^3\theta+F.\sin.^4\theta\}.$$ [9781]

The integrations of [9780, 9781], relative to θ, must be taken from $\theta=0$ to $\theta=2\pi$, π being the semi-circumference whose radius is unity; hence these expressions become respectively † [9782]

$$\tfrac{1}{4}\pi.\{3C+E\}.\int f^4df.\varphi(f),$$ [Tangential force in the direction x.] [9783]

$$\tfrac{1}{4}\pi.\{3F+D\}.\int f^4df.\varphi(f).$$ [Tangential force in the direction y.] [9784]

OB, equal to $fdf.\varphi(f).d\theta.\{Cx^3+Dxy^2+Exy^2+Fy^3\}$. To resolve this in the directions OA, OD, we must multiply it respectively by $\cos.\theta$, and $\sin.\theta$; and then it becomes as in [9777, 9778]. We may here remark, that the principle adopted in [9774] is equivalent to the supposition that all the particles of fluid, situated on the ordinate z, are projected on the tangent plane xy, in a single point, corresponding to the coordinates x, y. [9775c] [9775d]

* (4250) In the triangle OBA, fig. 141, page 810, we have $OA=OB.\cos.AOB$, $AB=OB.\sin.AOB$; and by substituting the symbols [9775a, &c.], we get the values of x,y [9779]. Substituting these in [9777], we obtain [9780], and by the same means [9778] changes into [9781]. [9779a]

† (4251) From [8], Int., we have

$$fd\theta.\cos.^4\theta=fd\theta.\{\tfrac{3}{8}+\tfrac{1}{2}\cos.2\theta+\tfrac{1}{8}\cos.4\theta\}=\tfrac{3}{8}\theta+\tfrac{1}{4}\sin.2\theta+\tfrac{1}{32}\sin.4\theta.$$ [9783a]

This vanishes when $\theta=0$; and when $\theta=2\pi$, it becomes $\int_0^{2\pi}d\theta.\cos.^4\theta=\frac{3}{8}.2\pi=\frac{3}{4}\pi$; and by changing θ into $\theta-90^{d.}$, we get $\int_0^{2\pi}d\theta.\sin.^4\theta=\frac{3}{4}\pi$. Again, by using [31, 1], Int. we have [9783b]

$$fd\theta.\cos.^2\theta.\sin.^2\theta=fd\theta.(\cos.\theta.\sin.\theta)^2=fd\theta.(\tfrac{1}{2}\sin.2\theta)^2=fd\theta.(\tfrac{1}{8}-\tfrac{1}{8}\cos.4\theta)=\tfrac{1}{8}\theta-\tfrac{1}{32}.\sin.4\theta,$$ [9783c]

The integral relative to f may be taken from $f=0$ to $f=\infty$, so that it is
[9785] independent of the dimensions of the attracting mass [9240a—l]. This is what characterizes this kind of attractions, which, being sensible only at insensible distances, allows us to notice or neglect, at pleasure, the attractions of bodies situated beyond their sphere of sensible activity. We shall put, as in [9233'],

[9786] $$\Pi(f)=c-\int_0^f df.\varphi(f),$$

the integral $\int df.\varphi(f)$ being taken from $f=0$, and c being its value when
[9787] f is infinite [9232']. $\Pi(f)$ will be a positive quantity, decreasing with extreme rapidity [9233'']; and we shall have, by taking the integrals from* $f=0$;

[9788] $$\int f^4df.\varphi(f)=-f^4\Pi(f)+4\int f^3df.\Pi(f).$$

$-f^4.\Pi(f)$ is nothing when $f=\infty$; for although f^4 then becomes infinite,
[9789] the extreme rapidity with which $\Pi(f)$ is supposed to decrease, renders $f^4.\Pi(f)$ nothing [9240k]. *The functions* $\varphi(f)$ *and* $\Pi(f)$ *may be very well*
[9790] *compared with exponentials like* $c^{-if.}$ [9240h, &c.]; c *being the number whose hyperbolic logarithm is unity, and* i *being a very great positive and*
[9791] *integral number.* For c^{-if} is finite when $f=0$, and becomes nothing when f is infinite; moreover it decreases with extreme rapidity, and in such a manner
[9792] that the product $f^n.c^{-if}$ always vanishes when f is infinite [9240k], whatever
[9793] be the value of the exponent n. We shall now put, as in [9241'],

[9794] $$\int_0^f fdf.\Pi(f)=c'-\Psi(f),$$

[9795] c' *being the value of that integral when* f *is infinite* [9240']. $\Psi(f)$ *will also be a positive quantity decreasing with extreme rapidity* [9241'']; and we shall have †

[9783d] which vanishes when $\theta=0$; and when $\theta=2\pi$, it becomes $\int_0^{2\pi}d\theta.\cos.^2\theta.\sin.^2\theta=\frac{1}{8}.2\pi=\frac{1}{4}\pi$. Moreover, we have $\int d\theta.\cos.^3\theta.\sin.\theta=-\int\cos.^3\theta.d.(\cos.\theta)=\frac{1}{4}-\frac{1}{4}\cos.^4\theta$, which vanishes
[9783e] when $\theta=0$, and also when $\theta=2\pi$; so that we have $\int_0^{2\pi}d\theta.\cos.^3\theta.\sin.\theta=0$; and if we change θ into $\theta-90^{d.}$, we get $\int_0^{2\pi}d\theta.\sin.^3\theta.\cos.\theta=0$. Substituting these integrals in [9780], it becomes as in [9783], and [9781] becomes as in [9784].

[9788a] * (4252) The differential of [9786] is $d.\Pi(f)=-df.\varphi(f)$; substituting this in
[9788b] the first member of [9788], we get $\int f^4df.\varphi(f)=-\int f^4.d.\Pi(f)$. Integrating by parts, it becomes as in the second member of [9788], as is easily proved by differentiation; and by
[9788c] neglecting the term $-f^4\Pi(f)$ [9789], it becomes $\int_0^\infty f^4df.\varphi(f)=4.\int_0^\infty f^3df.\Pi(f)$.

[9796a] † (4253) The differential of [9794] is $fdf.\Pi(f)=-d.\Psi(f)$; substituting this in the last term of [9788], and then integrating by parts, we get successively

$$4\int f^3 df.\Pi(f) = -4f^2.\Psi(f) + 8\int f df.\Psi(f).$$ [9796]

When f is infinite, $f^2.\Psi(f)$ becomes nothing; therefore we shall have, by taking the integral from $f=0$ to $f=\infty$, [9797]

$$4\int_0^\infty f^3 df.\Pi(f) = 8\int_0^\infty f df.\Psi(f).$$ [9798]

Lastly, if we put, as in [9253a'],

$$\frac{H}{2\pi} = \int_0^\infty f df.\Psi(f),$$ [9799]

we shall have,

$$\int_0^\infty f^4 df.\varphi(f) = 8\int_0^\infty f df.\Psi(f) = \frac{4H}{\pi}.$$ [9800]

Thus the two preceding tangential forces [9783, 9784], parallel to the axes of x and y, will become

$$(3C+E).H,$$ [Tangential force parallel to the axis of x.] [9801]

$$(3F+D).H.$$ [Tangential force parallel to the axis of y.] [9801']

Now, by observing that the axis of z is perpendicular to the surface of the fluid [9768], we shall have at the point O, as in [9769f], [9802]

$$\left(\frac{dz}{dx}\right) = 0, \qquad \left(\frac{dz}{dy}\right) = 0;$$ [9803]

the expression of z being developed in a series, according to the powers and products of x, y, by means of the theorem [610, 611], becomes *

$$\begin{aligned} z = & \left(\frac{ddz}{dx^2}\right).\frac{x^2}{2} + \left(\frac{ddz}{dxdy}\right).xy + \left(\frac{ddz}{dy^2}\right).\frac{y^2}{2} \\ & + \left(\frac{d^3z}{dx^3}\right).\frac{x^3}{6} + \left(\frac{d^3z}{dx^2dy}\right).\frac{x^2y}{2} + \left(\frac{d^3z}{dxdy^2}\right).\frac{xy^2}{2} + \left(\frac{d^3z}{dy^3}\right).\frac{y^3}{6}. \\ & + \&c.; \end{aligned}$$ [9804]

$$4\int f^3 df.\Pi(f) = -4\int f^2.d.\Psi(f) = -4f^2.\Psi(f) + 8\int f df.\Psi(f),$$ [9796b]

as in [9796]. The part $-4f^2.\Psi(f)$, vanishes at the limits of the integral, as in [9240i, k, &c.], and by neglecting it in the equation [9796b], we get [9798]; substituting it in [9788c], we obtain, by using $\frac{H}{2\pi}$ [9799], $\int_0^\infty f^4.df.\varphi(f) = 8\int_0^\infty f df.\Psi(f) = \frac{4H}{\pi}$, as in [9800]. Lastly, by substituting this integral in [9783, 9784], we obtain the tangential forces [9801, 9801']. [9796c] [9796d]

* (4254) The general development of z [610, 611], contains the terms in [9804]; the three first terms of the form $a+bx+cy$ being neglected, as in [9769a—f]. Comparing the expression of z [9804] with the assumed form [9769], we obtain the values of C, D, E, F [9805, 9806]; hence the tangential forces [9801, 9801'], become as in [9807, 9807'], respectively. [9804a] [9804b]

which gives

[9805] $$C=\tfrac{1}{6}.\left(\frac{d^3z}{dx^3}\right), \qquad D=\tfrac{1}{2}.\left(\frac{d^3z}{dx^2dy}\right),$$

[9806] $$E=\tfrac{1}{2}.\left(\frac{d^3z}{dxdy^2}\right), \qquad F=\tfrac{1}{6}.\left(\frac{d^3z}{dy^3}\right).$$

Consequently the preceding tangential forces [9801, 9801′], will become, as in [9804b],

[9807] $$\tfrac{1}{2}H.\left\{\left(\frac{d^3z}{dx^3}\right)+\left(\frac{d^3z}{dxdy^2}\right)\right\}, \qquad \text{[Tangential force parallel to the axis } x.]$$

[9807′] $$\tfrac{1}{2}H.\left\{\left(\frac{d^3z}{dy^3}\right)+\left(\frac{d^3z}{dx^2dy}\right)\right\}. \qquad \text{[Tangential force parallel to the axis } y.]$$

We shall put g for the force of gravity, and $-du$ for the element of its
[9807″] direction. Then the condition that the whole force acting at the surface must be perpendicular to it, or, in other words, that the resultant of the tangential forces is nothing, is reduced, as we have seen in [138, 138′], to the following formula, namely, *that the sum of the products of each force, by the element of its*
[9808] *direction, is nothing.* Multiplying, therefore, by dx the force parallel to the axis of x; by dy the force parallel to the axis of y; and the gravity g by $-du$; then taking the sum of these products, and putting it equal to nothing, we shall obtain the following equation:

[9809] $$\tfrac{1}{2}H.\left\{\left(\frac{d^3z}{dx^3}\right).dx+\left(\frac{d^3z}{dx^2dy}\right).dy+\left(\frac{d^3z}{dxdy^2}\right).dx+\left(\frac{d^3z}{dy^3}\right).dy\right\}-gdu=0.$$

[9809′] From the formula [9314], we have at the point O, where* $\mathrm{p}=\left(\frac{dz}{dx}\right)=0$,
[9809″] $\mathrm{q}=\left(\frac{dz}{dy}\right)=0$ [9312, 9803],

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[9810a] * (4255) Substituting $\mathrm{p}=0$, $\mathrm{q}=0$ [9809′, 9809″], in [9314], we get $\frac{1}{R}+\frac{1}{R'}=\mathrm{r}+\mathrm{t}$:
[9810b] and by substituting the values of r, t, [9313], we obtain $\left(\frac{ddz}{dx^2}\right)+\left(\frac{ddz}{dy^2}\right)=\frac{1}{R}+\frac{1}{R'}$. The differential of this equation is the same as in [9810], observing that z being a function of
[9810c] x, y, we have, by the usual rules of differentiation, $d.\left(\frac{ddz}{dx^2}\right)=\left(\frac{d^3z}{dx^3}\right).dx+\left(\frac{d^3z}{dx^2dy}\right).dy$;
[9810d] $d.\left(\frac{ddz}{dy^2}\right)=\left(\frac{d^3z}{dxdy^2}\right).dx+\left(\frac{d^3z}{dy^3}\right).dy$. Substituting [9810] in [9809], we get [9811]; multiplying this by $\frac{2}{H}$, we get [9811′]. Now the quantities K, b, b', which occur in the fundamental equation [9315], are constant; so that if we multiply this equation by $-\frac{2}{H}$,
[9810e] it may be put under the form $\left(\frac{1}{R}+\frac{1}{R'}\right)-\frac{2g}{H}.z=$ constant; or by changing gz [9309]
[9810f] into gu [9807″], to conform to the present notation, $\left(\frac{1}{R}+\frac{1}{R'}\right)-\frac{2g}{H}.u=$ constant, whose

$$\left(\frac{d^3z}{dx^3}\right).dx+\left(\frac{d^3z}{dx^2dy}\right).dy+\left(\frac{d^3z}{dxdy^2}\right).dx+\left(\frac{d^3z}{dy^3}\right).dy=d.\left(\frac{1}{R}+\frac{1}{R'}\right); \qquad [9810]$$

R and R' being the greatest, and the least, radii of curvature at that point. Substituting this in [9809], we get the following expression:

$$\tfrac{1}{2}H.d.\left\{\frac{1}{R}+\frac{1}{R'}\right\}-gdu=0, \quad \text{or} \qquad [9811]$$

$$d.\left\{\frac{1}{R}+\frac{1}{R'}\right\}-\frac{2g}{H}.du=0; \qquad [9811']$$

an equation which is evidently the differential of the fundamental equation of the theory of capillary attraction [9315, 9842p].

We may in the same manner determine the action perpendicular to the surface. This action depends upon the part* $Ax^2+\lambda xy+By^2$, of the value of z [9769]. [9812]
We shall put†

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differential is as in [9811′]. Hence it appears that the fundamental equation [9811′], deduced [9810g]
from this second method of investigation, is the same as the differential of that which is [9810h]
obtained by the first method in [9315], agreeing with what is stated in [9766].

* (4256) The chief part of the value of z [9769], corresponding to the surface of the fluid and to small values of x, y, is evidently that containing the terms of the second power of x, y; so that if we represent this by z', as in [9813″, 9823], we shall have very nearly $z'=Ax^2+\lambda xy+By^2$, as in [9812]; and if we change, as in [9772c], x into $-x$, y into [9812a]
$-y$, the action corresponding to the attracting point in this second position will conspire with that in the first position, in producing a force perpendicular to the surface. On the contrary, those depending on the *third and other uneven powers of* x, y, *will have different signs, and* [9812b]
their effects will generally have a tendency to counteract and balance each other; so that, on that account, as well as because of their comparative smallness relative to the retained terms of the second order, they may be neglected. Lastly, the terms of the fourth and higher orders [9812c]
of terms in x, y, may be neglected on account of their smallness, and thus we shall have the same value of z or z' as in [9812a, or 9824].

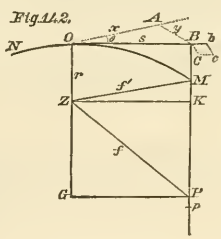
† (4257) To illustrate these definitions, we shall refer to the annexed figure 142, in which Z is the attracted point; P the attracting point, whose coordinates are [9814a]

$OA=x$, $AB=y$, $BP=z$; we have also $ZP=f$,

$OB=\sqrt{OA^2+AB^2}=s=\sqrt{x^2+y^2}$, [9814$b$]

$OZ=r$, angle $AOB=\theta$. Then, if we draw ZK perpendicular to BP, we shall have $ZK=OB=\sqrt{x^2+y^2}$ [9814], $PK=z-r$. Substituting these values in the equation [9814c]
$ZP^2=ZK^2+PK^2$, we get [9816].

Fig. 142.



[9813] $r =$ the distance OZ from the origin O of the coordinates to any point Z of the fluid situated in the axis of z, which is perpendicular to the surface at the point O [9768];

[9813'] $f =$ the distance ZP from the point Z to a particle P of the fluid whose coordinates are x, y, z; using for z the ordinate corresponding to any point whatever of the fluid;

[9813''] $f' =$ the distance ZM from the point Z to a particle of the fluid situated at the point M of the surface whose coordinates are x, y, z'; using for z' the ordinate BM, corresponding to the point M of the surface;

[9814] $s = \sqrt{x^2+y^2} =$ the hypotenuse OB of a right-angled triangle whose sides are x, y, situated in the plane of $x\,y$;

[9814'] $\theta =$ the angle AOB which the line s forms with the axis of x;

[9815] $\varphi(f)$ represents, as in [9776], the law of the corpuscular attraction at the distance f.

Then we shall have

[9816] $$f^2 = x^2 + y^2 + (z-r)^2,$$

[9816'] $sds.d\theta.dz =$ the element of the mass of the fluid* at the point P;

[9817] and the action of this element, situated at P, upon the point Z of the axis of z, resolved in a direction parallel to that axis, will be

[9818] $$sds.d\theta.dz.\frac{(z-r)}{f}.\varphi(f).$$ [Action of the particle at P on Z in the direction ZG.]

[9819] If we use the function $\Pi(f) = c - \int_0^f.df.\varphi(f)$ [9786], we may put the expression [9818] under the following form :†

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* (4258) If we change f [9775a] into s [9814b], the expression of $BCcb$, fig. 141,

[9818a] 142, pages 810, 815, will be $sds.d\theta$ [9775a']. Multiplying this by the height $Pp = dz$ of the attracting particle, we get its mass $sds.d\theta.dz$ [9816']. Now the attractive force, in

[9818b] the direction ZP, is $\varphi(f)$, and this force resolved in the direction ZG is $\varphi(f)\times\frac{(z-r)}{f}$. Multiplying this by the preceding expression of the mass [9818a], we get the same expression

[9818c] as in [9818], for the whole attraction of the particle situated at P, resolved in a direction parallel to the axis of z.

† (4259) From $f = \sqrt{x^2+y^2+(z-r)^2}$ [9816], we get

[9820a] $$\left(\frac{df}{dz}\right) = \frac{z-r}{\sqrt{x^2+y^2+(z-r)^2}} = \frac{z-r}{f};$$

$$-sds\,.\,d\vartheta\,.\,dz\,.\left(\frac{d\,.\,\Pi(f)}{dz}\right).$$ [Action of the particle at P on Z, in the direction ZG.] [9820]

Therefore the attraction of the whole solid upon the point under consideration, resolved in the direction of the axis of z, will be

$$-\textstyle\int\int\int sds\,.\,d\vartheta\,.\,dz\,.\left(\frac{d\,.\,\Pi(f)}{dz}\right).$$ [Action of the whole fluid on Z, in the direction ZG.] [9821]

Hence we find that this attraction, taken from $z=z'$ to $z=\infty$, which renders $\Pi(f)$ nothing, is [as in 9820f]

$$\textstyle\int\int sds\,.\,d\vartheta\,.\,\Pi(f');$$ [Action of the whole fluid on Z in the direction ZG.] [9822]

f' being the value of f, corresponding to the points of the surface, and z' the value of z relative to these points. Now we have, in [9812a], [9823]

$$z'=Ax^2+\lambda xy+By^2;$$ [9824]

moreover we have very nearly *

$$f'^2=x^2+y^2+r^2-2rz'=s^2+r^2-2rz';$$ [9825]

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consequently

$$\frac{(z-r)}{f}\cdot\varphi(f)=\varphi(f)\cdot\left(\frac{df}{dz}\right).$$ [9820b]

But from [9819], we have

$$\varphi(f)\,.\,df=-d\,.\,\Pi(f);$$

whence

$$\varphi(f)\cdot\left(\frac{df}{dz}\right)=-\left(\frac{d\,.\,\Pi(f)}{dz}\right);$$ [9820c]

hence

$$\frac{(z-r)}{f}\cdot\varphi(f)=-\left(\frac{d\,.\,\Pi(f)}{dz}\right),$$ [9820d]

and by substituting it in [9818], it becomes as in [9820]. Its triple integral gives the whole attraction, as in [9821]. Taking the integral, in the first place, relative to z, we get

$$-\textstyle\int dz\,.\left(\frac{d.\Pi(f)}{dz}\right)=-\,\Pi(f)+\Pi(f'),$$ [9820e]

the constant $\Pi(f')$ being added to make the integral vanish, at the first limit or point M, where $f=f'$. At the other limit, where $f=\infty$, we have $\Pi(f)=0$, and then the preceding expression becomes

$$-\textstyle\int dz\,.\left(\frac{d.\Pi(f)}{dz}\right)=\Pi(f'),$$ [9820f]

and by substituting it in [9821], we get [9822].

* (4260) Changing f into f', and z into z', in [9816], and then neglecting z'^2, we get the first expression in [9825], and by substituting $s^2=x^2+y^2$ [9814b], we get the second form of [9825]. Half its differential, considering r as constant, gives [9827]. [9825a]

[9826] therefore, by neglecting the square of z', in comparison with $x^2+y^2+r^2$, we shall have

[9827] $$f'df' = sds - rdz'.$$

[9828] If we substitute for z' its value, and observe that* $x = s.\cos.\theta$, $y = s.\sin.\theta$ [9779], the equation of the surface will give, by considering θ as constant,†

[9829] $$dz' = 2sds.\{A.\cos.^2\theta + \lambda.\sin.\theta.\cos.\theta + B.\sin.^2\theta\}.$$

[9829′] Thus, r *being extremely small*, while $\Pi(f')$ has a sensible value, we may suppose very nearly

[9830] $$sds = f'df'.\{1 + 2Ar.\cos.^2\theta + 2\lambda.r.\sin.\theta.\cos.\theta + 2Br.\sin.^2\theta\}.$$

By this means the integral $\int\int sds.d\theta.\Pi(f')$ [9822] is transformed into the following expression:

[9831] $$\int\int f'df'.d\theta.\{1+2Ar.\cos.^2\theta+2\lambda r.\sin.\theta.\cos.\theta+2Br.\sin.^2\theta\}.\Pi(f').$$ [Action of the whole fluid in the direction ZG.]

From the well-known principles of the transformation of double integrals, we may here integrate in the first place relative to θ, and then relative to f'. The [9832] integral relative to θ extends from the limits $\theta = 0$ to $\theta = 2\pi$; and, after making the integration relative to this quantity, we find that the double integral [9831] can be reduced to the following form:‡

[9833] $$2\pi.\int f'df'.\{1 + (A+B).r\}.\Pi(f').$$ [Attraction in the direction ZG.]

We shall now put, as in [9794],

[9834] $$\int_0^{f'} f'df'.\Pi(f') = c' - \Psi(f'),$$

[9828a] * (4261) These values of x, y, are deduced from those in [9779], by changing f [9773, 9774, 9775a], into s [9814], to conform to the present notation.

† (4262) Substituting the values of x, y [9828], in [9824], we get

[9829a] $$z' = s^2.\{A.\cos.^2\theta + \lambda.\sin.\theta.\cos.\theta + B.\sin.^2\theta\}.$$

In taking the differential relative to s, we must consider the independent variable quantity [9829b] θ as constant, and then the differential of this expression of z' gives dz' [9829]. Substituting this in [9827], it becomes

[9829c] $$f'df' = sds.\{1 - (2Ar.\cos.^2\theta + 2\lambda r.\sin.\theta.\cos.\theta + 2Br.\sin.^2\theta)\}.$$

Dividing this by the coefficient of sds, and neglecting terms of the second and higher powers [9829d] and products of the quantities Ar, λr, Br, which are of the very small order r^2 [9829′], we get the value of sds [9830], and by substituting it in [9822], we obtain [9831].

‡ (4263) We have, as in [1548a],

[9833a] $$\int_0^{2\pi}d\theta.\cos.^2\theta = \pi,\quad \int_0^{2\pi}d\theta.\sin.^2\theta = \pi,\quad \int_0^{2\pi}d\theta.\cos.\theta.\sin.\theta = 0,\ \text{also}\ \int_0^{2\pi}d\theta = 2\pi;$$

substituting these in [9831], we get [9833].

c' being the value of that integral when f' is infinite [9795]; hence the expression [9833] becomes* [9835]

$$2\pi.\{1+(A+B).r\}.\Psi(r). \qquad \left[\text{Attraction in the direction } ZO.\right]$$ [9836]

Now if we put R for the radius of curvature of the section of the surface, by a plane passing through the axes of x, z, we shall have† [9837]

$$A=\frac{1}{2R}.$$ [9838]

If we also put R' for the radius of curvature of the section of the surface by a plane passing through the axes of y, z, we shall have [9839]

$$B=\frac{1}{2R'}.$$ [9840]

Therefore we shall have, for the attraction of the body, upon a point placed within it, in the direction of the radius of curvature of the surface, and at the distance r from that surface,‡ [9841]

$$2\pi.\left\{1+\frac{r}{2}.\left(\frac{1}{R}+\frac{1}{R'}\right)\right\}.\Psi(r). \qquad \left[\text{Attraction in the direction } ZO.\right]$$ [9842]

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* (4264) If we change the limit of the integral f' [9834] into r, we shall have $\int_0^r f'df'.\Pi(f')=c'-\Psi(r)$. Subtracting this from $\int_0^\infty f'df'.\Pi(f')=c'$ [9835], we get [9834a] $\int_r^\infty f'df'.\Pi(f')=\Psi(r)$. Now in the integral [9833], the least value of f' is evidently [9834b] $ZO=r$, fig. 142, page 815, and its greatest value extends beyond the limits of the sphere of activity of the attracting fluid, and it may therefore be considered as infinite; so that we must substitute the integral [9834b] in [9833], and it will become as in [9836], r being constant in the integrations relative to f'. [9834c]

† (4265) The curve formed by the section of a plane passing through the axes of x, z, is evidently obtained by putting $y=0$, in the equation of the surface [9824], which gives [9837a] $z'=Ax^2$. Now if we suppose z', x, to be infinitely small, and R to be the radius of the circle of curvature corresponding to the point O, fig. 142, page 815, we shall have, by a well-known property of small circular arcs, $2Rz'=x^2$. Dividing the preceding value of z' [9837b] [9837c] [9837a] by this expression, we get $\frac{1}{2R}=A$ [9838]. In like manner, by putting $x=0$, in [9837d] [9824], we get the curve formed by the plane of z', y, represented by the equation $z'=By^2$; comparing this with the equation $2R'z'=y^2$, of a small circular arc, whose radius is R', we get the expression of B [9840]. [9837e]

‡ (4266) Substituting the values of A, B [9838, 9840], in [9836], it becomes as in [9842]. Multiplying this by dr, and integrating, as in [9843], it becomes [9841a]

$$2\pi.\int dr.\Psi(r)+\tfrac{1}{2}\left(\frac{1}{R}+\frac{1}{R'}\right).2\pi.\int rdr.\Psi(r);$$ [9841b]

[9842′] To obtain the whole action of the body, upon a fluid contained in an infinitely narrow canal perpendicular to the surface, whose base is taken for unity, we

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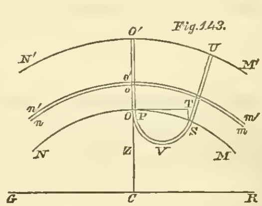
and by using the values [9844, 9844′], we get [9845], which corresponds with [9301]. The remarks in [9846] are conformable to those in [9301b].

[9841c] In all the calculations of this article, the author considers the fluid as being perfectly homogeneous, and wholly neglects the consideration of the change of density near the surface of the fluid. To notice this circumstance, we shall refer to fig. 143, which is marked with the

[9841d] same letters as fig. 142, page 815; and we shall, in the first place, for greater simplicity, suppose that the external surface $N'O'M'$ is spherical, and the density of the fluid, at that surface, nearly equal to nothing; moreover nom, $n'o'm'$, are infinitely near concentrical spherical surfaces, including a stratum of fluid of the uniform density D;



[9841e] lastly, NOM is another concentrical spherical surface, where the density is equal to unity, as in the interior of the fluid; the distance of the surfaces NOM, $N'O'M'$, being the insensible quantity $OO'=\lambda$ [9580f, g]; so that, if we suppose the radius of the

[9841f] surface NOM to be R, that of the surface $N'O'M'$ will be $R+\lambda$. The line $O'OC$ is supposed to be perpendicular to these surfaces, at the points O', O; and this line is taken for the axis of z, as in [9768, &c.]. Then the action of the stratum included between the surfaces nom, $n'o'm'$, upon the point Z, situated a little below the surface NOM, may be

[9841g] determined from the expression [9842]. For if we put, for a moment, $Zo=r$, and suppose nom to be the *external* surface of a spherical mass of fluid $RGnom$, of the uniform density unity, its action on the point Z of the axis OZ, in the interior of the fluid, will be given by the formula [9842], supposing $R=R'$ to be the radius of this external surface; so that we shall have for this action the following expression:

[9841h]
$$2\pi.\left\{1+\frac{1}{R}.r\right\}.\psi(r).$$

Now without altering the position of the point Z, we shall suppose the mass of the fluid to be increased by the addition of an infinitely small stratum of the fluid, of the same density unity, included between the concentrical surfaces nom, $n'o'm'$, whose distance oo' may

[9841i] be represented by dr. Then it is evident that the action of this additional stratum upon the point Z will be obtained, by taking the differential of the expression [9841h], supposing $Zo=r$ to be augmented by the quantity $oo'=dr$; and by putting, for brevity,

[9841k]
$$\left(\frac{d.\psi(r)}{dr}\right)=\psi'(r),$$

it becomes

[9841l]
$$2\pi dr.\psi'(r)+\frac{1}{R}.2\pi dr.\{\psi(r)+r.\psi'(r)\};$$

neglecting the term arising from the variation of the radius of curvature R, in passing from

must multiply the preceding expression by dr, and integrate it from $r=0$ to $r=\infty$. Then putting, as in [9253a'], [9843]

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one surface to another, on account of its smallness. For in this case we have $dR=dr$; and the part of the differential of [9841h] depending on it is [9841m]

$$-2\pi.\frac{1}{R^2}.rdr.\Psi(r)=\frac{1}{R}.2\pi dr.\left\{-\frac{r}{R}.\Psi(r)\right\}, \qquad [9841n]$$

which is of the order $\frac{r}{R}$ in comparison with the term depending on $\Psi(r)$ in [9841l]; and as r is supposed to be very small relative to R, and may be considered as of the order λ, this term may be neglected. Now multiplying [9841l] by the density D, considered as a function of r, we get for the action of *the stratum included between the surfaces* $n'o'm'$, nom, the following expression: [9841o]

$$2\pi.D.dr.\Psi'(r)+\frac{1}{R}.2\pi.D.dr.\{\Psi(r)+r.\Psi'(r)\}. \qquad [9841p]$$

To obtain the whole action of the fluid $RGn'o'm'$ upon the particle at Z, we must integrate [9841p], from the limit $r=-\lambda$, corresponding to a particle situated between the points Z and C, where the function $\Psi(r)$ becomes insensible, to the point o, where $r=r$; and it is evident that the first of these limits $r=-\lambda$, may be changed into $r=-\infty$; because the functions $\Psi(r)$, $\Psi'(r)$, &c, are supposed to be insensible between the limits $r=-\lambda$, and $r=-\infty$. Then, using for brevity the following symbols, [9841q]

$$\Psi_{\prime}(r)=\textstyle\int_{-\infty}^{r}D.dr.\Psi'(r), \qquad [9841r]$$

$$r.\Psi_2(r)=\textstyle\int_{-\infty}^{r}D.dr.\{\Psi(r)+r.\Psi'(r)\}, \qquad [9841s]$$

we shall evidently have, for the integral of [9841p], the following expression:

$$2\pi.\Psi_{\prime}(r)+\frac{1}{R}.2\pi r.\Psi_2(r)=\text{the action of the fluid } RGn'o\,m \text{ upon the particle at } Z. \qquad [9841t]$$

Now supposing the stratum $n'o'm'$ to coincide with the external surface of the fluid $N'O'M'$, and O' to be the origin of the line r, we shall have the same expression [9841t] for the action of the whole fluid $RGN'O'M'$, upon any point whatever of the axis $O'Z$, situated within the fluid, at the distance r below the surface at O'; so that, if we suppose a cylindrical column OO' to be formed, whose base at O is equal to unity, as in [9258a, b], and the attracted point to be at o, or $O'o=r$, the formula [9841t] will express the downward action of the whole fluid $RGN'O'M'$, upon a particle of the fluid at o. Multiplying this by Ddr, which expresses the quantity of fluid contained in the part $oo'=dr$ of the column, we get [9841u] [9841v] [9841w]

$$2\pi.D.dr.\Psi_{\prime}(r)+\frac{1}{R}.2\pi.D.rdr.\Psi_2(r), \qquad [9841x]$$

for the action of the whole fluid $RGN'O'M'$ upon the infinitely small column $oo'=dr$. Integrating this, so as to include the action upon the whole column OO', from $r=0$ to $r=\lambda$, we get, for the whole downward pressure of the column $O'O$ at O, [9841y]

$$2\pi.\textstyle\int_0^{\lambda}D.dr.\Psi_{\prime}(r)+\frac{1}{R}.2\pi.\textstyle\int_0^{\lambda}D.rdr.\Psi_2(r)=\text{pressure at } O. \qquad [9841z]$$

[9844] $$K = 2\pi . \int_0^\infty dr . \Psi(r),$$

[9844′] $$H = 2\pi . \int_0^\infty r dr . \Psi(r),$$

As the functions $\Psi_{\prime}(r)$, $\Psi_2(r)$, are supposed to become insensible at any sensible distances
[9842a] [9255, &c.], we may extend the second limit $r = \lambda$ to $r = \infty$ [9256]; so that, if we put for brevity, in like manner as we have done in [9253b, c],

[9842b] $$\mathrm{K} = 2\pi . \int_0^\infty D . dr . \Psi_{\prime}(r),$$

[9842c] $$\mathrm{H} = 2\pi . \int_0^\infty D . r dr . \Psi_2(r),$$

we shall have, for the pressure [9841z], the following value:

[9842d] $\mathrm{K} + \mathrm{H} . \frac{1}{R} =$ the pressure at the base O of the column $O'O$ of a variable density.

[9842e] Comparing this with the expression $K + H . \frac{1}{R}$ [9845] corresponding to a spherical surface where $R' = R$, which is computed by the author upon the supposition that the density of the fluid is equal to unity, we find that there is no difference whatever in the forms of these
[9842e′] functions [9842d, e], but merely a change of the constant quantities H, K, which become H, K, respectively. The same method of calculation may be used when we suppose the surfaces $N'O'M'$, $n'o'm'$, nom, NOM, to be concentrical ellipsoids, or surfaces which, in the case of uniform density, produce the expression [9842],

[9842f] $$2\pi . \left\{ 1 + \frac{r}{2} \left(\frac{1}{R} + \frac{1}{R'} \right) \right\} . \Psi(r),$$

instead of that depending on spherical surfaces,

$$2\pi . \left\{ 1 + \frac{1}{R} . r \right\} . \Psi(r) \quad [9841h];$$

the only change it produces is the substitution of $\frac{1}{2} \left(\frac{1}{R} + \frac{1}{R'} \right)$ for $\frac{1}{R}$; and by making this change in [9842d], we get

[9842g] $\mathrm{K} + \frac{\mathrm{H}}{2} . \left(\frac{1}{R} + \frac{1}{R'} \right) =$ the pressure at the base O of the column $O'O$ of a variable density.

This expression, corresponding to an ellipsoid, includes terms of the second order in x, y [9770, or 9812a], and may also be supposed to include those of the third order, as in [9812b]; so that, by neglecting terms of a higher order in x, y, we may suppose the function
[9842h] [9842g] to express the whole downward pressure at the base O of the tube $O'O$, whatever be the form of the limiting surface $N'O'M'$. Hence we may generally conclude, that the
[9842i] whole effect of the variation of density near the surface of the fluid, is merely to change, as in [9845, 9842g], H into H, and K into K. This change in the values of H, K, is, however, of no importance whatever, in a practical point of view, since these quantities must necessarily be determined by observation, the law of the variation of the corpuscular action being unknown, as has been already observed in the note on this fundamental theorem in [9261d—g]. We have, therefore, used, in conformity with the remarks in [9261n, &c.], the

we shall have, for the action of the body upon the canal,

$$K+\tfrac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right),$$ [Pressure at the point O.] [9845]

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formulas of the author, with his values of H, K, supposing them always to be those which are obtained from observation, and therefore such as would arise from the action of a fluid of a variable density. [9842k]

We shall suppose, in fig. 143, page 820, that the line OT is drawn tangent to the surface NOM, in the point O, and from any point S of the surface there is let fall upon this tangent the perpendicular $ST=z$. We shall also suppose that the cylindrical column $O'O$ is continued below the surface NOM, so as to form the canal $O'OVSU$, which, in its upward direction SU, meets the surface NOM at S, in a perpendicular direction, and passes on, in the same direction, to the external surface in U. Putting $R_{,}$ and $R'_{,}$, respectively, for the greatest and the least radii of curvature of the surface NOM at the point S, we shall have, in like manner as in [9842g], [9842l] [9842m]

$\mathrm{K}+\frac{1}{2}\mathrm{H}.\left(\frac{1}{R_{,}}+\frac{1}{R'_{,}}\right)=$ the pressure on the base S of the column US of a variable density. [9842n]

If the line $ST=z$ be vertical, it will express the elevation of the point O above the point S; and the pressure of a column of fluid of that height will be gz. If this line be inclined to the horizon by an angle whose cosine is represented by $\frac{u}{z}$, the vertical pressure will become $gz\times\frac{u}{z}$, or simply gu; u being the vertical elevation of the point O above the point S. Adding this quantity gu to the pressure at O [9842g], we get the whole pressure at S, which ought to balance the pressure in the canal US, given in [9842n], upon the principle of the equilibrium of the canals, which is so frequently used in this work; hence we have [9842o]

$$\mathrm{K}+\tfrac{1}{2}\mathrm{H}.\left(\frac{1}{R}+\frac{1}{R'}\right)+gu=\mathrm{K}+\tfrac{1}{2}\mathrm{H}.\left(\frac{1}{R_{,}}+\frac{1}{R'_{,}}\right).$$ [9842p]

If we suppose the situation of the point S to be successively varied, by infinitely small intervals, along the surface $NOSM$, while the point O remains the same, we shall have K, H, R, R', g, constant; and u, $R_{,}$, $R'_{,}$, variable; and then the differential of [9842p] will become, by transposing the term gdu, [9842q]

$$\tfrac{1}{2}\mathrm{H}.d.\left(\frac{1}{R_{,}}+\frac{1}{R'_{,}}\right)-gdu=0,$$ [9842r]

which is of the same form as [9811], changing H into H, as in [9842e'], and placing an additional accent on R, R', as in [9842m], to conform to the notation which we have here used. This equation may also be deduced from the same principle which is employed in [9808]; for the quantity g being multiplied by the element $-du$, is the same as in [9807''], [9842s]

which agrees with what we have before found in [9301]. This expression corresponds to a body which is terminated by a *convex* surface. *When the*
[9846] *surface is wholly concave, or convex in one direction, and concave in the other, we must suppose the radius of curvature negative which corresponds to the concavity.*

*We shall now consider a fluid contained in a capillary and prismatic tube**
[9846′] *dipped vertically by its lower end into a vessel of an indefinite extent, supposing the surface of the fluid to be concave*, and we shall define this surface by means of the following symbols:

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producing the second term of [9842r]; and we shall now show that the tangential forces
[9842t] produce the first term of that equation. For this purpose we shall suppose that the part of the canal OVS is bent along the surface OS, as in [9759, &c.]; then the pressure on the canal
[9842u] at the point O will be represented by the function [9842g], in the direction from O to S, because fluids press in every direction; and the contrary pressure at the opposite end S will be represented by the function [9842n]. The difference of these two pressures is

[9842v] $$\tfrac{1}{2}\mathrm{H}.\left(\frac{1}{R_{,}}+\frac{1}{R'_{,}}\right)-\tfrac{1}{2}\mathrm{H}.\left(\frac{1}{R}+\frac{1}{R'}\right);$$

and when the points O, S, are infinitely near to each other, this difference becomes equal to the differential of the pressure, or

[9842w] $$\tfrac{1}{2}\mathrm{H}.d.\left(\frac{1}{R}+\frac{1}{R'}\right).$$

Now this is true for every part of the surface NOM, taking for R, R', the radii of curvature corresponding to the proposed part; so that, if we take, on the arc SM, a point s infinitely near to S, we shall have, by using the values $R_{,}$, $R'_{,}$ [9842m],

[9842x] $\tfrac{1}{2}\mathrm{H}.d.\left(\frac{1}{R_{,}}+\frac{1}{R'_{,}}\right)=$ the increment of the pressure in the direction of the infinitely small arc Ss, in proceeding from S to s.

This expression of the tangential force depends chiefly on terms of the third order in x, y, treated of [9771, &c., 9772a, &c.]; and as it comprises the increment of pressure in the
[9842y] infinitely small arc Ss, it may be considered as representing the product of the tangential force by the element of its direction, being of exactly the same form as the first term of [9811], where the density is supposed to be constant. Hence it appears that the equation [9811], which is computed by the author on the supposition that the density is invariable, is
[9842z] also true when we notice the variation of density near the upper surface of the fluid $N'O'M'$, changing the constant coefficient H into H when the density is variable.

* (4267) The horizontal section of this prism may be considered as a curve of any
[9846a] form whatever, without any abrupt angular points which might cause an irregularity in its attraction.

x, y, z, are the three rectangular coordinates of any point of this concave surface of the fluid; x, y, being horizontal, and z a vertical line; the origin of these coordinates being at the lowest point of the concave surface of the fluid; [9847]

$h =$ the elevation of the lowest point of the concave surface of the fluid, above the level of the fluid in the vase; and from [9847] we have [9848]

$z =$ the elevation of any other point whatever of the concave surface of the fluid above the level of the lowest point of the same surface; [9849]
so that $h+z$ represents the elevation of the same point above the level of the fluid in the vessel; [9849′]

$D =$ the density of the fluid. [9850]

We shall suppose that a vertical canal* commences at the point of the surface corresponding to z, and continues downward below the tube, is there bent under it, and continues till it meets the surface of the fluid in the vessel; this tube being infinitely small in its diameter, and of the same dimensions throughout. Then the condition of the equilibrium of this canal will give the equation [9850′]

$$\tfrac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)=gD.(h+z). \qquad [9851]$$

[Changing H into H, as in [9849f], we may suppose the density to be variable near the surfaces of the fluid and tube.]

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* (4268) The canal is represented by $VRZ'QPO'$, in fig. 112, page 695, O' being the point of the concave surface which is under consideration, P the point at the same level with O, and Q the point at the same level with V; and we have, in the notation here used, $QP=h$, $PO'=z$, $QO'=h+z$ [9848, 9849]; so that, if we put $VR=Z'Q=k$, we shall have $Z'O'=k+h+z$. Then the capillary action at V, namely, K, is to be added to the gravity of the column VR, which is $gD.k$, to get the pressure $K+gD.k$ on the column VR at R. In like manner, the capillary action at O' for a *concave* surface [9845, 9846], being added to the gravity of the column $Z'O'$, namely, $gD.(k+h+z)$, gives the pressure on the column $O'Z'$, equal to [9849a] [9849b] [9849c] [9849c′]

$$K-\tfrac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)+gD.(k+h+z). \qquad [9849d]$$

These two columns ought to balance each other, as in [9315b]; hence we have

$$K+gDk=K-\tfrac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)+gD.(k+h+z). \qquad [9849e]$$

Rejecting the terms K, gDk, which destroy each other in both members, and transposing the term depending on H, we get [9851]; changing H into H, as in [9842z], the equation [9851] will correspond to the case where the density is supposed to be variable near the surfaces of the fluid and tube. [9849f]

[9852] Now if we put $\mathrm{p}=\left(\frac{dz}{dx}\right)$, $\mathrm{q}=\left(\frac{dz}{dy}\right)$ [9312], we shall have, by the theory of curve surfaces,*

[9853]
$$\frac{1}{R}+\frac{1}{R'}=\frac{(1+\mathrm{q}^2)\cdot\left(\frac{d\mathrm{p}}{dx}\right)-\mathrm{pq}\cdot\left\{\left(\frac{d\mathrm{p}}{dy}\right)+\left(\frac{d\mathrm{q}}{dx}\right)\right\}+(1+\mathrm{p}^2)\cdot\left(\frac{d\mathrm{q}}{dy}\right)}{(1+\mathrm{p}^2+\mathrm{q}^2)^{\frac{3}{2}}};$$

therefore we shall have

Differential equation of the concave surface in a capillary tube.

[9854]
$$\tfrac{1}{2}H.\left\{\frac{(1+\mathrm{q}^2)\cdot\left(\frac{d\mathrm{p}}{dx}\right)-\mathrm{pq}\cdot\left\{\left(\frac{d\mathrm{p}}{dy}\right)+\left(\frac{d\mathrm{q}}{dx}\right)\right\}+(1+\mathrm{p}^2)\cdot\left(\frac{d\mathrm{q}}{dy}\right)}{(1+\mathrm{p}^2+\mathrm{q}^2)^{\frac{3}{2}}}\right\}=gD.(h+z);$$

[9854′] which is evidently the same as the equation [9318], as in [9852c].

Multiplying this equation by $dxdy$, then taking its integral relative to dx and [9855] dy, observing also that the function multiplied by $\frac{1}{2}H$ may be put under the form †

[9856]
$$\left(\frac{d.\frac{\mathrm{p}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dx}\right)+\left(\frac{d.\frac{\mathrm{q}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dy}\right),$$

we shall have

[9857]
$$\tfrac{1}{2}H.\iint dx.dy.\left\{\left(\frac{d.\frac{\mathrm{p}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dx}\right)+\left(\frac{d.\frac{\mathrm{q}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dy}\right)\right\}=gD.\textstyle\iint(h+z).dx.dy.$$

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* (4269) Taking the partial differentials of p, q [9852], relative to x, y, and using the symbols [9312, 9313], we evidently have

[9852a]
$$\mathrm{r}=\left(\frac{ddz}{dx^2}\right)=\left(\frac{d\mathrm{p}}{dx}\right);\qquad \mathrm{s}=\left(\frac{ddz}{dxdy}\right)=\left(\frac{d\mathrm{p}}{dy}\right)=\left(\frac{d\mathrm{q}}{dx}\right);\qquad \mathrm{t}=\left(\frac{ddz}{dy^2}\right)=\left(\frac{d\mathrm{q}}{dy}\right).$$

[9852b] Putting these values of r, s, t, in [9314], we get [9853], by observing that $2\mathrm{s}=\left(\frac{d\mathrm{p}}{dy}\right)+\left(\frac{d\mathrm{q}}{dx}\right)$; substituting [9853] in [9851], we obtain [9854]; multiplying this by $\frac{2}{H}$, it becomes of the same form as [9318], gD being used for g, and the constant terms

[9852c] $\frac{1}{b}+\frac{1}{b'}$ taking the place of $\frac{2gDh}{H}$, agreeing with the remarks in [9854′]. We may observe

[9852d] that the length of the vertical column of air, of the density D', which presses on the point V, fig. 112, page 695, exceeds that at O' by $QO'=h+z$, and its weight $gD'.(h+z)$ is to be added to that in [9849c′], or to the first member of [9849e]. The effect of this

[9852e] correction is to decrease the second member of [9851] by $gD'.(h+z)$, or to change D into $D-D'$, as M. Poisson has done; but the density of the air D' being very small relative to D, this correction may be neglected.

† (4270) Developing the differentials in [9856], it becomes equal to the factor of $\frac{1}{2}H$

[9856a] in [9854]; substituting it in [9854], multiplying by $dxdy$, and prefixing the sign $\iint$, it becomes as in [9857].

These double integrals must be taken throughout the whole extent of the interior horizontal section of the prism. The double integral* $gD.\int\int(h+z).dx.dy$ [9858]
is therefore the weight of the fluid elevated by the capillary action above the level. Thus, by putting *V for the volume of this fluid*, we shall have [9859] V.

$$gD.\int\int(h+z).dx.dy=gD.V. \qquad [9860]$$

The double integral

$$\tfrac{1}{2}H.\int\int dx.dy.\left(\frac{d.\frac{\mathrm{p}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dx}\right) \qquad [9861]$$

becomes by integration, relative to x, of the following form,

$$\tfrac{1}{2}H.\int dy.\left(\frac{\mathrm{p}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}-\frac{(\mathrm{p})}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}\right), \qquad [9862]$$

(p) and (q) being what p and q become at the origin of the integral. In like [9863]
manner, the double integral

$$\tfrac{1}{2}H.\int\int dx.dy.\left(\frac{d.\frac{\mathrm{q}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dy}\right) \qquad [9864]$$

becomes by integration, relative to y,

$$\tfrac{1}{2}H.\int dx.\left(\frac{\mathrm{q}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}-\frac{(\mathrm{q})}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}\right). \qquad [9865]$$

To obtain a precise idea of these integrals and their limits, we shall observe that these limits are the horizontal section of the interior surface of the tube, [9866]
and that this section is a re-entering curve. We shall take the origin of x, y, outside of this curve, so that it may be wholly included in the right angle formed by the axes of x and y. In this case, *the values of dx and dy are evidently positive in the preceding double integrals, when* $gD.\int\int(h+z).dx.dy$ [9867]

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* (4271) If we suppose the whole volume of the prism to be divided into an infinite [9858a]
number of rectangular prisms, we shall have dx, dy, for the horizontal sides of the base of one of these prisms, and $h+z$ for its height; therefore its volume dV will be $dx.dy.(h+z)=dV$, [9858b]
its mass $D.dx.dy.(h+z)=D.dV$, and its gravity $gD.dx.dy.(h+z)=gD.dV$, whose integral is as in [9860]. If the surface of the fluid be supposed, as in [9319′], to [9858c]
be formed by the revolution of a curve about the axis of z, and u be taken for the distance of a particle of the fluid from the axis of z, the volume dV of the fluid, contained between [9858d]
the two concentrical cylindrical surfaces whose radii are u, $u+du$, will be found by multiplying the height $h+z$ by the area of the annulus $2\pi udu$, which forms its base; hence $dV=2\pi.(h+z).udu$. Multiplying this by the density D, and the gravity g, and then [9858e]
integrating, we get $gD.V=2\pi gD.\int(h+z).udu$, which will be of use hereafter.

[9868] *denotes the weight of the elevated fluid* [9858], *as we have here supposed; these differentials must therefore be considered as positive in the simple integrals.* This being premised, we may remark that the element

[9869] $\frac{\frac{1}{2}H.\mathrm{p}dy}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}=Pdy,$ corresponds to the branch $[GH]$ of the section which is *concave* * towards the axis of y;

[9870] $-\frac{\frac{1}{2}H.(\mathrm{p}).dy}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}=-(P).dy,$ corresponds to the branch $[EF]$ of the section which is *convex* towards the axis of y;

[9871] $\frac{\frac{1}{2}H.\mathrm{q}dx}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}=Qdx,$ corresponds to the branch $[FG]$ of the section which is *concave* towards the axis of x;

[9872] $-\frac{\frac{1}{2}H.(\mathrm{q}).dx}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}=-(Q).dx,$ corresponds to the branch $[EH]$ of the section which is *convex* towards the axis of x;

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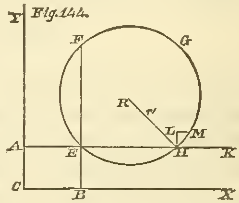
* (4272) We shall use for brevity the following symbols, which are inserted in [9869—9872], though they are not in the original work;

[9869a]
$$P=\frac{\frac{1}{2}H.\mathrm{p}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}; \qquad (P)=\frac{\frac{1}{2}H.(\mathrm{p})}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}};$$
$$Q=\frac{\frac{1}{2}H.\mathrm{q}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}; \qquad (Q)=\frac{\frac{1}{2}H.(\mathrm{q})}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}};$$



[9869b] and for illustration we shall suppose, in the annexed figure 144, that CBX is the axis of x, CAY that of y; $EFGH$ a section of the tube, by a plane parallel to the plane of xy,
[9869c] and intersecting the axis of z in the point C. Then, in taking the integral of [9861] relative to x, to obtain [9862], we must,
[9869d] as usual, consider y as constant, and take the integral in the direction EH parallel to the axis of x, from any point E corresponding to the element (P), to the point H corresponding to the element P; and we shall have for the integral formula [9861], the same expression as in [9862], namely,

[9869e]
$$\tfrac{1}{2}H.\int\!\!\int dx.dy.\left(\frac{d.\frac{\mathrm{p}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dx}\right)=\int Pdy-\int(P).dy.$$

In like manner the integral relative to y of the expression [9864], is taken in the direction BEF parallel to the axis of y, from any point E to the point F, making

[9869f]
$$\tfrac{1}{2}H.\int\!\!\int dx.dy.\left(\frac{d.\frac{\mathrm{q}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}}{dy}\right)=\int Qdx-\int(Q).dx,$$

as in [9865]. Substituting the values [9869e, f, 9860] in [9857], we get

[9869g]
$$\int Pdy-\int(P).dy+\int Qdx-\int(Q).dx=gD.V.$$

Now of the four elements of which the first member of this integral is composed, it is evident
[9869h] that, in the expression [9869e], the integral $\int Pdy$ corresponds to the part GH, which is *concave* towards the axis of y, as in [9869], and $\int(P).dy$ corresponds to EF, which is

therefore, by supposing that the elements [9873]

$$-\frac{\frac{1}{2}H.(\mathrm{q}).dx}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}} \; [9872], \qquad -\frac{\frac{1}{2}H.(\mathrm{p}).dy}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}} \; [9870],$$ [9874]

correspond to the same point [E] of the section, this point will appertain to the part of the section* which is convex both towards the axis of x [9870], and [9875]
the axis of y [9872]. In this part, the values of dx and dy, referred to the [9876]
curve, have contrary signs; *therefore, by supposing dx to be always positive, dy* will be negative, and the sum of the two preceding elements will be [9877]

$$\frac{\frac{1}{2}H.\{(\mathrm{p}).dy-(\mathrm{q}).dx\}}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}; \qquad [\text{Point } E.]$$ [9878]

the differentials dx, dy, being in this case those of the section.

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convex towards the axis of y, as in [9870]. In like manner, in the expression [9869f], the integral $\int Qdx$ corresponds to the part FG which is *concave* towards the axis of x, as in [9871], and $\int(Q).dx$ corresponds to the part EH which is *convex* towards the axis of x, as in [9869]. [9869i]

The integral expression in the first member of [9869g], contains both the differentials dx, dy, corresponding to all points of the circumference $EFGH$. This circumference may be considered as being divided into four distinct portions, corresponding to the parts near to E, F, G, H, respectively, where the elements dx, dy, can have different signs, when compared with each other. Now in proceeding along the curve $EFGH$, each element of the integral must contain both the differentials dx, dy; hence it is evident that there can be four distinct combinations of the four elements Pdy, $-(P).dy$, Qdx, $-(Q).dx$, [9869g, or 9869—9872], namely, [9869k] [9869l]

$+Pdy$ of the branch GH [9869], with $+Qdx$ of the branch FG [9871], at the point G, as in [9879]; [9869m]
$+Pdy$ of the branch GH [9869], with $-(Q).dx$ of the branch EH [9872], at the point H, as in [9883]; [9869n]
$-(P).dy$ of the branch EF [9870], with $+Qdx$ of the branch FG [9871], at the point F, as in [9885]; [9869o]
$-(P).dy$ of the branch EF [9870], with $-(Q).dx$ of the branch EH [9872], at the point E, as in [9874]; [9869p]

as will be evident by remarking that the point G is common to the two branches GH, FG [9869m]; the point H is common to the two branches GH, EH [9869n]; the point F is common to the two branches EF, FG [9869o]; and the point E is common to the two branches EF, EH [9869p]. [9869q]

* (4273) *This point corresponds to the point* E [9869p], in fig. 144, page 828, where the curve is convex towards the axes of x, y; and it is evident that, in proceeding along the curve EH, while x increases by the quantity dx, y will decrease; so that dy will be negative, as in [9876]; and by supposing the two elements [9874] to *correspond to the point* E, and taking dx positive, we must change the sign of dy, as in [9877], and then the sum of the two elements [9874] will become as in [9878]. [9876a] [9876b] [9876c]

[9879] In like manner, if the elements $\frac{\frac{1}{2}H.\mathrm{q}dx}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}$ [9871] and $\frac{\frac{1}{2}H.\mathrm{p}dy}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}$ [9869] refer to the same point [G] of the section, this point will be in a part of the
[9880] curve which is concave* to both the axes of x and y. In this part, the differentials dx, dy, referred to the curve, have contrary signs; therefore the sum
[9881] of the preceding elements will be, by supposing dx to be positive,

[9882] $$-\frac{\frac{1}{2}H.\{\mathrm{p}dy-\mathrm{q}dx\}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}, \qquad [\text{Point } G.]$$

the differentials dx and dy being in this case those of the section.

[9883] If the elements $-\frac{\frac{1}{2}H.(\mathrm{q}).dx}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}$ [9872] and $\frac{\frac{1}{2}H.\mathrm{p}dy}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}$ [9869] appertain to the same point [H], they will correspond to the part of the curve which is convex† towards the axis of x, and concave towards the axis of y; and then
[9884] dx, dy, have the same sign.

[9885] Lastly, if the elements $-\frac{\frac{1}{2}H.(\mathrm{p}).dy}{\sqrt{1+(\mathrm{p})^2+(\mathrm{q})^2}}$ [9870] and $\frac{\frac{1}{2}H.\mathrm{q}dx}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}$ [9871] correspond to the same point [F], they will appertain to the part of the curve which is convex‡ towards the axis of y, and concave towards the axis of x;
[9886] then dx, dy, will have the same sign.

[9887] Thus we see that, by expressing these elements generally by $\frac{\frac{1}{2}H.\mathrm{p}dy}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}$ and $\frac{\frac{1}{2}H.\mathrm{q}dx}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}$, whether they refer to the commencement or termination of the
[9888] integrals relative to x, y, *they will have a contrary sign in the same points of the*

[9880a] * (4274) *This corresponds to the point* G [9869*m*], in fig. 144, page 828, where the curve is concave towards the axis of x, y; and it is evident that, in proceeding along the curve GH, while x increases by the quantity dx, y will decrease, so that dy will be negative;
[9880b] and we must change the sign of dy, as in the last note; and then the sum of the two elements [9879] becomes as in [9882].

[9883a] † (4275) *This corresponds to the point* H [9869*n*], in fig. 144, page 828, where the curve is convex towards the axis of x, and concave towards the axis of y; so that x
[9883b] and y increase together along the curve HG; consequently dx, dy, have the same signs as in [9884].

[9885a] ‡ (4276) *This corresponds to the point* F [9869*o*], in fig. 144, page 828, where the curve is convex towards the axis of y, and concave towards the axis of x; so that x, y,
[9885b] increase together along the curve FG; consequently dx, dy, have the same sign as in [9886].

curve, when the differentials dx, dy, are those of the curve itself; therefore their sum will be, supposing always dx to be positive,* [9888′]

$$\pm \frac{\frac{1}{2}H.\{\mathrm{p}dy - \mathrm{q}dx\}}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}; \qquad [9889]$$

the sign + taking place in the part of the curve which is convex towards the axis of x, and the sign — taking place in the concave part.* [9890]

Now it is easy to prove, by the theory of curve surfaces, that, if we suppose the tube to be placed in a vertical direction, and then put [9891]

ϖ = the angle which the tangent plane of the fluid, at the surface of the tube near the point of the limit of the sphere of activity of the tube, forms with the vertical side of the tube, the origin of the angle ϖ being on the lower part of the tube, or that which is in the direction of gravity [this angle ϖ being the same as that which is named θ in 9601″], we shall have † [9892] [9892′]

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* (4277) The positive factor $+\frac{1}{2}H.(\mathrm{p}dy - \mathrm{q}dx)$ [9889] corresponds to the terms [9878, 9883], or to the points E, H [9876a, 9883a], where the curve EH is convex towards the axis of x, as in [9890]. In like manner the negative factor $-\frac{1}{2}H.(\mathrm{p}dy - \mathrm{q}dx)$ [9889] corresponds to the terms [9882, 9885], or to the points G, F [9880a, 9885a], where the curve GF is concave towards the axis of x, as in [9890]. Prefixing the sign $\int$ to [9889], we get the expression [9869g], under the form [9889a] [9889b]

$$gD.V = \pm \int \frac{\frac{1}{2}H.(\mathrm{p}dy - \mathrm{q}dx)}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}, \qquad [9889c]$$

where the elements dx, dy, correspond to the curve itself.

† (4278) We have seen, in note (15a), vol. 1, pages 15, 16, that if $du=0$ be the differential element of the equation of a curve surface, whose rectangular coordinates are x, y, z, and $dr=0$ the differential of the equation of a right line drawn perpendicular to this element, we shall have [9892a]

$$dr = \frac{du}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}. \qquad [9892b]$$

Moreover, if we denote the cosines of the angles which the line r makes with the axes of x, y, z, by $\cos.(r,x)$, $\cos.(r,y)$, $\cos.(r,z)$, respectively, we shall have, as in the notes (15a), (55a), vol. 1, pages 15, 85, [9892c]

$$\left(\frac{dr}{dx}\right) = \frac{x-a}{r} = \cos.(r,x), \qquad \left(\frac{dr}{dy}\right) = \frac{y-b}{r} = \cos.(r,y), \qquad \left(\frac{dr}{dz}\right) = \frac{z-c}{r} = \cos.(r,z). \qquad [9892d]$$

Now z being a function of x, y, we have $dz = \left(\frac{dz}{dx}\right).dx + \left(\frac{dz}{dy}\right).dy = \mathrm{p}dx + \mathrm{q}dy$ [9852], or $dz - \mathrm{p}dx - \mathrm{q}dy = 0$; putting this for the assumed differential equation of the surface [9892e]

[9893] $$\pm\frac{(\mathrm{p}dy-\mathrm{q}dx)}{ds\cdot\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}=\cos.\varpi,$$

[9894] ds being the element of the section; therefore we shall have, *by observing that the angle* ϖ *is constant, as we have seen in* [9197],

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of the fluid $du=0$, we get $du=dz-\mathrm{p}dx-\mathrm{q}dy=0$; whence

[9892f] $$\left(\frac{du}{dz}\right)=1,\qquad\left(\frac{du}{dy}\right)=-\mathrm{q},\qquad\left(\frac{du}{dx}\right)=-\mathrm{p};$$

[9892f′] substituting these in [9892b], we get $dr=\frac{dz-\mathrm{p}dx-\mathrm{q}dy}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}$; and, if we put for brevity $\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}=\rho$, we shall have, from this general differential dr, the following partial differential coefficients, namely,

[9892g] $$\left(\frac{dr}{dz}\right)=\frac{1}{\rho},\qquad\left(\frac{dr}{dy}\right)=-\frac{\mathrm{q}}{\rho},\qquad\left(\frac{dr}{dx}\right)=-\frac{\mathrm{p}}{\rho};$$

substituting these in [9892d], we obtain

[9892h] $$\cos.(r,x)=-\frac{\mathrm{p}}{\rho},\qquad\cos.(r,y)=-\frac{\mathrm{q}}{\rho},\qquad\cos.(r,z)=\frac{1}{\rho}.$$

[9892h′] Now in fig. 144, page 828, we shall suppose the line $HR=r'$ to be drawn in the plane of this figure, perpendicular to the arc $HM=ds$, to which correspond the differentials of the coordinates $HL=dy$, $LM=dx$; then we evidently have

[9892i] $$\cos.LHM=\frac{dy}{ds},\qquad\cos.LMH=\frac{dx}{ds}.$$

Moreover the angles which the line r' makes with the lines HK, HL, drawn parallel to the axes of x, y, and in their positive directions, are KHR, LHR; and by using a notation similar to that in [9892c], we have

[9892k] $$\cos.(r',x)=\cos.KHR=-\cos.RHE=-\cos.LHM=-\frac{dy}{ds},$$
$$\cos.(r',y)=\cos.RHL=\cos.LMH=\frac{dx}{ds},\qquad\cos.(r',z)=\cos.90^{\mathrm{d}}=0,$$

observing that the axis of z, being vertical, is perpendicular to the plane of the figure. From

[9892l] the formula [172h] we get $\cos.(r,r')$, or the cosine of the angle which the lines r, r', form with each other, as in [9892m]; substituting in it the values [9892h, k], it becomes, as in [9892n],

[9892m] $$\cos.(r,r')=\cos.(r,x).\cos.(r',x)+\cos.(r,y).\cos.(r',y)+\cos.(r,z).\cos.(r',z);$$

[9892n] $$=\left(-\frac{\mathrm{p}}{\rho}\right).\left(-\frac{dy}{ds}\right)+\left(-\frac{\mathrm{q}}{\rho}\right).\left(\frac{dx}{ds}\right)=\frac{\mathrm{p}dy-\mathrm{q}dx}{\rho ds}=\frac{\mathrm{p}dy-\mathrm{q}dx}{ds.\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}.$$

[9892n′] Now the line r being drawn perpendicular to the surface of the fluid [9892a], and the line r' perpendicular to the surface of the tube, it is evident that the inclination of the lines r, r', is equal to the angle ϖ [9892], which the tangent to the surface of the fluid makes with the

[9892o] side of the tube; therefore $\cos.(r,r')=\cos.\varpi$; and, by substituting it in [9892n], it becomes as in [9893], the double sign $\pm$ being prefixed on account of the radical $\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}$.

$$\pm\int\frac{(\mathrm{p}dy-\mathrm{q}dx)}{\sqrt{1+\mathrm{p}^2+\mathrm{q}^2}}=c.\cos.\varpi,$$ [9895]

c *being the whole circumference of the section;** therefore [9896]

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* (4279) Multiplying [9893] by ds, and integrating through the whole extent of the circumference of the tube, we find that the first member becomes like that of [9895], and the second member of the integral is $\cos.\varpi.\int ds$; but $\int ds=c$ [9896], hence this becomes [9896a] $c.\cos.\varpi$, as in the second member of [9895]. Now multiplying [9895] by $\frac{1}{2}H$, we find that [9896b] its first member becomes like the integral of [9889], which is shown, in [9861—9889], to be equivalent to the integral expression in the first member of [9857]; hence we get [9897], [9896c] being equal to $gD.V$ [9857, 9860]; therefore we have $gD.V=\frac{1}{2}Hc.\cos.\varpi$, as in [9898]. [9896d] This equation, which is computed upon the supposition of a fluid of uniform density, will also correspond to a fluid of variable density near its surface, by changing H into H, as in [9261m], [9896d'] or taking for H the value deduced from actual observation of the effect of the capillary action [9896e] on the fluid, and substituting it in [9851] and in the resulting equation [9898].

For the purpose of illustrating the preceding calculation, we may observe that the process [9896f] which is explained in [9855—9898] becomes much more simple when we restrict ourselves to the case where the tube is supposed to be a vertical cylinder of revolution. For then the [9896g] equation [9854] becomes integrable, its first member being equal to the first member of [9318 or 9324] multiplied by $\frac{1}{2}H$; and, by substituting it in [9854], we get

$$\tfrac{1}{2}H.\left\{\frac{\frac{ddz}{du^2}+\frac{1}{u}.\frac{dz}{du}.\left(1+\frac{dz^2}{du^2}\right)}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}\right\}=gD.(h+z).$$ [9896h]

Multiplying this by $2\pi.udu$, and integrating, we obtain

$$\tfrac{1}{2}H.\frac{2\pi u.\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}=2\pi.gD.\int(h+z).udu,$$ [9896i]

which is easily proved by differentiation, as we have already seen in [9329a, 9329]. Multiplying the numerator and denominator of the first member by du, and substituting, in the second member, its value $gD.V$ [9858e], we get

$$\tfrac{1}{2}H.2\pi u.\frac{dz}{\sqrt{du^2+dz^2}}=gD.V.$$ [9896k]

Now when u becomes equal to the radius of the tube, we evidently have $\frac{dz}{\sqrt{du^2+dz^2}}=\cos.\varpi$, ϖ being the angle formed by the arc $\sqrt{du^2+dz^2}$ and the vertical dz; moreover the whole [9896l] circumference c of the tube becomes $c=2\pi u$; hence the equation [9896k] becomes

$$\tfrac{1}{2}Hc.\cos.\varpi=gD.V,$$ [9896m]

as in [9898].

For greater simplicity it is supposed, in the calculations of this article, that the ordinate

[9897] $$\tfrac{1}{2}H.\iint dx.dy.\left\{\left(\frac{d.\frac{p}{\sqrt{1+p^2+q^2}}}{dx}\right)+\left(\frac{d.\frac{q}{\sqrt{1+p^2+q^2}}}{dy}\right)\right\}=\tfrac{1}{2}Hc.\cos.\varpi,$$

which gives

[9898] $$gD.V=\tfrac{1}{2}Hc.\cos.\varpi. \quad (o)$$ [This equation can be used when the fluid varies in density near the surface, taking for H its value H deduced from observation, as is observed in [9896c].]

[9899] *Thus the mass of the fluid elevated above the level by the capillary action, is proportional to the circumference of the section of the inner surface of the tube. We may obtain this remarkable equation, by considering the effects of the capillary action under the following point of view.*

NEW MANNER OF CONSIDERING THE CAPILLARY ACTION.

The manner in which we have heretofore considered the capillary phenomena, is founded upon the consideration of the surface of a fluid contained in a capillary space, and upon the conditions of the equilibrium of the fluid, in an infinitely narrow canal, terminating at one end in that surface, and at the other
[9900] end in the level surface of the indefinite fluid in which the sides of the capillary tube are dipped. We shall now consider directly the forces which elevate or depress the fluid in that space. This process will lead to several general results, which could not be obtained without some difficulty by the
[9900′] preceding method; and the comparison of these two methods will furnish us with the means of comparing exactly the affinities of several bodies for fluids.

We shall suppose that there is a prismatic tube whose sides are perpendicular
[9901] to the base, [without having any angular points, 9846a]. Then, if it be dipped vertically, by its lower end, into a fluid [like water, alcohol, &c.], the fluid will rise in the tube above the level of the fluid in the vessel, and it is

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$AEHK$, fig. 144, page 828, intersects the curve in no more than *two points* E, H; so that,
[9896n] in finding the integral relative to x in [9869e], we obtain $\int Pdy-\int(P).dy$, consisting of only the *two portions* depending on P and (P). But if the curve be of a serpentine form,
[9896o] there may be supposed, between H and K, other branches, as $E'H'$, similar to EH, and falling, like it, partly below the line AK. Now, if we suppose $P'dy$, $(P').dy$, to be the
[9896p] elements corresponding to the points E', H', and respectively similar to Pdy, $(P).dy$, the integral [9869e] will be increased by the two additional terms $\int P'dy-(P').dy$; but it is evident that this will produce no difference in the resulting equation [9898], because we may proceed with these new terms in exactly the same manner as we have done with the
[9896q] former ones. Similar remarks may be made relative to the integration corresponding to y in [9869f].

evident that this must arise from the combined action of the sides of the tube upon the fluid, with that of the fluid upon its own particles. *The first lamina of the fluid, contiguous to the sides, is elevated by this action; this raises a* [9902]
second, that a third, and so on, until the weight of the elevated fluid balances the attractive forces which tend to raise it still higher. To determine this quantity in the state of equilibrium, we shall suppose that, at the lower end of this tube, a *second* imaginary tube is placed, whose infinitely thin surface is formed by the continuation of the sides of the interior part of the *first* tube, and [9903]
that this surface has no action upon the fluid, and does not impede the reciprocal action of the particles of the first tube and of the fluid. We shall suppose that this second tube is vertical in its first branch, that then it is bent horizontally, [9904]
and at last resumes its vertical direction in a second branch, retaining, throughout its whole extent, the same figure and the same width. It is evident that, in the state of equilibrium of the fluid, the pressure must be the same in the two [9905]
vertical branches of the canal, composed of the first and second tube. But as there is more fluid in the first vertical branch, composed of the first tube and a part of the second, than in the second vertical branch, it is necessary that the excess of pressure resulting from it should be balanced by the vertical [9906]
attractions of the prism and fluid upon the fluid contained in this first branch. We shall analyze with care these different attractions, and shall consider, in the first place, those *which take place near the lower part of the first tube.**

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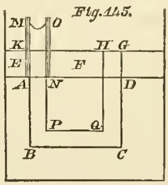
* (4280) To illustrate this, we have drawn the annexed figure 145, in which $MANO$ is the vertical or *first tube*, whose inner surface is supposed to be continued so as to form a *second* or bent tube $ABCGHQPN$, of uniform diameter, having a horizontal branch $BCQP$, and a *second* vertical branch or tube CH, terminated at the level surface of the *homogeneous* fluid KHG; lastly, through the lower part AN of the first tube is drawn the horizontal plane AND parallel to the level surface of the fluid KHG. Then it is evident [9907a] [9907b]



that the fluid in the *second* tube $ABCD$, and that which is contiguous to it below the level surface AD, balance each other by their mutual attractions. The fluid situated above [9907c]
the plane AD, and surrounding the first tube at E and F, is without the sphere of attraction [9907d]
of the fluid in the first tube $MANO$, on account of the thickness of the glass, so that this [9907e]
action may be neglected. The matter of the first tube, at its lower part AN, draws the fluid of the second tube upwards with a force Q [9911]; moreover the *first* tube, at its upper part [9907f]
MO, draws the fluid of the *first* tube upwards with the same force Q, as will be seen in the [9907g]
next note [9920g]; the sum of these two forces is $2Q$. Lastly, the fluid surrounding the [9907h]

[9907] The prism being supposed to be a *right* one, and in a vertical position, its base will be horizontal. The fluid contained in the second tube is drawn downwards vertically, *first*, by its own particles; *second*, by the fluid
[9908] surrounding this tube. But these two attractions are destroyed by the similar attractions which the fluid contained in the second vertical branch of the canal suffers, near the level surface of the fluid; we may therefore, in this case, neglect them. The fluid in the first vertical branch of the second tube is drawn
[9909] vertically upwards by the fluid of the first tube; but this attraction is destroyed by the attraction which it exerts upon this last fluid; we may therefore neglect these
[9910] two reciprocal attractions. Lastly, the fluid in the second tube is drawn
Q. vertically upwards by the first tube, *thus producing an action upon this fluid*
[9911] *which we shall denote by* Q, contributing in part to balance the excess of pressure arising from the elevation of the fluid in the first tube.

We shall now examine the forces by which the fluid of the first tube is urged.
[9912] It suffers in its lower part the following attractions: *First:* It is attracted by its own particles; but the reciprocal attractions of a body do not give it any motion if it is solid, and we may, without disturbing the equilibrium, conceive the fluid in the first tube to become solid. *Second:* This fluid is attracted by
[9913] the fluid within the second tube; but we have just seen that the reciprocal attractions of these two fluids destroy each other, and that it is not necessary
[9914] to take any notice of them. *Third:* It is attracted by the external fluid which
Q'. surrounds the second tube, and from this attraction *a vertical force is produced,*
[9915] *which draws downwards, and which we shall denote by* $-Q'$. We have prefixed to it the sign —, to denote that its direction is contrary to that of the force Q. We shall now observe that, if the laws of attraction, relative to the
[9916] distance, are the same for the particles of the first tube and for those of the fluid, so that they only differ with respect to their intensities, and if we put ρ and ρ' for these intensities corresponding to an equal volume [the density of the fluid being considered as uniform], the forces Q and Q' will be in the same
[9917] proportion to each other as ρ to ρ'. For the inner surface of the fluid which surrounds the second tube, is the same as the inner surface of the first tube; therefore the two masses differ only by their thickness; but the attraction of

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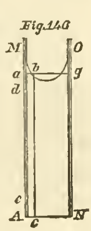
[9907i] second tube near AN, and below the level surface AD, draws the fluid of the *first* tube downwards with the force Q', or upwards with the force $-Q'$ [9915]. Adding this to
[9907k] the force $2Q$ [9907h], we obtain the whole force acting on the homogeneous fluid in the first tube $2Q-Q'$ [9923], drawing upwards.

the masses becomes insensible at sensible distances; therefore the difference in [9918]
their thickness produces no difference in their attractions, provided these thicknesses are sensible. *Fourth:* The fluid in the first tube is attracted vertically upwards by the tube. [9919]
For, if we suppose this fluid to be divided into an infinite number of small vertical columns, and through the upper end of one of them we draw a horizontal plane, the part of the tube below this plane will not produce any vertical force in the column;* therefore there will not be [9920]



* (4281) In the annexed figure 146, $MANO$ represents the first tube, [9920a]
as in fig. 145; AN is a horizontal plane, representing the bottom part of the tube; abg another horizontal plane drawn through any point b of the surface; [9920b]
and bc a vertical canal passing through that point. Now if we take any two points d, e, of the tube, situated vertically over each other, so that e may be as [9920c]
much above the plane AN as d is below the plane abg, it will be evident that the particle at e will draw the column bc *downwards* with the same force [9920d]
that the particles at d will draw the same column *upwards*, so that these forces [9920e]
will destroy each other; and the same may be proved of all such similarly situated particles in the part Aa of the tube; therefore the action of this part of the tube upon the column bc is destroyed; but the other part of the tube aM draws the column bc [9920f]
with the same force that the whole tube MA draws upwards a similarly placed column in the second tube. Hence we easily perceive that the whole action of the matter of the *first* tube, upon the fluid contained within it, is the same as that of the same tube upon the fluid [9920g]
contained in the *second* tube, and this is equal to Q [9911], agreeing with [9922]; and this principle is used in the preceding note, the fluid being always considered as homogeneous.

If we take into consideration the change of density of the fluid near its surface, and near [9920h]
the sides of the tube, we may suppose the three forces treated of in [9911, 9915, 9922], to become respectively Q_0, $-Q'_1$, Q_2. We have not put $Q_0 = Q_2$, as is done by the author in [9920i]
[9922], because the density of the fluid, near the surface of the second tube, is the same as that of the internal mass of the fluid, except in the very small portion in the immediate vicinity of the lower part of the tube A; whereas, near the surface of the first tube, the [9920k]
density may be very different from that of the internal mass of the fluid, on account of the action of the solid tube; we cannot, therefore, consider the action Q_0 [9911] of the first tube upon the fluid in the second tube, as being exactly equal to the force Q_2 [9922] of the top part of [9920l]
the first tube upon the fluid in the first tube. Now taking the sum of the three forces [9920i], and putting them equal to $gD.V$, as in [9924], we get $gD.V = (Q_0 + Q_2) - Q'_1$; so [9920m]
that, if we put, for the sake of symmetry, $Q_0 + Q_2 = 2Q_1$, it becomes [9920n]

$$gD.V = 2Q_1 - Q'_1. \qquad [9920o]$$

This is to be used instead of [9925]; and if we put, in like manner as in [9927', 9929], $Q_1 = \rho_{/}.c$, $Q'_1 = \rho'_{/}.c$, we shall get, as in [9930], [9920p]

$$gD.V = (2\rho_{/} - \rho'_{/}).c; \qquad [9920q]$$

any vertical force produced, but that which depends upon the part of the tube above the plane; and it is evident that the vertical attraction of this part
[9921] of the tube upon the column, will be the same as that of the whole tube upon an equal and similar column placed in the second tube [neglecting the variation of density of the fluid near its surface and near the tube]. Therefore *the whole vertical force produced by the attraction of the first tube upon the fluid contained in it, will be equal to that produced by the attraction of this tube upon*
[9922] *the fluid contained in the second tube; consequently this force will be*, as in [9911], *equal to* Q [the fluid being homogeneous].

Connecting together all the vertical attractions which the fluid contained in the first vertical branch of the canal suffers, we shall have a vertical force, drawing
[9923] upwards, and equal to $2Q-Q'$ [the fluid being homogeneous]. This force ought to balance the excess of the pressure arising from the weight of the fluid which is elevated above its level. Then putting, as above, V for its volume
[9924] [9859], D its density [9850], and g the force of gravity, $gD.V$ will be its weight; therefore we shall have

[9925] $$gD.V=2Q-Q'. \qquad \text{[The fluid being supposed homogeneous.]}$$

The sides of the tube may be considered as being developed on a plane.

Now, the action being sensible only at insensible distances, the first tube acts sensibly only upon some columns extremely near to its sides; *we may therefore neglect the consideration of the curvature of these sides, and consider them as*
[9926] *being developed upon a plane*. The force Q will be proportional to the width of this plane, or, in other words, to the circumference of the interior base of

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remarking, however, that the quantities $\rho_{,}$, $\rho'_{,}$, in this formula, may have different values
[9920r] from ρ, ρ' in [9930, 9927′, 9929], where the fluid is supposed to be homogeneous. Putting the expressions of $gD.V$ [9898, 9920q], equal to each other, we get

[9920s] $$gD.V=\tfrac{1}{2}Hc.\cos.\varpi=(2\rho_{,}-\rho'_{,}).c;$$

dividing the two last forms of this eouation by c, we obtain

[9920t] $$2\rho_{,}-\rho'_{,}=\tfrac{1}{2}H.\cos.\varpi;$$

[9920u] *H being deduced from observation, or it is that value which corresponds to the case of nature*, as in [9896e], where it is named H; and we may here incidentally observe, that this value of
[9920v] H is denoted in [9354, 9261m] by $\mathrm{H}=gbq$, for a concave surface whose radius is b, and as
[9920w] the capillary elevation q is found by observation [9364] to be positive, *the value of* H *resulting from observation must therefore be positive*. The same result may be deduced from the
[9920x] expressions of a convex surface, considering, in the preceding value of $\mathrm{H}=gbq$, the quantities b and q as being negative, which is equivalent to the result of the separate calculation for a convex surface in [9369—9375].

the prism; so that if we put this circumference equal to c, we shall have [9927]
$Q = \rho . c$. ρ *is a constant quantity, which may represent the intensity of the* [9927'] c.
attraction of the matter of the first tube upon the fluid, in the case where the ρ.
attractions of the different bodies are expressed by the same function of the
distance, but which, in all cases, expresses a quantity depending on the [9928]
attraction of the matter of the tube, independent of its figure and magnitude. ρ'.
We shall likewise have $Q' = \rho' . c$, ρ' *being, as it respects the attraction of the fluid* [9929]
upon its own particles, what we have denoted by ρ, *relative to the attraction of the*
tube upon the fluid; therefore we shall have, from [9925, 9927', 9929],

$$gD.V = (2\rho - \rho').c, \qquad (p) \qquad \left[\begin{smallmatrix}\text{Correct only with a}\\ \text{homogeneous fluid.}\end{smallmatrix}\right]$$ [9930]

All these equations correspond to a homogeneous fluid, neglecting the variation of density at the surfaces.

which becomes the same as [9898], by putting *

$$2\rho - \rho' = \tfrac{1}{2}H.\cos.\varpi. \qquad \left[\begin{smallmatrix}\text{Correct only with a}\\ \text{homogeneous fluid.}\end{smallmatrix}\right]$$ [9931]

We have seen, in [9626m, n], that $\varpi = 0$ when $\rho = \rho'$; therefore the preceding [9932]
equation will give

$$\rho' = \tfrac{1}{2}H; \qquad \left[\begin{smallmatrix}\text{Correct only with a}\\ \text{homogeneous fluid.}\end{smallmatrix}\right]$$ [9933]

hence, in the general case, where ρ differs from ρ', we shall have

$$2\rho - \rho' = \rho'.\cos.\varpi, \qquad \left[\begin{smallmatrix}\text{Correct only with a}\\ \text{homogeneous fluid.}\end{smallmatrix}\right]$$ [9934]

or

$$\rho = \rho'.\cos.^2\tfrac{1}{2}\varpi; \qquad \left[\begin{smallmatrix}\text{Correct only with a}\\ \text{homogeneous fluid.}\end{smallmatrix}\right]$$ [9935]

therefore, if we know the angle ϖ, it will give the ratio of ρ to ρ', and the
contrary. We may demonstrate directly that [in the case of a homogeneous [9936]
fluid] $\rho' = \frac{1}{2}H$ [9933], in the following manner:

We shall suppose that there is *a solid vertical plane*, of a sensible thickness, [9937]
whose base is horizontal, and that there is, at the distance a from this plane, a

---

* (4282) Putting the expressions of $gD.V$ [9898, 9930] equal to each other, and [9931a]
dividing by c, we get [9931]. Now if we suppose the matter of the tube to vary, while the
homogeneous fluid remains the same, the quantities ρ, ϖ, will vary, but H [9844'] and ρ'
[9929] will be unaltered; then the equation [9931] will give the relation between the variable [9931b]
quantities ρ, ϖ. Now we have seen in [9626f, m, &c.], that, when $\rho = \rho'$, the curve
surface will be a *concave hemisphere;* and as ϖ [9892] represents the angle which this surface [9931c]
forms with the lower side of the tube, we shall have, in this case, $\varpi = 0$. Substituting the [9931d]
values $\rho = \rho'$, $\varpi = 0$, in [9931], it becomes $2\rho' - \rho' = \frac{1}{2}H$, or, $\rho' = \frac{1}{2}H$, as in [9933]; and
by substituting this value of $\frac{1}{2}H$ in [9931], it becomes as in [9934]; or, as it may be written,

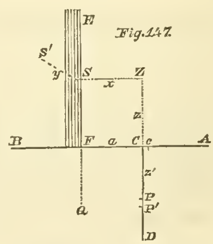
$$2\rho = \rho'.(1 + \cos.\varpi) = 2\rho'.\cos.^2\tfrac{1}{2}\varpi \quad [6], \text{ Int.};$$ [9931e]

hence we easily deduce the value of ρ [9935]; all these equations being grounded on the [9931f]
supposition that the fluid has the same density at the surface as in the interior of the mass.

[9937'] *vertical line*, infinite in length, parallel to the plane, and attracted by it; this
[9938] vertical line being terminated, at the top, by a *second plane*, which is drawn horizontally through the base of the *solid vertical plane*. We shall take this
[9938'] termination of the line for the origin of the coordinates x, y, z, of any point of the solid plane; supposing the axis of x to be situated upon the line a, which
[9939] represents the least distance from the origin of the coordinates to the solid plane;
[9940] and the axis of y to be horizontal and perpendicular to the axis of x. Lastly, we shall represent by z', the depression of any attracted point of the vertical
[9940'] line below the origin of the coordinates. Then *the vertical attraction of the solid plane upon that point will be* *

[9941] $$\textstyle\iiint dx\,.\,dy\,.\,dz\,.\,\frac{(z+z')}{s}\,.\,\varphi(s);$$ [Attraction on the point P in the vertical direction PZ.]

[9942] $\varphi(s)$ *being the law of the attraction at the distance* s, *and* s *being the distance*



* (4283) In the annexed figure 147, C is the origin of the
[9941a] coordinates; CZ the axis of z; CB the axis of x; the axis of y being perpendicular to the plane of the figure. The attracted line is CD, in which is placed the attracted point P,
[9941b] so that $CP=z'$. BFE represents a section of the solid plane by the plane of xz, which is perpendicular to the surface FE
[9941c] of the solid. If we now suppose that any point S', situated above the plane of the figure, at the distance y, be projected perpendicularly upon the plane of xz, in the point S, and then, from the point S, we let fall upon CZ the perpendicular SZ,
[9941d] we shall have $SZ=x$, $CZ=z$, $S'S=y$, $CP=z'$, lastly the distance $S'P=s$. The
[9941e] particle of matter placed at S' is represented by $dx.dy.dz$; this attracts the point P, in the direction PS' or s, with the force $\varphi(s)$ [9942], multiplied by the particle $dx.dy.dz$;
[9941f] so that this force is represented by $dx.dy.dz.\varphi(s)$. Now as the line $PZ=z+z'$ is the projection of the line $S'P=s$, upon the vertical axis z, we shall have for this force, when resolved in the direction PZ, or parallel to the axis of z, the expression [9941f] multiplied
[9941g] by $\frac{PZ}{S'P}$ or $\frac{z+z'}{s}$, so that it becomes equal to $dx.dy.dz.\frac{(z+z')}{s}.\varphi(s)$; and by prefixing the sign of integration $\iiint$, so as to include all the particles of the solid plane, it becomes as in
[9941h] [9941]. The value of s^2 or $S'P^2$ is evidently represented by $S'P^2=SZ^2+S'S^2+PZ^2$;
[9941i] and by substituting $SZ=x$, $S'S=y$, $PZ=z+z'$, it will give s^2, as in [9943], which is to be substituted in [9941]; then, multiplying it by $PP'=dz'$, and integrating relative to z', from $z'=0$ to $z'=\infty$, we obtain the whole attraction of the solid upon the line CD, which
[9941k] will be $\iiint dx.dy.dz.\int dz'.\frac{(z+z')}{s}.\varphi(s)$, as in [9941, 9944].

from an attracting point in the plane, to the attracted point of the line; so that we shall have, as in [9941*i*],

$$s^2 = x^2 + y^2 + (z+z')^2. \qquad [9943]$$

To obtain the vertical attraction of the solid plane upon the whole line, we must multiply the preceding triple integral by dz', and integrate it relatively to z' from $z'=0$ to $z'=\infty$. Putting, therefore, as in [9232′—9233′], [9944]

$$c = \int_0^\infty ds\,.\,\varphi(s)\,, \qquad [9945]$$

$$\Pi(s) = \int_s^\infty ds\,.\,\varphi(s) \qquad [9946]$$

$$= c - \int_0^s ds\,.\,\varphi(s)\,, \qquad [9946']$$

we shall have *

$$\int_{s_{\prime}}^\infty dz'\,.\,\frac{(z+z')}{s}\,.\,\varphi(s) = \Pi(s_{\prime})\,; \qquad [9947]$$

---

* (4284) From [9943] we have $s = \sqrt{x^2+y^2+(z+z')^2}$; whence

$$\left(\frac{ds}{dz'}\right) = \frac{z+z'}{\sqrt{x^2+y^2+(z+z')^2}} = \frac{z+z'}{s}. \qquad [9947a]$$

Substituting this value of $\frac{z+z'}{s}$ in the first member of [9947], it becomes

$$\int dz'\,.\,\frac{(z+z')}{s}\,.\,\varphi(s) = \int dz'\,.\left(\frac{ds}{dz'}\right).\,\varphi(s)\,; \qquad [9947b]$$

but from [9946′] we get $ds.\varphi(s) = -d.\Pi(s)$; and if we consider z' as the only variable quantity in the value of s, this will become

$$dz'.\left(\frac{ds}{dz'}\right).\varphi(s) = -\left(\frac{d.\Pi(s)}{dz'}\right).dz'; \qquad [9947c]$$

hence the expression [9947*b*] becomes

$$\int dz'.\frac{(z+z')}{s}.\varphi(s) = -\int\left(\frac{d.\Pi(s)}{dz'}\right).dz' = \text{constant} - \Pi(s). \qquad [9947d]$$

At the origin C of the integral, where $s = s_{\prime}$ [9948], or $s_{\prime}^2 = x^2+y^2+z^2$ [9943], it becomes $0 = \text{constant} - \Pi(s_{\prime})$, or $\text{constant} = \Pi(s_{\prime})$. Substituting this in the general equation, we get [9947d′]

$$\int dz'.\frac{(z+z')}{s}.\varphi(s) = \Pi(s_{\prime}) - \Pi(s). \qquad [9947d'']$$

When $s=\infty$, we have $\Pi(s)=0$ [9233″]; hence [9947*d*″] becomes

$$\int_{s_{\prime}}^\infty dz'.\frac{(z+z)}{s}.\varphi(s) = \Pi(s_{\prime}), \qquad [9947e]$$

as in [9947]; substituting this in [9941*k*], it becomes as in [9949]. We may here remark, that, to avoid confusion in the use of the same symbol for different expressions, we have, in [9948, &c.], changed the author's symbol s into $s_{\prime}$; also in [9956] we have changed his s into $s_{\prime\prime}$; we have also written $\varpi_{\prime}$, $\theta_{\prime}$, for ϖ, θ, in [9949′, &c.]. [9947f]

[9948] $s_{\prime}$ *being, in the second member of this equation, what* s [9942] *becomes at the origin of the coordinates, or when* z' *is nothing.* Therefore the attraction of the solid plane upon the whole line will be

[9949] $$\int\int\int dx.dy.dz.\Pi(s_{\prime}).$$ [Attraction of the solid plane upon the line CD, in the vertical direction CZ.]

We shall now put

[9949′] $\varpi_{\prime}$ = the angle which the line $s_{\prime}$ forms with the horizontal plane drawn through the origin of the coordinates;

[9950] $\theta_{\prime}$ = the angle which the projection of $s_{\prime}$ upon the horizontal plane of the coordinates, makes with the axis of y.

Then we shall have*

[9950′] $$x = s_{\prime}.\sin.\theta_{\prime}.\cos.\varpi_{\prime},$$

[9951] $$y = s_{\prime}.\cos.\theta_{\prime}.\cos.\varpi_{\prime}.$$

We may, instead of the rectangular element $dx.dy.dz$, substitute the

[9951′] corresponding polar element† $s_{\prime}^2 ds_{\prime}.d\theta_{\prime}.d\varpi_{\prime}.\cos.\varpi_{\prime}$; and then the preceding triple integral becomes

[9952] $$\int\int\int s_{\prime}^2 ds_{\prime}.d\theta_{\prime}.d\varpi_{\prime}.\cos.\varpi_{\prime}.\Pi(s_{\prime}).$$ [Attraction of the parallelopiped BFE upon the line CD, in the vertical direction CZ.]

It is a matter of indifference, in this kind of attraction, whether we suppose the thickness of the solid plane to be finite or infinite, when the thickness is sensible; therefore we shall suppose it to be infinite. Then putting, as in [9241′],

[9953] $$\int_0^{s_{\prime}} s_{\prime} ds_{\prime}.\Pi(s_{\prime}) = c' - \Psi(s_{\prime}),$$

[9950a] * (4285) The line $s_{\prime}$ [9948] represents the distance of the point S' from C; and if we put, as in [9949′], $\varpi_{\prime}$ for the angle which this line $s_{\prime}$ makes with the horizontal plane xy,
[9950b] its projection upon this plane will evidently be represented by $s_{\prime}.\cos.\varpi_{\prime}$, which, for brevity, we
[9950c] shall denote by $\mathrm{s}_{\prime}$; so that we shall have $\mathrm{s}_{\prime} = s_{\prime}.\cos.\varpi_{\prime}$. Then the line $\mathrm{s}_{\prime}$ forms the angle $\theta_{\prime}$ [9950] with the axis of y, or $90^{d.} - \theta_{\prime}$ with the axis of x; therefore the projection of
[9950d] this line $\mathrm{s}_{\prime}$ upon these axes, will give $y = \mathrm{s}_{\prime}.\cos.\theta_{\prime}$, $x = \mathrm{s}_{\prime}.\sin.\theta_{\prime}$; and by substituting the preceding value of $\mathrm{s}_{\prime} = s_{\prime}.\cos.\varpi_{\prime}$, they become as in [9951, 9950′].

[9951a] † (4286) If we change, in the notation [9226, 9227], u into $s_{\prime}$, r into z, θ into $90^{d.} - \varpi_{\prime}$, and ϖ into $\theta_{\prime}$, it will conform to the notation which is used in [9949′, &c.], as is
[9951b] evident by considering that $\varpi_{\prime}$ [9949′] is the complement of the angle which the line $s_{\prime}$ forms with the axis z, and $\theta_{\prime}$ is as in [9950]. Making the same changes in the element
[9951c] $u^2 du.d\varpi.d\theta.\sin.\theta$ [9227], it becomes, by neglecting the sign, $s_{\prime}^2 ds_{\prime}.d\theta_{\prime}.d\varpi_{\prime}.\cos.\varpi_{\prime}$, as in
[9951d] [9951′]. Substituting this for $dx.dy.dz$, in [9949], it becomes as in [9952].

c' being the value of the integral when $s_{,}=\infty$ [9240'], we shall have * [9954]

$$\int s_{,}^{2}ds_{,}.\Pi(s_{,})=-s_{,}.\Psi(s_{,})+\int ds_{,}.\Psi(s_{,})+\text{constant}. \qquad [9955]$$

To determine the constant quantity, we shall observe that the two integrals of this last equation are taken from $s_{,}=s_{,,}$ to $s_{,}=\infty$ [$s_{,,}$ being the least value of $s_{,}$, or the distance from the origin of the coordinates to the point where the line $s_{,}$ first meets the solid plane]; hence we get [9956]

$$\text{constant}=s_{,,}.\Psi(s_{,,}). \qquad [9957]$$

Now as $s_{,}.\Psi(s_{,})$ vanishes when $s_{,}=\infty$, we shall have

$$\int_{s_{,,}}^{\infty}s_{,}^{2}ds_{,}.\Pi(s_{,})=s_{,,}.\Psi(s_{,,})+\int_{s_{,,}}^{\infty}ds_{,}.\Psi(s_{,}). \qquad [9958]$$

We shall put

$$c''=\int_{0}^{\infty}ds_{,}.\Psi(s_{,}), \qquad [9959]$$

$$\Gamma(s_{,})=\int_{s_{,}}^{\infty}ds_{,}.\Psi(s_{,}), \qquad [9960]$$

$$=c''-\int_{0}^{s_{,}}ds_{,}.\Psi(s_{,}); \qquad [9960']$$

and then we shall have †

$$\int_{s_{,,}}^{\infty}s_{,}^{2}ds_{,}.\Pi(s_{,})=s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,}). \qquad [9961]$$

---

* (4287) The differential of [9953] being multiplied by $s_{,}$ gives

$$s_{,}^{2}ds_{,}.\Pi(s_{,})=-s_{,}d.\Psi(s_{,}). \qquad [9954a]$$

Integrating this by parts, it becomes as in [9955], as is easily proved by taking its differential. This integral commences with the least value of $s_{,}$, *corresponding to the surface FE of the solid plane which is nearest to the origin C*, fig. 147, page 840; *putting this equal to* $s_{,,}$ [9956], and the terms under the sign $\int$ equal to nothing at that point, we find that the expression [9955] becomes $0=-s_{,,}.\Psi(s_{,,})+\text{constant}$, as in [9957]. Substituting this in [9955], it becomes [9954b]

$$\int_{s_{,,}}^{s_{,}}s_{,}^{2}ds_{,}.\Pi(s_{,})=-s_{,}.\Psi(s_{,})+\int_{s_{,,}}^{s_{,}}ds_{,}.\Psi(s_{,})+s_{,,}.\Psi(s_{,,}); \qquad [9954c]$$

and when $s_{,}=\infty$, $-s_{,}.\Psi(s_{,})$ becomes nothing [9240k, &c.]; so that when $s_{,}=\infty$, the preceding expression becomes

$$\int_{s_{,,}}^{\infty}s_{,}^{2}ds_{,}.\Pi(s_{,})=\int_{s_{,,}}^{\infty}ds_{,}.\Psi(s_{,})+s_{,,}.\Psi(s_{,,}), \qquad [9954d]$$

as in [9958].

† (4288) From [9960] we have $\int_{s_{,,}}^{\infty}ds_{,}.\Psi(s_{,})=\Gamma(s_{,,})$; substituting this in [9954d], we obtain [9960a]

$$\int_{s_{,,}}^{\infty}s_{,}^{2}ds_{,}.\Pi(s_{,})=\Gamma(s_{,,})+s_{,,}.\Psi(s_{,,});$$

and by introducing this into the integral expression [9952], it becomes

$$\int\int d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\{\Gamma(s_{,,})+s_{,,}.\Psi(s_{,,})\}, \qquad [9960b]$$

agreeing with [9962].

Hence the triple integral [9952] can be reduced to the following double integral;

[9962] $$\int\!\!\int d\theta_{,}.\,d\varpi_{,}.\cos.\varpi_{,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\}.$$

[Attraction of the parallelopiped *BFE* upon the line *CD*, in the vertical direction *FE*.]

[9962′] We shall now suppose that an indefinite vertical plane *surface* passes through the attracted line, and falls perpendicularly upon the *solid* attracting plane; and we shall compute the vertical attraction of this *solid* upon that *surface*. It is evident that we must multiply the preceding function by da [9937], and

[9963] integrate it relative to a, from $a=0$, to $a=\infty$; now we have*

[9964] $$a=s_{,,}.\sin.\theta_{,}.\cos.\varpi_{,},$$

[9964′] which gives, by supposing $\theta_{,}$ and $\varpi_{,}$ constant,

[9965] $$da=ds_{,,}.\sin.\theta_{,}.\cos.\varpi_{,};$$

[9965′] the preceding double integral, being multiplied by this expression of da, and then integrated relatively to $s_{,,}$, will therefore become

[9966] $$\int\!\!\int\!\!\int ds_{,,}.\,d\theta_{,}.\,d\varpi_{,}.\sin.\theta_{,}.\cos.^2\varpi_{,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\}.$$

[Attraction of the parallelopiped *BFE* upon the plane *QFC* in the vertical direction *FE*.]

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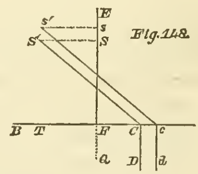
* (4289) Having found, in [9962], the expression of the attraction of the solid plane

[9964a] BFE upon the vertical line CD of an infinite length, we shall now suppose that, on the continuation of the line FC, fig. 148, we have $Cc=da$, and that

[9964b] the infinite line cd is drawn parallel to CD. Then it is evident that the attraction of the solid plane BFE upon the plane surface $DCcd$, will be found by multiplying the function [9960b] by da; and if we integrate this, relative to a, from $a=0$ to $a=a$, we shall evidently get for the whole action of the solid BFE upon the plane $QFCD$, the following expression;



[9964c] $$\int\!\!\int\!\!\int da.\,d\theta_{,}.\,d\varpi_{,}.\cos.\varpi_{,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\};$$

[9964d] $s_{,,}$ being the distance from the point C to the point S', where the line CS' first meets the nearest surface of the solid plane [9956]; and since the differentiation or integration relative to a is considered as independent of the quantities $\theta_{,}$, $\varpi_{,}$, so that they are considered as constant [9964′], we must find $ds_{,,}$ upon the same principle. Now this is done by supposing the line cs' to be drawn parallel to CS', to meet the nearest surface of the solid plane in s';

[9964e] since, by [9950, 9949′], $\theta_{,}$, $\varpi_{,}$, will be the same for the lines CS', cs'; and if we put $CS'=s_{,,}$, we shall have $cs'=s_{,,}+ds_{,,}$. Moreover it is evident, as CF, FS, SS', are the

[9964f] rectangular coordinates of the point S', corresponding to x, z, y [9938′, &c.], the value $FC=a$ will be found by writing $s_{,,}$ for $s_{,}$ in the expression of x [9950′], so that by this means it becomes as in [9964]. Its differential, supposing a, $s_{,,}$, to be the only variable quantities, is

[9964g] $da=ds_{,,}.\sin.\theta_{,}.\cos.\varpi_{,}$ [9965]; hence [9964c] becomes as in [9966].

The integral must here be taken from $s_{\prime\prime}=0_{\prime\prime}$ to $s_{\prime\prime}=\infty$. Now in this case* [9967]

$$\int_0^\infty s_{\prime\prime}ds_{\prime\prime}.\Psi(s_{\prime\prime})=-s_{\prime\prime}.\Gamma(s_{\prime\prime})+\int_0^\infty ds_{\prime\prime}.\Gamma(s_{\prime\prime})=\int_0^\infty ds_{\prime\prime}.\Gamma(s_{\prime\prime}), \quad [9968]$$

because, $s_{\prime\prime}$ being infinite, $s_{\prime\prime}.\Gamma(s_{\prime\prime})$ is nothing [9240k, &c.]; moreover we have, as in [9253a'], $2\int_0^\infty s_{\prime\prime}ds_{\prime\prime}.\Psi(s_{\prime\prime})=\frac{H}{\pi}$; hence [9968′]

$$\int_0^\infty ds_{\prime\prime}.\{s_{\prime\prime}.\Psi(s_{\prime\prime})+\Gamma(s_{\prime\prime})\}=\frac{H}{\pi}; \quad [9969]$$

therefore the preceding triple integral [9966], will become

$$\frac{H}{\pi}.\iint d\varpi_{\prime}.d\theta_{\prime}.\sin.\theta_{\prime}.\cos.^2\varpi_{\prime}. \quad [9970]$$

[Attraction of the parallelopiped BFE upon the plane QFC in the vertical direction FE.]

The integral relative to $\varpi_{\prime}$ must be taken from $\varpi_{\prime}=0$ to $\varpi_{\prime}=$ to a right angle; the integral relative to $\theta_{\prime}$ must be taken from $\theta_{\prime}=0$ to $\theta_{\prime}=$ to two right angles, which gives† [9971]

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* (4290) From [9960′] we have $\int_0^{s_{\prime\prime}} ds_{\prime\prime}.\Psi(s_{\prime\prime})=c''-\Gamma(s_{\prime\prime})$, whose differential is $ds_{\prime\prime}.\Psi(s_{\prime\prime})=-d.\Gamma(s_{\prime\prime})$. Multiplying this by $s_{\prime\prime}$ and integrating, we get

$$\int s_{\prime\prime}ds_{\prime\prime}.\Psi(s_{\prime\prime})=-\int s_{\prime\prime}.d.\Gamma(s_{\prime\prime})=-s_{\prime\prime}.\Gamma(s_{\prime\prime})+\int ds_{\prime\prime}.\Gamma(s_{\prime\prime}), \quad [9967a]$$

as is easily proved by differentiation. This agrees with [9968], the limits of the integrals being from $s_{\prime\prime}=0$ to $s_{\prime\prime}=\infty$ [9967]; and at both these limits we have $s_{\prime\prime}.\Gamma(s_{\prime\prime})=0$ [9240k, &c.]; hence $\int_0^\infty s_{\prime\prime}ds_{\prime\prime}.\Psi(s_{\prime\prime})=\int_0^\infty ds_{\prime\prime}.\Gamma(s_{\prime\prime})$ [9968]. Substituting this value of $\int_0^\infty ds_{\prime\prime}.\Gamma(s_{\prime\prime})$, in the first member of [9969], we get [9967b]

$$\int_0^\infty ds_{\prime\prime}.\{s_{\prime\prime}.\Psi(s_{\prime\prime})+\Gamma(s_{\prime\prime})\}=2\int_0^\infty s_{\prime\prime}ds_{\prime\prime}.\Psi(s_{\prime\prime}); \quad [9967c]$$

but by [9253a'], we have $2\int_0^\infty s_{\prime\prime}ds_{\prime\prime}.\Psi(s_{\prime\prime})=\frac{H}{\pi}$, as in [9968′]; hence the preceding integral [9967c] becomes as in [9969], and by substituting it in [9966], we get [9970]. [9967d]

† (4291) We have $\int d\theta_{\prime}.\sin.\theta_{\prime}=1-\cos.\theta_{\prime}$, which vanishes when $\theta_{\prime}=0$; and at the second limit, where $\theta_{\prime}=180^{d}=\pi$ [9971], it becomes $\int_0^\pi d\theta_{\prime}.\sin.\theta_{\prime}=2$; hence the first member of [9972] becomes $\int d\varpi_{\prime}.2\cos.^2\varpi_{\prime}=\int d\varpi_{\prime}.(1+\cos.2\varpi_{\prime})=\varpi_{\prime}+\frac{1}{2}\sin.2\varpi_{\prime}$. This vanishes when $\varpi_{\prime}=0$; and at the second limit, where $\varpi_{\prime}=\frac{1}{2}\pi$, it becomes equal to $\frac{1}{2}\pi$, as in the second member of [9972]. Substituting this in [9970], it becomes $\frac{1}{2}H$, as in [9973]. In this calculation the attracting body and the attracting plane are considered as being of the uniform density unity; and as the symbol H is sometimes used instead of H' [10032, &c.], we shall, to avoid confusion, suppose that, when the fluid is homogeneous, we shall have [9972a] [9972b] [9972c] [9972d]

$$\tfrac{1}{2}H=q, \quad \text{or} \quad H=2q; \quad [9972e]$$

and in this case, we get

The vertical action of the homogeneous rectangular solid BFE upon the rectangular plane $QFCD=q$. [9972f]

[9972] $$\int_0^{\pi}\int_0^{\frac{1}{2}\pi} d\theta_{,}.d\varpi_{,}.\sin.\theta_{,}.\cos.^2\varpi_{,}=\tfrac{1}{2}\pi;$$

therefore we shall have

[9973] $\frac{1}{2}H=$ the whole vertical attraction of a solid parallelopiped upon a plane surface.

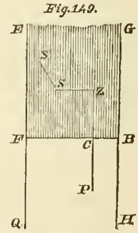
[9973′] This attraction is what we have before called ρ, or ρ', if the plane is of the same nature as the [homogeneous] fluid; therefore we shall have *

[9973a] * (4292) The expression $\frac{1}{2}H=q$ [9972e, f], represents the action of the rectangular solid BFE, fig. 148, page 844, upon the rectangular plane $QFCD$, extended infinitely in the directions FC, FQ; and if we suppose its thickness, in the direction perpendicular to the
[9973b] plane $QFCD$, to be dc, the whole action of the solid upon the plane will be $\frac{1}{2}H.dc$; and its
[9973c] integral $\frac{1}{2}H.c$ represents the whole action of the solid upon the plane whose thickness is c,
[9973d] this action being in the vertical direction FE [9940′]. This is the quantity which is called Q' or $\rho'.c$ [9915, 9929]; hence we have $\rho'.c=\frac{1}{2}H.c$. Dividing by c, we get $\rho'=\frac{1}{2}H$, as in [9974].

[9973e] We have supposed, in the formula [9973], that the dimensions of the attracting solid, in the directions FB, FE, fig. 148, page 844, are infinite; and in like manner that the dimensions of the attracted plane, in the directions FQ, FC, are infinite; but it is evident that we may decrease these quantities very much, without affecting that formula, and that we can suppose
[9973f] the limits, in the directions FB, FC, to be equal to the insensible quantity λ [9173n], which expresses the distance to which the corpuscular action extends; so that, in the formula
[9973f′] [9973], we may suppose $BF=FC=\lambda$; FC being on the continuation of BF, and the angle $BFE=QFC=90^{d}$. What we have said relative to the limits of the attracting solid, *in the direction FB, holds good relative to the direction FE, and also in the direction perpendicular to the plane of the figure, either above or below it;* so that we may consider
[9973g] the attracting solid as being composed of two cubes, whose sides are represented by λ, supposing the base to be included in the angle BFE; one of the cubes being above the plane of the figure, the other below it. We shall hereafter have occasion to ascertain the action of an attracting solid in the form of a wedge upon a plane, or triangle, &c.; we shall, therefore, in this note, investigate several formulas of this kind.

Instead of supposing the line FC to be on the continuation of the line BF, as in the preceding calculation, fig. 148, page 844, we shall now suppose these
[9973h] lines to coincide, as in fig. 149; then, taking in the plane of the figure the horizontal line $BF=\lambda$, we shall consider the vertical lines EFQ, GBH, as being drawn in that plane perpendicular to the line BF, and continued
[9973i] infinitely, on both sides of it. We shall also suppose that the attracting solid is bounded by planes drawn through BF, BG, FE, perpendicular to the plane of the figure, and continued infinitely above and below it; the
[9973k] attracted plane being the rectangular parallelogram $HBFQ$, situated in the plane of the figure, and having the infinite sides BH, FQ. We shall suppose, as in the preceding calculation [9941d], that the rectangular

Fig. 149.



$$\rho' = \tfrac{1}{2} H,$$ [Vertical action of a parallelopiped upon a homogeneous plane surface.] [9974]

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coordinates of any point S' of the attracting solid, are $CZ = z$, $SZ = x$, $SS' = y$; $CP = z'$, $PS' = s$; the origin of the coordinates x, y, z, being the point C, corresponding to $FC = a$. The calculation of the action of the solid upon the plane differs but very little from that in [9941—9973]. In the first place, the vertical action of a particle, $dx.dy.dz$, situated at the point S', upon a particle at the point P, is represented in like manner as in [9973l]

[9941g], by $dx.dy.dz.\frac{(z+z')}{s}.\varphi(s)$, whose triple integral gives the whole action of the solid [9973m]

upon the point P, as in [9941]. In the next place, integrating, relative to z', it becomes, as in [9949], [9973n]

$$\int\int\int dx.dy.dz.\Pi(s_{,});$$ [9973n']

$s_{,}$ being the value of s, when the point P falls in C, making $CS' = s_{,}$; and putting, in like [9973o]
manner as in [9946'], $\Pi(s_{,}) = c_{,} - \int_0^{s_{,}} ds_{,}.\varphi(s_{,})$. Then supposing, as in [9949'], that $\varpi_{,}$ [9973p]
represents the angle which the line $s_{,}$ forms with the horizontal plane drawn through the origin of the coordinates C, and $\theta_{,}$ the angle which the projection of $s_{,}$ upon the horizontal plane, makes with the axis of y, as in [9950], we shall have the same values of x, y, as in [9950', 9951]. Substituting them and the expression of $dx.dy.dz$ [9951'], in [9973n'], we get, as in [9952], for the action of the solid upon the fluid situated upon the line CP, [9973p']

$$\int\int\int s_{,}^2 ds_{,}.d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\Pi(s_{,}).$$ [9973q]

Assuming the expression of the function $\Psi(s_{,}) = c' - \int_0^{s} s_{,} ds_{,}.\Pi(s_{,})$, as in [9953], and then integrating [9973q], relative to $s_{,}$, we get, as in [9955],

$$\int_0^{s} s_{,}^2 ds_{,}.\Pi(s_{,}) = -s_{,}.\Psi(s_{,}) + \int_0^{s_{,}} ds_{,}.\Psi(s_{,});$$ [9973r]

the constant quantity in its second member being neglected, because $s_{,}.\Psi(s_{,})$ vanishes at the commencement of the integral where $s_{,} = 0$. Substituting the value of

$$\int_0^{s} ds_{,}.\Psi(s_{,}) = c'' - \Gamma(s_{,}) \quad [9960'],$$ [9973s]

and putting $s_{,,}$ for the value of $s_{,}$ at the second limit of the integral, corresponding to points of the solid which are situated on the vertical side of it whose section is EF, we get

$$\int_0^{s_{,,}} s_{,}^2 ds_{,}.\Pi(s_{,}) = -s_{,,}.\Psi(s_{,,}) - \Gamma(s_{,,}) + c''.$$ [9973t]

Hence the expression [9973q] becomes

$$\int\int d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\{-s_{,,}.\Psi(s_{,,}) - \Gamma(s_{,,}) + c''\}.$$ [9973u]

The part of this expression depending on c'' may be cleared from the signs of integration, because $\int_0^{\pi} d\theta_{,} = \pi$, $\int d\varpi_{,}.\cos.\varpi_{,} = \sin.\varpi_{,}$, and $\int_0^{\frac{1}{2}\pi} d\varpi_{,}.\cos.\varpi_{,} = 1$; hence the expression [9973v]
[9973u] becomes

$$c''\pi - \int\int d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\{s_{,,}.\Psi(s_{,,}) + \Gamma(s_{,,})\}.$$ [9973w]

Multiplying this by da, and integrating from $a = 0$, at the point F, to $a = \lambda$, at the point B, [9973x]
we get the action of that part of the solid corresponding to values of $s_{,,}$, which terminate at the surface EF; and as the values of $s_{,,}$, terminating at the surface GB, must produce a similar expression, their sum will be doubled, and by this means the whole action becomes [9973y]

[9974'] as we have found, in [9936], by the comparison of the results of

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[9973z] $$2c''.\pi.\int da - 2\int\int\int da.d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\}.$$

Substituting in the first term $\int_0^{\lambda} da=\lambda$, and in the second $da=ds_{,,}.\sin.\theta_{,}.\cos.\varpi_{,}$ [9965], it becomes

[9974a] $$2c''\pi\lambda - 2\int\int\int d\theta_{,}.d\varpi_{,}.\sin.\theta_{,}.\cos.^2\varpi_{,}.ds_{,,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\}.$$

We may here observe, that, while the angles $\theta_{,}$, $\varpi_{,}$, remain unaltered, we can suppose the origin C to move from the point F, where $a=0$, and $s_{,,}=0$, to the point B, or to the
[9974b] second limit of $s_{,,}$, where $a=\lambda$, and $s_{,,}$ is equal to, or exceeds λ. Now as the corpuscular action vanishes at the distances which are equal to λ, or exceed λ, we may extend the second limit of $s_{,,}$ to $s_{,,}=\infty$; and by substituting $H=2q$ [9972e] in [9969], we get

[9974b'] $$\int_0^{\infty} ds_{,,}.\{s_{,,}\Psi(s_{,,})+\Gamma(s_{,,})\}=\frac{2q}{\pi};$$

hence the expression [9974a] becomes

[9974b''] $$2c''\pi\lambda-\frac{4q}{\pi}.\int\int d\theta_{,}.d\varpi_{,}.\sin.\theta_{,}.\cos.^2\varpi_{,};$$

and, by using the integral [9972], we finally get, for the whole action of the solid upon the plane, the expression

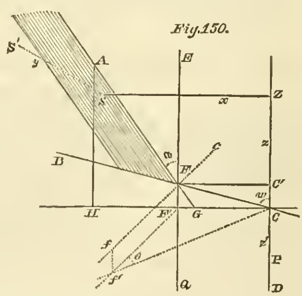
[9974c] $2c''\pi\lambda-2q=$ the action of the solid $EFBG$ upon the plane $QFBH$.

Now by [9973s], we have $c''=\int_0^{\infty} ds_{,}.\Psi(s_{,})=\int_0^{\infty} dz.\Psi(z)=\frac{K}{2\pi}$ [9253a']; whence we get $2c''\pi=K$; substituting this in [9974c], we obtain

[9974d] $\lambda K-2q=$ the action of the parallelopiped $EFBG$ upon the plane $QFBH$, in a vertical direction parallel to FE or BG.

In the preceding calculations [9937—9973], the author supposes the upper line FC of the plane $QFCD$, fig. 148, page 844, to be horizontal, and the surface EF of the attracting
[9974e] body BFE to be situated in a vertical plane perpendicular to the line FC. We shall now suppose that the attracting body is in the form of a wedge AFB, fig. 150, limited by
[9974f] two infinite plane surfaces drawn through the lines FA, FB, perpendicular to the plane of the figure, and continued infinitely above and below it; the attracted plane being the surface $QFCD$, which is limited
[9974g] by the vertical line FQ, and the line FC on the continuation of BF. We shall also suppose that the vertical lines QF, DC, are continued upwards towards E and Z; that the lines FC', $CGF'H$, are horizontal, and that the line AF, being continued, meets CH in G. We shall take C, for the origin of the rectangular coordinates; the vertical line CZ, for
[9974g'] the axis of z; the horizontal line CH, for the axis of x; and for the axis of y, the horizontal line drawn through C, perpendicular to the plane of

Fig. 150.



the two methods. We see evidently by both, not only the identity of [9974″]

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the figure, and in an upward direction; so that if an *attracting* particle be at S', we shall have for its rectangular coordinates the vertical line $CZ=z$, and the horizontal lines $ZS=x$, [9974h] $SS'=y$. We shall also put $CP=z'$, P being the place of an *attracted* point as in [9974i] [9941e]; the angle $FCH=90^{d}-w$, the angle $FCZ=w$, the angle $EFA=\alpha$, [9974k] $CF'=a$, and $PS'=s$. Then the *vertical* action of the attracting particle at S' upon the attracted particle at P, is the same as in [9941]; the value of s^2 is as in [9943]; and the integration of this attraction relative to z', being taken as in [9944—9947], produces the expression [9947]; so that the whole of this attraction finally becomes, as in [9949], equal [9974l] to $\int\int\int dx.dy.dz.\Pi(s_{,})$; $s_{,}$ being, as in [9948], the value of CS', corresponding to the origin C. Substituting in this the values of x, y [9950′, 9951], it becomes, as in [9952],

$$\int\int\int s_{,}^2 ds_{,}.d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\Pi(s_{,}), \qquad [9974m]$$

$\varpi_{,}$, $\theta_{,}$, being defined as in [9949′, 9950]; considering C as the origin of the line $s_{,}$, and the horizontal plane to which $\theta_{,}$, $\varpi_{,}$, are referred, as that drawn through CH, perpendicular to the [9974m′] plane of the figure. Integrating this relative to $s_{,}$, as in [9953—9962], we finally get, as in [9962], by putting $s_{,,}$ equal to that part of the line $CS'=s_{,}$, which is contained between [9974m″] the point C and the surface FA,

$$\int\int d\theta_{,}.d\varpi_{,}.\cos.\varpi_{,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\}. \qquad [9974n]$$

By continuing the same process of calculation as in [9963, &c.], it becomes necessary to find the value of a in terms of $s_{,,}$, $\theta_{,}$, $\varpi_{,}$. Now in the rectangular triangle $CF'F$, we have, by using the notation [9974k], $FF'=CF'.\text{tang}.F'CF=a.\cot.w$; and in the rectangular [9974o] triangle $FF'G$ we have $F'G=FF'.\text{tang}.F'FG=FF'.\text{tang}.\alpha=a.\cot.w.\text{tang}.\alpha$. We [9974p] shall now suppose that the extreme point of the line $s_{,,}$, where it meets the surface of the wedge whose section is FA, is projected perpendicularly upon the line FA in the point A, and upon the line CH in the point H; then we shall have, by using the notation [9974m′, m″, 9949′, 9950],

$$CH=s_{,,}.\sin.\theta_{,}.\cos.\varpi_{,}, \qquad AH=s_{,,}.\sin.\varpi_{,}, \qquad HG=AH.\text{tang}.\alpha=s_{,,}.\sin.\varpi_{,}.\text{tang}.\alpha; \qquad [9974q]$$

and since a or CF' is represented by $CF'=CH-HG+F'G$, we shall have

$$a=s_{,,}.\sin.\theta_{,}.\cos.\varpi_{,}-s_{,,}.\sin.\varpi_{,}.\text{tang}.\alpha+a.\cot.w.\text{tang}.\alpha. \qquad [9974r]$$

Transposing the last term of this expression, and then dividing by the coefficient of a, we get the value of a [9974t], using for brevity the expression of m [9974s],

$$m=\frac{\sin.\theta_{,}.\cos.\varpi_{,}-\sin.\varpi_{,}.\text{tang}.\alpha}{1-\cot.w.\text{tang}.\alpha}, \qquad [9974s]$$

$$a=m.s_{,,}. \qquad [9974t]$$

The differential of this expression of a, considering a, $s_{,,}$, as the only variable quantities, is $da=m.ds_{,,}$, which is to be used instead of [9965], when multiplying [9974n], to obtain [9974u] the expression of the attraction similar to that in [9966], which we shall represent by Z; so that we shall have

$$Z=\int\int\int ds_{,,}.d\theta_{,}.d\varpi_{,}.m.\cos.\varpi_{,}.\{s_{,,}.\Psi(s_{,,})+\Gamma(s_{,,})\}. \qquad [9974v]$$

[9975] the forces ρ' and $\frac{1}{2}H$, upon which the capillary phenomena [of a

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[9974w] The integration of this expression relative to $s_{\prime\prime}$ is made as in [9966—9969]; and by substituting the result of this integration [9969] in [9974v], we get

[9974x] $$Z=\frac{H}{\pi}.\int\int d\theta_{\prime}.d\varpi_{\prime}.m.\cos.\varpi_{\prime}.$$

Resubstituting the value of m [9974s], it becomes

[9974y] $$Z=\frac{H}{\pi.(1-\cot.w.\tang.\alpha)}.\int\int d\theta_{\prime}.d\varpi_{\prime}.\{\sin.\theta_{\prime}.\cos.^2\varpi_{\prime}-\sin.\varpi_{\prime}.\cos.\varpi_{\prime}.\tang.\alpha\}.$$

[9974z] Substituting $\cos.^2\varpi_{\prime}=\frac{1}{2}+\frac{1}{2}\cos.2\varpi_{\prime}$, $\sin.\varpi_{\prime}.\cos.\varpi_{\prime}=\frac{1}{2}\sin.2\varpi_{\prime}$, then integrating relative to $\varpi_{\prime}$ from *its least value* $\varpi_{\prime}=\varpi_2$ to its *greatest value* $\varpi_{\prime}=\varpi_3$; we get

[9975a] 1, 2
$$Z=\frac{H}{\pi.(1-\cot.w.\tang.\alpha)}.\left\{\begin{array}{l}+\int d\theta_{\prime}.\sin.\theta_{\prime}.(+\frac{1}{2}\varpi_3+\frac{1}{4}\sin.2\varpi_3)+\frac{1}{4}\tang.\alpha.\int d\theta_{\prime}.\cos.2\varpi_3\\+\int d\theta_{\prime}.\sin.\theta_{\prime}.(-\frac{1}{2}\varpi_2-\frac{1}{4}\sin.2\varpi_2)-\frac{1}{4}\tang.\alpha.\int d\theta_{\prime}.\cos.2\varpi_2\end{array}\right\}.$$

The value of ϖ_2 evidently corresponds to the angular edge of the wedge, or the line fFc,
[9975b] drawn through F perpendicular to the plane of the figure; and if we draw the line ff' parallel and equal to FF', we shall have the angle $fCf'=\varpi_2$, the angle $Cf'F'=\theta_{\prime}$,

[9975c]
$$ff'=FF'=a.\cot.w \quad [9974o], \qquad Cf'=\frac{CF'}{\sin.Cf'F'}=\frac{a}{\sin.\theta_{\prime}},$$
$$\tang.\varpi_2=\frac{ff'}{Cf'}=a.\cot.w\times\frac{\sin.\theta_{\prime}}{a}=\sin.\theta_{\prime}.\cot.w;$$

whence we get, by using [30″], Int.,

[9975d] $$\varpi_2=\text{arc}.(\tang.=\sin.\theta_{\prime}.\cot.w),$$

[9975e] $$\sin.2\varpi_2=\frac{2\tang.\varpi_2}{1+\tang.^2\varpi_2}=\frac{2\sin.\theta_{\prime}.\cot.w}{1+\sin.^2\theta_{\prime}.\cot.^2w},$$

[9975f] $$\cos.2\varpi_2=\sqrt{1-\sin.^2 2w_2}=\frac{1-\sin.^2\theta_{\prime}.\cot.^2w}{1+\sin.^2\theta_{\prime}.\cot.^2w}.$$

[9975g] The greatest value of $\varpi_{\prime}$, represented by ϖ_3, is evidently found by putting $\frac{s_{\prime\prime}}{a}=\infty$, or $\frac{a}{s_{\prime\prime}}=0$, which gives $m=0$ [9974t]; and by substituting the value of m [9974s], we obtain

[9975g′] $$0=\sin.\theta_{\prime}.\cos.\varpi_3-\sin.\varpi_3.\tang.\alpha.$$

Dividing by $\cos.\varpi_3.\tang.\alpha$, we get $\tang.\varpi_3=\sin.\theta_{\prime}.\cot.\alpha$, or

[9975h] $$\varpi_3=\text{arc}.(\tang.=\sin.\theta_{\prime}.\cot.\alpha).$$

[9975h′] Comparing this with [9975d], we find that ϖ_3 *can be derived from* ϖ_2, *by merely changing* w *into* α; and the same substitutions can be made in [9975e, f]. From the expression of ϖ_2 [9975d], we easily perceive, that, at the upper extremity f of the line cFf, where
[9975i] $Ff=\infty$, $\theta_{\prime}=0$, we have $\varpi_2=0$; and at the point F, where $\theta_{\prime}=\frac{1}{2}\pi$, we have $\varpi_2=90^{d}-w$. At the lower extremity c of the line cFf, where $Fc=-\infty$, and $\theta=\pi$, we have $\varpi_2=0$. Hence we see that, after substituting the values [9975d, e, f] in [9975a, line 2],
[9975k] and taking the integral relative to $\theta_{\prime}$ from $\theta_{\prime}=0$ to $\theta=\pi$, we shall have, at both these limits, $\varpi_2=0$. Now substituting the values [9975d, e], in the first member of [9975l], we

homogeneous fluid] depend, but also their derivation from the attractive [9975′]

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get its second member; and its integral is expressed by the function [9975m], as we shall soon show;

$$fd\theta_{,}.\sin.\theta_{,}.(-\tfrac{1}{2}\varpi_2-\tfrac{1}{4}\sin.2\varpi_2)=fd\theta_{,}.\sin.\theta_{,}.\left\{-\tfrac{1}{2}\text{arc}.(\text{tang.}=\sin.\theta_{,}.\cot.w)-\frac{\frac{1}{2}\sin.\theta_{,}.\cot.w}{1+\sin.^2\theta_{,}.\cot.^2w}\right\} \quad [9975l]$$

$$=\tfrac{1}{2}\cos.\theta_{,}.\text{arc}.(\text{tang.}=\sin.\theta_{,}.\cot.w)+\tfrac{1}{2}\cos.w.\text{arc}.(\text{tang.}=\cot.\theta_{,}.\sin.w)-\frac{\pi}{4}.\cos.w. \quad [9975m]$$

That the expression [9975m] is the integral of [9975l], is easily perceived by observing, that, if we consider $\cos.\theta_{,}$ as the only variable quantity in the first term of [9975m], its differential will be the same as the first term of the second member of [9975l], so that it only remains to show that the differentials of the two arcs in [9975m], namely, $\text{arc}.(\text{tang.}=\sin.\theta_{,}.\cot.w)$, and $\text{arc}.(\text{tang.}=\cot.\theta_{,}.\sin.w)$, considering $\theta_{,}$ as the variable [9975n] quantity, produce the second term of the second member of [9975l]. Now the differentials of these two arcs produce in the differential of [9975m] the following terms, using [54], Int., and the similar expression $d.\cot.\theta_{,}=-d\theta_{,}.(1+\cot.^2\theta_{,})$,

$$\tfrac{1}{2}d\theta_{,}.\left\{\frac{\cos.^2\theta_{,}.\cot.w}{1+\sin.^2\theta_{,}.\cot.^2w}-\frac{(1+\cot.^2\theta_{,}).\sin.w.\cos.w}{1+\cot.^2\theta_{,}.\sin.^2w}\right\}. \quad [9975o]$$

Multiplying the numerator and denominator of the second of these two terms between the braces by $\frac{\sin.^2\theta_{,}}{\sin.^2w}$, or $\sin.^2\theta_{,}.(1+\cot.^2w)$, it becomes

$$-\frac{(\sin.^2\theta_{,}+\cos.^2\theta_{,}).\cot.w}{\sin.^2\theta_{,}.(1+\cot.^2w)+\cos.^2\theta_{,}}=-\frac{(\sin.^2\theta_{,}+\cos.^2\theta_{,}).\cot.w}{1+\sin.^2\theta_{,}.\cot.^2w}; \quad [9975p]$$

substituting this in [9975o], and then reducing, it becomes $-\tfrac{1}{2}d\theta_{,}.\frac{\sin.^2\theta.\cot.w}{1+\sin.^2\theta_{,}.\cot.^2w}$, being the same as the second term of [9975l]. The integral [9975m] vanishes at the first limit $\theta_{,}=0$, where $\text{arc}.(\text{tang.}=\sin.\theta_{,}.\cot.w)=0$; and $\text{arc}.(\text{tang.}=\cot.\theta_{,}.\sin.w)=\tfrac{1}{2}\pi$. [9975q] Then, while the arc $\theta_{,}$ *increases* from $\theta_{,}=0$ to $\theta_{,}=\pi$, the $\text{arc}.(\text{tang.}=\cot.\theta_{,}.\sin.w)$ decreases from $\tfrac{1}{2}\pi$ to 0, and then becomes negative, so that, at the second limit of $\theta_{,}=\pi$, it becomes equal to $-\tfrac{1}{2}\pi$; moreover as $\varpi_2=\text{arc}.(\text{tang.}=\sin.\theta_{,}.\cot.w)=0$, at this second [9975r] limit of $\theta_{,}$, the first term of [9975m] will vanish, and the complete integral [9975m], depending on the two remaining terms, will become

$$-\tfrac{1}{2}\cos.w.\frac{\pi}{2}-\frac{\pi}{4}.\cos.w=-\tfrac{1}{2}\pi.\cos.w; \quad [9975r′]$$

hence the expression [9975m] gives

$$\textstyle\int_0^\pi d\theta_{,}.\sin.\theta_{,}.(-\tfrac{1}{2}\varpi_2-\tfrac{1}{4}\sin.2\varpi_2)=-\tfrac{1}{2}\pi.\cos.w. \quad [9975s]$$

Again, if we substitute the value of $\cos.2\varpi_2$ [9975f], in the first member of [9975t], and then integrate it, we shall get

$$fd\theta_{,}.\cos.2\varpi_2=fd\theta_{,}.\frac{(1-\sin.^2\theta_{,}.\cot.^2w)}{1+\sin.^2\theta_{,}.\cot.^2w}=-\theta_{,}+2\sin.w.\text{arc}.(\text{tang.}=\text{tang}.\theta_{,}.\text{cosec}.w), \quad [9975t]$$

as is easily proved by taking the differential of its last member, relative to $\theta_{,}$; for it becomes,

[9975''] forces of the particles of the bodies which produce their affinities. *The*

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by successive reductions, and putting $\sin.w.\mathrm{cosec}.w=1$, $\mathrm{cosec}.^2 w=1+\cot.^2 w$, &c.

[9975u]
$$-d\theta_{,}+2\sin.w.\frac{d\theta_{,}.(1+\mathrm{tang}.^2\theta_{,}).\mathrm{cosec}.w}{1+\mathrm{tang}.^2\theta_{,}.\mathrm{cosec}.^2w}=-d\theta_{,}+\frac{2d\theta_{,}.(1+\mathrm{tang}.^2\theta_{,})}{1+\mathrm{tang}.^2\theta_{,}.(1+\cot.^2w)}$$
$$=-d\theta_{,}+\frac{2d\theta_{,}}{1+\sin.^2\theta_{,}.\cot.^2w}=d\theta_{,}.\frac{(1-\sin.^2\theta_{,}.\cot.^2w)}{1+\sin.^2\theta_{,}.\cot.^2w},$$

which is the same as the proposed differential in [9975t]. At the first limit $\theta_{,}=0$ of the integral [9975t], it vanishes. Moreover, as $\theta_{,}$ increases from 0 to π, the $\mathrm{arc}.(\mathrm{tang}.=\mathrm{tang}.\theta_{,}.\mathrm{cosec}.w)$ also increases from 0 to π; so that, at the second limit of the integral, we shall have $\theta_{,}=\pi$, and $\mathrm{arc}.(\mathrm{tang}.=\mathrm{tang}.\theta_{,}.\mathrm{cosec}.w)=\pi$. Substituting these values in [9975t], we get

[9975v]
$$\textstyle\int_0^\pi d\theta_{,}.\cos.2\varpi_2=\pi.(-1+2\sin.w).$$

Multiplying this by $-\frac{1}{4}\mathrm{tang}.\alpha$, and adding the product to the integral [9975s], we obtain

[9975w]
$$\textstyle\int_0^\pi d\theta_{,}.\sin.\theta_{,}.(-\frac{1}{2}\varpi_2-\frac{1}{4}\sin.^22\varpi_2)-\frac{1}{4}\mathrm{tang}.\alpha.\int_0^\pi d\theta_{,}.\cos.2\varpi_2=\frac{1}{2}\pi.(-\cos.w+\frac{1}{2}\mathrm{tang}.\alpha-\mathrm{tang}.\alpha.\sin.w).$$

If we change ϖ_2 into ϖ_3 in the integrals [9975s, v], the effect will be to change w into α in the second members of these expressions, as we have seen in [9975h']. Making these changes, and multiplying the integral derived from [9975v] by $-\frac{1}{4}\mathrm{tang}.\alpha$, we get for the sum

[9975x]
$$\textstyle\int_0^\pi d\theta_{,}.\sin.\theta_{,}.(-\frac{1}{2}\varpi_3-\frac{1}{4}\sin.^22\varpi_3)-\frac{1}{4}\mathrm{tang}.\alpha.\int_0^\pi d\theta_{,}.\cos.2\varpi_3=\frac{1}{2}\pi.(-\cos.\alpha+\frac{1}{2}\mathrm{tang}.\alpha-\mathrm{tang}.\alpha.\sin.\alpha).$$

Subtracting [9975x] from [9975w], and multiplying the remainder by $\frac{H}{\pi.(1-\cot.w.\mathrm{tang}.\alpha)}$, we get the expression of Z [9975a], which becomes

[9975y]
$$Z=\frac{\frac{1}{2}H.\{(\cos.\alpha-\cos.w)+(\mathrm{tang}.\alpha.\sin.\alpha-\mathrm{tang}.\alpha.\sin.w)\}}{1-\cot.w.\mathrm{tang}.\alpha}.$$

Multiplying the numerator and denominator of this expression by $\sin.w.\cos.\alpha$, and reducing by means of [24, 22, 1, 31, 34'], Int., we get, by successive operations,

[9975z]
$$Z=\frac{\frac{1}{2}H.\sin.w.\{(\cos.^2\alpha+\sin.^2\alpha)-(\cos.w.\cos.\alpha+\sin.w.\sin.\alpha)\}}{\sin.w.\cos.\alpha-\cos.w.\sin.\alpha}=\frac{\frac{1}{2}H.\sin.w.\{1-\cos.(w-\alpha)\}}{\sin.(w-\alpha)}$$

[9976a]
$$=\frac{\frac{1}{2}H.\sin.w.\{2\sin.^2\frac{1}{2}(w-\alpha)\}}{2\sin.\frac{1}{2}(w-\alpha).\cos.\frac{1}{2}(w-\alpha)}=\tfrac{1}{2}H.\sin.w.\mathrm{tang}.\tfrac{1}{2}(w-\alpha).$$

This expresses the upward vertical action of the wedge AFB, included by the angle $w-\alpha$, upon the plane QFC, corresponding to the angle w.

If we change α into α', in the expression of Z [9976a], it will become

[9976b]
$$Z=\tfrac{1}{2}H.\sin.w.\mathrm{tang}.\tfrac{1}{2}(w-\alpha');$$

subtracting this from the expression [9976a], we get the value of Z, corresponding to the action of a wedge included by the angle $\alpha'-\alpha$; the vertex of the wedge being the line drawn through F, perpendicular to the plane of the figure, and continued infinitely above and below it; this value is

[9976c]
$$Z=\tfrac{1}{2}H.\sin.w.\{\mathrm{tang}.\tfrac{1}{2}(w-\alpha)-\mathrm{tang}.\tfrac{1}{2}(w-\alpha')\}.$$ [Action of a wedge $\alpha'-\alpha$ upon a plane $w-o$.]

capillary forces are only modifications of these attractive forces, depending [9976]

Changing w into w', we get the corresponding value of Z relative to the angle w', namely,

$$Z = \tfrac{1}{2}H.\sin.w'.\{\text{tang}.\tfrac{1}{2}(w'-\alpha) - \text{tang}.\tfrac{1}{2}(w'-\alpha')\}. \quad \left[\begin{smallmatrix}\text{Action of a wedge } \alpha'-\alpha \\ \text{upon a plane } w'-o.\end{smallmatrix}\right] \qquad [9976d]$$

Subtracting the expression [9976c] from [9976d], we get, for the action of the wedge $\alpha' - \alpha$ upon the part of the plane included by the angle $w' - w$, the following expression,

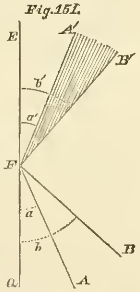
$$Z = \tfrac{1}{2}H. \left\{ \begin{array}{l} +\sin.w'.\{+\text{tang}.\tfrac{1}{2}(w'-\alpha) - \text{tang}.\tfrac{1}{2}(w'-\alpha')\} \\ +\sin.w.\{-\text{tang}.\tfrac{1}{2}(w-\alpha) + \text{tang}.\tfrac{1}{2}(w-\alpha')\} \end{array} \right\}, \quad \left[\begin{smallmatrix}\text{Action of a wedge } \alpha'-\alpha \\ \text{upon a plane } w'-w.\end{smallmatrix}\right] \qquad [9976e]$$

this action being upwards in a vertical direction. The angles $w' - w$, $\alpha' - \alpha$, are here considered as positive quantities. The positive values of the angles w, w', relative to the attracted plane, are counted from the line FQ towards the *right,* and the positive values of the angles α, α', relative to the attracting wedge, are counted from the line FE towards the *left.* If we wish to count the angles α, α', from the line FE towards the *right,* we must change the signs, putting $\alpha = -\mathrm{b}'$, $\alpha' = -\mathrm{a}'$, considering a', b', as positive quantities. If we also, for the sake of symmetry, put $w = \mathrm{a}$, $w' = \mathrm{b}$, and then substitute $\tfrac{1}{2}H = q$ [9972e], we shall find that the expression [9976e] will become [9976e'] [9976f]

$$Z = q. \left\{ \begin{array}{l} +\sin.\mathrm{b}.\{+\text{tang}.\tfrac{1}{2}(\mathrm{b}+\mathrm{b}') - \text{tang}.\tfrac{1}{2}(\mathrm{b}+\mathrm{a}')\} \\ +\sin.\mathrm{a}.\{-\text{tang}.\tfrac{1}{2}(\mathrm{a}+\mathrm{b}') + \text{tang}.\tfrac{1}{2}(\mathrm{a}+\mathrm{a}')\} \end{array} \right\}, \quad \left[\begin{smallmatrix}\text{Action of a wedge } \mathrm{b}'-\mathrm{a}' \\ \text{upon a plane } \mathrm{b}-\mathrm{a}.\end{smallmatrix}\right] \qquad [9976g]$$

representing the action of a wedge $A'FB'$, fig. 151, included by the angle $\mathrm{b}' - \mathrm{a}'$ upon a triangular part of the plane AFB included by the angle $\mathrm{b} - \mathrm{a}$; the angles a, b, a′, b′, being considered as *positive* when falling to the right of the line EFQ, otherwise negative; the origin of the angles a′, b′, being the *upward vertical* line FE, and the origin of the angles a, b, being the line FQ on the continuation of EF, or in the direction of gravity. If we use, for abridgment, the values of A, B, A', B' [9976k], and make a slight change in the arrangement of the terms of the expression [9976g], we shall find that it will become of the form [9976l], from which we may easily derive the equivalent expression [9976m], as we shall soon show. [9976h] [9976i]

Fig. 151.



$$A = \tfrac{1}{2}(\mathrm{a}'+\mathrm{a}), \qquad B = \tfrac{1}{2}(\mathrm{b}'+\mathrm{b}), \qquad A' = \tfrac{1}{2}(\mathrm{a}'+\mathrm{b}), \qquad B' = \tfrac{1}{2}(\mathrm{b}'+\mathrm{a}); \qquad [9976k]$$

$$Z = q.\{\sin.\mathrm{a}.(\text{tang}.A - \text{tang}.B') + \sin.\mathrm{b}.(\text{tang}.B - \text{tang}.A')\}; \qquad [9976l]$$

$$Z = q.\{\sin.\mathrm{a}'.(\text{tang}.A - \text{tang}.A') + \sin.\mathrm{b}'.(\text{tang}.B - \text{tang}.B')\}; \quad \left[\begin{smallmatrix}\text{Action of a wedge } \mathrm{b}'-\mathrm{a}' \\ \text{upon a plane } \mathrm{b}-\mathrm{a}.\end{smallmatrix}\right] \qquad [9976m]$$

The identity of the two expressions [9976l, m], is easily proved; for if we change b into $-\mathrm{a}'$, in [38], Int., we can easily deduce $\cos.\mathrm{a}' - \cos.\mathrm{a} = \text{tang}.\tfrac{1}{2}(\mathrm{a}'+\mathrm{a}).(\sin.\mathrm{a} - \sin.\mathrm{a}')$, [9976n] which, by using the notation [9976k], becomes as in [9976o]; and in like manner we get [9976p, q, r]. The sum of these four equations gives [9976s]; observing that the first member of this sum vanishes. Multiplying this last expression by $-q$, and adding the product to [9976l], we get [9976m].

[9976′] *upon the curvature of the fluid surface in the first method, and upon*

[9976o] $$\cos.\mathrm{a}' - \cos.\mathrm{a} = \operatorname{tang}.A.(\sin.\mathrm{a} - \sin.\mathrm{a}');$$

[9976p] $$\cos.\mathrm{b}' - \cos.\mathrm{b} = \operatorname{tang}.B.(\sin.\mathrm{b} - \sin.\mathrm{b}');$$

[9976q] $$\cos.\mathrm{b} - \cos.\mathrm{a}' = \operatorname{tang}.A'.(\sin.\mathrm{a}' - \sin.\mathrm{b});$$

[9976r] $$\cos.\mathrm{a} - \cos.\mathrm{b}' = \operatorname{tang}.B'.(\sin.\mathrm{b}' - \sin.\mathrm{a});$$

[9976s] $$\begin{aligned}0 = &-\sin.\mathrm{a}'.(\operatorname{tang}.A - \operatorname{tang}.A') - \sin.\mathrm{b}'.(\operatorname{tang}.B - \operatorname{tang}.B') \\ &+ \sin.\mathrm{a}.(\operatorname{tang}.A - \operatorname{tang}.B\,) + \sin.\mathrm{b}.(\operatorname{tang}.B - \operatorname{tang}.A').\end{aligned}$$

We may also deduce the equation [9976*m*] from [9976*l*], by observing that the action of a
[9976t] solid wedge, included between the angular space $\mathrm{b}-\mathrm{a}$, upon a triangular plane included in the angular space $\mathrm{b}'-\mathrm{a}'$, is equal and contrary to the reaction of the plane upon the wedge; we may therefore change the symbols a, b, into a′, b′, reciprocally, without altering the
[9976u] value of Z. This change does not alter the values of A, B [9976*k*], but changes A' into B', and B' into A'; and by making the same changes in [9976*l*], we get [9976*m*]. These two values of Z are equivalent to those given by M. Poisson, in page 89 of his *Nouvelle Théorie, &c.*, where they are marked (5) and (4); a′, b′, &c., being changed into a, b, &c.

We may in this connection remark that whenever any one of the angles A, B, A', B', becomes equal to a right angle, which can happen when either of the lines
[9976v] FA, FB, fig. 151, is in the same direction as one of the lines $A'F$, or $B'F$, so that the plane and solid can be in contact with each other upon an infinitely long line, the expression of Z may become infinite by means of the tangent of this angle; but when this tangent is multiplied by the sine of an angle a, b, a′, or b′, which then becomes equal to nothing, the value of Z
[9976w] may be finite, though it may appear under the illusory form of $Z=\frac{0}{0}$. An example of this kind occurs when we suppose the lines FA, FA', FQ, to coalesce, as in fig. 152, and the line FB to be on the continuation of BF'. In this case, a′, b′, will be negative, and we shall have

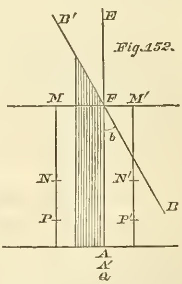


Fig. 152.

[9976x] $$\mathrm{a}' = -180^{\mathrm{d}.}, \qquad \mathrm{a}=0, \qquad \mathrm{b}' = -\mathrm{b};$$

hence we get from [9976*k*]

[9976x′] $$A = -90^{\mathrm{d}.}, \quad B=0, \quad A' = -90^{\mathrm{d}.} + \tfrac{1}{2}\mathrm{b}, \quad B' = -\tfrac{1}{2}\mathrm{b}.$$

Here $\operatorname{tang}.A$ becomes negative and infinite, but being multiplied by $\sin.\mathrm{a}$, or $\sin.\mathrm{a}'$, in the expressions of Z [9976*l*, *m*], it becomes illusory. To ascertain its true value, we shall, in the first place, suppose $\mathrm{b} = \frac{1}{2}\pi$, making the line $B'FB$ coincide with the horizontal
[9976y] line MFM'. In this case, the vertical action of the body MFQ upon the plane $M'FQ$ vanishes, as we may prove in the following manner: Suppose the *vertical* lines $M'N'P'$, MNP, to be drawn parallel to FQ, the line $M'N'P'$ being in the attracted plane, and the line MNP in the attracting body; this last line being situated either in the plane of the figure, or at any distance above
[9976z] or below it. Then, through any points N, P, of the line MNP, draw the lines NN', PP', parallel to the horizontal plane, passing through the line MM'; and if we consider the action

the position of the attracting planes in the second method; whereas the [9976″]

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of two equal particles of the attracting body situated at the points N, P, upon the two equal particles of the attracted plane at the points N', P', we shall evidently perceive that the *upward* or vertical action of the particle N upon P' is balanced by the *downward* action of the particle P upon N'; and as the same result holds good for all the particles, the vertical action of MFQ upon $M'FQ$ must vanish. Now [9977a]

$$\text{The triangle } QFB = \text{plane } M'FQ - \text{triangle } M'FB. \qquad [9977b]$$

And as the vertical action of the body MFQ upon the plane $M'FQ$ vanishes, it is evident that we shall have

$$\text{The action of } MFQ \text{ upon the triangle } QFB = -\text{the action of } MFQ \text{ upon the triangle } M'FB. \qquad [9977c]$$

Adding to this the action of the part $B'FM$ upon the triangle QFB, we get for the whole vertical action of the body $B'FQ$ upon the triangle QFB, the following expression, namely,

$$\text{The action of } B'FQ \text{ upon } QFB = \text{the action of } B'FM \text{ upon } QFB - \text{action of } MFQ \text{ upon } M'FB. \qquad [9977d]$$

We shall now compute the values of these two portions. In the first place, in finding the action of $B'FM$ upon QFB, we have

$$\mathrm{a} = 0, \qquad \mathrm{a}' = -90^{\mathrm{d.}}, \qquad \mathrm{b}' = -\mathrm{b}; \qquad [9977e]$$

and then, from [9976k], we get

$$A = -45^{\mathrm{d.}}, \qquad B = 0, \qquad A' = -(45^{\mathrm{d}} - \tfrac{1}{2}\mathrm{b}), \qquad B' = -\tfrac{1}{2}\mathrm{b}. \qquad [9977e']$$

Substituting these in [9976l], we get

$$\text{The action of } B'FM \text{ upon } QFB = q.\sin.\mathrm{b}.\mathrm{tang}.(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}). \qquad [9977f]$$

In computing the action of MFQ upon $M'FB$, we have

$$\mathrm{a}' = -180^{\mathrm{d.}}, \qquad \mathrm{b}' = -90^{\mathrm{d.}}, \qquad \mathrm{b} = 90^{\mathrm{d.}}, \qquad [9977f']$$

and a must be changed into b; hence, from [9976k], we have

$$A = -90^{\mathrm{d.}} + \tfrac{1}{2}\mathrm{b}, \qquad B = 0, \qquad A' = -45^{\mathrm{d.}}, \qquad B' = -(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}). \qquad [9977g]$$

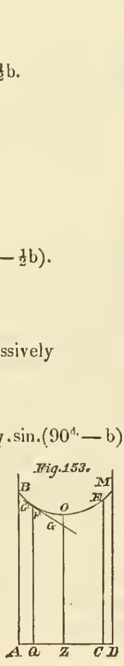
Substituting these in [9976m], we get

$$\text{The action of } MFQ \text{ upon } M'FB = -q.\mathrm{tang}.(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}). \qquad [9977h]$$

Substituting [9977f, h], in [9977d], and using [6, 34′, 31], Int., we get successively

$$\begin{aligned}\text{The action of } B'FQ \text{ upon } QFB &= q.\mathrm{tang}.(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}).\{1 + \sin.\mathrm{b}\} \\ &= q.\mathrm{tang}.(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}).\{2\cos.^2(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b})\} \\ &= 2q.\sin.(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}).\cos.(45^{\mathrm{d.}} - \tfrac{1}{2}\mathrm{b}) = q.\sin.(90^{\mathrm{d.}} - \mathrm{b}) \\ &= q.\cos.\mathrm{b}.\end{aligned} \qquad [9977i]$$

Fig. 153.



We may apply this formula to the calculation of the action of the successive concentrical lamina of a *homogeneous* fluid, elevated in a cylindrical capillary tube, treated of by the author in [9902, &c.]. For this purpose we shall suppose $BADM$, fig. 153, to be the tube, whose vertical axis is OZ; *AZD the level of the surface of the fluid in the vase in which the tube is dipped*; $ZA = ZD$ the inner radius of the tube; $ZQ = ZC$ the inner [9977k]

[9976‴] *affinities seem to me to be the attractive forces themselves, acting with*

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radius of the concentrical stratum, whose section by the plane of the figure is represented by
[9977k] the parts $ABFQ$ and $DMEC$. If we draw a tangent $G'FG$ to the point F, and neglect, as in [9926], the curvature of the sides of the tube, we may consider the fluid contained in the part $ABFQ$ as being limited by two planes, drawn through QF, FB, perpendicular
[9977l] to the plane of the figure, and continued infinitely above and below it; and if we put the angle $QFG=b$, we shall have, from [9977i],

[9977m] The vertical action of the external mass QFB upon the internal plane $GFQC=q.\cos.b$;

so that, if we suppose the thickness of this attracted plane to be dc, the action upon it will be
[9977n] $q.\cos.b.dc$. Integrating this relative to the whole circumference c of the stratum whose radius is ZQ, we get

[9977o] The whole action of the external stratum upon the internal mass of the fluid $FQCE\,O=qc.\cos.b$.

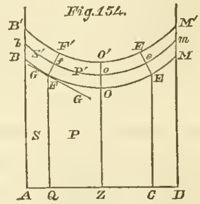
At the side of the tube, or rather at an insensible distance from it, where $b=\varpi$ [9892], this
[9977p] action becomes $qc.\cos.\varpi$; and by resubstituting $q=\frac{1}{2}H$ [9972e], it is reduced to the following expression;

[9977q] Vertical action of the annulus $=\frac{1}{2}Hc.\cos.\varpi$,

being the same as the weight of the internal mass of the homogeneous fluid $FQCE$, as in [9898].

It is easy to extend this demonstration to the case of nature, where the fluid, near its surface, is covered with an excessively thin pellicle of variable density, whose upper surface
[9977r] $B'F'O'M'$, fig. 154, is so rare that its density may be considered as nearly equal to nothing. Through all the points of this surface, we shall suppose perpendiculars $O'oO$, $F'fF$, &c. to be erected. Then, taking, on any one of these perpendiculars, as, for example, on the vertical axis $O'oOZ$, a point o, *at an insensible distance below*
[9977s] *the surface, but sufficiently far from it to have the density equal to that of the internal mass of the fluid, which is represented by unity;* we shall draw through o the surface $bfom$, cutting all the
[9977t] perpendiculars $O'oO$, $F'fF$, &c., at right angles in o, f, &c. In like manner, we shall draw the surface $BFOM$, cutting the same perpendiculars at right angles in O, F, &c.; *this surface being at an insensible distance below $bfom$, but sufficiently*
[9977u] *far from it to render the corpuscular actions of the fluid, which are actually situated in the surfaces $BFOM$, $bfom$, wholly insensible upon each other.* Taking any point F of the
[9977v] surface $BFOEM$, and drawing through it the vertical line FQ parallel to the axis OZ, we shall suppose the line FQ *to be at such a distance from the side AB of the tube, that the*
[9977v′] *fluid near the tube, whose density is variable, can have no corpuscular action upon the fluid*
[9977w] *which is situated beyond FQ, or in the plane $QFEC$.* Then the figure $AB'F'FQ$, being supposed to revolve about the axis $O'Z$, will form a figure, somewhat like an annulus, whose action on the fluid contained within it will be exactly equal to the weight of the internal mass

Fig. 154.



all their energy. [9976'''']

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of the fluid $QFF'O'E'EC$, which is elevated *above the level AD of the fluid in the vase where the tube is dipped.* To prove this, we shall, for the sake of distinctness, denote the different portions of the fluid, and of the plane, in the following manner: [9977x]

$S=$ the portion of the fluid formed by the revolution of the figure $ABFQ$ about the axis $O'Z$; [9977y]

$S'=$ the portion of the fluid formed by the revolution of the figure $BB'F'F$ about the axis $O'Z$; [9977z]

$P=$ the portion $QFOEC$ of the attracted plane, whose thickness is supposed to be extremely small, and represented by unity; [9978a]

$P'=$ the portion $FF'O'E'EOF$ of the same attracted plane; [9978b]

$b=$ the angle QFG, formed at the point F by the vertical line FQ, and the tangent FG to the section of the surface FOE. [9978c]

In calculating the action of the annulus $S+S'$ upon the attracted plane $P+P'$, it is convenient to take separately into consideration the four distinct portions. *First*, the action of S upon P; *second*, the action of S upon P'; *third*, the action of S' upon P; *fourth*, the action of S' upon P'. *In computing these attractions, we shall neglect*, as in [9926], *the curvature of the side of the tube, and consider it as being developed in a plane passing through AB', perpendicular to the plane of the figure.* Moreover we may consider the mass S, in its action upon P, as being of the uniform density unity, because the part of the plane P which is nearest to the side of the tube AB, or nearest to the surface $bfoc$, is beyond the sphere of any sensible action upon P; therefore we shall have the action of S upon P by the same process as in [9977i], which gives [9978d] [9978e] [9978f] [9978g]

The vertical action of S upon $P=q.\cos.b$. [9978h]

In calculating the action of S on P', we may, as in [9978f], consider the masses S and P' as being of the uniform density unity, because the part of S near the side of the tube is beyond the sphere of any sensible corpuscular action upon P'; and the same holds good relative to the upper part of P', which is beyond the sphere of any corpuscular action of S. We may therefore calculate the action of S upon P' by either of the formulas [9976l, m]. Then, according to the notation in [9976i, k], we have [9978i] [9978k]

$$\mathrm{a}=b, \quad \mathrm{b}=90^{\mathrm{d}}+b, \quad \mathrm{a}'=-180^{\mathrm{d}}, \quad \mathrm{b}'=-b;$$
$$A=-90^{\mathrm{d}}+\tfrac{1}{2}b, \quad B=45^{\mathrm{d}}, \quad A'=-(45^{\mathrm{d}}-\tfrac{1}{2}b), \quad B'=0. \qquad [9978l]$$

Substituting these in the value of Z [9976m], it becomes $-q.\sin.b$; so that we shall have

The vertical action of S upon $P'=-q.\sin.b$. [9978m]

In like manner, in calculating the action of S' upon P, we may consider the masses S' and P as being of the uniform density unity, because the part S' near the side of the tube, and [9978n]

[9977] We shall now resume the equation [9930], observing that, if, *with*

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near its upper surface, is beyond the sphere of any sensible action upon P. Then, according to the notation in [9976i, k], we have

[9978o]
$$\mathrm{a}=0, \quad \mathrm{b}=b, \quad \mathrm{a}'=-b, \quad \mathrm{b}'=90^{\mathrm{d.}}-b;$$
$$A=-\tfrac{1}{2}b, \quad B=45^{\mathrm{d.}}, \quad A'=0, \quad B'=45^{\mathrm{d.}}-\tfrac{1}{2}b.$$

Substituting these in [9976l], we get

[9978o']
$$Z=q.\sin.b.\tang.B=q.\sin.b;$$

so that we shall have

[9978p]
$$\text{The vertical action of } S' \text{ upon } P=q.\sin.b.$$

[9978q] Lastly, if we represent the direct action of S' upon P' by $q_{,}$, this direction being inclined to the vertical by the angle b, we shall have, by resolving it in a vertical direction, the following expression:

[9978r]
$$\text{The vertical action of } S' \text{ upon } P'=q_{,}.\cos.b.$$

Adding together the four parts of this action [9978h, m, p, r], and neglecting the terms which
[9978s] destroy each other, we finally get, for the vertical action of $S+S'$ upon $P+P'$, the following expression:

[9978t]
$$\text{The vertical action of the external mass } B'F'FQA \text{ upon the internal plane } QFF'O'E'EC=(q+q_{,}).\cos.b.$$

In computing that part of the mass of the fluid which is elevated by the capillary action, and surrounded by the mass $S+S'$; and whose section is represented by $P+P'$, or, in the figure 154, page 856, by $QFF'O'E'EC$; we may, on account of the insensible thickness of P',
[9978u] neglect wholly the consideration of this pellicle or mass of fluid of variable density near the surface; and then, taking FOE for the limiting surface where the density becomes equal to unity, we may consider the internal elevated mass to be limited by the space whose section is $FQCEO$, being the same as in fig. 153, page 855 [9977o, &c.]. Now if we put, in like manner as in [9977p],

[9978v]
$$q=\tfrac{1}{2}H, \qquad q+q_{,}=\tfrac{1}{2}\mathrm{H},$$

and then compare together the expressions [9977m, 9978t], we shall find that the effect of noticing the variation of density near the surface of the fluid, in computing the vertical action of the mass $S+S'$ upon the internal mass whose section is $P+P'$, is merely to change q into $q+q_{,}$, or H into H [9978v], as in [9261e, f]; and by making this change in [9977q], we get

[9978w]
$$\text{The vertical action of the annulus}=\tfrac{1}{2}\mathrm{H}c.\cos.\varpi,$$

being the same as the weight of the internal mass of the elevated fluid $FQCEO$ [9898], changing H into H, as above.

By the same process which we have used in computing the action of the *external* annulus $B'F'FQA$, fig. 154, p. 856, upon the *internal* plane $QFF'O'E'EC$ [9978t], we may find the
[9978x] action of the *internal mass* of fluid whose section is $QFF'O'E'EC$ upon the *external plane*, or upon the section of the annulus $B'F'FQA$. The only change will be to insert in the

a cylindrical tube whose internal radius is l, we put q for the mean [9977′]

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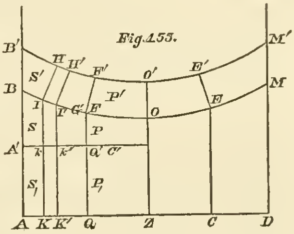
formula [9978t] the angle $QFG'=w$, instead of its supplement $QFG=b=180^{\mathrm{d}}-w$ [9978y]
hence we get

The vertical action of the internal mass $QFF'O'E'EC$ upon the external plane $B'F'FQA=(q+q_{,}).\cos.w$ [9978z]

In these expressions of the vertical action, we have supposed the line $AQCD$ [9977x] to be on the horizontal level of the surface of the fluid in the vase in which the tube is dipped; but the same formulas hold good when we suppose the masses S and P to be limited by any horizontal plane $A'Q'C'$, fig. 155, parallel to AQ, but much nearer to the surface of the fluid, taking care, however, to have it sufficiently distant, so that the corpuscular action of the surface BFO shall not extend to the plane $A'Q'$. In this case, the formula [9978z] will give [9979a]



The vertical action of the internal mass $C'Q'FF'O'E'E$ upon the external plane $B'F'FQ'A'=(q+q_{,}).\cos.w.$ [9979b]

We may here remark that the whole mass of the fluid $B'ADM'$ which is situated above the level AD of the mass of fluid in the vase in which the tube is dipped, is in equilibrium by the mutual action and reaction of the fluid and the tube; and we shall now [9979c] proceed to show how we may make use of this circumstance to investigate the relation between the angle w and the forces which act on the fluid in the plane $B'A'Q'FF'$, which, for brevity, we shall represent by N; supposing the thickness of this plane to be infinitely [9979d] small, and represented by unity. The forces acting on the plane N, supposing them all to be resolved *in a vertical or upward direction*, are the following: [9979e]

First. The upward action of the part of the tube near $B'A'$, which is within the sphere of the corpuscular action on the plane N. This force we shall represent by T. [9979f]

Second. The upward action of the mass $S+S'$ contained in the space which is formed by the revolution of the plane N, or $B'A'Q'FF'$, about the axis $O'Z$. This force [9979g] we shall represent by S.

Third. The upward action of the mass $S_{,}$, which is included in the space formed by the revolution of the plane $AA'QQ'$ about the axis $O'Z$. This force we shall [9979h] represent by $\varpi_{,}$.

Fourth. The upward action of the mass $P_{,}$, formed by the revolution of the plane $C'Q'QZ$ about the axis $O'Z$. This force we shall denote by $\mathrm{P}_{,}$. [9979i]

Fifth. The upward action of the internal mass $P+P'$, formed by the revolution of the plane $C'Q'FF'O'$ about the axis $O'Z$. The value of this action has been [9979k] computed in [9979b], and found to be equal to $(q+q_{,}).\cos.w$, or $\frac{1}{2}\mathrm{H}.\cos.w$.

The sum of these five forces must be put equal to nothing, upon the principle adopted in

[9977″] *height to which the fluid rises above the level,* the volume of the

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[9979c], namely, that there is an equilibrium in the actions of the fluid and tube upon any portion $S+S'$ of the mass of the fluid; hence we have

[9979l] $$0=\mathrm{T}+\mathrm{S}+\varpi_{,}+\mathrm{P}_{,}+(q+q_{,})\cdot\cos.w;$$

and it now remains to compute the values of T, S, P, observing that $\varpi_{,}$ cannot be determined, because the law of the corpuscular action on the part of the fluid of variable density near the side of the tube is unknown.

[9979m] If we suppose the plane N to be divided into filaments parallel to $A'Q'$, or perpendicular to the side of the tube, the vertical action of the part of the tube which is situated above any filament, will evidently be balanced by the contrary action of the part of the tube situated below the same filament; therefore the sum of all these actions will give

[9979n] $$\mathrm{T}=0.$$

In computing the force S, we shall suppose the plane N to be divided into vertical filaments kI, $k'I'$, &c., and which, at the points I, I', &c., are bent, in the plane of the figure, into
[9979o] the directions IH, $I'H'$, &c., perpendicular to the curve $BII'F$, at the points I, I', &c., respectively. If this vertical plane, with its filaments, revolve about the axis $O'Z$, it will divide the whole space $S+S'$ [9979g] into similar and equal filaments. To distinguish the
[9979p] plane and filaments in their new positions, we shall suppose that, after the plane N has revolved about the axis $O'Z$, by any given angle α, it is denoted by (N) in its new position; and in like manner the filament kIH changes into (kIH), $k'I'H'$ into $(k'I'H')$, &c. Then it is evident that, as the filaments kIH, (kIH), are at the same distance from the side of the tube, they must have the same density, and be similar and equal to each other; and
[9979q] this holds good even for the filaments which are extremely near to the tube, supposing always that their bases are infinitely small and equal to each other; similar remarks may be made relative to any other corresponding filaments, as $k'I'H'$, $(k'I'H')$, &c. As the action of the plane (N) upon the plane N must be insensible when their distance is of any sensible
[9979q′] magnitude, the angle α may be considered as excessively small; so that, without any sensible error, we may consider the planes N, (N), as parallel, and the side of the tube corresponding to $A'B'$ as a plane, as is supposed by the author in [9926, &c.]. Then, on account of the symmetry of the situation of the four filaments kIH (kIH), $k'I'H'$ $(k'I'H')$, and having, moreover, the mass $kIH=$ mass (kIH), the mass $k'I'H'=$ the mass $(k'I'H')$, we shall have, by considering only their vertical actions upon each other,

[9979r] $$\text{Action of }(kIH)\text{ upon }k'I'H'=\text{action of }kIH\text{ upon }(k'I'H),$$
[9979s] $$=-\text{action of }(k'I'H')\text{ upon }kIH;$$

observing that the equation [9979r] can be demonstrated by a method similar to that which is used in [9976y, z, &c.], and that the expression [9979s] can be derived from the second
[9979t] member of [9979r], by considering that the action of kIH upon $(k'I'H')$ is equal and contrary to the action of $(k'I'H')$ upon kIH. Transposing the expression [9979s] to its first member, we get

[9979u] $$\text{Action of }(kIH)\text{ upon }k'I'H'+\text{action of }(k'I'H')\text{ upon }kIH=0.$$

elevated mass will be $\pi l^2 q$ [as in 9985*a*], and the circumference c of [9978]

From this equation it follows, that the vertical action of the two filaments (kIH), $(k'I'H')$, of the plane (N), upon the two corresponding filaments kIH, $k'I'H'$, of the plane N, mutually destroy each other; and, as this must hold good for the filaments into which the plane N or (N) can be divided, the whole sum of these actions will give [9979*v*]

The vertical action of the plane (N) upon the plane $N=0$. [9979*w*]

Now this equation takes place for every one of the planes (N) in its action upon the plane N; hence it follows that the sum of all these actions is equal to nothing; consequently we have

$$\mathrm{S}=0. \qquad [9979x]$$

In computing the force $\mathrm{P}_{,}$ [9979*i*], or the action of the mass $P_{,}$ upon the plane N, we may remark that the corpuscular action of $P_{,}$ does not extend to the surface BO, nor does it approach so near to the side of the tube as to act on the part of the fluid whose density differs from unity; we may therefore compute the action by means of the formula [9973*a*], supposing the mass $P_{,}$ and the fluid in the plane N to be of the uniform density unity; observing that the angles $QQ'C$ and $A'Q'F$ are right angles; therefore the action $\mathrm{P}_{,}$ is equal to $\frac{1}{2}H$ [9973*a*], and by substituting its value q [9978*v*], we get $P_{,}=q$. This action of the mass $P_{,}$ is in the direction of gravity, and to reduce it to a vertical or upward direction [9979*c*], we must change its sign; hence we finally obtain [9979*y*] [9979*z*]

$$\mathrm{P}_{,}=-q. \qquad [9980a]$$

Substituting the values of T, S, $\mathrm{P}_{,}$ [9979*n*, *x*, 9980*a*], in [9979*l*], we get

$$0=\varpi_{,}-q+(q+q_{,}).\cos.w. \qquad [9980b]$$

Now putting, as in [9978*v*],

$$q+q_{,}=\tfrac{1}{2}\mathrm{H}, \quad \text{also} \quad q-\varpi_{,}=\tfrac{1}{2}\mathrm{F}, \qquad [9980c]$$

it becomes

$$0=-\tfrac{1}{2}\mathrm{F}+\tfrac{1}{2}\mathrm{H}.\cos.w; \qquad [9980d]$$

whence we get

$$\cos.w=\frac{q-\varpi_{,}}{q+q_{,}}=\frac{\mathrm{F}}{\mathrm{H}}. \qquad [9980e]$$

This formula is the same as the formula (3), page 99 of M. Poisson's *Nouvelle Théorie, &c.*, and the equation [9980*b*] is equivalent to the equation (8), page 93 of the same work. We may here remark that, in the equation [9980*d*], the term $\frac{1}{2}\mathrm{H}.\cos.w=(q+q_{,}).\cos.w$ [9980*c*] *represents the upward vertical action of the mass* $P+P'$ *upon the plane* N, as appears from [9979*k*]. Moreover the term $-\frac{1}{2}\mathrm{F}$ of the same equation [9980*d*], or the equivalent expression $\varpi_{,}-q$ [9980*c*], which is equal to $\varpi_{,}+\mathrm{P}_{,}$ [9980*a*], *represents, as in* [9979*h*, *i*], *the upward action of the mass* $S_{,}+P_{,}$ *upon the same plane* N. [9980*f*] [9980*g*]

In this calculation, we have supposed the tube to be of a cylindrical form, with a circular base; but as the corpuscular action extends only to an insensible distance, it is easy to apply this demonstration to the case where the base is any regular curve whatever, without abrupt [9980*h*]

[9978′] the base will be $2\pi l$; therefore the equation [9930] will give, in this

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[9980i] flexures, using, instead of the axis $O'Z$, the vertical line drawn through the point Z of the base, which is the centre of curvature of that part of the base or circumference A particularly under consideration; so that the vertical plane $AZO'B'$ may contain the plane N, which is supposed to be in equilibrium in this calculation. In like manner, instead of supposing the
[9980k] tube to be limited by a vertical line $B'A$, we may suppose that, below the point A', it is any regular curve whatever, whose tangent is on the continuation of the line BA'. For, the
[9980l] distance to which the corpuscular action extends being wholly insensible, this curvature of the tube will have no effect whatever on the values of q, $q_{,}$, $\varpi_{,}$; so that the equation [9980e] holds good for all cases.

[9980m] The angle $w=180^{\text{d.}}-b$ [9978y], which is used by M. Poisson, is the same as
[9980m′] $180^{\text{d.}}-\varpi$ of La Place's notation [9892]; so that we have $w=180^{\text{d.}}-\varpi$; substituting this in [9980d, e], we get

[9980n]
$$0=\tfrac{1}{2}\mathrm{F}+\tfrac{1}{2}\mathrm{H}.\cos.\varpi;$$

[9980o]
$$\cos.\varpi=\frac{\varpi_{,}-q}{q+q_{,}}=-\frac{\mathrm{F}}{\mathrm{H}};$$

ϖ being, as in [9892], the angle contained between the lower side of the tube and the surface
[9980p] of the fluid, or rather the angle contained by that side of the tube and the tangent to the surface of the fluid, at a sensible distance from it. Finally, it follows from the equation
[9980p′] [9980o], that, as the forces q, $q_{,}$, $\varpi_{,}$, are independent of the curvature of the tube [9980l],
[9980q] *the angle of inclination ϖ of the fluid to the side of the tube must also be independent of that curvature*, as in [9197].

If the matter of the tube has no action on the strata of the fluid $S_{,}$, S, S', adjacent to the side
[9980r] of the tube [9979h], the stratum $S_{,}$ will be in a similar situation to the stratum at the surface of the fluid S' [9978q] in contact with the atmosphere. These two strata, of an insensible
[9980s] thickness, transmit the internal pressure, the one to the surface of the air $B'F'$, and the other to the surface of the tube AA'; and these pressures are destroyed by the actions, in contrary directions, upon these strata, from the action of the caloric in the particles of the tube, fluid,
[9980t] and air. In this case, if we suppose that the thicknesses of the two strata $F'F$, $A'Q'$, are equal to each other, the densities at the corresponding points of the lines $F'F$, $A'Q'$, must be very nearly equal, and then it is evident that the action of S' upon P', which we have represented by $q_{,}$ [9978q], in the direction towards the interior of the mass S', must be equal to that of $S_{,}$ upon the plane N, which, in [9979h], is represented by $\varpi_{,}$ in a vertical direction [9979e], or
[9980u] $-\varpi_{,}$ in a *downward direction towards the interior of the mass* $S_{,}$; so that, in this case, we shall have

[9980v]
$$\varpi_{,}=-q_{,}, \quad \text{or} \quad q-\varpi_{,}=q+q_{,};$$

[9980v′] whence $\mathrm{F}=\mathrm{H}$ [9980c]. Substituting this in [9980o], we obtain

[9980w]
$$\cos.\varpi=\frac{-q-q_{,}}{q+q_{,}}=-1, \quad \text{or} \quad \varpi=180^{\text{d.}},$$

[9980w′] *which corresponds to the surface of a convex hemisphere*, and agrees with what we have

particular case [as in 9985b], [9979]

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already found in [9654a, b]. From this it appears that, *when the tube has no action on the fluid, the angle ϖ will attain its maximum value* $180^{d.}$. [9980x]

If the matter of the tube has the same action on the particles of the fluid as the fluid has on its own particles, the density of the mass $S_{,}$ will be nearly equal to unity, and its action on [9980y] the plane N will be represented by the function $\lambda K - 2q$ [9974d], in the direction of gravity; and by changing its sign, we get its value in a vertical direction $\varpi_{,} = 2q - \lambda K$ [9979h]; [9980z] substituting this in [9980o], we get $\cos.\varpi = \frac{q - \lambda K}{q + q_{,}} = 1 - \left(\frac{\lambda K + q_{,}}{q + q_{,}}\right)$; and by neglecting [9981a] the quantity $\lambda K + q_{,}$, supposing it to be very small in comparison with q, we get $\cos.\varpi = 1$, or $\varpi = 0^{d.}$. Hence it follows that, *when the particles of the tube and of the fluid act upon each other with equal forces, we shall have very nearly* $\varpi = 0$, *which corresponds to a concave spherical surface*, as we have already seen in [9625, &c.] When we have accurately $\varpi = 0^{d.}$, the equation [9980o] gives [9981a′]

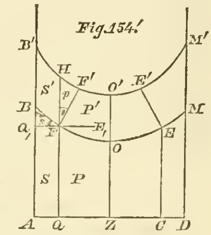
$$\varpi_{,} - q = q + q_{,}, \quad \text{or} \quad \varpi_{,} = 2(q + q_{,}) - q_{,} = \text{H} - q_{,} \quad [9978v]. \qquad [9981b]$$

Hence it appears that, while the angle ϖ varies from $\varpi = 180^{d.}$ to $\varpi = 0^{d.}$, the force $\varpi_{,}$ increases from $\varpi_{,} = -q_{,}$ [9980v] to $\varpi_{,} = \text{H} - q_{,}$ [9981b], the increment being H. Now [9981c] it is found by experiment, that H is a positive quantity [9920w]; therefore, while the density of the fluid near the sides of the tube increases from nothing till it is equal to, or exceeds [9981d] that of the internal mass of the fluid, the *upward* action of the mass $S_{,}$ will increase by the *positive* quantity H, which indicates that the repulsive force of the caloric in the successive [9981e] increments of the mass $S_{,}$ exceeds that of the attractive force of the corpuscular action.

We may finally observe, that, in changing successively the matter of which the tube is [9981f] composed, so that its action upon the fluid may always increase, it will finally become greater than that of the fluid on its own particles, and then the part of the fluid which is almost in [9981g] contact with the sides of the tube, may, in consequence of this strong action of the tube, become more dense than the fluid in the interior of its mass. In this case, a very thin [9981h] stratum of the fluid will be gathered round the sides of the tube, in the manner mentioned by the author in [9220], so as to form an internal tube, whose inner surface has the same density [9981i] as that of the internal mass of the fluid; and it is found by experiment that, when this happens, the angle ϖ is nearly equal to nothing. If a fluid forms in this way a stratum about the sides [9981k] of a tube, the action of the part which is gathered round the tube above the capillary surface, may have some action upon the plane N; but this action may be considered as constituting a [9981l] part of the force $\varpi_{,}$, without altering any of the preceding formulas, or affecting the results which have been obtained from them. [9981m]

The formulas [9976k, l, m] may be used in finding the correction of the computed pressures [9981n] on opposite sides of either one of two vertical and parallel planes dipped into a fluid, of which we have already spoken in [9580o, &c.] For this purpose we shall refer to the annexed [9981o]

[9980] $$2\rho - \rho' = \tfrac{1}{2} g D . lq ;$$ [This is correct even for a fluid whose density varies near the surface, using the approximate values of ρ, ρ' [9920r], namely $\rho_{,}$, $\rho_{,}'$.]



[9981p] figure 154′, which is the same as fig. 154, page 856, with the addition of the horizontal line $E_{,}FQ_{,}$, and the vertical line QFH,
[9981q] drawn through the point F, meeting respectively the surface of the plane AB' in $Q_{,}$, and the surface of the fluid in H; the line
[9981r] FF' being perpendicular to the surface of the fluid at F'. We shall suppose, as in [9977t—w], that the surface BOM is at an
[9981s] insensible distance from the surface $B'O'M'$, but so great, however, as not to have the action of the fluid on the surface BOM
[9981t] extend to the fluid of variable density near the surface $B'O'M'$; so that *the fluid above the surface BOM, at an insensible distance,*
[9981u] *may have the same density unity as in the interior mass of the fluid. In like manner, FQ is at an insensible distance from AB', but the part of the fluid $FQAB'$ which is near to*
[9981v] *FQ, and at an insensible distance from it, has the same density unity as in the interior of the fluid mass, so that the action of the fluid situated on the line FQ does not extend to the*
[9981w] *fluid of variable density near AB'.* For abridgment we shall denote the different sections of the fluid plane or base $ADM'B'$ by the following symbols, which are similar to those in [9977y, &c.];

[9981x] $S = QFB$, $\quad S' = FF'B'B$, $\quad P = MOFQD$, $\quad P' = MOFF'O'M'$,

[9981y] $p = HFF'$, $\quad s = BFQ_{,}$, $\quad S - s = AQFQ_{,}$, $\quad S' + s = FF'B'Q_{,}$.

[9981z] We may suppose that the whole mass of the fluid is contained within the space which is included by surfaces drawn perpendicular to the plane of the base $M'O'B'AD$, and continued infinitely above and below it; and this mass may be separated into portions corresponding to the parts of the base S, S', &c., by similar perpendicular surfaces drawn through the borders of these portions, and continued infinitely above and below the plane of the figure. We shall denote any one of these portions of the fluid by enclosing in a parenthesis
[9982a] the symbol which represents that part of the base; so that, for example, (S) represents the part of the whole mass (M) which corresponds to the portion of the base $S = QFB$. Then, as the whole base $M'O'B'AD$ can be separated into the four portions P, P', $S'+s$, $S-s$, we shall have

[9982b] $$(M) = (P) + (P') + (S'+s) + (S-s).$$

And we shall also for abridgment use the following symbols;

[9982c] $F =$ the whole horizontal action of the mass (M) upon the fluid which is contained in the portion of the plane or base $AB'F'FQ = S + S'$; the thickness of this plane being considered as extremely small, and represented by unity;

[9982c′] $(A).[B] =$ the horizontal action of any portion (A) of the mass (M) upon any portion B of the plane $S + S'$; the attracting mass being included in the parentheses ( ), and the attracted part of the plane in the brackets [ ];

hence the equation [9930] becomes generally [as in 9985c], [9980′]

$$S_{,}=(P).[S']=\text{the action of } (MOFQD) \text{ upon } FF'B'B; \quad [9982d]$$

$$T_{,}=(P').[S']=\text{the action of } (MOFF''O'M') \text{ upon } FF'B'B; \quad [9982e]$$

$$Q_{,}=(P+P'+p).[s]=\text{the action of } (M'O'HQD) \text{ upon } BFQ_{,}; \quad [9982f]$$

$$R_{,}=(p).[S]=\text{the action of } (HFF'') \text{ upon } QFB. \quad [9982g]$$

The horizontal action F of the whole mass (M) upon the plane $S+S'$ may be separated into two portions, $(M).[S-s]$ and $(M).[S'+s]$; hence we have [9982h]

$$F=(M).[S-s]+(M).[S'+s]. \quad [9982h']$$

The first term of the second member of this expression $(M).[S-s]$, represents the horizontal action of the whole mass (M) upon the plane $S-s$, or $AQFQ_{,}$, and this must evidently be equal to the horizontal pressure on the plane $AQ_{,}$. Now we have already [9982i] computed this pressure, in [9580a—n], by means of the first method of considering the capillary action, founded on the equilibrium of the fluid in a canal; and we have found it to be equal to $\frac{1}{2}gk^2$ [9580n], neglecting the pressure on the outside of the plane, which [9982i′] depends on $k_{,}$ [9580m], which is noticed afterwards in [9983t, &c.]. Substituting this value of $(M).[S-s]=\frac{1}{2}gk^2$, in [9982h′], we get

$$F=\tfrac{1}{2}gk^2+(M).[S'+s]; \quad [9982k]$$

and by substituting the value of (M) [9982b], we get

$$F=\tfrac{1}{2}g.k^2+(P).[S'+s]+(P').[S'+s]+(S'+s).[S'+s]+(S-s).[S'+s]. \quad [9982l]$$

Now we have, by separating $[S'+s]$ into its parts $[S']+[s]$, and using $S_{,}$, $T_{,}$, [9982d, e],

$$(P).[S'+s]=(P).[S']+(P).[s]=S_{,}+(P).[s], \quad [9982m]$$

$$(P').[S'+s]=(P').[S']+(P').[s]=T_{,}+(P').[s]. \quad [9982n]$$

The expression $(S'+s).[S'+s]$ [9981y, 9982l], may be computed in the same manner as we have ascertained the value of S in [9979o—x], and we shall obtain a similar result to that in [9979x], namely,

$$(S'+s).[S'+s]=0; \quad [9982o]$$

substituting this, and the values [9982m, n], in [9982l], we get

$$F=\tfrac{1}{2}g.k^2+S_{,}+T_{,}+(P+P').[s]+(S-s).[S'+s]. \quad [9982p]$$

The action $(S-s)$ upon $[p]$, being equal and contrary to that of (p) upon $[S-s]$, gives

$$0=(S-s).[p]+(p).[S-s]; \quad [9982q]$$

subtracting this from the last term of the equation [9982p], we get

$$(S-s).[S'+s]=(S-s).[S'+s-p]-(p).[S-s]=(S-s).[S'+s-p]-(p).[S]+(p).[s] \quad [9982r]$$

$$=(S-s).[S'+s-p]-R_{,}+(p).[s] \quad [9982g]; \quad [9982s]$$

substituting this in [9982p], and putting

$$(P+P').[s]+(p).[s]=Q_{,} \quad [9982f],$$

[9981] $$V = \tfrac{1}{2} lq.c.$$ [This equation is correct whether we consider the fluid as homogeneous, or variable in density near its surface.]

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we get

[9982t] $$F = \tfrac{1}{2} g.k^2 + S_{,} + T_{,} - R_{,} + Q_{,} + (S-s).[S'+s-p].$$

Now the horizontal action of the mass $(S-s)$ upon the plane $[S'+s-p]$, or of
[9982u] $(AQFQ_{,})$ upon $[B'HFQ_{,}]$, vanishes, as is evident from the consideration that the distance to which the corpuscular action extends is insensible, and that the density of the fluid is
[9982v] equal at an insensible distance above and below $FQ_{,}$ in any vertical column; so that the action of the mass is in a vertical direction, and the horizontal action vanishes, as we may easily perceive by proceeding in the same manner as in [9976y, z, &c.]; hence we shall have very nearly

[9982v'] $$(S-s).[S'+s-p] = 0;$$

consequently the expression [9982t] becomes

[9982w] $$F = \tfrac{1}{2} g.k^2 + S_{,} + T_{,} - R_{,} + Q_{,},$$

[9982x] which may be much simplified by computing the values of $S_{,}$, $T_{,}$, $R_{,}$, $Q_{,}$, by means of the formulas [9976k, l, m].

[9982y] In computing the value of $Q_{,}$, or the action of $(M'O'HQD)$ upon $BFQ_{,}$ [9982f], we have, by referring the angles to the horizontal axis $Q_{,}FE_{,}$, and using the symbols [9976h, i],

[9982z] $\mathrm{a}' = E_{,}FQ = -90^{\mathrm{d.}}$, $\mathrm{b}' = E_{,}FH = 90^{\mathrm{d.}}$, $\mathrm{a} = Q_{,}FQ_{,} = 0$, $\mathrm{b} = Q_{,}FB = 90^{\mathrm{d.}} - \varpi$;

substituting these in [9976k], we get

[9983a] $$A = -45^{\mathrm{d.}}, \qquad B = 90^{\mathrm{d.}} - \tfrac{1}{2}\varpi, \qquad A' = -\tfrac{1}{2}\varpi, \qquad B' = 45^{\mathrm{d.}};$$

and, with these values, the expression of Z [9976l] gives the value of $Q_{,}$ under the following form;

[9983b] $$Q_{,} = q.\cos.\varpi.\{\cot.\tfrac{1}{2}\varpi + \mathrm{tang}.\tfrac{1}{2}\varpi\}.$$

This may be reduced, by observing that the sum of the two formulas [41', 42'], Int., gives

[9983c] $$\cot.\tfrac{1}{2}\varpi + \mathrm{tang}.\tfrac{1}{2}\varpi = \frac{2}{\sin.\varpi};$$

hence [9983b] becomes

[9983d] $$Q_{,} = q.\cos.\varpi.\frac{2}{\sin.\varpi} = 2q.\cot.\varpi.$$

To determine $R_{,}$, or the action of (HFF') upon QFB [9982g], we have, as in [9976h—k],

[9983e] $\mathrm{a}' = E_{,}FF' = \varpi$, $\mathrm{b}' = E_{,}FH = 90^{\mathrm{d.}}$, $\mathrm{a} = QFQ_{,} = -90^{\mathrm{d.}}$, $\mathrm{b} = BFQ_{,} = 90^{\mathrm{d.}} - \varpi$,

[9983f] $$A = -(45^{\mathrm{d.}} - \tfrac{1}{2}\varpi), \qquad B = 90^{\mathrm{d.}} - \tfrac{1}{2}\varpi, \qquad A' = 45^{\mathrm{d.}}, \qquad B' = 0;$$

substituting these in [9976l], we get the value of $R_{,}$, namely,

[9983g] $$R_{,} = q.\{\mathrm{tang}.(45^{\mathrm{d.}} - \tfrac{1}{2}\varpi) + \cos.\varpi.(\cot.\tfrac{1}{2}\varpi - 1)\}.$$

From this it follows, that, in all prismatic tubes which have the [9982]

In finding the value of $S_{,}$, or the action of $(MOFQD)$ upon $FF'B'B$ [9982d], we have, as in [9976h—k],

$$\mathrm{a}' = E_{,}FQ = -90^{\mathrm{d.}}, \qquad \mathrm{b}' = E_{,}FO = -90^{\mathrm{d.}} + \varpi,$$ [9983h]

$$\mathrm{a} = BFQ_{,} = 90^{\mathrm{d.}} - \varpi, \qquad \mathrm{b} = Q_{,}FF' = 180^{\mathrm{d.}} - \varpi,$$ [9983h]

$$A = -\tfrac{1}{2}\varpi, \qquad B = 45^{\mathrm{d.}}, \qquad A' = 45^{\mathrm{d.}} - \tfrac{1}{2}\varpi, \qquad B' = 0.$$ [9983i]

Substituting these in [9976m], we get the corresponding value of $S_{,}$ under the form

$$S_{,} = q.\{\mathrm{tang.}\tfrac{1}{2}\varpi + \mathrm{tang.}(45^{\mathrm{d.}} - \tfrac{1}{2}\varpi) - \cos.\varpi\}.$$ [9983k]

Substituting the values of $Q_{,}$, $R_{,}$, $S_{,}$ [9983d, g, k], in the first member of [9983l], and making successive reductions, using also [9983c] with [41], Int., we get

$$Q_{,} - R_{,} + S_{,} = q.\left\{\begin{array}{l} +2\cot.\varpi - \mathrm{tang.}(45^{\mathrm{d.}} - \tfrac{1}{2}\varpi) - \cos.\varpi.(\cot.\tfrac{1}{2}\varpi - 1) \\ +\mathrm{tang.}\tfrac{1}{2}\varpi + \mathrm{tang.}(45^{\mathrm{d.}} - \tfrac{1}{2}\varpi) - \cos.\varpi \end{array}\right\}$$ [9983l]

$$= q.\{2\cot.\varpi - \cos.\varpi.\cot.\tfrac{1}{2}\varpi + \mathrm{tang.}\tfrac{1}{2}\varpi\}$$

$$= q.\left\{\cos.\varpi.\left(\frac{2}{\sin.\varpi} - \cot.\tfrac{1}{2}\varpi\right) + \mathrm{tang.}\tfrac{1}{2}\varpi\right\}$$

$$= q.\{\cos.\varpi.\mathrm{tang.}\tfrac{1}{2}\varpi + \mathrm{tang.}\tfrac{1}{2}\varpi\} = q.\mathrm{tang.}\tfrac{1}{2}\varpi.\{1 + \cos.\varpi\}$$

$$= q.\sin.\varpi.$$ [9983m]

The direct action of P' upon S' being represented by $q_{,}$ [9978q], its horizontal action may be obtained in like manner as in [9978r], and will be represented, as in [9982e], by

$$T_{,} = q_{,}.\sin.\varpi.$$ [9983n]

Adding this to the expression [9983m], and using the value of H [9978v], we get

$$Q_{,} - R_{,} + S_{,} + T_{,} = (q + q_{,}).\sin.\varpi = \tfrac{1}{2}\mathrm{H}.\sin.\varpi;$$ [9983o]

substituting this in [9982w], we finally obtain

$$F = \tfrac{1}{2}g.k^2 + \tfrac{1}{2}\mathrm{H}.\sin.\varpi.$$ [9983p]

If we use the value of $\mathrm{H} = g.\alpha^2$ [9323p], this expression of F will become [9983q]

$$F = \tfrac{1}{2}g.k^2 + \tfrac{1}{2}g.\alpha^2.\sin.\varpi,$$ [9983r]

which represents the horizontal action of the whole mass (M) of the fluid upon that in the plane $F'B'AQ$, which is equal to the pressure of the internal fluid upon the inner side of [9983s] the plane AB'. In like manner we may represent the action of the external fluid upon the opposite side of the same plane by

$$\tfrac{1}{2}g.k_{,}^2 + \tfrac{1}{2}g.\alpha^2.\sin.\varpi_{,},$$ [9983t]

$k_{,}$ and $\varpi_{,}$ being, for the external fluid, the quantities corresponding respectively to k, ϖ, for the internal fluid. [9983u] The difference of the two expressions [9983r, t] represents the difference of the pressures on the opposite sides of the solid plane AB', which will therefore be [9983v] represented, as in [9580q], by

$$\tfrac{1}{2}g.(k^2 - k_{,}^2) + \tfrac{1}{2}g.\alpha^2.(\sin.\varpi - \sin.\varpi_{,}) = [\text{difference of the pressures on opposite sides of a plane } AB'],$$ [9983w]

[9982′] *same internal base, the hollow cylinder is that in which the mass*

[9983w′] instead of being equal to $\frac{1}{2}g.(k^2-k_{,}^2)$, as is supposed by La Place in [9580, 9580c, &c.]; so that the correction of his computed value is

[9983x] $$\tfrac{1}{2}g.\alpha^2.(\sin.\varpi-\sin.\varpi_{,}).$$ [Correction of La Place's computation of the difference of pressures on opposite sides of a plane.]

[9983y] The extreme value of z is put equal to q in [9416r], and this is changed into k in [9580i];
[9983z] now if we make the same change of q into k, in the formula [9416s], we shall get the
[9984a] following expression of k^2;

[9984b] $$k^2=h^2+\alpha^2.(1-\sin.\varpi);$$

where $k=GN$, fig. 127, page 772, $h=PO$, and ϖ is equal to the angle formed at N by
[9984c] the curve NO, and the vertical line NG. In like manner, by putting a mark below the letters h, k, ϖ, to obtain the similar quantities for the external curve $ZZ'V$, we have $h_{,}=0$, $k_{,}=GZ$, and $\varpi_{,}$ equal to the angle formed at Z by the curve ZZ' and the vertical line ZG; and in this case the equation corresponding to [9984b] becomes

[9984d] $$k_{,}^2=\alpha^2.(1-\sin.\varpi_{,}).$$

Subtracting [9984d] from [9984b], we get

[9984e] $$k^2-k_{,}^2=h^2-\alpha^2.(\sin.\varpi-\sin.\varpi_{,});$$

and by substituting this in the expression [9983w], and neglecting the quantities which destroy each other, it is reduced to the following simple form;

[9984f] $$\tfrac{1}{2}g.h^2=\text{the difference of pressures on opposite sides of a solid plane } AB';$$

[9984g] where h represents the elevation of the lowest point of the surface of the fluid between the planes above the level of the fluid in the vase into which the planes are dipped.

When the distance of the planes is infinite, this elevation h will vanish, and then the difference
[9984h] of pressures on opposite sides of the plane AB' [9984f] will become equal to nothing, whatever difference there may be in the values of the angles ϖ, $\varpi_{,}$. This will not, however, be the case if we suppose the difference of the pressures to be the same as La Place estimated it. For if we put $h=0$ in [9984e], we shall get

[9984i] $$k^2-k_{,}^2=\alpha^2.(\sin.\varpi_{,}-\sin.\varpi);$$

and by substituting this value of $k^2-k_{,}^2$ in La Place's estimate of the difference of the pressures $\frac{1}{2}g.(k^2-k_{,}^2)$ [9983w′], it becomes

[9984k] $$\tfrac{1}{2}g.\alpha^2.(\sin.\varpi_{,}-\sin.\varpi),$$

which does not vanish except $\varpi_{,}=\varpi$. This circumstance was brought forward by Dr. Young, as an important and almost fatal objection to the theory of La Place; because
[9984l] this difference in the pressures would produce a perpetual horizontal motion in a broad parallelopipedon floating on the surface of a fluid of infinite extent, supposing the opposite
[9984m] parallel surfaces of the floating plane to be more or less moistened, or to be of different substances, so that the angles $\varpi_{,}$, ϖ, corresponding to these opposite surfaces, may differ from each other. This objection fails when we apply the correction [9983x], because we have

*of elevated fluid is the least possible, since it has the least circumference.** [9982″]

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seen in [9984*f*], that the corrected differences of the pressures on the opposite sides of the floating solid vanish when $h=0$, whatever be the values of the angles ϖ, $\varpi_{,}$. [9984*n*]

After this long digression on the subject of the capillary action, we shall resume the general commentary, by the insertion of the following note, which is referred to in [9978, &c.]. [9984*o*]

4293. The radius of the base of the cylinder of the fluid being l [9977′], its circumference will be $c=2\pi l$ [9978′], and the area of the base πl^2. Multiplying this by the height of the cylinder q, we get *its volume* $V=\pi l^2 q$ [9924, 9978]. Substituting these values of c, V, in [9930], we get $gD.\pi l^2 q=(2\rho-\rho').2\pi l$; then, dividing by $2\pi l$, we get [9980], which is correct even when we notice the change of density near the surface of the fluid; observing, however, to change the values of ρ, ρ', into $\rho_{,}, \rho'_{,}$, respectively, as in [9920*p*—*r*]. Again, substituting the value of $2\rho-\rho'$ [9980] in [9930], we get $gDV=\frac{1}{2}gD.lqc$; dividing this by gD, we obtain $V=\frac{1}{2}lqc$ [9981], corresponding to prisms whose bases are of any form; so that, if we determine the values of l, q, corresponding to a cylindrical tube, we may ascertain the volume V of the elevated fluid in a prism of the same matter and of any form, by multiplying *the given factor* $\frac{1}{2}lq$ by the circumference c of the proposed prism; or, in other words, the elevated mass gDV *is proportional to the circumference of the base* **c**. We may remark, as in [9846*a*], that these prisms ought not to have any abrupt angular corners, which might produce some irregularity in the attraction. We may also deduce [9981] directly from [9898], by substituting in this last expression, divided by V, $c=2\pi l$, $V=\pi l^2 q$ [9985*a*], corresponding to a cylindrical tube, which gives generally $gD=\frac{H.\cos.\varpi}{lq}$, and by substituting this value of gD in [9898], and then multiplying by $\frac{lq}{H.\cos.\varpi}$, we get for a prism of any form $V=\frac{1}{2}lq.c$, as in [9981]. This, being deduced from [9898], which is correct when the density near the surfaces is variable, must also be true when the fluid is variable near its surface and near the sides of the tube. [9985*a*] [9985*b*] [9985*b*′] [9985*c*] [9985*d*] [9985*e*] [9985*f*] [9985*g*] [9985*h*] [9985*i*]

* (4294) The elevated mass of the fluid is proportional to the circumference of the base c [9985*e*], and if the area of the base be a given quantity, it will follow that the elevated mass is the least when this circumference is a minimum, and this corresponds to a circular base, or to a hollow cylinder, as in [9982′]; it being a well known theorem, that a circle encloses a greater area than any other figure of equal circumference. This theorem may be demonstrated by the method in [36*k*], observing that, if x, y, be the rectangular coordinates of a plane curve, whose arc is s, we shall have $ds=\sqrt{dx^2+dy^2}$, for the differential of the arc, and ydx for the differential of the area; therefore the whole circumference c of the curve will be represented by $\int ds=c$, and its whole area by $\int ydx$. The variation of the first of these expressions is evidently $\delta.\int ds=0$, and that of the second, when a maximum, [9986*a*] [9986*b*] [9986*c*] [9986*d*] [9986*e*] [9986*f*]

[9983] *If b be the base of a prismatic tube, and h the mean height of all the parts of the fluid included within it above the level surface, we shall have $V=hb$;** therefore

[9984] $$h=\frac{lqc}{2b}.$$ [This is correct even when the density is supposed to be variable near the surfaces.]

[9985] We may here observe that, when the fluid is depressed instead of being elevated, q, V, and h, will be negative;† [which is also correct when the fluid is considered as variable at an insensible distance from the surface].

Elevation of fluids in prismatic tubes.

[9986] *The preceding formulas will generally hold good even when the curvature of the circumference of the interior part of the base is discontinuous; as, for example, in the case where this circumference is a rectilinear polygon.* For there could be no error except near the angles of these polygons, and within the limit of the sphere of sensible activity of the particles of the tube; but, this extent being supposed to be imperceptible, the whole error must be entirely insensible.‡ *We*

[9986′] *may therefore apply these formulas to bases of any figure whatever.*

[9986f′] is $\delta.\int ydx=0$, as in [36k]; and by supposing $d\delta x=0$, as in [36g], it gives $\int \delta y.dx=0$. Multiplying this last variation by the indeterminate constant quantity a, and adding it to the

[9986g] first, we get, in the case of a maximum, $\delta.\int ds+a.\int\delta y.dx=0$. Now as we have $d\delta x=0$,

[9986h] we shall get $\delta.\int ds=\delta.\int\sqrt{dx^2+dy^2}=\int\frac{dy.d\delta y}{\sqrt{dx^2+dy^2}}$; so that if we put, for brevity,

[9986i] $p=\frac{dy}{\sqrt{dx^2+dy^2}}=\frac{dy}{ds}$, we shall have $\delta.\int ds=\int p.d\delta y$. Integrating this by parts, we get $\delta.\int ds=p\delta y-\int\delta y.dp$, as in [36h]; hence the equation [9986g] becomes

[9986k] $p\delta y+\int\delta y.(adx-dp)=0$. The part $p\delta y$ without the sign $\int$, may be neglected, because the beginning and end of the re-entering curve which encloses this base, is at the same point,

[9986l] which may be considered as a given point, and then we shall have $\int\delta y.(adx-dp)=0$, which must hold good whatever be the value of δy; therefore we must have $adx-dp=0$ for

[9986m] the equation of the required curve. Its integral is $ax-p=b$, b being a constant quantity;

[9986n] and by substituting p [9986i], we get $ax-\frac{dy}{ds}=b$, or $x-\frac{b}{a}=\frac{1}{a}.\frac{dy}{ds}$, which is, as in

[9986o] [49, &c.] Int., the equation of a circular arc, whose radius is $\frac{1}{a}$, sine $=y$, and cosine $=x-\frac{b}{a}$.

* (4295) The area of the base being b, and the height h, the volume V must evidently

[9987a] be $V=hb$, as in [9983]; substituting this in [9981], and then dividing by b, we get [9984].

[9988a] † (4296) This is evident by reasoning as in [9369, &c.—9379].

‡ (4297) The circumstance of the irregular variations in the density near these

When the bases are similar, they are proportional to the squares of their homologous sides; and as their circumferences c are proportional to their sides, *the mean heights* h *will be inversely proportional to the homologous sides.** [9987]

If the circumferences of the bases are polygons circumscribing a circle, they will be equal to the product of the circumference, by half the radius of the [9988] inscribed circle; therefore the heights will be inversely as the radius; and by putting the radius equal to r, we shall have†

$$h=\frac{lq}{r};$$ [This is true even when the density of the fluid is considered as variable near its surface.] [9989]

whence it follows that, in all right prismatic tubes whose bases are polygons circumscribed about the same circle, the fluid will rise to the same mean height. [9990]

Supposing two bases to be equal, but that one of them is a square, and the other [9991] *an equilateral triangle, the values of* h *will be to each other as* 2 *to* $3^{\frac{3}{4}}$, *or very* [9992] *nearly as* 7 *to* 8.‡

Gellert has made some experiments upon the elevation of water in glass tubes, of prismatic forms, with rectangular or triangular bases (Commentarii Acad. Petrop. Tom. xii.). They confirm the law that the elevations are

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angular points is not here expressly noticed by the author; but it is probable that this would not produce any sensible effect, except the angular points were quite abrupt [9988b] and very near to each other.

* (4298) If we put s for one of these homologous sides, we shall have $c \propto s$, $b \propto s^2$; hence $\frac{c}{2b} \propto \frac{1}{s}$; and as lq *is a given quantity* [9985d], we shall have, from [9984], $h \propto \frac{1}{s}$. [9989a]

† (4299) In the case here treated of, we have $2b=rc$; substituting this in [9984], we get the same value of h as in [9989]. All the quantities l, q, r [9977′, 9988], are [9990a] constant in the hypothesis mentioned in [9990]; therefore h must also be of the same value for all the polygons.

‡ (4300) The area of a square whose side is x, is x^2, and its circumference $4x$. An [9991a] equilateral triangle whose side is y, will have for the perpendicular let fall from the vertex to the base, $y\sqrt{\frac{3}{4}}$; therefore the area will be $\frac{1}{2}y \times y\sqrt{\frac{3}{4}}$, or $\frac{1}{4}y^2\sqrt{3}$. If we suppose this to be [9991b] equal to the area of the square x^2 [9991a], we shall have $x^2=\frac{1}{4}y^2.\sqrt{3}$, or $y=2x.3^{-\frac{1}{4}}$, [9991c] and the circumference of the triangle $3y$ becomes $3y=2x.3^{\frac{3}{4}}$. Therefore the circumference of the square $4x$ is to that of the triangle as $4x$ to $2x.3^{\frac{3}{4}}$, or as 2 to $3^{\frac{3}{4}}$, as [9991d] in [9992]; and as the bases are equal, this will represent the ratio of the values of h.

[9993] inversely proportional to the homologous sides of the similar bases. He also concluded from these experiments, that, in rectangular and triangular prisms whose bases are equal, the elevations of the fluid are the same. But he admits that he is not so sure of this last law, as he is of that of the heights being inversely proportional to the homologous lines of the similar bases. In fact, we see that there is a difference of an eighth part between the elevations of the fluid in a
[9994] rectangular and a triangular prism, whose bases are equal, the one being a square, the other an equilateral triangle. The experiments given by Gellert do not furnish sufficient data for an accurate comparison of the results with the preceding formulas.

If the base of the prism be a rectangular parallelogram, whose greatest side
[9995] is a, and least side l, we shall have $b=al$, $c=2a+2l$; therefore*

[9996] $$h=\frac{(2a+2l).lq}{2al}=\left(1+\frac{l}{a}\right).q.$$

[This is accurate even when we suppose the density to be variable near the surfaces of the fluid.]

If a be very large in comparison with l, we shall have $h=q$; now this corresponds very nearly with the case of two parallel planes which are distant
[9997] from each other by the quantity† l. *Therefore the mean height of the fluid*
[9998] *elevated between these planes, is very nearly the same as in a cylindrical tube whose radius is l.* This agrees with the result of the former method in [9410—9412].

[9998′] *The height of the lowest point of the surface of the fluid which rises in a vertical, cylindrical, and very narrow tube, is not exactly in the inverse ratio of the diameter of the tube. If the fluid completely moisten the sides of the tube,*
[9999] *as alcohol and water do glass, we must add a sixth part of the diameter to that height, to obtain a quantity which is inversely as that diameter. For if we put*
[10000] *this height equal to q, and l for the semi-diameter of the tube,* the volume of the elevated fluid which is situated below the lowest point of its surface, will be
[10001] $\pi l^2.q$ [9985a]. To obtain the whole volume of the column, we must add to the

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[9996a] * (4301) Substituting the values of b, c [9995], in [9984], we get [9996].

† (4302) In a cylindric tube whose radius is l, the elevation of the fluid by the capillary
[9998a] action is supposed to be q [9977′]. In the rectangular prism, the sides of whose base are a, l [9995], the mean height is h [9983]; and when a very much exceeds l, we shall have, from
[9998b] [9996], $h=q$ nearly; now this case corresponds very nearly with that of two indefinite parallel planes whose distance is l; hence we find that the mean ascent h of a fluid between two parallel
[9998c] planes whose distance is l, is equal to the ascent q in a capillary tube whose radius is l, being the same result as is found in [9410] by the former method.

preceding quantity the meniscus which is cut off by the horizontal plane drawn through the lowest point of the surface; now this surface is very nearly hemispherical, as in [9631]; therefore the volume of the meniscus is* $\frac{1}{3}\pi l^3$; [10002]
consequently the whole volume of the column is $\pi l^2.(q+\frac{1}{3}l)$. But we have already shown that this volume must be proportional to the circumference $2\pi l$ [10003]
of the base; therefore $l.(q+\frac{1}{3}l)$ is a constant quantity in the different capillary tubes; so that, *to obtain the quantities which are inversely proportional to the diameters of the capillary tubes, we must add to the height of the lowest* [10004]
point of the surface of the fluid, the third part of the radius of the tube, or the sixth part of the diameter.

Suppose now that we have a curved glass tube, in which the shortest branch is of a capillary form, but the longest branch very large, and forming a vase of great capacity. If we pour alcohol into the vase, the fluid will rise in the [10005]
capillary branch above its level in the vase, and, by continuing to pour in the alcohol, it will rise more and more in the capillary branch;† but in the state of equilibrium of the fluid, the difference between its level in the two branches will [10006]
be always the same, until the fluid has risen to the top of the capillary branch. *If we still continue to pour more alcohol into the vase, the surface of the fluid in* [10006']
the capillary branch will become less and less concave [10012g, &c.], *and when the surface in the vase is upon a level with the top of the capillary branch, the surface in that branch will become horizontal* [10012h]. [10007]

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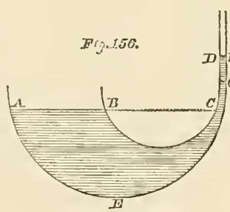
* (4303) If we put $\varpi=0$, in the equation $u=b.\cos.\varpi$ [9336m], it becomes $u=b$; [10002a]
hence the expression of the hemispherical annulus [9336o] becomes equal to $\frac{1}{3}\pi b^3$; and by changing the radius b [9336k] into l [10000], it becomes $\frac{1}{3}\pi l^3$, as in [10002]; whence we get the correction of q [10003, &c.], agreeing with [9372x, &c.]. The same result is [10002b]
obtained in another way, in [10296i].

† (4304) We have drawn figure 156 to represent this case, where the fluid is alcohol, water, or any other liquid which completely moistens the tube. Then the surface of the capillary part DF is concave, and it is elevated by the quantity CD above the level ABC of the fluid in the vase ABE. This figure does not require any particular explanation; we may, however, observe that the author supposes, in [10006], that the capillary part of the tube FG has the same diameter throughout that part of it where the fluid rises as it is poured into the vase. In the case of mercury, the surface DF is convex, and it falls below the level AB. [10005a] [10005b] [10005c] [10005d]

Fig. 156.

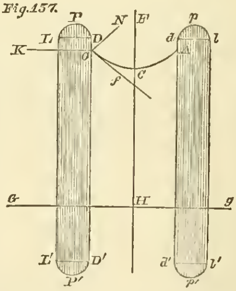


We have observed, in [9630—9631], that, if the action of the glass upon a
[10008] fluid exceeds that of the fluid upon its own particles, a stratum of the fluid will adhere to the sides of the glass, and thus form with those sides another body, whose action upon the fluid is the same as that of the fluid upon its own particles; *we may therefore, as it regards those fluids which exactly moisten the*
[10009] *glass, suppose the action of the glass upon the fluid to be equal to the action of the fluid upon its own particles.* Thus, in the preceding case [10007], the alcohol is in the same state as it would be in, upon the supposition that an indefinite mass of this fluid, in equilibrium in a vessel, were in part congealed, so as to form a capillary tube, having a communication with the other part, which still remains in its fluid state. It is evident that this supposition does not
[10010] affect the equilibrium; and therefore the surface of the fluid in the capillary tube will remain horizontal, as it was before. It is not, therefore, accurate to say,
[10011] generally, that the surface of a fluid cuts always under the same angle the sides of the vessel which contains it; this is not true when the fluid has risen to the
[10012] extremities of these sides; for it is evident that the action of the sides of the tube upon the fluid is not then the same.*

Inaccurate remarks of the author relative to a fluid at the top of a tube.

[10012a] * (4305) The remarks of La Place relative to the action of a tube upon a fluid at its upper extremity require some modification, as has been well observed by M. Poisson, in his *NouvelleThéorie, &c.*, and we shall here give the substance of his method of considering the action at these points. For this purpose we shall suppose the tube, fig. 157, to be of a cylindrical form, and generated by the revolution of the figure
[10012b] $D'DPLL'P'$ about the axis FCH, so that a section of the tube, by a plane passing through the axis FCH, may form the two equal and similar figures $D'DPLL'P'$, $d'dpll'p'$, composed of the lines LL', DD', dd', ll', parallel to CH, and of the regular curves LPD, $L'P'D'$, lpd, $l'p'd'$.
[10012c] Moreover OCA is a section of the concave surface of the fluid, whose lowest point C is elevated, by the quantity CH, above the level GHg of the fluid in the vase. The line Of is the tangent to the surface of the fluid, at the point O, this point being just beyond the sphere of activity of the tube
[10012d] upon the fluid; ON is perpendicular to Of, and is drawn outside of the fluid; OK is perpendicular to the inner line or surface OD' of the tube, and is drawn in an outward direction, or towards the external surface of the tube. Then, according to the notation in [9892], we have the angle $D'Of=\varpi$; and as $NOf=90^{d.}$,
[10012e] we get $DON=90^{d.}-\varpi$, $KON=180^{d.}-\varpi=DOf$; so that if, for brevity, we put
[10012e'] $\varpi_{,}=180^{d.}-\varpi$, we shall have the angle $KON=\varpi_{,}=180^{d.}-\varpi$; and this angle is constant for tubes of the same matter, acting upon the same fluid, as is observed in [9894, 9980o—q];

Fig. 157.



If we continue still to pour water into the before-mentioned tube, this fluid will form, at the extremity of the capillary branch, an external drop, which will become [10013]
more and more convex, until it is hemispherical [10012l]. At this limit, the

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and we may here incidentally remark, that this angle $\varpi_{,}$ is denoted by M. Poisson by w, in [10012f]
the work above mentioned. Now if we continue to immerse the tube, more and more, in the fluid, the point O will approach towards D; and until it has attained the point D, the figure of the surface OCA of the fluid will not change, nor the length of the ordinate CH vary. After passing the point D, the normal OK will cease to be horizontal, and will become inclined to the horizon by an angle i [9359e]. Then the concavity of the surface [10012g]
OCA will decrease, as the angle i increases, and the point O ascends towards the summit P of the tube, or to the point where the tangent to the surface DPL becomes horizontal. [10012h]
If the fluid be water or alcohol, and the tube of glass, we shall have $\varpi=0$ [9359k], or $\varpi_{,}=180^{d.}$ [10012e']; and then the line ON will be on the continuation of the line KO. [10012i]
In this case, when the point O has ascended to the summit P of the tube, the angle i will become equal to $90^{d.}$, the line KON will be vertical, and the surface OCA will become [10012k]
horizontal [9359o]; as the point O proceeds from P towards L, and the angle i still increases, the surface of the fluid will become convex, and finally hemispherical at L [9359q]; and if we still continue to immerse the tube into the vase, the fluid will run over at the top of the tube. [10012l]
What we have here stated as occurring when the top of the tube DPL is a regular curve, must take place, though in a somewhat irregular manner, when the top of the tube is terminated by the horizontal line DL, having a sharp angular point at D. [10012m]
For, in fact, this point is somewhat rounded, and though it may be for a very small extent, yet this small distance is extremely great in comparison with the radius of activity of the corpuscular [10012n]
attraction, which is wholly insensible; so that even this small curvature is quite sufficient to render the preceding results applicable to the case of a tube which is terminated at the top by the plane surface DL. [10012o]
For when the point O has arrived at the upper part D of the cylindrical surface of the tube, the line KON will turn about the corner at D; the normal OK to the surface of the tube will pass gradually from a horizontal to a vertical direction, while the angle KON remains invariable. [10012p]
During this change, the point O will move very little, but the point C will alter considerably; and the elevation CH above the external level, as well as the form of the surface OCA, will depend, at any moment, on the direction of the [10012q]
line OK. While these variations are experienced in the value of i, the point O will vary so little that we cannot determine the value of this angle at any moment by actual [10012r]
measurement, but it may be deduced from the formula [9350]. For the extreme value of z, which occurs in that formula, represents the difference in the elevations of the points D, C, [10012s]
above the horizontal line, which can be measured; we can also measure the radius of the tube l from the axis FH to the extreme point O where the water has arrived. With these values of z, l, we can obtain from [9350] the corresponding value of θ', and then from [9359g] we have [10012t]
$i=90^{d.}-\varpi-\theta'$. If we suppose, as in [10012i], that $\varpi=0$, this will become $i=90^{d.}-\theta'$. [10012u]
With these values of θ', l, we may obtain from [9351] the value of $\frac{1}{b}$; and finally, from

fluid will be as much elevated in the vase, above the top of the capillary branch, as it was depressed below its level in that branch, before it had arrived at the
[10014] top. For the pressure depending upon the convexity of the drop in the first case, is equal to the suction arising from the concavity of the surface in the second case. Lastly, *a little more alcohol being added to that in the vase will*
[10015] *cause the drop to disappear, since, by lengthening it, it must burst in those points of the surface where the radius of curvature increases by this extension.*

Similar results are obtained when we hold a column of alcohol vertically suspended in a glass capillary tube. This fluid forms, at the lower end of the
[10016] *tube, a drop which becomes more and more convex, as the length of the column is increased; and when the drop is of a hemispherical form, the length of the column is equal to double the elevation of the fluid in the tube, when its lower end is*

[10012v] [9358 or 9360], we can obtain the value of q, or the elevation of the point C of the curve above the level of the fluid in the vase GHg.

The influence of the angle i upon the elevation of the centre of the capillary surface, furnishes also the explanation of the different values of such elevations deduced from
[10012w] experiments with the same fluid and the same vertical cylindric tube, when it has not been completely moistened by the fluid. For, whatever be the degree of polish of the interior surface of the tube, there will be some sinuosities whose elevations are always incomparably
[10012x] greater than the radius of activity of the corpuscular attraction, and whose normals, instead of being horizontal, are inclined to the horizon by an angle which is denoted by i in [9359f, &c.]. Then the angle θ', which the tangent to the extreme side of the surface of the fluid forms
[10012y] with the horizon, is represented by $\theta'=90^{d.}-\varpi-i$ [9359g], instead of $\theta'=90^{d.}-\varpi$, which would be its value if there were no sinuosities, or $i=0$. Therefore, instead of putting θ' equal to the constant angle $90^{d.}-\varpi$ [9359c], in the expression of the elevation q of the
[10012z] fluid [9358 or 9360], we must put θ' equal to the angle $90^{d.}-\varpi-i$, which varies with the angle i; and this variation produces a corresponding change in the elevation q of the fluid in the tube, which, instead of being represented, as in [9360], by

[10012z′]
$$q=\frac{H}{gl}.\sin.\theta'=\frac{H}{gl}.\cos.\varpi,$$

must be expressed by

[10013a]
$$q=\frac{H}{gl}.\sin.(90^{d.}-\varpi-i)=\frac{H}{gl}.\cos.(\varpi+i);$$

and from this we evidently see the great changes which must arise in the elevation of the fluid in consequence of the sinuosities depending on the angle i.

[10013b] The remarks we have here made relative to the curvature of a fluid at the top of a tube, may be applied, without any modification, to the case treated of in [10017, &c.], where a column of water is suspended in a capillary tube, and the weight of the column is so great
[10013c] that the water spreads over the angular point D' at the bottom of the tube $D'P'L'$.

dipped into a vessel filled with the same fluid. If we increase the length of the* [10017]

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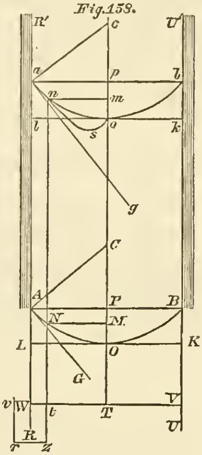
* (4306) To obtain the differential equations of the upper and lower surfaces of a fluid in a capillary tube, we shall refer to the annexed figure 158; supposing the tube to be a cylinder $R'ABU'$, whose vertical axis is Tc, sides AR', BU', upper surface of the fluid aob, lower surface of the fluid AOB. Then, taking any point T of the axis for the origin of the coordinates, and drawing the horizontal line TtW for the axis of u, and the vertical line Tc for the axis of z; also the line tNn, parallel to the axis of z, cutting the surfaces in N, n, we shall put [10016a] [10016b] [10016c]

Fig. 158.



$$TO=h, \quad Oo=\varepsilon, \quad To=h+\varepsilon, \quad Tt=u, \quad tn=z,$$ [10016d]

$$tN=\mathrm{z}, \quad mo=z-h-\varepsilon, \quad MO=\mathrm{z}-h;$$ [10016d']

R, R', the greatest and the least radii of curvature at the point n; [10016e]

$b=b'$, the corresponding radii of curvature at the lowest point o of the surface aob; [10016f]

R, R', the greatest and the least radii of curvature at the point N; [10016g]

$\mathrm{b}=\mathrm{b}'$, the corresponding radii of curvature at the lowest point O of the surface AOB. [10016h]

Then, if we suppose a canal nso to be drawn, similar to NSO, in fig. 116, page 713, we may find the equation of the upper surface aob in the same manner as in [9309, &c.], the resulting equation in [9315] being [10016i]

$$K-\tfrac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)+g\times mo=K-\tfrac{1}{2}H.\left(\frac{1}{b}+\frac{1}{b'}\right). \quad \text{[Upper concave surface.]}$$ [10016k]

Substituting $mo=z-h-\varepsilon$ [10016d'], $g=H\alpha$ [9328], dividing by $-\frac{1}{2}H$, and reducing, we get [10016l]

$$\frac{1}{R}+\frac{1}{R'}=\frac{1}{b}+\frac{1}{b'}+2\alpha.(z-h-\varepsilon);$$ [10016l']

and if, for brevity, we put

$$\frac{1}{b}+\frac{1}{b'}=2\alpha.(h+\varepsilon-c),$$ [10016m]

the preceding equation will become

$$\frac{1}{R}+\frac{1}{R'}=2\alpha.(z-c).$$ [10016n]

Substituting the values of $\frac{1}{R}$, $\frac{1}{R'}$ [9326, 9326'], then multiplying by $\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$, we get the following differential equation of the upper surface of the fluid aob;

$$\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)=2\alpha.(z-c).\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}; \quad \text{[Upper concave surface.]}$$ [10016o]

being the same as in [9324], changing z into $z-c-\frac{2}{2\alpha b}$.

In like manner we may obtain the equation of the convex surface at the bottom of the tube,

[10017′] *column, the drop will burst, and spread over the lower base of the tube, where it*

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which we shall suppose to be represented by AOB, in figure 158, considering the points A, B, as the lowest parts of the tube, and supposing the inner lines of the tube $R'A$, $U'B$, to be continued so as to meet the line WTV in the points W and V. In this case, the fluid being situated *above* the surface AOB, the corpuscular action at any point N of the surface

[10016p] [9315b], will change from $K-\frac{1}{2}H.\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}\right)$ to $K+\frac{1}{2}H.\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}\right)$, as we have seen

[10016q] in [9301a, b]; and the corpuscular action at O will be $K+\frac{1}{2}H.\left(\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}\right)$. Then, if an infinitely small canal NMO, similar to nso, be drawn, with a vertical branch MO, and a

[10016r] horizontal branch NM, the capillary action at N will be $K+\frac{1}{2}H.\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}\right)$, and this is continued along the horizontal branch of the canal NM to M; and in descending in the branch MO, the pressure is increased at O by the weight of the column MO, which is $g\times MO=g.(\mathrm{z}-h)$ [10016d']; so that the action at O will be

[10016s] $$K+\tfrac{1}{2}H.\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}\right)+g.(\mathrm{z}-h);$$

and this must be equal to the expression of that action in [10016q]; hence we have

[10016t] $$K+\tfrac{1}{2}H.\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}\right)+g.(\mathrm{z}-h)=K+\tfrac{1}{2}H.\left(\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}\right);$$

so that, if we substitute $g=H\alpha$ [10016l], and then divide by $\frac{1}{2}H$, we shall get the following equation of the surface AOB;

[10016u] $$\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}=\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}-2\alpha.(\mathrm{z}-h).$$

We may eliminate $\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}$ from this equation, by observing that the corpuscular action at o,

[10016v] at the summit of the vertical canal Oo, is $K-\frac{1}{2}H.\left(\frac{1}{b}+\frac{1}{b'}\right)$; and by adding to this the weight of the column Oo, which is represented by $g\times Oo=g.\varepsilon$ [10016d], we get the

[10016w] pressure at O, equal to $K-\frac{1}{2}H.\left(\frac{1}{b}+\frac{1}{b'}\right)+g.\varepsilon$. Now this must be equal to the

[10016w'] corpuscular action at O, which is $K+\frac{1}{2}H.\left(\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}\right)$ [10016q]; hence we have

[10016x] $$K+\tfrac{1}{2}H.\left(\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}\right)=K-\tfrac{1}{2}H.\left(\frac{1}{b}+\frac{1}{b'}\right)+g.\varepsilon.$$

Rejecting K from both members of this equation, and then dividing by $\frac{1}{2}H$, or $\frac{1}{2}g\alpha^{-1}$ [10016l], we get

[10016x'] $$\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}=-\left(\frac{1}{b}+\frac{1}{b'}\right)+2\alpha\varepsilon;$$

and by substituting [10016m], we obtain

[10016y] $$\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{b}'}=2\alpha.(c-h);$$

will form a new drop, which will become more and more convex, until it forms a hemisphere whose diameter is equal to the exterior diameter of the tube. Then, if the column be in equilibrium, its length will be equal to the [10017″]

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substituting this in [10016u], it becomes

$$\frac{1}{R}+\frac{1}{R'}=-2\alpha.(z-c);$$ [10016z]

and by using the values of $\frac{1}{R}$, $\frac{1}{R'}$ [9326, 9326′], we finally obtain the following differential equation of the lower convex surface of the fluid AOB;

$$\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)=-2\alpha.(z-c).\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}};$$ [Lower convex surface.] [10017a]

which differs from the equation of the upper surface aob [10016o], only in the sign of α so that the integrals of the equations of both these surfaces may be obtained by the methods [10017b] pointed out by the author in [9328—9379], and it is unnecessary to enter into any further explanation of them. We may remark that similar results are obtained when we suppose the upper surface to be convex, or the lower surface concave.

The volume $aobVW$, included between the upper surface aob and the base WTV, is evidently represented by $2\pi.\int zudu$; and the similar volume $AOBVW$, included between [10017c] the lower surface AOB and the same base, is $2\pi.\int zudu$; the difference of these two integrals is equal to the volume V of the fluid $aobBOA$; hence we have

$$V=2\pi.\int zudu-2\pi.\int zudu.$$ [10017d]

The integrals of these expressions may be obtained by approximation, in a series arranged according to the powers of α, by methods similar to those which are used in [9328—9379], restricting the calculation to terms of the first order in α, in the value of V. We have not [10017e] thought it expedient to insert these general calculations, which are of no practical use in comparing with experiments; we shall therefore restrict our remarks to the notice of two of the most simple and remarkable cases. First, when the elevation of water in a tube Po, is exactly equal to that arising from the capillary action of the same tube, when well wet and [10017f] dipped by its lower end AB into a vessel of water. In this case, it is evident that the figure of the lower surface AOB will coincide very nearly with that of the plane APB. If some [10017g] of the water is taken from the tube, so as to decrease the altitude Po, the point O will ascend above P, and the lower external surface will become *concave*. On the contrary, if more water is inserted in the tube, so as to increase the altitude Po, the lower surface will become *convex;* and this convexity will increase by the addition of more fluid, until the surface AOB becomes nearly hemispherical, like the upper surface aob. If we suppose the [10017h] upper and lower surfaces to be hemispheres whose radius is l, the elevation Po, produced by the capillary action at the upper surface, will be $q=\frac{H}{gl}$ [9360], and the similar action of [10017i] the lower surface ABO will likewise be $q=\frac{H}{gl}$; hence their combined action will produce [10017k] the double elevation $2q$, spoken of in [10016, &c.].

sum of the elevations of the fluid in two other glass tubes dipped into the vessel
[10018] *by their lower ends, their diameters being respectively equal to the internal and external diameters of the first tube. Lastly, if the column be of a greater*

In like manner the volume V' of a drop of the fluid, formed at the bottom of the tube
[10017l] AB, is represented by $V'=\pi.\int u^2dz$, taken between the limits $z=TO$ and $z=TP$. This may likewise be reduced to a series of terms in α, as in [10017e, &c.]. If the figure
[10017m] AOB be a hemisphere, we shall have $V'=\frac{2}{3}\pi.l^3$, which is the value of the actual volume, if we neglect terms of the order α; but if we retain terms of the order α, the volume of the
[10017n] drop will become $V'=\frac{2}{3}\pi l^3.(1+\frac{1}{2}\alpha l^2)$, as we may easily prove by means of the formula [10017l], using values of dz, u, &c., similar to those in [9339—9350], and differing only by writing $-\alpha$ for α; but we have not thought it worth the trouble to insert this calculation, as it is of no importance, on account of the difficulty of ascertaining the mass of a drop with
[10017o] any great degree of accuracy [10017v]. If the density of the drop be D, its gravity g, its mass m, supposing it to be a hemisphere, we shall have

[10017p] $$m=\tfrac{2}{3}\pi.gD.l^3.$$

According to this theory, the mass of the drop m must be proportional to the density of the
[10017q] fluid; but this does not agree with the observations of Gay-Lussac. For, by some experiments made with the same tube, and in similar circumstances, he found that 100 drops
[10017r] of water, whose density was 1,0000, weighed $8^{\text{gram.}},9875$; and 100 drops of alcohol, whose
[10017s] density was 0,8453, weighed only $3^{\text{gram.}},0375$; so that the drop of alcohol was about one third of that of water, while their densities differed only one fifth part. The radius of the
[10017t] tube used in these experiments was $l=3^{\text{mi.}},09$; hence $V'=62$ cubic millimetres nearly [10017m], or $V'=0,062$ cubic centimetres. Multiplying this successively by the densities
[10017u] 1,0000, 0,8453, we get for the mass of the drop of water $m=0^{\text{gram.}},06$, nearly; and for that of alcohol, $m=0^{\text{gram.}},05$ nearly. These differ very much from the experiments [10017r, s]; and the differences will not be avoided by introducing the term of the order α [10017n];
[10017v] for, though it will decrease the error of the mass of the drop of water, it will increase that of the drop of alcohol; hence we see that the uncertainty of the observations makes it unnecessary to notice the terms of the order α. We may finally remark, that the figure of
[10017w] the drop is calculated when in a state of equilibrium, but the drop separates from the tube only when the equilibrium is destroyed; and it is probable that from this source arises the great difference between the calculated and observed mass in a drop.

In the preceding calculations, it is supposed that the water is poured into the top of the tube;
[10017x] but we may suppose the tube to be inserted in the bottom of a large vessel of water. Then the fluid will not run out from the bottom of the tube, if the distance from the level of the
[10017y] fluid in the vessel to the bottom of the tube be less than the quantity q [10017i], corresponding to the elevation in a capillary tube; but a drop will begin to form at the bottom
[10017z] of the tube when that distance exceeds q; finally, when the distance exceeds $2q$ [10017k], the water may flow out from the tube, neglecting in these estimates terms of the order α, as in [10017m, &c.].

length, part of the fluid will fall from the tube. All these results of the theory are confirmed by experiment. [10019]

We shall now consider an indefinite vase filled with any number of fluids, placed horizontally, the one above the other. "*If we dip vertically into the vase the lower* [10020] *end of a right prismatic tube, the excess of the weight of the fluids contained in the tube, above the weight of the fluids which it would have contained independent of the capillary action, is the same as the weight of the fluid which would rise above the level, in case there was in the vessel only that fluid in which the lowest part of the tube is dipped.*"* In fact, the action of the prism, and of this fluid, [10021] upon the same fluid contained in the tube, is evidently the same as in this last case. The other fluids contained in the prism being sensibly *elevated above its* [10022] *lower base*, the action of the prism upon each of them can neither elevate nor depress them. As to the reciprocal action of these fluids upon each other,

Action of several fluids in a tube is the same as if only the lower fluid were contained in it.

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* (4307) For the purpose of illustration, we may take the case in which there are only two fluids included in the vertical cylindrical tube $RR'U'U$, fig. 158, page 877, where aob is [10020a] the upper surface; AOB, the common surface of the two fluids; $WtTV$, a horizontal plane, drawn within the tube, and at a sensible distance above its lower extremities R, U, and at a [10020b] finite distance below the surface AOB. To estimate the upward pressure upon any point t of this plane, arising from the action of the fluid in the vase, and in the lower part of the tube $RWVU$, we shall suppose a slender filament or canal $nNtzrv$ to be drawn, having [10020c] two vertical branches rv, zn, connected by the horizontal branch rz; the termination v being at the level surface of the fluid in the vase. Then the pressure at the bottom of the canal vr, upon a unit of surface, will be represented, as in [9555], by $P+K+g\times vr$. [10020d] This same pressure is continued in the horizontal direction of the canal rz; and in ascending in the vertical branch zt, it is decreased by taking away the pressure of the fluid in that branch, corresponding to the ascent; so that, at the point t, we have for the expression of the *upward pressure*, the function $P+K+g.(vr-zt)$. Now this quantity remains the same [10020e] whatever changes may be made in the number or nature of the fluids in the tube above the surface AOB; always supposing the fluid in the lower part of the tube and in the vase to [10020f] remain unaltered, as well as its point of level v. For, in these changes, no alteration is made in the process for finding the expression of the pressure [10020e], which depends wholly on the elevation of the fluid in the vase, and on that of the fluid in the tube below the plane WV; so that its action upon the fluid situated immediately above WV, and within its sphere [10020g] of action, must remain the same; and the action of the tube upon the same part of the fluid must also remain unaltered. Hence it follows that, whatever be the number of fluids in the tube, the mass in the branch nt must remain unaltered; and as this is true for every canal [10020h] passing through the surface WV, the whole mass above the plane WV must remain invariable, whatever changes may take place in the number of the fluids; therefore this mass must remain the same as if there were only one fluid in the tube and in the vase, as in [10024]. [10020i]

[10023] they will evidently destroy each other, if they form together a solid mass, which may be supposed without affecting the equilibrium.

Hence it follows that, *if we dip into a fluid the lower end of a prismatic tube,*
[10024] *and then pour into the tube another fluid which remains above the first,* the weight*

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* (4308) If the volume of the upper fluid be given, as well as the densities of the two
[10024a] fluids, we may thence deduce the increment of the altitude of the central point of the upper surface, in consequence of the introduction of the upper fluid into the tube. For this purpose we shall suppose the surfaces of the two fluids to be spherical, as in [10027, 10027a, p', &c.], and we shall use the following symbols, referring to fig. 158, page 877;

[10024b] ϖ, ϖ', θ, are the same angles as those which are defined in [10028—10030];

[10024c] $l=$ the internal radius of the tube, as in [10035']; its circumference being $2\pi l$;

[10024c'] $h=TO=$ the elevation of the lowest point O of the lower spherical surface AOB above the horizontal plane WTV;

[10024d] $h+\varepsilon'=To=$ the elevation of the lowest point o of the upper spherical surface aob above the horizontal plane WTV; hence $Oo=To-TO=\varepsilon'$;

[10024e] $D=$ the density of the lower fluid, or the fluid in the vase;

[10024f] $D'=$ the density of the upper fluid; and for brevity we shall put

[10024g] $$F(\varpi)=\frac{\cos.^2\varpi+\frac{2}{3}\sin.^3\varpi-\frac{2}{3}}{\cos.^3\varpi}; \qquad F(\theta)=\frac{\cos.^2\theta+\frac{2}{3}\sin.^3\theta-\frac{2}{3}}{\cos.^3\theta};$$

[10024h] $\pi l^2\varepsilon=$ the volume of the upper fluid included between the spherical surfaces aob, AOB, this quantity being supposed to be given.

Now it is evident that this volume of fluid, contained between the spherical surfaces aob, AOB, is composed of the three following parts, namely,

[10024i] The cylinder $lkKL$ + the annulus $lkboa$ — the annulus $LKBOA$;

[10024k] and we have the cylinder $lkKL=\pi l^2\varepsilon'$, the annulus $lkboa=\pi l^3.F(\varpi)$ [9336o, 10024g]; and in like manner, the annulus $LKBOA=\pi l^3.F(\theta)$. Substituting these values in the expression [10024i], and putting the resulting quantity equal to $\pi l^2\varepsilon$ [10024h], we get

[10024l] $$\pi l^2\varepsilon=\pi l^2\varepsilon'+\pi l^3.F(\varpi)-\pi l^3.F(\theta).$$

Dividing by the common factor πl^2, we obtain

[10024m] $$\varepsilon=\varepsilon'+l.F(\varpi)-l.F(\theta).$$

Again, since the volume $WAOBV$ is equal to the sum of the volumes of the cylinder
[10024n] $WLKV$, and of the annulus $LKBOA$, we shall have, by substituting the values of these quantities, in symbols,

[10024o] $$\text{The volume } WAOBV=\pi l^2.h+\pi l^3.F(\theta).$$

Multiplying the volume of the upper fluid $\pi l^2\varepsilon$ [10024h] by its density D', and the volume of the lower fluid [10024o] by its density D, then taking the sum of the two products, we obtain the expression of the whole mass of the fluid in the tube, above the plane WTV, namely,

[10024p] $$\text{Mass } WaobV=D'.\pi l^2\varepsilon+D.\pi l^2.\{h+l.F(\theta)\}.$$

of the two fluids contained in the tube will be the same as that of the fluid which it contained before [10020c]. It is evident, by the first method, that the surface of [10025]
the upper fluid will be the same as in the case where the lower end of the tube is dipped into that fluid. *At the points of contact of the two fluids, they will*

The value of this mass must remain the same as when there is only one fluid in the tube [10020i]; and if we suppose, in this case, that the elevation of the point o above the plane WV is H, and the angle formed by the fluid at the surface and near the side of the tube is ϖ', as in [10029], the volume $WaobV$ will become $\pi l^2 H+\pi l^3.F(\varpi')$, as is evident from [10024q]
[10024o]; multiplying this by the density D, we get another expression of the same mass, namely,

$$\text{The mass of the fluid } WaobV = D.\pi l^2.\{H+l.F(\varpi')\}. \qquad [10024r]$$

Putting this equal to the expression [10024p], and then dividing by $D.\pi l^2$, we get by successive reductions, and using [10024m],

$$H+l.F(\varpi')=\frac{D'}{D}.\varepsilon+h+l.F(\theta) \qquad [10024s]$$

$$=\varepsilon-\frac{(D-D')}{D}.\varepsilon+h+l.F(\theta)$$

$$=\varepsilon'+l.F(\varpi)-\frac{(D-D')}{D}.\varepsilon+h; \qquad [10024t]$$

whence we obtain

$$(h+\varepsilon')-H=\frac{(D-D')}{D}.\varepsilon-l.\{F(\varpi)-F(\varpi')\}; \qquad [10024u]$$

and by putting, for brevity, x equal to the increment of the elevation of the point o, in consequence of pouring into the tube the column of the upper fluid, whose volume is $\pi l^2\varepsilon$, and density D', over the mass of the lower fluid, we shall have $x=(h+\varepsilon')-H$, and [10024v]
the expression [10024u] will give

$$x=\frac{(D-D')}{D}.\varepsilon-l.\{F(\varpi)-F(\varpi')\}. \qquad [10024w]$$

If $\varpi=\varpi'$, this value of x will become

$$x=\frac{(D-D')}{D}.\varepsilon; \qquad [10024x]$$

and as $D>D'$, the expression of x will be positive; so that, in this case, the elevation of the point o of the upper surface, will be increased by pouring into the tube the upper fluid, whose density is D'. If we put $\varpi=0$, $\varpi'=60^{d}$, we shall have, from [10024g], [10024y]
$F(\varpi)=0{,}333$, $F(\theta)=0{,}131$; substituting these in [10024w], we shall get very nearly,

$$x=\frac{(D-D')}{D}.\varepsilon-\tfrac{1}{5}l. \qquad [10024z]$$

The expressions of x [10024w, z] agree with those which are given by M. Poisson, in pages 140, 141, of his *Nouvelle Théorie, &c.*, changing D into ρ, D' into ρ', l into α,

[10026] *have a common surface; but this surface will be different from that which the two fluids would have separately, and it is interesting to determine the nature of it.*

For this purpose, we shall suppose the interior surface of the prism to be a right cylinder, of a very small diameter, and placed vertically. In this case, it is evident, by what has been said in the preceding theory, that *the common surface*
[10027] *of the two fluids, and those which they would separately have in this tube, will be spherical surfaces of different radii.** We shall put

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[10025a] and putting $\cos.\varpi' = -\frac{F}{H}$, $\cos.\varpi = -\frac{F'}{H'}$, to conform to his notation; and if we suppose $D' = \frac{9}{10}D$, we shall have, from [10021z], $x = \frac{1}{10}\varepsilon - \frac{1}{5}l$, which may become *negative* if $\varepsilon < 2l$; and M. Poisson remarks, that this result may serve to explain the phenomenon
[10025b] observed by Dr. Young, who dipped a capillary tube into a vessel containing water, and suffered the fluid to ascend in it; he afterwards poured into the top of the tube a small drop of
[10025c] oil, and found that the upper surface of the oil finally settled at a less elevation above the level of the water in the vessel, than that which the water in the tube had before the addition of the oil. M. Poisson supposes, in this experiment, that the tube had not been previously
[10025d] moistened by the water in its upper part; and upon pouring in the oil, it became moistened by the oil, and by this means the upper angle, or that formed by the lower side of the tube and the tangent to the surface of the oil, became $\varpi = 0$. On the contrary, as the tube had
[10025e] not been previously moistened by the water, he supposes that the surface of this fluid was not
[10025f] a tangent to the side of the tube, so that ϖ' might have a finite value; and in this case, x could become negative, as in the experiment of Dr. Young. M. Poisson finally observes, that this
[10025g] experiment was brought by Dr. Young as an objection to La Place's theory of the capillary action, supposing it to be incompatible with this theory; instead of which it serves to furnish an interesting verification of its accuracy.

* (4309) When there is only one fluid in a cylindrical tube $RR'U'U$, fig. 158,
[10027a] page 877, the surface will be nearly spherical, as is proved in [9336g, &c.], from the general equation [9342], or rather from [9334]; so that the surface corresponding to the angles ϖ, ϖ', in the hypothesis [10027, &c.], must be nearly spherical. The same may be proved relative
[10027b] to the common surface of the two fluids AOB, fig. 158, by pursuing the same method as in [9309—9342], and using the figure in like manner as we have used fig. 116, page 713, or fig. 112, page 695; that is, by drawing an infinitely slender and uniform canal $nNtTOMo$,
[10027c] with the vertical branches tNn, $TOMo$, and the horizontal branches tT, NM; the area of a section of this canal perpendicular to its length, being taken for unity. Then it is evident
[10027d] that the fluid in the canal $nNMo$ must be in equilibrium; so that the vertical pressure of the fluid in the branch nN, at the point N, must be the same as at the point M, in the branch
[10027e] oM, being equal to that in the horizontal branch NM; and in like manner, from the equilibrium of the fluid in the canal $nNtTOMo$, the vertical pressure at t must be the
[10027e'] same as at T. Hence it follows that the part of the pressure at the point t, arising from the capillary action at N, and the weight of the column Nt, must be equal to the part of the

$\varpi =$ the angle which the surface of the upper fluid forms with the lower surface of the tube, supposing no other fluid to be used; [10028]

$\varpi' =$ the similar angle when the lower fluid only is used; [10029]

$\theta =$ the angle which the common surface of the two fluids forms with the lower surface of the tube. [10030]

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pressure at T arising from the capillary action at O, and the weight of the column MT; and from this principle we shall deduce the equation of the surface AOB in the following manner:

Putting, as in [9310], R, R', *for the greatest and the least radii of curvature at the point* N of the common surface $ANOB$ of the two fluids; and, as in [9316], b, b', *for the greatest and the least radii of curvature at the point* O; also [10027f]

$$tN = TM = z, \qquad tn = z', \qquad MN = Tt = u, \qquad TO = h, \text{ \&c.}; \qquad [10027g]$$

we shall have, as in [10032, 9275], $K' - \frac{1}{2}H'.\left(\frac{1}{R}+\frac{1}{R'}\right)$, for the downward action of the fluid below the surface AOB, upon a portion of the same fluid in the canal at N; and the upper fluid will attract the external lower fluid at N with the upward force $K_1 - \frac{1}{2}H_1.\left(\frac{1}{R}+\frac{1}{R'}\right)$ [10033, 9254]. The difference of these two forces gives the whole downward force at N equal to $K_{\prime\prime} - \frac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right)$, G being the measure of the capillary intensity; and at the point O, the preceding expression will become $K_{\prime\prime} - \frac{1}{2}G.\left(\frac{1}{b}+\frac{1}{b'}\right)$. Then, the density of the lower fluid being represented by D, and that of the upper by D', we shall have $Dg \times tN = Dgz$ for the mass of the fluid in the part tN of the canal tn. Adding to this the capillary action at N [10027k], it becomes [10027h] [10027i] [10027k] [10027l]

$$K_{\prime\prime} - \tfrac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right) + Dgz. \qquad [10027m]$$

Again, the mass of the column MO is represented by $D'g \times MO = D'g.(z-h)$, and that of the column TO by Dgh; their sum being added to the capillary action at O [10027l], gives [10027n]

$$K_{\prime\prime} - \tfrac{1}{2}G.\left(\frac{1}{b}+\frac{1}{b'}\right) + D'g.(z-h) + Dgh, \qquad [10027o]$$

for the action in the column MT, which, as in [10027e'], is to be put equal to the action in the column tN [10027m]. Hence we have

$$K_{\prime\prime} - \tfrac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right) + Dgz = K_{\prime\prime} - \tfrac{1}{2}G.\left(\frac{1}{b}+\frac{1}{b'}\right) + D'g.(z-h) + Dgh; \qquad [10027o']$$

and if we put, for abridgment, $c = \frac{1}{2}G.\left(\frac{1}{b}+\frac{1}{b'}\right) - (D-D').gh$, we shall get [10027o'']

$$\tfrac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right) = (D-D').gz + c. \qquad \left[\text{Equation of the common surface } AOB.\right] \qquad [10027p]$$

From this equation we may deduce others, like those in [9318, 9334, 9442, &c.]; whence

We may here observe, that *these angles are not those which the surfaces form at*
[10030′] *their points of contact with the tube; but they are formed by the planes which are tangents to those surfaces at the limit of the sphere of sensible activity of the tube*, as we have said several times. We shall suppose that

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[10027p′] we may conclude, as in [9336g], that the common surface of the two fluids, or that which corresponds to the angle θ [10030], will also be very nearly spherical.

The whole mass of fluid contained in the vertical branch nNt is evidently represented by
[10027q] $D'g\times nN+Dg\times Nt$, or, in symbols, $D'g.(z'-z)+Dgz=(D-D').gz+D'gz'$; and as the ordinates z, z', are measured from the level surface of the fluid in the vessel, this mass will be elevated by means of the two capillary actions at N, n, namely,
[10027q′] $\frac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right)$ [10027k], and $\frac{1}{2}H.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)$, μ, μ', *being the greatest and the least radii of curvature at the point* n; hence we have

[10027r] $$\tfrac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right)+\tfrac{1}{2}H.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=(D-D').gz+D'gz'.$$

Subtracting the equation [10027p] from [10027r], we obtain the following equation *of the upper surface;*

[10027s] $$\tfrac{1}{2}H.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=D'gz'-c.$$ [Upper concave surface.]

We have supposed, in this figure, that both the surfaces AOB, aob, are concave; but it is
[10027t] evident that, if the upper surface be convex, we must change the sign of the term depending on H, in [10027q′—s], as is done in [9275, 9276], and then the equation of the upper surface [10027s] will become

[10027u] $$-\tfrac{1}{2}H.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=D'gz'-c.$$ [Upper convex surface.]

The equations [10027p, s], are of the same forms as the equations (a), which are given by M. Poisson, in page 136 of his *Nouvelle Théorie, &c.*, for the solution of this problem; the
[10027u′] symbols D, D', R, R', being changed respectively into $\rho, \rho', \lambda, \lambda'$, to conform to his notation;
[10027v] and from these equations he deduces results which are similar to those given by La Place in this part of the work.

In the preceding calculations, the origin of z is supposed to be on a level with the surface of the fluid in the vessel [10027q]. If this origin be taken *at the distance n below that*
[10027v′] *level*, z will be changed into z, and z' into z'; and we shall have

[10027w] $$z=\mathrm{z}-n, \qquad z'=\mathrm{z}'-n;$$

substituting these in [10027p, s, u], they become respectively

[10027x] $$\tfrac{1}{2}G.\left(\frac{1}{R}+\frac{1}{R'}\right)=(D-D').g\mathrm{z}-(D-D').gn+c,$$ [Lower concave surface.]

[10027y] $$\tfrac{1}{2}H.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=D'g\mathrm{z}'-D'gn-c,$$ [Upper surface, if concave.]

[10027z] $$-\tfrac{1}{2}H.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=D'g\mathrm{z}'-D'gn-c.$$ [Upper surface, if convex.]

K, H, represent, for the upper fluid, the same quantities as in [9253, 9253′]; [10031]

K', H', are the similar quantities for the lower fluid; [10032]

K_1, H_1, represent the values which K, H, respectively become, when, instead of considering the action of the upper fluid upon its own particles, we consider the action of the *upper* fluid upon the *lower* fluid. Moreover, as the action is always equal and contrary to the reaction, K_1 and H_1 will also represent what K', H', become when we consider the action of the *lower* fluid upon the *upper* fluid. [When we take into consideration the change of density near the surfaces of the fluid, we must use the values of K, H, K', H', K_1, H_1, as they are deduced from experiment.] [10033] [10034]

This being premised, we shall suppose that an infinitely narrow canal is drawn through the axis of the tube, till it gets below it, where it is bent in a horizontal direction, and then continued upwards till it meets the level surface of the fluid in the vessel. The upper fluid contained in this canal, will be drawn downwards, near its upper surface, by the force* $K-\frac{H.\cos.\varpi}{l}$, *$l$ being the radius of the hollow part of the tube.* [10035] [10035′]

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* (4310) In the cylindrical tube $RR'U'U$, fig. 158, page 877, we shall suppose aob to be the surface of the upper fluid; AOB, the common surface of the two fluids; ag, AG, tangents to these surfaces, in the points a, A; these points being just beyond the sphere of activity of the tube upon the fluid; c, C, the centres of these surfaces, supposing them to be spherical, as in [10027]; then the angle $Aag=$ the angle $pac=\varpi$; the angle $RAG=\theta$; $ca=b$; $ap=l$; and $ca=\frac{ap}{\cos.pac}$, which in symbols is $b=\frac{l}{\cos.\varpi}$. We shall now compute according to the method of the author, the action of both fluids upon the part oO of the infinitely narrow canal oOT; and, in the first place, we have for the action of the *upper* fluid upon this canal near o, $K-\frac{H}{b}$ [9275], drawing *downwards;* and if we substitute the preceding value of b, it becomes $K-\frac{H.\cos.\varpi}{l}$, as in [10035′]. In like manner, we have $CA=\frac{AP}{\cos.PAC}=\frac{l}{\cos.\theta}$; substituting this for b, in the expression $K+\frac{H}{b}$ [9276], which represents the action of the upper fluid upon the same canal Oo, near the point O, but sufficiently above the common surface of the two fluids to be clear of the variable density of the upper fluid, near its junction with the lower fluid, it becomes $K+\frac{H.\cos.\theta}{l}$, drawing *upwards*, as in [10036], or, by changing the signs, $-K-\frac{H.\cos.\theta}{l}$, drawing *downwards*. Now if the upper fluid were to become of the same nature as the lower, K, H [10031], [10035a] [10035b] [10035c] [10035d] [10035e] [10035f] [10035g] [10035h]

At the common surface of the two fluids, the fluid in the canal will be drawn

[10036] upwards by the force $K+\frac{H.\cos.\theta}{l}$, in virtue of the action of the fluid in the upper part of the tube upon its own particles; it will be drawn downwards by

[10036′] the force $K_1+\frac{H_1.\cos.\theta}{l}$, in virtue of the fluid in the lower part of the tube; the upper fluid of the canal will therefore be urged downwards by the force

[10037] $$K_1+\frac{(H_1-H).\cos.\theta}{l}-\frac{H.\cos.\varpi}{l}.$$ [Downward pressure of the upper fluid in the canal.]

The lower fluid in the canal will be urged downwards, in virtue of the action

[10038] of the lower fluid in the tube, by the force* $K'-\frac{H'.\cos.\theta}{l}$; and by the action

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would change into K', H' [10032], respectively; and then the action of the upper fluid upon

[10035i] the part of the canal Oo, near O, would be $K'+\frac{H'.\cos.\theta}{l}$, drawing *upwards;* and as the whole fluid is supposed to be homogeneous, this action must be balanced by the equal and opposite or downward action of the lower fluid upon the same canal. Therefore the downward action of the lower fluid upon the part Oo of the upper fluid in the canal, near

[10035k] O, will be represented by $K'+\frac{H'.\cos.\theta}{l}$, supposing the upper fluid to be the same as the lower. But, the upper fluid being different from the lower, we must, as in [10034], change

[10035l] K' into K_1, and H' into H_1, to obtain the real action of the lower fluid upon the upper

[10035m] fluid in the canal Oo near O, which will therefore be represented by $K_1+\frac{H_1.\cos.\theta}{l}$, as in [10036′]. Adding this to the parts computed in [10035e, h], we get the whole force

[10035n] acting on the canal Oo, and drawing it downwards, as in [10037]. The action on the lower part OT of the canal remains yet to be computed, as in [10038—10040].

* (4311) This is found in exactly the same manner as the expression [10035′], changing ϖ into θ, K into K', and H into H', so as to correspond with the notation in [10032];

[10038a] hence we get $K'-\frac{H'.\cos.\theta}{l}$ for the action of the lower fluid upon the column OT, fig. 158, page 877, near O. Again, it follows from [9257b], that, by conforming to the

[10038b] notation in [10033], we shall have $K_1-\frac{H_1}{CA}$, for the action of the upper fluid upon the canal OT near O, drawing it upwards; substituting the value of CA [10035f], it becomes

[10038c] $K_1-\frac{H_1.\cos.\theta}{l}$, as in [10039]. Subtracting this from the force [10038a], we get the whole action on the tube OT, drawing downwards, as in [10039′]. Adding this to the action of the

[10038d] part oO of the tube [10037], we get the whole action on the canal, as in [10040], drawing downwards.

of the upper fluid in the tube, it will be drawn upwards by the force $K_1 - \frac{H_1 . \cos.\theta}{l}$; therefore it will be drawn downwards by the force [10039]

$$K' - K_1 + \frac{(H_1 - H') . \cos.\theta}{l}.$$ [Downward pressure of the lower fluid in the canal.] [10039′]

Thus the whole force of the fluids in the canal, depending upon the reciprocal actions of the fluids in the tube, and in a downward direction, will be represented by

$$K' + \frac{(2H_1 - H - H') . \cos.\theta}{l} - \frac{H . \cos.\varpi}{l}.$$ [Downward pressure of the whole canal.] [10040]

If the tube contain no other fluid except the lower one, this force will become *

$$K' - \frac{H' . \cos.\varpi'}{l};$$ [10041]

and as we have seen in [10024], that the weights of the fluid contained in the canal, must be the same in both cases, these two forces must be equal to each other; therefore we shall have

$$\frac{H . \cos.\varpi}{l} - (2H_1 - H - H') . \frac{\cos.\theta}{l} = \frac{H' . \cos.\varpi'}{l},$$ [10042]

which gives

$$\cos.\theta = \frac{H' . \cos.\varpi' - H . \cos.\varpi}{H + H' - 2H_1}.$$ [Correct value of cos.θ, even when we notice the change in the densities near the surfaces of the fluids.] [10043]

We may eliminate the angles ϖ, ϖ', from this expression of $\cos.\theta$, by using the following equations, which can be easily deduced from what has been said; † [10044]

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* (4312) This is of the same form as the expression [10035′], corresponding to the upper fluid, changing ϖ, K, H, into ϖ', K', H', respectively, to conform to the notation in [10028—10033]. Now it has been shown in [10024], that the expression must be equal to that in [10040]. Putting them, therefore, equal to each other, rejecting the terms K', K', which mutually destroy each other, and changing the signs of all the other terms, we get [10042]; whence we easily deduce [10043]. This value of $\cos.\theta$, being deduced from the equilibrium of the pressures in the canal [10024], is true, even when we notice the change of density near the surfaces of the fluid. [10041a] [10041b] [10041c] [10041d]

† (4313) In [9936], we have $\rho' = \frac{1}{2}H$, neglecting the consideration of the change of density of the fluid near its surface; and in like manner from [9927—9929, 10047], $\rho = \frac{1}{2}\overline{H}$. Substituting these in [9931], we get $2\overline{H} - H = H . \cos.\varpi$; and if we change ϖ, H, $\overline{H}$, into ϖ', H', $\overline{H'}$, so as to conform to the action of the lower fluid, and to the notation in [10029, 10032, 10047′], we shall get $2\overline{H'} - H' = H' . \cos.\varpi'$. These values of $H . \cos.\varpi$, [10045a] [10045b]

[10045] $\overline{H}=H.(\tfrac{1}{2}+\tfrac{1}{2}\cos.\varpi)=H.\cos.^2\tfrac{1}{2}\varpi;$

[10046] $\overline{H'}=H'.(\tfrac{1}{2}+\tfrac{1}{2}\cos.\varpi')=H'.\cos.^2\tfrac{1}{2}\varpi';$

[These expressions must be considered as definitions of the values of $\overline{H}$, $\overline{H'}$, to be used instead of those given by the author in [10047, 10047'].]

[10047] $\overline{H}$ being what H becomes, when we consider the action of the *upper fluid* upon the matter of the tube; [or, more correctly, $\overline{H}$ is given in terms of H, ϖ, as in [10045];

[10047'] $\overline{H'}$ being what H' becomes, when we consider the action of the *lower fluid* upon the matter of the tube; [or, more correctly, $\overline{H'}$ is given in terms of H', ϖ', as in [10046].

Hence we shall have

[10048] $$\cos.\theta=\frac{2\overline{H'}-2\overline{H}+H-H'}{H+H'-2H_1}.$$

[This is correct if we use the values of $\overline{H}$, $\overline{H'}$ [10045, 10046], and deduce H, H', H_1, from observation.]

$H'.\cos.\varpi'$, are given in the original work instead of the equivalent values of $\overline{H}$, $\overline{H'}$, which we have inserted in [10045, 10046]. The author supposes that these values satisfy the [10045c] definitions of the values of $\overline{H}$, $\overline{H'}$ [10047, 10047']; but as he has neglected the consideration of the change of the density of the fluid near its surface, we have thought it [10045d] would be more accurate to define the values of $\overline{H}$, $\overline{H'}$, in terms of H, H', by means of the formulas [10045, 10046], or by those in [10045a, b]. Finally, if we substitute those values [10045e] in [10043], we shall obtain [10048]; which is correct if we use the values of $\overline{H}$, $\overline{H'}$, given by the formulas [10045, 10046], after substituting in them the values of H, H', as ascertained by observation.

The expression [10043] is equivalent to those given by M. Poisson in page 137, &c., of his *Nouvelle Théorie, &c.* For by putting, as in [9980m, &c.], in order to conform to his notation,

[10045f] $$\theta=180^{d.}-\varphi,\qquad \varpi=180^{d.}-w,\qquad \varpi'=180^{d.}-w',$$

the expression [10043] will become, by multiplying by $-(H+H'-2H_1)$,

[10045g] $$(H+H'-2H_1).\cos.\varphi=H'.\cos.w'-H.\cos.w;$$

we must also change reciprocally H into H', because La Place supposes, in [10031, 10032], [10045h] that the accented letter H' corresponds to the lower fluid, but Poisson uses the unaccented H, in page 136 of his work. Making these changes in [10045g], it becomes

[10045i] $$(H+H'-2H_1).\cos.\varphi=H.\cos.w'-H'.\cos.w;$$

so that, if we put for brevity

[10045k] $$H+H'-2H_1=G,\qquad H'.\cos.w=F',\qquad H.\cos.\varpi'=F,$$

it becomes

[10045l] $$G.\cos.\varphi=F-F';$$

and as Poisson supposes $F-F'=K$, in page 136 of his work, we finally obtain

[10045m] $$K=G.\cos.\varphi,\qquad F'=H'.\cos.w,$$

which are the same as his equations (b), page 137 of his work.

The angle θ being considered as a known quantity, we shall easily obtain, by means of the theory of capillary attraction, the differential equation of the common surface of the two fluids, whatever be the width of the tube or its figure [10027x, &c.]. We must observe that this angle is that which *the tangent plane of the surface, at the limits of sensible activity of the sides of the tube, forms with those sides* [10030']. [10049] [10050]

In the preceding formulas, it is supposed that the fluids do not perfectly moisten the sides of the tube. We have observed, in [9630, &c.], that, if the action of the tube upon the fluid exceed that of the fluid upon its own particles, an extremely thin lamina of the fluid will cover the sides of the tube, and form a new tube, in which the fluids will rise or fall. Thus, in case the tube contains several fluids which exactly moisten the tube, they will form, within it, a series of different tubes, to which we cannot therefore apply the preceding formulas. We shall here consider only two fluids, water and mercury; supposing the tube to be of glass which has been well moistened, so that its inner surface is covered with a very thin pellicle of water adhering to the glass. In this case, we must consider the tube as an aqueous one, and we shall have* [10051] [10052] [10053] [10054]

$$H_1 = \bar{H}', \qquad \bar{H} = H; \qquad \text{[Correct values.]} \qquad [10055]$$

therefore we shall have $\cos.\theta = -1$, consequently $\theta = \pi$. Then the surface of the mercury is convex, and very nearly hemispherical, if the tube is very slender. We may otherwise prove, by applying to this case the reasoning in [10056]

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* (4314) When the tube is moistened with water, or, in other words, when the tube is an aqueous one, and the upper fluid is water, we shall have $\varpi = 0$ [9655i, 9981a', &c.]; and by substituting it in [10045], we get $\bar{H} = H$, as in [10055]. The same holds good when we define $\bar{H}$ as the author has done in [10047], where it represents the action of water upon its own particles; observing also that, in [10031], it is represented by H; so that we have also, in this way of considering the subject, $\bar{H} = H$. Moreover, since the surface of mercury in a glass tube, moistened with water, is very nearly a convex hemisphere [9744], we shall have $\varpi' = 180^{d.}$ [10029]; substituting this in [10046], we get $\bar{H}' = 0$. The surface being a convex hemisphere, the action of the aqueous tube on the mercury will be very nearly equal to nothing [9654a]; hence we have $H_1 = 0$ [10034]; consequently $\bar{H}' = H_1$, as in [10055]. The same result holds good when we define $\bar{H}'$ as the author has done in [10047'], supposing it to correspond to the action of mercury on water; for this same quantity is represented by H_1 in [10031]; hence $\bar{H}' = H_1$, as in [10055e]. Substituting the values [10055] in [10048], we get $\cos.\theta = \frac{2H_1 - H - H'}{H + H' - 2H_1} = -1$, as in [10056]; whence we evidently have $\theta = \pi$, and the common surface of the two fluids corresponds nearly to that of a convex hemisphere. [10055a] [10055b] [10055c] [10055d] [10055e] [10055f] [10055g] [10055h]

[10057] [9654], that the surface of the fluid, in a very slender tube whose action is insensible, is convex and hemispherical.

[10058] The depression of mercury is, by what has been said,* $-\frac{H'.\cos.\varpi'}{gl}$, or $\frac{H'-2H_1}{gl}$, neglecting the little column of water which rests upon its surface; so that, if we put

[10059] $b=$ the height of this column of water, and

[10059′] $\frac{1}{D}=$ the density of water, that of mercury being taken for unity,

it is evident that the depression of the mercury will be represented by †

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* (4315) If we compare the definition of θ' [9346] with that of ϖ' [10029], we shall get

[10058a] $\theta'=90^{\text{d.}}-\varpi'$, whence $\sin.\theta'=\cos.\varpi'$; substituting this in [9379], we get $\frac{H.\cos.\varpi'}{gl}$ for

[10058b] the *elevation* of the mercury; and by changing its sign, we get its *depression*, as in [10058].

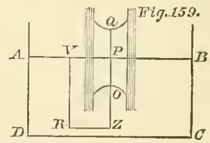
[10058c] Now from [10045b], we have $-H'.\cos.\varpi'=H'-2\overline{H'}=H'-2H_1$ [10055]; substituting

[10058d] this in the first expression [10058], it becomes $\frac{H'-2H_1}{gl}$, as in [10058], which is accurate, if we use the value of $H'-2H_1$ deduced from experiment.

† (4316) The annexed figure 159 is similar to fig. 112, page 695. The vase $ABCD$ is filled with mercury, which ascends in the capillary tube to the height O; and upon it

[10060a] rests the small column of water OQ; $AVPB$ being the level of the mercury in the vase. Then the pressure at the point

[10060b] R of the column VR, is $K+g.VR$ [9355b]; K being changed into K', as in [10032]. In like manner, the pressure at Z, in the column QZ, is equal to the gravity of the column QZ added to the capillary action



[10060c] of that column [10040]. Now the gravity of the column QZ is composed of the column of mercury OZ, whose mass is nearly equal to $g.OZ$, and of the column of water OQ, whose mass is $\frac{g}{D}.OQ$, nearly; the sum of these two expressions is

[10060d] $$g.OZ+\frac{g}{D}.OQ=g.OZ+\frac{gb}{D} \quad [10059,\ 10059'].$$

Now substituting $\theta=\pi$ [10055h], and $\varpi=0$ [10055a], in the expression of the capillary

[10060e] action [10040], it becomes $K'+\frac{H'-2H_1}{l}$. Adding this to the gravity of the column QZ [10060d], we get the whole action at Z in the canal QZ, namely,

[10060f] $$K'+\frac{H'-2H_1}{l}+g.OZ+\frac{gb}{D}.$$

Putting this equal to the action at R in the canal VR, namely, $K'+g.VR$ [10060b],

$$\frac{H'-2H_1}{gl}+\frac{b}{D}.$$ [Depression of the mercury in a tube moistened with water. If we determine $H'-2H_1$ by experiment, this depression will be correct, even when we consider the density near the surfaces as variable.] [10060]

We shall now suppose the same tube to be moistened by alcohol, and shall put [10060′]

$'H$ for the action of alcohol upon mercury; [10061]

$'b$ for the height of the column of alcohol above the surface of the mercury; [10062]

$\frac{1}{'D}$ for the density of alcohol, that of mercury being taken for unity. [10063]

Then the depression of the mercury will become

$$\frac{H'-2'H}{gl}+\frac{'b}{'D}.$$ [Depression of mercury in a tube moistened with alcohol. This depression is correct even when we notice the change of density near the surfaces, provided $H'-2'H$ is determined by experiment.] [10064]

The action of water upon its own particles being much greater than that of the mutual action of the particles of alcohol, as we shall see in [10322, 10333], it is very probable that the action of water upon mercury exceeds that of alcohol upon the same liquor, so that $'H$ is less than* H_1. This [10065] difference ought, therefore, to be sensible by experiment.

M. Gay-Lussac was willing to undertake the examination of this point. After having well moistened a glass tube, whose internal diameter, measured with great accuracy by means of the weight of a column of mercury which filled up the tube, was equal to $1^{mi.},29441$, he dipped the lower end of it into a [10066] vessel filled with mercury; and found, by the mean of ten experiments, which differed but little from each other, that the depression of the mercury was equal to $7^{mi.},4148$. The mercury, rising in the tube, had raised above its surface part [10067] of the water which had adhered to the sides of the tube in moistening it, and the length of the column of water formed in this way, was $7^{mi.},730$. The [10068] temperature was $17°,5$ during the experiments. The depression of the mercury, decreased by the weight of this column of water, is therefore equal to $6^{mi.},8464$; † hence we have [10069]

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and making a slight reduction, we get $g.(VR-OZ)$, or $g.OP=\frac{H'-2H_1}{l}+\frac{gb}{D}$. [10060g]
Dividing this by g, we get the depression of the mercury OP, as in [10060]. The formula [10064] is similar to [10060], changing H_1, b, D, into $'H$, $'b$, $'D$, respectively, to conform to the case mentioned in [10061—10064].

* (4317) Observing that $'H$ [10065, 10061], refers to alcohol, and H_1 [10060, 10034], [10065a] to water, as found by actual experiment.

† (4318) The density of water $\frac{1}{D}$ [10059′] being put equal to $\frac{1}{13{,}6}$, we shall find that [10069a]

[10070]
$$\frac{H'-2H_1}{gl}=6^{\text{mi.}},8464.$$

Then, moistening the same tube with alcohol whose specific gravity, compared
[10071] with that of water, was 0,81971, he found, by the mean of ten experiments, differing but little from each other, that the depression of the mercury was
[10072] $8^{\text{mi.}},0261$, and the length of the column of alcohol, which rested above the
[10073] surface of the mercury, was equal to $7^{\text{mi.}},4735$. The temperature was also 17°,5 during these experiments. Whence we conclude *

[10074]
$$\frac{H'-2'H}{gl}=7^{\text{mi.}},5757.$$

[10075] Therefore this value is sensibly greater than that of $\frac{H-2H_1}{gl}$ [10070], as is supposed in [10065, 10074*b*].

M. Gay-Lussac observed the versed sine of the convexity of the mercury in the tube before mentioned, and found it to be the same as the concavity of
[10076] the upper surface of the column of water or alcohol; all these surfaces are therefore equal to each other, and they have the form of a hemisphere whose diameter is the same as that of the tube, conformably to the preceding theory.

We shall suppose that there is an indefinite vase, containing only two [homogeneous] fluids, in which a right prism or tube is wholly immersed in a
[10077] vertical position, with its lower end in the lower fluid, and its upper end in the upper fluid. Then the weight of the lower fluid, which is raised above its level by means of the capillary action, is equal to the weight of a like volume of the
[10078] upper fluid, *increased* by the weight of the lower fluid, which could be raised in

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the column of water $b=7^{\text{mi.}},730$ [10068] is equivalent in weight to a column of mercury of the height $\frac{7^{\text{mi.}},730}{13,6}=0^{\text{mi.}},5684=\frac{b}{D}$; and as the value of the expression [10060] was found in this experiment to be equal to $7^{\text{mi.}},4148$ [10067], we shall have

[10069*b*]
$$\frac{H'-2H_1}{gl}=7^{\text{mi.}},4148-0^{\text{mi.}},5684=6^{\text{mi.}},8464,$$

as in [10070].

* (4319) Proceeding as in the last note, we get

[10074*a*]
$$\frac{H'-2'H}{gl}=8^{\text{mi.}},0261-7^{\text{mi.}},4735\times\frac{0,81971}{13,6}=8^{\text{mi.}},0261-0^{\text{mi.}},4504=7^{\text{mi.}},5757,$$

as in [10074], which is sensibly greater than the expression $\frac{H'-2H_1}{gl}$ [10070]; hence we
[10074*b*] have $H'-2'H>H'-2H_1$, or $-2'H>-2H_1$, and by transposition $H_1>'H$, as in [10065].

the same prism, if there were only that one fluid in the vessel, and *decreased* by the weight of the upper fluid, which would rise in the same prism above the [10078′]
level, if this fluid only were in the vase, and the prism were dipped into this fluid by its lower end.*

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* (4320) In the case which is taken into consideration in [10077—10084], it is supposed by the author that the fluids are perfectly homogeneous, and that their actions and that of the prism are expressed as in [10079—10083′]. The results of these forces are equivalent to the theorem in [10078, &c.], as is easily proved in the following manner: We shall take the common surface of the two fluids *in the vase* for the *horizontal fixed plane* from which the [10078a]
elevations of the common surface of the fluids *in the tube* are counted, supposing the positive values to fall above this plane, and the negative ones below it, so that, when this common surface of the two fluids is depressed below this plane, we may consider its elevation and the corresponding volume as negative. We shall also put

L = the volume of *the lower fluid in the tube* above the horizontal *fixed plane*, this value being negative when it falls below this plane; [10078a′]

U = the volume of *the upper fluid in the tube* above the mass L; [10078b]

L' = the volume of the *lower fluid* which would be raised in the same tube by the capillary action, supposing the tube to be dipped vertically into a vase containing only the *lower* fluid; [10078c]

U' = the volume of the upper fluid which would be raised in the same tube by the capillary action, supposing the tube to be dipped vertically into a vase containing only the *upper* fluid; [10078d]

l = the density of the *lower* fluid; [10078e]

u = the density of the *upper* fluid. [10078f]

We shall now substitute this notation in the expressions of the *four* different pressures which are computed in [10079—10083]; considering the *downward* pressures as *positive*, and the [10078g]
upward pressures as negative; and they will become respectively

First, The action of the tube and lower fluid is represented, as in [10080], by $\quad -L'l$; [10078h]

Second, The action of the tube and upper fluid is represented, as in [10082], by $\quad +U'u$; [10078i]

Third, The pressure of the column of the two fluids in the tube mentioned in [10083], is equal to $\quad +Ll+Uu$; [10078k]

Fourth, The pressure of the external column of the upper fluid, whose mass is $L+U$, mentioned in [10083′], is equal to $\quad -Lu-Uu$. [10078l]

The sum of these four pressures being put equal to nothing, because they are in equilibrium [10084], gives $-L'l+U'u+Ll-Lu=0$; whence $Ll=Lu+L'l-U'u$; which [10078m]
is the same theorem as in [10078], the expressions Ll, Lu, $L'l$, $U'u$, being respectively the [10078n]
gravities of the four columns of the homogeneous fluids mentioned in that theorem.

To demonstrate it, we shall observe, *First*, that the action of the prism and
[10079] the lower fluid upon the part of the lower fluid contained in it, is the same as if the fluid existed alone in the vessel; therefore this fluid is, in both these two cases, urged vertically upwards in the same manner; and it is evident that the
[10080] force which urges it in this last case, is *equivalent to the weight of the mass of this fluid which is then elevated above its level. Second:* In like manner the upper fluid, contained in the upper part of the prism, is urged vertically downwards by the action of the prism and of the fluid which surrounds this
[10081] part, as it would be urged upwards by the same action, if the vessel contained only the upper fluid, and the prism were moistened at its lower end by that fluid; and in this case the combined force of these actions is *equivalent to the*
[10082] *weight of the upper fluid, which would rise in the prism above its level in the vessel. Third:* The column of the fluids within the prism, which is above the
[10083] level of the lowest fluid in the vessel, is urged vertically downwards *by its own*
[10083'] *weight. Fourth:* This last column is also urged *upwards by the weight of an equal column of the upper external fluid.* By uniting all these forces, which
[10084] must be in equilibrium with each other, we shall obtain the theorem just mentioned [10078, &c.]. We may determine by the same principles what must take place when a hollow prism is wholly immersed in a vessel containing any number of [homogeneous] fluids.

In all that precedes, we have supposed the lower base of the prism to be
[10085] *horizontal. But, whatever be its inclination and the figure of the lower extremity of the tube, the vertical attraction of the tube, and that of the external fluid upon the fluid contained within it, will be the same as if the base were horizontal;* consequently the volume of the fluid which is raised above the level, will be the same in both cases. To prove this, we shall suppose, as in [9903], that the
[10086] interior surface of the prismatic tube is prolonged into the fluid, so as to form an additional or *second tube*, whose sides, being infinitely thin, do not affect the action of the surrounding fluid upon the fluid in the tube. It is evident that,
[10087] if we decompose the first tube into vertical and infinitely narrow columns, the action of each of these columns, to raise the fluid within the two prisms, will be the same as if the base were horizontal; therefore the sum of these actions
[10088] will be equal to* $2\rho c$.

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* (4321) The forces depending on the action of the tube are Q, computed in [9911],
[10088a] and the same quantity Q, computed in [9922]; their sum is $2Q=2\rho c$ [9927], as in [10088]; always supposing the density of the fluid to be uniform, in making these estimates, in conformity with the hypothesis of the author.

"If the prism, which by its lower end is dipped into a fluid contained in an indefinite vessel, is inclined to the horizon, *the volume of the fluid which is raised in the prism above the level of the fluid in the vessel, being multiplied by the sine of the inclination of the sides of the prism to the horizon, is always the same, whatever this inclination may be.*"* [10089]

For this product expresses the weight of the column of the fluid raised above the level, and resolved in the direction parallel to the sides of the prism. [10090] The weight thus resolved must balance the action of the prism and the external fluid upon the fluid contained in it; and this action is evidently the same, [10091] whatever be the inclination of the prism; therefore the mean vertical height above the level is always the same [10089g, h]. [10092]

"If we place a prism vertically within another hollow vertical prism of the same substance, and then dip their lower ends into a fluid, *we shall have for the volume V of the fluid, elevated above its level in the space included between these two prisms, the following expression;*† [10093]

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* (4322) Supposing the density of the fluid to be uniform, and referring to fig. 145, page 835, we have, for the actions of the first and second tubes, and of the fluid within and about them, $2Q-Q'$ [9907k, or 9923]; and when the tube is vertical, this is put equal to [10089a] the gravity of the elevated column gDV, as in [9925]. But when the tube is inclined to [10089b] the horizon by the angle A, the capillary action $2Q-Q'$, in a direction parallel to the axis [10089c] of the tube, must be put equal to the gravity gDV, resolved in the same direction, which is $gDV.\sin.A$; hence we get $gDV.\sin.A=2Q-Q'$. Dividing this by gD, and [10089d] substituting $Q=\rho c$ [9927], $Q'=\rho' c$, we obtain

$$V.\sin.A=\frac{(2\rho-\rho').c}{gD}. \qquad [10089e]$$

As all the terms of the second member of this expression are constant, whatever be the angle A, it follows that $V.\sin.A$ must be a constant quantity, as in [10089]. Substituting [10089f] $V=hb$ [9983], in [10089e], and dividing by the constant quantity b, we get $h.\sin.A$, equal [10089g] to a constant quantity; and as $h.\sin.A$ represents the mean vertical height above the level, this height must be constant, as in [10092].

The same result holds good when we notice the change of density near the surface of the fluid and near the sides of the tube, using the expression of gDV [9920o, or 9920q], from [10089h] which we can deduce similar expressions to those in [10089e, g], ρ being changed into $\rho_{,}$, and ρ' into $\rho'_{,}$, as in [9920p, &c.].

† (4323) The capillary action of the outer tube is represented by $2Q-Q'$ [10089a], or $(2\rho-\rho').c$ [10089d], supposing the fluid to be of uniform density. In like manner the [10094a]

[10094] $$V = \frac{(2\rho - \rho')}{gD} \cdot (c + c')$$ [This is correct even when we notice the change of density near the surfaces, using $\rho_{,}$, $\rho'_{,}$, as in [9920r].]

[10094'] $$= \tfrac{1}{2} lq \cdot (c + c');$$

[10095] *c being the circumference of the inner base of the great prism, and c' that of the outer base of the least prism."*

[10096] This theorem may be easily demonstrated by means of the principles explained above. If the bases of the two prisms are similar polygons, whose
[10097] homologous sides are parallel, and placed at the same distance l from each other, we shall have $\frac{1}{2} l . (c + c')$ for the area of the base of the space included
[10097'] between the two prisms; and as h is the mean height of the elevated fluid [9983], we shall also have

[10098] $$V = \tfrac{1}{2} hl . (c + c');$$ [This is correct even when the density is supposed to be variable near the surfaces.]

consequently *

[10099] $$h = q;$$

that is, *the mean height h of the elevated fluid is the same as that of the fluid*
[10100] *which is elevated in a cylindrical tube whose radius l [9977'] is equal to the interval of the two prisms* [10097]. If we suppose the prisms to be cylinders,
[10101] we shall obtain the theorem in [9410]. We may also determine, by the same principles, what must take place in case the tubes are dipped wholly or in part into a vessel filled with any number of fluids, and we may also suppose that these prisms are inclined to the horizon.

The same things being supposed as in the preceding theorem, if the two
[10102] prisms are of different substances, we shall put ρ for the greatest tube, and

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capillary action of the inner tube, whose circumference is c' [10095], is $(2\rho - \rho') . c'$. The
[10094b] sum of these two capillary forces $(2\rho - \rho') . (c + c')$, is balanced by the gravity gDV of the
[10094c] vertical column of the fluid whose volume is V; hence we have $gDV = (2\rho - \rho') . (c + c')$. Dividing this by gD, we get the value of V [10094]; and by substituting the value of
[10094d] $2\rho - \rho'$ [9980], it becomes as in [10094']. Now changing, as in [10089h, &c.], ρ into $\rho_{,}$, and ρ' into $\rho'_{,}$, we get an expression of V similar to [10094], corresponding to the case
[10094e] of nature where the fluid varies in density near its surface and near the sides of the tube; and this is easily reduced to the form [10094'], supposing q to be determined by actual experiments.

[10099a] * (4324) The expressions of the value of the base [10097], and that of V [10098], correspond with the usual rules of mensuration. Putting this value of V equal to that in
[10099b] [10094'], we get $\frac{1}{2} hl . (c + c') = \frac{1}{2} lq . (c + c')$. Dividing this by $\frac{1}{2} l . (c + c')$, we get $h = q$, as in [10099].

ρ_1 for the least tube, what we have before denoted by ρ; and then we shall have *

$$V = \frac{(2\rho - \rho')}{gD}\,c + \frac{(2\rho_1 - \rho')}{gD}\,.\,c';$$ [This is correct even when we notice the change of density near the surfaces of the fluid, using appropriate values of ρ, ρ', ρ_1, as in [9920r, &c.].] [10103]

so that, if we put q and q_1 *for the elevations of the fluid in two very slender cylindrical tubes, having the same interior radius* l, and made respectively out of these substances, we shall have [10104]

$$V = \tfrac{1}{2}l\,.\,(qc + q_1c');$$ [This may be used when the density is supposed to be variable near the surfaces.] [10105]

consequently †

$$h = \frac{qc + q_1c'}{c + c'}.$$ [This may be used when the density is supposed to be variable near the surfaces.] [10105′]

This theorem is also easily demonstrated by the preceding principles. We ought to make q, q_1, negative, if the substances to which they correspond depress, instead of elevating, the fluid. We may obtain, by the same principles, the volume of the fluid which is elevated above the level, in a space included by any number of vertical planes of different substances. [10106]

It follows from the preceding theorem, that *the volume* V *of a fluid*

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* (4325) Making the calculation as in [10094a—e], we find as in [10094a], that the force $2Q - Q'$, corresponding to the outer tube, is $(2\rho - \rho').c$, and the similar force for the inner tube evidently becomes $(2\rho_1 - \rho').c'$, using the notation [10102]. The sum of these two expressions being put equal to gDV, as in [10094c], gives [10103a] [10103b]

$$gDV = (2\rho - \rho').c + (2\rho_1 - \rho').c'. \qquad [10103c]$$

Dividing this by gD, we get V [10103]. Now by [9980], we have

$$2\rho - \rho' = \tfrac{1}{2}gD.lq, \qquad [10103c']$$

corresponding to the outer tube; and in like manner for the inner tube, we have

$$2\rho_1 - \rho' = \tfrac{1}{2}gD.lq_1, \qquad [10103d]$$

using the notation in [10102, 10104]. Substituting these values in [10103], we get [10105]. These formulas are correct even when we notice the change of density near the surface of the fluid and near the sides of the tube, using in this case the corresponding numerical values of ρ, ρ', ρ_1, spoken of in [9920r, &c.], or the values of q, q_1, deduced from actual observation. [10103e]

† (4326) The mean height of the elevated fluid being represented by h [10097′], its volume will be as in [10098]; and by putting this equal to the expression [10105], we get

$$\tfrac{1}{2}hl.(c + c') = \tfrac{1}{2}l.(qc + q_1c'); \qquad [10105a]$$

dividing this by $\tfrac{1}{2}l.(c + c')$, we get the value of h [10105′].

*elevated by the capillary action, about a solid prism which is dipped, by its lower end, into the fluid, is**

[10107] $$V = \frac{(2\rho - \rho')}{gD}.c$$ [This may be used when the density is supposed to be variable near the surfaces.]

[10107′] $$= \tfrac{1}{2}lq.c;$$

[10108] *c being the horizontal circumference of the prism.* This volume expresses the augmentation of the weight of the prism depending upon the capillary action. In general, the augmentation of the weight of a body of any figure depending on that action, is equal to the weight of the mass of fluid which is raised
[10109] by that action above the level; and if the fluid is depressed below, the augmentation will be changed into a diminution of the weight; and the whole diminution of the weight of the body is represented by the weight of a mass of
[10110] the fluid which is equal to that displaced by the body, whether by the space it occupies below the level, or by the void space produced by the capillary action.

[10111] This principle includes the known principle of hydrostatics relative to the diminution of the weight of a body immersed in a fluid; it is only necessary to suppress what relates to the capillary action, which wholly disappears when the body is entirely immersed in the fluid.

To demonstrate the principle we have just mentioned, we shall consider a vertical canal, sufficiently wide to include the body and all the mass of fluid

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[10107a] * (4327) If we suppose the fluid to be homogeneous, and the matter of which the outer tube is composed to be such that the surface of the fluid near it is horizontal, we shall have,
[10107b] as in [9618], $2\rho - \rho' = 0$; which also follows from [9931]; because, in this hypothesis, ϖ [9892] becomes 90^{d}, whose cosine is 0. This value of $2\rho - \rho'$ being substituted in
[10107c] [10103], we get $V = \frac{(2\rho_1 - \rho')}{gD}.c'$; and if we suppose the outer tube to be at an infinite distance from the inner tube, this value of V will represent the effect of the inner prism. If
[10107d] we now suppose that ρ_1 is changed into ρ, and c' into c, or, in other words, that ρ, c,
[10107e] correspond to the inner prism, the expression [10107c] will become $V = \frac{(2\rho - \rho')}{gD}.c$.
[10107f] Substituting in this the value of $2\rho - \rho'$ [9980], it beoomes as in [10107′]; which may be used even when we notice the change of density near the surface of the fluid and near the
[10107f′] sides of the tube, as in [10089h, 10094e, &c.], changing ρ into ρ_1, and ρ' into ρ'_1 [9920p, &c.].
[10107g] In the case of water or alcohol elevated by a glass tube, we have very nearly $lq = \alpha^2$
[10107h] [9372l, &c.]; substituting this in [10107′], we get $V = \frac{1}{2}\alpha^2 c$; and if the prism is a cylinder whose radius is l, the circumference will be $c = 2\pi l$, and by substituting it, the preceding
[10107i] expression becomes $V = \pi . l\alpha^2$, which will be of use hereafter.

which is sensibly elevated by it, or the void space produced by the capillary action, and shall suppose that this canal, after having penetrated into the fluid, [10112] is bent horizontally, and then vertically upwards, retaining the same width throughout its whole extent. It is evident that, in the state of equilibrium, the weights contained in the two vertical branches of this canal must be equal to [10113] each other; consequently, the body, by its weight, must compensate for the void space produced by the capillary action; or, if it raises the fluid by this action, [10114] it must compensate by the decrease of specific gravity, for the weight of the elevated fluid. In the first case, this action raises the body, which might, by that means, be supported upon the surface, although specifically heavier than [10115] the fluid. In the second case, it tends to draw the body down into the fluid.

We shall now consider a solid, rectangular, and very thin prism, whose width is l, height h, and length a, placed horizontally upon a fluid, so that its [10116] greatest side a may be horizontal. *We shall suppose that it depresses the fluid about it*, and we shall put

$q =$ the mean capillary depression of the fluid below the level, in a [10117] cylindrical tube of the same matter as the prism, the radius of the tube being equal to l;

$D =$ the density of the fluid; [10117']

$iD =$ the density of the prism; [10118]

$x =$ the depth of the lower surface or base of the prism below the level of [10118'] the surface of the fluid.

Then, in a state of equilibrium, we shall have, by the preceding theorem,*

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* (4328) We shall, as an example, suppose the prism to be of glass, floating upon a surface of mercury contained in a vase; then it is evident that the weight of the glass is equal [10119a] to the weight of the mercury displaced by the prism [10114, &c.]. Now the area of the base of the prism being al, and its height h [10116], its volume must be ahl. Multiplying [10119b] this by the density iD [10118], and the gravity g, we get the whole weight or pressure of the prism, equal to $igD.ahl$. Again, as the base of the prism is depressed below the level [10119c] surface of the mercury by the quantity x, it displaces a volume of this fluid which is represented by $al \times x$. Multiplying this by the density D [10117'], and the gravity g, we get $gD.alx$, for the weight of this quantity of mercury. To this we must add the [10119d] weight of the volume $V = \frac{1}{2}lq.c$ [10107'], depressed around the prism by the capillary action; and as the circumference $c = 2(a+l)$ [10108, 10116], this weight becomes [10119e] $V.gD = gD.lq.(a+l)$; adding this to the other part $gD.alx$ [10119d], we get the [10119f] whole weight of the displaced mercury, equal to $gD.alx + gD.lq.(a+l)$. Putting this equal to the weight of the prism $igD.ahl$ [10119c], we obtain the equation [10119]. [10119g]

[10119] $$gD.alx+gD.lq.(a+l)=igD.ahl,$$

which gives

[10120] $$x=ih-q.\left(1+\frac{l}{a}\right)=h+(i-1).\left\{h-\frac{q}{i-1}.\left(1+\frac{l}{a}\right)\right\}.$$ [This is correct even when the density is variable near the surfaces.]

[10121] *Therefore, by supposing* h *to be less than* $\frac{q}{i-1}.\left(1+\frac{l}{a}\right)$, *the prism will not be wholly immersed in the fluid, although* i *exceed* 1; *that is, although the prism is denser than the fluid. It is in this way that a very slender steel cylinder, whose*
[10122] *contact with the water is prevented, either by a varnish, or by a small stratum of air which surrounds it, is supported at the surface of the fluid. If we place in this*
[10123] *manner, horizontally upon the surface of the water, two equal and parallel cylinders, which touch so that their ends project beyond each other, it is found that they will instantly slide upon each other, so as to bring their ends together.* The fluid being more depressed by the capillary action, at the end of each of them which is in contact with the other cylinder, than at the opposite end,* the base of this last part is more pressed than the other base; consequently each cylinder
[10124] has a tendency to unite more closely with the other; and since accelerative forces, acting upon a system of bodies, which is displaced a little from the situation of equilibrium, always carry them beyond that situation, the two cylinders must slide backwards and forwards upon each other, making

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Dividing this by $gDal$, we obtain the first value of x [10120]. The second value is easily deduced from the first, by a slight reduction; this is not inserted in the original work. From
[10119h] this last expression we see that, if $i>1$, and $h<\frac{q}{i-1}.\left(1+\frac{l}{a}\right)$, we shall have $x<h$, as in [10121, &c.].

* (4329) Supposing the cylinders $ABCD$, $abcd$, fig. 160, to be floating on the surface of the fluid,
[10123a] which is repelled from them to the distance represented by the dotted lines encompassing them, it will be evident, from the figure, that the cylinder $abcd$ is not

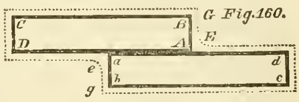

Fig. 160.

[10123b] so much pressed by the fluid near the corner a, as it is at the opposite end d, because the
[10123c] part of the cylinder $ABCD$ between aD repels the fluid from the corner a, and this does not take place at the opposite corner d. A similar effect is produced at the corner A of the cylinder $ABDC$. Therefore the cylinder $abcd$ is pressed at the end cd towards
[10123d] a more than it is pressed at the end ab towards d in an opposite direction. The resultant of these pressures is therefore equivalent to a pressure of the cylinder $abcd$, from A towards
[10123e] a; and in like manner the cylinder $ABCD$ will be pressed from D towards a; agreeing with the remarks in [10123, &c.]

oscillations which incessantly decrease, by the resistance they suffer, until they [10125]
entirely cease. These cylinders, having then attained their state of equilibrium, will touch each other at their ends.

We see, by what has been said, that the manner in which we have just
considered the capillary action, leads, in a very simple manner, to the principal [10126]
results of the theory of that action [9171—9756]. But the method employed in that theory has some advantages peculiar to itself. It makes known the
nature of the surfaces of the fluids contained within the capillary spaces, and [10127]
evidently proves that, in a slender cylindrical tube, the surface of the fluid is very nearly spherical [9336g]; and therefore the heights of its different points above the level differ but very little from each other. Moreover, we may thence conclude that in several tubes, formed of the same substance, dipped by their
lower ends into the same fluid, if their figure, at the upper surface to which the [10128]
fluid is elevated, is the same, the fluid will rise in all of them to the same height, whatever may be the figure of the lower parts of the tubes. This evidently
follows from the equilibrium of the fluid in an infinitely narrow canal passing [10129]
through the axis of each tube, below which it is bent, in order to pass upwards towards the level surface of the fluid. For it is plain that, if the figure of the
tubes be the same, in the parts where the fluid is raised, the surface of the fluid [10130]
will there be the same; consequently, also, the action of the fluid in the tube upon
that of the canal, will be the same in these tubes; the one of the canals being [10131]
supposed to be in equilibrium, the others will of course be so.

We shall here observe that there are several states of equilibrium in the same [10132]
tube, if its diameter is not uniform. Thus, by supposing two capillary tubes to communicate with each other,* so that the tube which has the least diameter may be placed vertically over that which has the greatest, we may conceive

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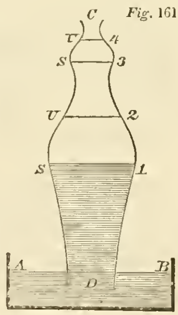
* (4330) To illustrate the remarks of the author relative
to the successive states of stability, we have drawn the annexed [10132a]
figure 161, where AB is the level surface of the fluid in a vase into which the curved tube CD is dipped; the lines
$S1$, $S3$, &c., represent the places in the tube where the fluid [10132b]
will rest in a stable equilibrium; and $U2$, $U4$, &c., the lines of unstable equilibrium. The diameters of the tube at the points
$S1$, $U2$, $S3$, $U4$, &c., form a decreasing series; these quantities [10132c]
are reciprocally proportional to the elevations of the fluid at those
points above the level of the fluid in the vase into which the tube [10132d]
is dipped.

Fig. 161.



their diameters and lengths to be such that the fluid may be at first in equilibrium
[10133] above the level in the greatest, and that, by pouring in more of the fluid, until it reaches the second tube, and fills up a part of it, the fluid will still maintain its equilibrium at this greater elevation.
[10134] *When the figure of a capillary tube decreases by moderate degrees*, the several states of equilibrium will be alternately stable and unstable. At first the fluid tends to rise in the tube, and this tendency decreases, and finally becomes nothing, in the state of equilibrium; beyond this point, this tendency becomes negative, and of course the fluid will
[10135] descend; therefore this first state is stable, because the fluid, when a little disturbed from it, tends to return to the same state. If we continue to elevate the fluid, its tendency to descend diminishes, and becomes nothing, in the second state of equilibrium; beyond that, it becomes positive, and the fluid tends to rise, consequently to move from that state which is not stable. By continuing in
[10136] this manner, we find that the third state will be stable, the fourth unstable, and so on.

Lastly, the comparison of the two methods [supposing the fluid to be
[10137] homogeneous] makes known the ratio of the quantities* ρ and ρ', or, in other words, the ratio of the quantities $\frac{1}{2}H$, $\frac{1}{2}H'$, by means of the angle ϖ, which the side of the tube forms with the tangent plane of the surface of the fluid, at the limits of the sphere of the sensible activity of the tube.
[10138] *These quantities represent the forces upon which the capillary action depends; they arise from the attractive forces of the particles of bodies, of which they are but modifications;*
[10139] *but they are incomparably less than these attractive forces, which, when they act with all their energy, are the chemical affinities themselves.*† If the law of the attraction, relatively to the distance, were the same for different bodies [and for

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[10137a] * (4331) The formula [9935] gives $\frac{\rho}{\rho'} = \cos.^2 \frac{1}{2}\varpi$, supposing the fluid to be of uniform
[10137b] density. Now we have in [9933], $\rho' = \frac{1}{2}H$, supposing H to correspond to the action of the fluid, or to the quantity ρ'; so that, for the sake of symmetry, we may accent H, and put
[10137c] $\rho' = \frac{1}{2}H'$ for the fluid, and, by a similar notation, $\rho = \frac{1}{2}H$ for the matter of the tube;
[10137d] substituting these in [10137a], we get $\frac{H}{H'} = \frac{\rho}{\rho'} = \cos.^2 \frac{1}{2}\varpi$ [10137].

† (4332) This is in conformity with what has already been stated by the author in
[10139a] [9257], relative to the comparative values of K, H, using for K its value corresponding to the interior of the fluid mass; observing, however, that the whole corpuscular action near the surface may be very small, or even negative, in the case of a fluid of variable density, as in [9174a].

a fluid of uniform density], the values of ρ and ρ' would be, as we have already observed [9927', 9929], proportional to the respective intensities of their attractions; that is, ρ and ρ' would be proportional to the constant coefficients [10140]
by which we must multiply the common function of the distance, which represents the law of these attractions; and then the values of ρ *and* ρ' *would* [10141]
correspond to equal volumes, and not to equal masses. To prove this, we shall suppose that there are two capillary tubes of the same diameter, made of different substances, but in which a fluid rises to the same height. It is evident, from what has been said, that, if we take, in these tubes, two equal and infinitely small volumes, similarly situated relative to the interior fluid, their [10141']
action upon the fluid will be the same, and we may substitute the one for the other; therefore we must, to obtain the ratio of their attractions with equal masses, divide the values of ρ by the respective densities of the different [10142]
bodies.

Hence it follows that the values of ρ, ρ', and ϖ, must vary with the temperature; and for an example we shall consider the case of a fluid which exactly moistens the glass, like *alcohol;* supposing the lower end of a glass [10143]
capillary tube to be dipped into that fluid, and that, at the temperature *zero*, the fluid rises to the height q above the level. We shall also suppose that, as the temperature increases, the density of the fluid decreases in the ratio of $1-\alpha$ [10144]
to 1; and, as in [9222], we shall take into consideration an infinitely narrow canal, which passes through the axis of the tube. Then the action of the fluid [10145]
meniscus formed by a horizontal plane passing through the lowest point of the surface of the fluid in the tube, will be decreased by the two following causes. *First:* Its density being less, its attraction will be decreased in the same ratio. For it is reasonable to suppose that this corpuscular attraction, in the same substance, is in proportion to its density; and it has been proved to be so, [10146]
relative to the action of gas upon light; since this action has been found, by very accurate experiments upon the same gas, to be strictly proportional to the density of the gas. *Second:* The action of the fluid meniscus upon the canal, decreases also with the density of the fluid in the canal, as is very evident. [10146']
The combined effect of these two causes decreases the value of H in the same proportion as the square of the density of the fluid is decreased, that is, in the ratio of $(1-\alpha)^2$ to 1. But the value of H, divided by the radius of the [10147]
tube l, which expresses the action of the meniscus upon the canal, must [10148]
balance the weight of the fluid which is drawn up into the canal by the capillary action; and this weight is equal to the product of the elevation of the fluid by its density and its gravity. If we represent this elevation by q', [10148']

[10149] the density of the fluid at the temperature *zero* by unity, and the force of gravity by g, we shall have the two equations,*

[10150a] * (4333) We have very nearly $\frac{1}{b}=\frac{\sin.\theta'}{l}$ [9351]; and as $\theta'=90^{d.}$ for alcohol [9362'], we get $b=l$, nearly. The effect of the variation of the measure of the diameter of the tube by the temperature, is noticed in [10153]. Now the action of the meniscus is represented in [9260, or 9354], by $\frac{H}{b}=gq$; which, by substituting the preceding value of
[10150b] b, becomes $\frac{H}{l}=gq$, as in [10150]. Again, when the temperature increases, H becomes
[10150c] $H.(1-\alpha)^2$, and the product of the altitude q by the density 1, changes into $q'.(1-\alpha)$ [10148, &c.]; making these alterations in [10150], it becomes as in [10151]. Substituting
[10150d] the value of $\frac{H}{l}$ [10150] in [10151], and then dividing by $g.(1-\alpha)$, we get the expression of q' [10152], which must evidently hold good even when we notice the change of density of the fluid near its surface, q and q' being the values corresponding to the case of nature.

[10150e] The principle made use of in [10147, &c.], relative to a fluid whose density varies in consequence of a change of temperature, may be applied to a mixture of two fluids, as water and alcohol. For this purpose we shall suppose that two infinitely small and equal
[10150f] volumes V, V', are taken in a fluid, so near to each other as to be within the limits of their mutual corpuscular attraction. If these spaces V, V', be filled with water, we shall suppose
[10150g] the capillary action or intensity H [9262c] to be represented by $H_{w,w}$; if they be both filled with alcohol, we shall represent it by $H_{a,a}$; and if one be filled with water, and the other
[10150h] with alcohol, we shall represent it by $H_{a,w}$. If we now denote by w, a, two positive fractions, whose sum is equal to unity, and suppose the two fluids to be mixed together in the
[10150i] proportion of a volume w of water to that of a volume a of alcohol, the *mutual action* or intensity of the water contained in the spaces V, V', will be represented by $\mathrm{w}^2.H_{w,w}$, that
[10150k] of the alcohol by $\mathrm{a}^2.H_{a,a}$; the mutual action of the water in the space V upon the alcohol
[10150l] in the space V', will be represented by $\mathrm{aw}.H_{a,w}$, and in like manner that of the alcohol in the space V upon the water in the space V', by $\mathrm{aw}.H_{a,w}$. The sum of these four quantities
[10150m] gives the whole mutual action or intensity of the mixed mass of water and alcohol in the spaces V, V', namely,

[10150n] $$\mathrm{w}^2.H_{w,w}+2\mathrm{wa}.H_{a,w}+\mathrm{a}^2.H_{a,a};$$

[10150o] supposing always that the temperatures of the two fluids are equal to each other, and that they are the same before and after the mixture; neglecting always the change of intensity of the repulsive forces which may result from the absorption of the heat which accompanies the
[10150o'] mixture of the two fluids, and observing that $H_{w,w}$, $H_{a,w}$, $H_{a,a}$, are independent of w, a. Substituting the expression of the capillary intensity [10150n], instead of H, in the formula [9372z], and putting for brevity

[10150p] $$\frac{1}{gl}.H_{w,w}=f, \qquad \frac{2}{gl}.H_{a,w}=f_{\prime}, \qquad \frac{1}{gl}.H_{a,a}=f',$$

$$\frac{H}{l}=gq,$$ [10150]

Correct even when we notice the variable density of the fluid near its surfaces, using H as it is given by observation.

$$\frac{H}{l}\cdot(1-\alpha)^2=gq'\cdot(1-\alpha);$$ [10151]

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it becomes

$$D\cdot(q+\tfrac{1}{3}l)=\mathrm{w}^2\cdot f+\mathrm{wa}\cdot f_{\prime}+\mathrm{a}^2\cdot f',$$ [10150q]

$f, f_{\prime}, f'$, being independent of w, a; and if we put successively $\mathrm{w}=1$, $\mathrm{a}=0$, and $\mathrm{w}=0$, $\mathrm{a}=1$, we shall evidently see that f is the value of $D\cdot(q+\frac{1}{3}l)$, corresponding to water, and f' the value of $D\cdot(q+\frac{1}{3}l)$, corresponding to alcohol. [10150r]

The formula [10150q] is founded upon the supposition that the loss of heat which takes place during the mixture of the two fluids, when the temperature has become the same as before, has no influence on the integral value of H [9253', &c.], relative to the corpuscular attraction upon which the capillary phenomena depend. This agrees with the experiment which makes the variation of the height q proportional to the increment of density, in the case of the same fluid at different temperatures; and it is interesting to ascertain whether this hypothesis agrees equally well in the mixture of two fluids. For this purpose we shall apply the formula [10150q] to some experiments of M. Gay-Lussac, published by M. Poisson, and computed by him in page 294 of his *Nouvelle Théorie, &c.* [10150s] [10150t] [10150u]

The first set of these experiments were made with mixtures of water and alcohol, the temperature being between 8 and 9 degrees, and the radius of the tube $l=0^{\mathrm{mi.}},648$. The results of the experiments are given in a tabular form in [10150y]. The first column of the table contains the volume of water w, the second the volume of alcohol a, making the whole volume $\mathrm{w}+\mathrm{a}=1$; the third column contains the density D of the mixture, as found by observation; the fourth column, the observed elevation q of the lowest point of the capillary surface; the fifth and sixth columns are the results of calculations of the values of q made in [10151d]; the density of water being taken for unity, and a millimetre for the unit of measure. [10150v] [10150w] [10150x]

Volume of the water w.	Volume of the alcohol a.	Observed density of the mixture D.	Observed elevation q.	Computed elevation q.	Differences of the observed and computed values of q.	
1	0	1,0000	$23^{\mathrm{mi.}},16$			1
$\frac{4}{5}$	$\frac{1}{5}$	0,9779	13 ,77	$16^{\mathrm{mi.}},608$	$+2^{\mathrm{mi.}},838$	2
$\frac{2}{3}$	$\frac{1}{3}$	0,9657	11 ,31	13 ,131	+1 ,821	3
$\frac{1}{2}$	$\frac{1}{2}$	0,9115	10 ,00			4 [10150y]
$\frac{1}{3}$	$\frac{2}{3}$	0,9068	9 ,56	8 ,235	−1 ,325	5
$\frac{1}{5}$	$\frac{4}{5}$	0,8726	9 ,40	7 ,861	−1 ,539	6
0	1	0,8196	9 ,182			7

The observations in line 1 of this table, namely, $\mathrm{w}=1$, $\mathrm{a}=0$, $D=1$, $q=23^{\mathrm{mi.}},16$, being substituted in [10150q], together with $\frac{1}{3}l=0^{\mathrm{mi.}},216$ [10150v], give $f=23^{\mathrm{mi.}},376$. [10150z]
The observations in line 7, namely, $\mathrm{w}=0$, $\mathrm{a}=1$, $D=0,8196$, $q=9^{\mathrm{mi.}},182$, $\frac{1}{3}l=0^{\mathrm{mi.}},216$, [10151a]

whence we deduce

[10152] $$q' = q.(1 - \text{a}).$$ [Correct even when we notice the change of density near the surfaces.]

Thus the elevation of the fluid in the same tube, at different temperatures, is in

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[10151b] being substituted in the same formula [10150q], give $f' = 7^{\text{mi.}},703$. To find $f_{,}$, we shall take the observations in line 4, namely, $\text{w} = \frac{1}{2}$, $\text{a} = \frac{1}{2}$, $D = 0{,}9415$, $q = 10^{\text{mi.}},00$, and
[10151b′] $\frac{1}{3}l = 0^{\text{mi.}},216$; substituting these, together with the preceding values of f, f', in [10150q], we get $f_{,} = 7^{\text{mi.}},395$. With these values of $f, f', f_{,}$, we deduce from [10150q] the following equation for all values of w, a;

[10151c] $$D.(q + \tfrac{1}{3}l) = 23^{\text{mi.}},376.\text{w}^2 + 7^{\text{mi.}},395.\text{wa} + 7^{\text{mi.}},703.\text{a}^2.$$

[10151d] Substituting in this equation the values of w, a, given in [10150y, lines 2, 3, 5, 6], we get successively the corresponding values of q in the fifth column of the table; the differences between these and the numbers in column 4 represent the errors of the values of q deduced from the formula [10150q], and they show that the hypothesis upon which it is founded does not agree well with mixtures of water and alcohol. Now this is so much the more singular,
[10151e] because the formula agrees perfectly well with mixtures of water and nitric acid, as we shall see in [10151g, m], and there is in both cases a similar loss of heat and a concentration of the fluids.

The second set of experiments were made with mixtures of water and nitric acid, the
[10151f] temperature being between 10 and 12 degrees, and the radius of the tube $l = 0^{\text{mi.}},6565$. The results of the experiments are given in the table [10151g], which is similar to that in [10150y]; the fifth and sixth columns of the table [10151g] are the results of the calculations made in [10151m].

[10151g]

Volume of water w.	Volume of nitric acid. a.	Observed density of the mixture D.	Observed elevation q.	Computed elevation q.	Differences of the observed and computed values of q.	
1	0	1,0000	$22^{\text{mi.}},68$			1
$\frac{4}{5}$	$\frac{1}{5}$	1,0891	20 ,52	$20^{\text{mi.}},495$	$-0^{\text{mi.}},025$	2
$\frac{2}{3}$	$\frac{1}{3}$	1,1474	19 ,17	19 ,167	-0 ,003	3
$\frac{1}{2}$	$\frac{1}{2}$	1,2151	17 ,66	17 ,677	$+0$,017	4
$\frac{1}{3}$	$\frac{2}{3}$	1,2751	16 ,35	16 ,357	$+0$,007	5
0	1	1,3691	14 ,08			6

[10151h] The values of f, f', are found as in [10150z, &c.]; first by putting $\text{w} = 1$, $\text{a} = 0$, $D = 1$, $q = 22^{\text{mi.}},68$ [10151g, line 1], and $\frac{1}{3}l = 0^{\text{mi.}},219$ [10151f], in the equation [10150q], whence
[10151i] we get $f = 22^{\text{mi.}},899$; then, substituting in the same equation the values $\text{w} = 0$, $\text{a} = 1$, $D = 1{,}3691$, $q = 14^{\text{mi.}},08$ [10151g, line 6], and $\frac{1}{3}l = 0^{\text{mi.}},219$, we get $f' = 19^{\text{mi.}},576$. The value of $f_{,}$ may be found by combining the four observations in [10151g, lines 2, 3, 4, 5],
[10151k] each of which gives an equation like [10150q], containing $f, f_{,}, f'$. Taking the sum of these four equations, and substituting in this sum the preceding values of f, f', we finally obtain

the ratio of its density. We have not here noticed the dilation of the tube, which, by increasing its interior diameter, decreases its elevation. By noticing [10153] it, we shall have the following theorem, which must hold good with all fluids which, like alcohol, appear to possess a perfect fluidity: "*The elevation of a fluid which perfectly moistens the sides of a capillary tube, is, at different* [10154] *temperatures, in the direct ratio of the density of the fluid, and in the inverse ratio of the interior diameter of the tube.*"

Effect of a change of temperature.

ON THE APPARENT ATTRACTION AND REPULSION OF SMALL BODIES WHICH FLOAT UPON THE SURFACE OF A FLUID.

We have reduced to analytical expressions, in [9552—9586], the apparent mutual attraction of two homogeneous, vertical, and parallel planes of a sensible thickness, dipped by their lower ends into a fluid; and we have shown that this [10155] capillary action tends to bring them together, whether the fluid be elevated or depressed in the interval between them. Each plane then experiences a pressure * towards the other, which is represented by the weight of a prism of the same fluid, whose height is equal to the half sum of the elevations of the fluid above the level, or the depressions below it, of the extreme parts of contact [10156]

$f_{,}=44^{\text{mi.}},510$. Now substituting these values of $f, f_{,}, f'$, in [10150q], we get the following expression for all values of w, a;

$$D.(q+\tfrac{1}{3}l)=22^{\text{mi.}},899.\text{w}^2+44^{\text{mi.}}510.\text{wa}+19^{\text{mi.}},576.\text{a}^2. \qquad [10151l]$$

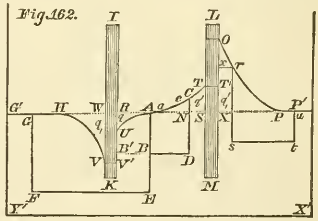
Substituting in this equation the values of w, a, given in [10151g, lines 2, 3, 4, 5], we get successively the values of q contained in the fifth column of the table; the differences between these and the corresponding numbers in column 4, are given in column 6, as the errors of the formula, which, in this case, are extremely small. [10151m]

If we compare the densities of these mixtures, we shall find that those of water and alcohol suffer nearly the same condensation as those of the acid and water. Thus the mean of the densities 1,0000 and 0,8196 of water and alcohol [10150y, lines 1, 7] is 0,9098, that of the mixture of equal parts of the fluids being 0,9115 [10150g, line 4], which is greater by 0,0317. In like manner, the mean of the densities 1,0000 and 1,3691 [10151g, lines 1, 6] is 1,1845; the difference between this and 1,2151 [10151g, line 4] is 0,0306, which is the excess of the density of the mixture of the two fluids. [10151n] [10151o]

* (4334) This supposes that both sides of the planes have the same action on the fluid, or that the angles of contact ϖ, ϖ', of the planes with the fluid are the same on opposite sides of the planes respectively. If this be not the case, it will be necessary to apply a correction to the expression of the pressure given in [10156], as in [9580q, &c.]. [10156a]

of the inner and outer surfaces of the fluid with the plane, and whose base is the part of the plane included between the two horizontal lines drawn through those points [9580, 9585]. This theorem contains the true cause of the apparent attraction of bodies which swim upon a fluid, when it is elevated or depressed near them. But it is found by experiment that the bodies repel each other
[10157] when the fluid is elevated near to one of the planes, and is depressed near to the other. To account for this phenomena, we shall here consider generally the apparent repulsion of two vertical and parallel planes of different substances, dipped at their lower ends into the same fluid.

We shall suppose the fluid to be depressed near the first plane, and elevated
[10158] *near the second; the section of the surface of the fluid included between them*
The surface has a point of contrary flexure. *will have at first a point of contrary flexure, if the two planes are very distant from each other;*
[10158′] *this point is upon a level with the indefinite surface of the fluid in which we suppose the planes to be dipped;* for, if we suppose an
[10159] infinitely narrow canal to pass through this point, and then to be bent so as to pass below one of the planes, and to terminate far from them, at the surface
[10160] of the exterior fluid, the radii of curvature of the surface of the fluid being infinite at both extremities of this canal, it must be upon a level in both
[10161] branches. This being premised, we shall put*



* (4335) To illustrate this, we have given the annexed figure 162, in which IK is the first plane,
[10161a] LM the second plane; and the surface of the fluid is limited by the level parts $G'G$, PP', and the curved parts HV, UAT, OP. C is the point of the surface corresponding to the rectangular
[10161b] coordinates $y = RN$, $z = NC$ [10162, 10163]; A is the point of inflexion [10158]. $AEFG$ is an extremely slender cylindrical canal, the area of whose base is equal to unity, having two vertical branches EA, FG, and a horizontal branch
[10161c] EF; $ABDC$, a similar canal, with the two vertical branches BA, DC, and a horizontal branch BD; lastly, the level of the fluid in the vase is represented by the horizontal line
[10161d] $GHWRASXPP'$. If the planes are at a great distance from each other, the curvature of the surface, near the planes at U, T, will extend only to a small distance; and at some intermediate point, as at A, the surface will have a point of inflexion, as is evident from the
[10161e] consideration that the concavity turns upwards near T, and downwards near U. This also
[10161f] follows from the principle, that, at a point of inflexion, the *radius of curvature must be infinite*, as at the point G of the level surface without the planes; therefore, the capillary action at
[10161g] both these points must be equal; and then, from the equilibrium of the canal $AEFG$, we

$z =$ the elevation of any point of the section of the surface of the interior fluid, *above the level of the surface of the fluid*; considering z as negative when the elevation changes into a depression; [10162]

$y =$ the horizontal distance of the same point from the first plane. [10163]

Then we shall have *

Differential equation of the surface between two planes.

$$\frac{\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}} = 2\alpha z, \qquad \left[\begin{array}{c}\text{Differential equation of the curve} \\ \text{surface between two planes.}\end{array}\right] \qquad [10164]$$

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shall have $FG = EA$; consequently the point of inflexion A must be upon the level of the fluid in the vase, as is observed in [10158']; so that the value of z corresponding to this point of inflexion will be [10161h]

$$z = 0. \qquad [10161i]$$

We may finally remark that, when $z = 0$, the equation [10164] gives $\frac{ddz}{dy^2} = 0$, which is also a well known condition appertaining to points of inflexion; and this value of z corresponds to $R = \infty$, in the equation [10164c], given in the next note. [10161k]

* (4336) To conform to the notation which is used in [10162, 10163, 10168, 10175], and in [10184', 10188], we shall have in fig. 162, page 910,

$$NC = z, \qquad RN = y, \qquad RU = q, \qquad ST = q', \qquad WV = q_{\prime}, \qquad OX = q'_{\prime}. \qquad [10164a]$$

If we suppose the radii of curvature of the surface at C to be R, R'; and at A, to be b, b'; the equation [9315] will give

$$\frac{H}{2}\cdot\left(\frac{1}{R}+\frac{1}{R'}-\frac{1}{b}-\frac{1}{b'}\right) = gz. \qquad [10164b]$$

Now we have seen, in [10161f, &c.], that b, b', are infinite. Moreover, R' [9327', &c.], which corresponds to the plane drawn through C, perpendicular to Cc or to the plane of the figure, is infinite; hence the preceding equation becomes simply

$$\frac{H}{2}\cdot\frac{1}{R} = gz, \quad \text{or} \quad \frac{1}{R} = 2\frac{g}{H}\cdot z = 2\alpha z \quad [9328]. \qquad [10164c]$$

Substituting the value of $\frac{1}{R}$ [9326], it becomes as in [10164]; observing that, in the present notation, y takes the place of u; du or dy being supposed constant, as in [9327c]. If we multiply the equation [10164] by α^2, and then put, as in [9323p], $\alpha\alpha^2 = 1$, we shall get [10164d] [10164e]

$$\frac{\alpha^2\cdot\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}} = 2z, \qquad [10164f]$$

which is the same as given by M. Poisson, in page 174 of his *Nouvelle Théorie, &c.*, for the solution of this problem, changing y into x, and α into a, to conform to his notation.

This equation being multiplied by $-dz$, and then integrated, gives

[10165]
$$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}=\text{constant}-\alpha z^2.$$

ϖ
[10166] To determine the constant quantity, *we shall put* ϖ *equal to the acute angle formed by a tangent to the part of the section, placed at the limit of the sphere of sensible activity of the first plane, and a vertical plane.* Then we shall have at that point*

[10167]
$$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}=\sin.\varpi.$$

q
[10168] *Putting* q *for the depression of this point below the level*, we shall have at this point, $\alpha z^2=\alpha q^2$; therefore†

[10167a] * (4337) The angle ϖ [10166] is the same as the complement of θ' [9346]; and u being changed into y, as in [10164e], the expression [9389] will give

[10167b]
$$\cos.\varpi=\frac{\frac{dz}{dy}}{\sqrt{1+\frac{dz^2}{dy^2}}};$$

whence we easily deduce the value of $\sin.\varpi$ [10167]. This is also more easily obtained by the usual well known rule,

[10167c]
$$\sin.\varpi=\frac{\text{differential of the absciss}}{\text{differential of the curve}}=\frac{dy}{ds}=\frac{dy}{\sqrt{dz^2+dy^2}}=\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}.$$

† (4338) At the point U, fig. 162, page 910, where $y=0$, and $z=-q$, the equation [10168a] [10165] becomes $\sin.\varpi=\text{constant}-\alpha q^2$, as is evident from [10167]. This gives for the constant quantity, the value [10169]; and, by substituting it in [10165], we get [10170]; which, by using the abridged symbol Z [10171], becomes

[10168b]
$$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}=Z, \quad \text{or} \quad 1=Z^2.\left(1+\frac{dz^2}{dy^2}\right);$$

whence we easily deduce the value of dy [10172]. Again, the preceding value of Z may

[10168c] be put under the form $Z=\frac{dy}{\sqrt{dz^2+dy^2}}$; and if we put

[10168d] $\varpi_{,}=$ the angle formed at C by the infinitely small part Cc of the arc CA and the ordinate CD,

we shall have, as in [10167c],

[10168e]
$$\frac{dy}{\sqrt{dz^2+dy^2}}=\sin.\varpi_{,};$$

and then, from [10168c, e], we obtain

consequently

$$\text{constant} = \sin.\varpi + \alpha q^2;$$ [10169]

$$Z = \sin.\varpi_{\prime},$$ [10168f]

which will be of use hereafter. Now substituting this last value of Z in [10171], we get

$$\sin.\varpi_{\prime} = \sin.\varpi + \alpha q^2 - \alpha z^2,$$ [10168g]

or

$$\sin.\varpi_{\prime} + \alpha z^2 = \sin.\varpi + \alpha q^2,$$ [Equation of the curve surface between two planes.] [10168h]

which expresses the equation of the curve surface of the fluid, in terms of $\varpi_{\prime}$, z, to which we may have occasion to refer in the course of these notes. [10168i]

The equation [10172] is similar to [9421], and like it may be integrated by means of elliptical functions, as in [9415m—9416n]; a somewhat different process must, however, be used, where there is a point of inflexion A, fig. 162, page 910; and we shall now proceed to give in detail the process of integration corresponding to this case. For greater simplicity, we shall change the origin of y, and instead of supposing it to be at the point R, near the first plane, we shall suppose it to be at the point of inflexion A, and we shall consider separately each of the two parts of the curve which terminate at this point, supposing the radical in [10168v, &c.] to have the same sign as dz. We shall also use the following additional symbols, referring to fig. 162; [10168k] [10168l]

Investigation of the surface of the fluid, when there is a point of inflexion.

$\alpha_{\prime} = RA =$ the distance of the point of inflexion from the first plane; [10168m]

$\alpha_{\prime}' = AS =$ the distance of the point of inflexion from the second plane; [10168n]

$\mathrm{y} = y - \alpha_{\prime} =$ the new absciss AN, corresponding to the ordinate $NC = z$; [10168o]

$i =$ the angle SAa, formed by the horizontal line AS, and the surface Aa, at A, supposing the point a to fall above the horizontal line AS. [10168p]

The differential of [10168o] gives $dy = d\mathrm{y}$; substituting this in [10165], multiplying by α^2, and substituting $\alpha\alpha^2 = 1$ [9323p], we get [10168q]

$$\frac{\alpha^2}{\sqrt{1 + \frac{dz^2}{d\mathrm{y}^2}}} = \text{constant} - z^2.$$ [10168r]

The constant quantity in the second member of this equation may be determined by observing that, at the point A, we have $z = 0$ [10161i], and $\varpi_{\prime} = 90^{d} - i$ [10168d, p]. Substituting this value of $\varpi_{\prime}$ in [10168e], we get, at the point A, [10168s]

$$\frac{1}{\sqrt{1 + \frac{dz^2}{d\mathrm{y}^2}}} = \cos.i;$$ [10168s']

multiplying this by α^2, we find that the first member of the product is the same as the first member of [10168r]; so that, by putting $z = 0$, as in [10168s], we get $\alpha^2 . \cos.i = \text{constant}$; hence the general expression of the equation [10168r] becomes

$$\frac{\alpha^2}{\sqrt{1 + \frac{dz^2}{d\mathrm{y}^2}}} = \alpha^2 . \cos.i - z^2.$$ [Equation of the curve surface between two planes.] [10168t]

[10170] $$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}=\sin.\varpi+\alpha q^2-\alpha z^2;$$

Squaring this equation, multiplying also by dy^2+dz^2, we get, by a slight reduction,

[10168u] $$(\alpha^2.\cos.i-z^2)^2.dz^2=\{\alpha^4-(\alpha^2.\cos.i-z^2)^2\}.dy^2$$
$$=\{\alpha^2-(\alpha^2.\cos.i-z^2)\}.\{\alpha^2+(\alpha^2.\cos.i-z^2)\}.dy^2;$$

whence we obtain

[10168v] $$dy=\frac{(\alpha^2.\cos.i-z^2).dz}{\sqrt{\{z^2+\alpha^2.(1-\cos.i.)\}.\{\alpha^2.(1+\cos.i)-z^2\}}}.$$

To reduce this to elliptical functions, we shall put $c=\cos.\frac{1}{2}i$, and then, from [1, &c.], Int., we obtain

[10168w] $$\cos.\tfrac{1}{2}i=c,\qquad \cos.i=2c^2-1=1-2b^2,\qquad \sin.^2\tfrac{1}{2}i=1-c^2=b^2,$$
$$1-\cos.i=2-2c^2=2b^2,\qquad 1+\cos.i=2c^2,\qquad b^2+c^2=1.$$

We shall then suppose that the relation between z and φ is expressed by means of the first form of the equation [10168x], which, by dividing the numerator and denominator by $\cos.^2\varphi=(1+\tang.^2\varphi)^{-1}$, becomes of the form [10168y]; from which we easily deduce the expression of $\tang.^2\varphi$ [10168z];

[10168x] $$z^2=\frac{2\alpha^2c^2.(1-c^2).\sin.^2\varphi}{1-c^2.\sin.^2\varphi}=\frac{2\alpha^2b^2c^2.\sin.^2\varphi}{1-c^2.\sin.^2\varphi}$$

[10168y] $$=\frac{2\alpha^2b^2c^2.\tang.^2\varphi}{(1+\tang.^2\varphi)-c^2.\tang.^2\varphi}=\frac{2\alpha^2b^2c^2.\tang.^2\varphi}{1+b^2.\tang.^2\varphi};$$

[10168z] $$\tang.^2\varphi=\frac{z^2}{(2\alpha^2c^2-z^2).b^2}=\frac{z^2}{(2\alpha^2c^2-z^2).(1-c^2)}.$$

In considering separately the two branches AT, AU, of the curve, fig. 162, page 910, and supposing the radical in [10168v] to have the same sign as dz, we shall have, in [10168v],
[10169a] $z^2<\alpha^2.\cos.i$, and in [10168w], $\cos.i<2c^2$, or $\alpha^2.\cos.i<2\alpha^2c^2$; hence $z^2<2\alpha^2c^2$; therefore the expression of $\tang.^2\varphi$ [10168z] must be positive, and φ a real quantity.

[10169b] We shall use for brevity the symbols $\Delta=\sqrt{1-c^2.\sin.^2\varphi}$, $X=\frac{\sin.\varphi.\cos.\varphi}{\Delta}$ [9416f];

then, substituting the values of $1\mp\cos.i$ [10168w], and the second expression of z^2 [10168x], in the first members of [10169e, g], we obtain their reduced values [10169f, h];
[10169b′] observing that $c^2.\sin.^2\varphi+\Delta^2=1$ [10169b], and

[10169c] $$\Delta^2-b^2.\sin.^2\varphi=1-(b^2+c^2).\sin.^2\varphi=1-\sin.^2\varphi=\cos.^2\varphi.$$

[10169d] Multiplying the two expressions [10169f, h] together, and taking their square root, we get [10169i];

[10169e] $$z^2+\alpha^2.(1-\cos.i)=z^2+2\alpha^2b^2=\frac{2\alpha^2b^2c^2.\sin.^2\varphi}{\Delta^2}+2\alpha^2b^2=\frac{2\alpha^2b^2}{\Delta^2}.(c^2.\sin.^2\varphi+\Delta^2)$$

[10169f] $$=\frac{2\alpha^2b^2}{\Delta^2};$$

and if we put [10170′]

$$\alpha^2.(1+\cos.i)-z^2=2\alpha^2c^2-z^2=2\alpha^2c^2-\frac{2\alpha^2b^2c^2.\sin.^2\varphi}{\Delta^2}=\frac{2\alpha^2c^2}{\Delta^2}.(\Delta^2-b^2.\sin.^2\varphi)$$ [10169g]

$$=\frac{2\alpha^2c^2}{\Delta^2}.\cos.^2\varphi;$$ [10169h]

$$\sqrt{\{z^2+\alpha^2.(1-\cos.i)\}.\{\alpha^2.(1+\cos.i)-z^2\}}=\frac{2\alpha^2bc.\cos.\varphi}{\Delta^2}.$$ [10169i]

Taking the differential of the second expression of z^2 [10168x], and dividing it by 2, we get, by successive reductions, its value [10169m]; dividing this by $z=\frac{\sqrt{2}.\alpha bc.\sin.\varphi}{\Delta}$ [10168x], [10169k] we obtain dz [10169m]; dividing this by the expression [10169i], we get [10169o],

$$zdz=\alpha^2b^2d.\left(\frac{c^2.\sin.^2\varphi}{1-c^2.\sin.^2\varphi}\right)=\alpha^2b^2.\left(\frac{2c^2d\varphi.\sin.\varphi.\cos.\varphi}{(1-c^2.\sin.^2\varphi)^2}\right)$$ [10169l]

$$=\frac{2\alpha^2b^2c^2.d\varphi.\sin.\varphi.\cos.\varphi}{\Delta^4}$$ [10169m]

$$dz=\frac{\sqrt{2}.\alpha bc.d\varphi.\cos.\varphi}{\Delta^3}$$ [10169n]

$$\frac{dz}{\sqrt{\{z^2+\alpha^2.(1-\cos.i)\}.\{\alpha^2.(1+\cos.i)-z^2\}}}=\frac{1}{\sqrt{2}.\alpha}.\frac{d\varphi}{\Delta}.$$ [10169o]

Substituting $\cos.i=1-2b^2$ [10168w], and z^2 [10168x], in the first member of [10169p], and making successive reductions, we get [10169q]; multiplying this by [10169o], we get the expression of dy [10168v], under the form [10169r];

$$\alpha^2.\cos.i-z^2=\alpha^2.(1-2b^2)-\frac{2\alpha^2b^2c^2.\sin.^2\varphi}{\Delta^2}=\alpha^2-2\alpha^2b^2.\left(1+\frac{c^2.\sin.^2\varphi}{\Delta^2}\right)$$ [10169p]

$$=\alpha^2-2\alpha^2b^2.\frac{1}{\Delta^2};$$ [10169q]

$$dy=\frac{\alpha}{\sqrt{2}}.\frac{d\varphi}{\Delta}-\frac{\sqrt{2}.\alpha b^2.d\varphi}{\Delta^3}.$$ [10169r]

Now we have, in [9416i], the identical equation $\frac{dX}{d\varphi}+\frac{b^2}{c^2\Delta^3}=\frac{\Delta}{c^2}$; multiplying this by $-\sqrt{2}.\alpha c^2.d\varphi$, and transposing the first term, we obtain [10169s]

$$\frac{-\sqrt{2}.\alpha b^2.d\varphi}{\Delta^3}=-\sqrt{2}.\alpha.d\varphi.\Delta+\sqrt{2}.\alpha c^2.dX;$$ [10169t]

substituting this in [10169r], we get, by reinstating the value of X [10169b],

$$dy=\frac{\alpha}{\sqrt{2}}.\frac{d\varphi}{\Delta}-\sqrt{2}.\alpha d\varphi.\Delta+\sqrt{2}.\alpha c^2d.\left(\frac{\sin.\varphi.\cos.\varphi}{\Delta}\right).$$ [10169u]

Integrating this expression, and using the elliptical symbols [8910k, l], we finally obtain

$$y=\frac{\alpha}{\sqrt{2}}.\mathrm{F}(c,\varphi)-\sqrt{2}.\alpha.\mathrm{E}(c,\varphi)+\sqrt{2}.\alpha c^2.\frac{\sin.\varphi.\cos.\varphi}{\Delta},$$ [10169v]

no constant quantity being added, because, at the point of inflexion A, we have $y=0$, [10169w] $z=0$, $\varphi=0$ [10168z]; finally, if we multiply [10169v] by $\frac{\sqrt{2}}{\alpha}$, we shall get

Z [10171]
$$Z = \sin.\varpi + \alpha q^2 - \alpha z^2,$$

[10169x]
$$\frac{y\sqrt{2}}{\alpha} = \mathrm{F}(c,\varphi) - 2\mathrm{E}(c,\varphi) + \frac{2c^2.\sin.\varphi.\cos.\varphi}{\sqrt{1-c^2.\sin.^2\varphi}}.$$

This is the same as the equation (11), in page 186 of M. Poisson's *Nouvelle Théorie, &c.*; and from it we may deduce the same relations which he has obtained between the values of

[10169y] the quantities i, q, $q_{,}$, ϖ, ϖ', &c., and the angles φ corresponding to the surface of the fluid near the two planes. Thus, by substituting the expression [10174], and $z = q'$ [10175], corresponding to the second plane, in the equation [10168t], we get successively, by using

[10169z] $\cos.i = 2c^2 - 1$ [10168w],

[10170a]
$$q'^2 = \alpha^2.(\cos.i - \sin.\varpi')$$

[10170b]
$$= \alpha^2.(2c^2 - 1 - \sin.\varpi').$$

[10170b'] In like manner we have at the first plane $q^2 = \alpha^2.(\cos.i - \sin.\varpi)$. Substituting the value of $z^2 = q'^2$ [10170b], in the last expression of [10168z], and putting Θ' for the value of φ,

[10170c] corresponding to this point, we shall get, by using $2c^2 - 1 = \cos.i$, and $1 - c^2 = \sin.^2\frac{1}{2}i$ [10168w],

[10170c']
$$\tan g.^2\Theta' = \frac{2c^2 - 1 - \sin.\varpi'}{(1-c^2).(1+\sin.\varpi')} = \frac{\cos.i - \sin.\varpi'}{\sin.^2\frac{1}{2}i.(1+\sin.\varpi')};$$

and the equation [10169x] will then become, by putting $y = \alpha'_{,}$ [10168n, o], which is its value near the second plane,

[10170d]
$$\frac{\alpha'_{,}\sqrt{2}}{\alpha} = \mathrm{F}(c,\Theta') - 2\mathrm{E}(c,\Theta') + \frac{2c^2.\sin.\Theta'.\cos.\Theta'}{\sqrt{1-c^2\sin.^2\Theta'}}.$$

[10170d'] If we put $\alpha_{,}$, ϖ, Θ, for what $\alpha'_{,}$, ϖ', Θ', become relative to the other extremity, we shall obtain another equation, which can be derived from [10170d], by changing $\alpha'_{,}$ into $\alpha_{,}$, and Θ' into Θ, namely,

[10170e]
$$\frac{\alpha_{,}\sqrt{2}}{\alpha} = \mathrm{F}(c,\Theta) - 2\mathrm{E}(c,\Theta) + \frac{2c^2.\sin.\Theta.\cos.\Theta}{\sqrt{1-c^2.\sin.^2\Theta}}.$$

Taking the sum of these two equations, and putting $2l$ [10210], for the distance of the two planes, we shall get

[10170f]
$$\frac{2l.\sqrt{2}}{\alpha} = \mathrm{F}(c,\Theta) + \mathrm{F}(c,\Theta') - 2\mathrm{E}(c,\Theta) - 2\mathrm{E}(c,\Theta') + \frac{2c^2.\sin.\Theta.\cos.\Theta}{\sqrt{1-c^2.\sin.^2\Theta}} + \frac{2c^2.\sin.\Theta'.\cos.\Theta'}{\sqrt{1-c^2.\sin.^2\Theta'}}$$

The equations [10170a, d, f] are equivalent to the equations (12), (13), (14), given by M. Poisson, in pages 186, 187, of his work. They serve to determine the form of the surface, supposing some of the quantities to be given. Thus, if we know the values of

[10170g] c, α, ϖ, ϖ', we shall have, from [10170c'], the value of Θ'; and from [10170c', d'], the value of Θ; then, from [10170f], we get the value of $2l$; finally, for any proposed angle φ, we get the absciss y [10169x], and the ordinate z [10168x]. This process requires the use of

[10170h] Le Gendre's tables of elliptical functions; but, the calculation being very simple, we shall not give any examples. If $2l$ is given instead of c, and it is required to find the value of c from

we shall have [10171']

$2l$, α, ϖ, ϖ', we must proceed as in [9417d—g], and compute for values of c, from $c=0$ to $c=1$, with small intervals, the corresponding values of Θ', Θ [10170c', d'], and those of $2l$ [10170f]; then, by inspection in this table, we may find the value of c, corresponding to any proposed value of the distance of the two planes $2l$. [10170i] [10170k]

We may change the origin of the coordinates, from the point of inflexion A, fig. 162, page 910, to the point S, near the second plane, by subtracting the expression [10169x] from that in [10170d], and then putting, in like manner as in [9417h], $y'=\alpha'_{\prime}-y$; hence we get [10170l]

$$\frac{y'\sqrt{2}}{a}=\mathrm{F}(c,\Theta')-\mathrm{F}(c,\varphi)-2\mathrm{E}(c,\Theta')+2\mathrm{E}(c,\varphi) + \frac{2c^2.\sin.\Theta'.\cos.\Theta'}{\sqrt{1-c^2.\sin.^2\Theta'}}-\frac{2c^2.\sin.\varphi.\cos.\varphi}{\sqrt{1-c^2.\sin.^2\varphi}}; \qquad [10170m]$$

y' being the absciss SN, counted from the point S in the second plane; and by changing Θ' into Θ, &c., we get a similar equation, where the origin of the abscisses is at the point R of the first plane. The formula [10170m] is the complete integral of the proposed differential equation [10161], and it can be used even in cases where methods of approximation would wholly fail. In the extreme cases, when the distance of the planes is either very great or very small, we may develop the formula in a series of terms of an approximative form. The first of these cases has already been treated of in another way in [9435, &c.]; and we shall now take into consideration the other case, where the distance of the planes is very small. [10170n] [10170o]

When $\varpi=\varpi'$, the point of inflexion A, fig. 162, page 910, will be in the middle of the line RS [10195, &c.]; consequently $\alpha_{\prime}=\alpha'_{\prime}$ [10168m, n]; and the branch AT, above the horizontal plane RS, will be exactly similar and equal to the branch AU, which falls below that plane; and this holds good however near the planes may be brought to each other. In the investigation of the figure of the surface of the fluid, corresponding to the case where ϖ, ϖ', are either equal, or differ but very little from each other, it will be convenient to have the second member of the expression [10170d] developed in a series, according to the powers of Θ', supposing Θ' to be small. This development we shall now make, neglecting terms of the order Θ'^4. [10170p] [10170q] [10170q']

Surface of the fluid where the two branches are equal and similar, with a point of inflexion between them.

In this case, we easily deduce from [43, 44, &c.], Int., the following expressions, using for abridgment the symbol $\Delta=\sqrt{1-c^2.\sin.^2\Theta'}$; [10170r]

$$\sin.\Theta'=\Theta'-\tfrac{1}{6}\Theta'^3; \qquad \cos.\Theta'=1-\tfrac{1}{2}\Theta'^2; \qquad \sin.\Theta'.\cos.\Theta'=\Theta'-\tfrac{2}{3}\Theta'^3; \qquad [10170s]$$

$$\Delta=\sqrt{1-c^2.\Theta'^2}=1-\tfrac{1}{2}c^2\Theta'^2; \qquad \Delta^{-1}=1+\tfrac{1}{2}c^2\Theta'^2; \qquad \frac{\sin.\Theta'.\cos.\Theta'}{\Delta}=\Theta'-\tfrac{2}{3}\Theta'^3+\tfrac{1}{2}c^2\Theta'^3. \qquad [10170t]$$

Substituting these values in the first members of [10170u, v, w], and making the necessary developments and integrations, we obtain respectively the second members of these three expressions. The sum of them being substituted in the second member of [10170d], gives [10170x],

[10172] $$dy=\frac{Zdz}{\sqrt{1-Z^2}};\qquad (i).$$

[10170u] $$\mathrm{F}(c,\Theta')=\int\frac{d\Theta}{\triangle}=\Theta'+\tfrac{1}{6}c^2\Theta'^3;$$

[10170v] $$-2\mathrm{E}(c,\Theta')=-2\int d\Theta'.\triangle=-2\Theta'+\tfrac{2}{6}c^2\Theta'^3;$$

[10170w] $$\frac{2c^2.\sin.\Theta'.\cos.\Theta'}{\triangle}=2c^2\Theta'-\tfrac{8}{6}c^2\Theta'^3+c^4\Theta'^3;$$

[10170x] $$\frac{\alpha'_{\prime}\sqrt{2}}{\alpha}=(2c^2-1).\Theta'-\tfrac{5}{6}c^2\Theta'^3+c^4\Theta'^3.$$

We may develop in the same manner the second member of [10169x]; for the equation [10169x] can be derived from [10170d], by changing $\alpha'_{\prime}$ into y, and Θ' into φ; and by making the same changes in [10170x], we get

[10170y] $$\frac{\mathrm{y}\sqrt{2}}{\alpha}=(2c^2-1).\varphi-\tfrac{5}{6}c^2\varphi^3+c^4\varphi^3.$$

[10170z] If we put $\varpi'=\frac{1}{2}\pi-\mu'$, $1+\sin.\varpi'=1+\cos.\mu'=2\cos.^2\frac{1}{2}\mu'$, in the second expression of $\tang.^2\Theta'$ [10170c'], we shall get, by neglecting Θ'^4,

[10171a] $$\Theta'^2=\frac{\cos.i-\cos.\mu'}{2\sin.^2\frac{1}{2}i.\cos.^2\frac{1}{2}\mu'}.$$

As Θ' is supposed to be very small [10170q'], the numerator of this last expression must also
[10171b] be very small, which requires that i should be nearly equal to μ'; therefore, $2\sin.\frac{1}{2}i.\cos.\frac{1}{2}\mu'$ is very nearly equal to $2\sin.\frac{1}{2}\mu'.\cos.\frac{1}{2}\mu'=\sin.\mu'$; so that, if we multiply [10171a] by the
[10171c] square of this expression of $\sin.\mu'$, we shall get very nearly $\Theta'^2.\sin.^2\mu'=2.(\cos.i-\cos.\mu')$, or $\cos.i=\cos.\mu'+\frac{1}{2}\Theta'^2.\sin.^2\mu'$; and as the second member of this expression is nearly equal to $\cos.(\mu'-\frac{1}{2}\Theta'^2.\sin.\mu')$, [61], Int., we shall have very nearly

[10171d] $$i=\mu'-\tfrac{1}{2}\Theta'^2.\sin.\mu'.$$

The angle φ, which enters into the equation [10168x, 10169x], being always less than Θ', we may neglect the cube of φ; and then the second expression in [10168x] gives, by taking
[10171e] its square root, $z=\sqrt{2}.abc.\varphi$, and [10170y] becomes $\frac{\mathrm{y}\sqrt{2}}{\alpha}=(2c^2-1).\varphi$. Eliminating φ from these two expressions, we get by successive reductions, and using the values [10168w],

[10171f] $$z=\mathrm{y}.\frac{2bc}{2c^2-1}=\mathrm{y}.\frac{2\sin.\frac{1}{2}i.\cos.\frac{1}{2}i}{\cos.i}=\mathrm{y}.\frac{\sin.i}{\cos.i}=\mathrm{y}.\tang.i;$$

and by substituting the value of i [10171d], and neglecting Θ'^2, we get very nearly

[10171g] $$z=\mathrm{y}.\tang.\mu';$$

so that the curve, in this case, will be very nearly a right line. This equation becomes, however, inaccurate when the divisor $2c^2-1$, or $\cos.i$ [10168w], is small; then we must
[10171h] retain the third power of φ and Θ', as in [10170x, y]. In taking this case into consideration,

*Putting ϖ' for the acute angle formed by a vertical plane and the tangent to a point of the section placed at the limit of the sphere of sensible activity of the second plane, we shall have** ϖ' [10173]

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we shall suppose, for greater simplicity, that $\varpi=\varpi'=0$, or $\mu'=\frac{1}{2}\pi$ [10170z]; and then [10171d] becomes

$$i=\tfrac{1}{2}\pi-\tfrac{1}{2}\Theta'^2; \qquad [10171i]$$

hence $\cos.i=\sin.(\frac{1}{2}\Theta'^2)=\frac{1}{2}\Theta'^2=2c^2-1$ [10168w]; therefore we have

$$c^2=\tfrac{1}{2}+\tfrac{1}{4}\Theta'^2, \qquad b^2=1-c^2=\tfrac{1}{2}-\tfrac{1}{4}\Theta'^2. \qquad [10171k]$$

Substituting this value of c^2 in [10170x], we get

$$\frac{\alpha'_{\prime}\sqrt{2}}{a}=\tfrac{1}{3}\Theta'^3; \qquad [10171k']$$

consequently

$$\tfrac{1}{2}\Theta'^2=\tfrac{1}{2}\sqrt[3]{\frac{18\alpha'^2_{\prime}}{a^2}}=(2c^2-1) \quad [10171k]; \qquad [10171l]$$

using this value of $2c^2-1$ in the first term of [10170y], and putting $c^2=\frac{1}{2}$ in the second and third terms, we get

$$\frac{y\sqrt{2}}{a}=\tfrac{1}{2}\varphi.\sqrt[3]{\frac{18\alpha'^2_{\prime}}{a^2}}-\tfrac{1}{6}\varphi^3; \qquad [10171m]$$

observing that the coefficient of $\frac{1}{2}\varphi$, in the first term of the second member, is of the order Θ'^2 [10171l], so that this term may be considered as of the same order φ^3 as the second term [10171m'] of the second member of the same equation. Substituting $bc=\frac{1}{2}$ nearly [10171k], in z [10171e], we get

$$z=\frac{a\varphi}{\sqrt{2}}, \quad \text{or} \quad \varphi=\frac{z\sqrt{2}}{a}; \qquad [10171n]$$

substituting this value of φ, in [10171m], and multiplying by $\frac{a}{\sqrt{2}}$, we obtain the following equation of the surface of the fluid;

$$y=\tfrac{1}{2}z.\sqrt[3]{\frac{18\alpha'^2_{\prime}}{a^2}}-\frac{z^3}{3a^2}; \qquad [10171o]$$

which corresponds to a cubic parabola, both terms being of the same order [10171m']. This [10171p] equation is the same as is given by M. Poisson, in his *Nouvelle Théorie, &c.*, page 192, for the case of $\varpi=\varpi'$, where there is a point of inflexion A, fig. 162, page 910, between the [10171q] two equal and similar branches AT, AU; changing y, $\alpha'_{\prime}$, a, into x, α, a, respectively, to conform to his notation.

* (4339) At the point T, fig. 162, page 910, where $z=ST=q'$ [10175], the expression of

$$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}} \quad [10167c], \qquad [10174a]$$

becomes $\sin.\varpi'$, as in [10174], which is exactly similar to [10167]; and the expression [10170] becomes as in [10176]; then, changing the signs, and transposing $\sin.\varpi$, we get [10177]. [10174b]

[10174] $$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}}=\sin.\varpi'.$$

q [10175] *Putting q' for the value of z corresponding to that point,* we shall have

[10176] $$\sin.\varpi'=\sin.\varpi+\alpha q^2-\alpha q'^2;$$

therefore

[10177] $$\sin.\varpi-\sin.\varpi'=\alpha q'^2-\alpha q^2; \qquad (r).$$

[10178] *Z cannot exceed unity;** and if the section has a point of inflexion, z will vanish† at that point, and then Z will be equal to $\sin.\varpi+\alpha q^2$ [10171].
[10179] Therefore $\sin.\varpi+\alpha q^2$ *is equal to or less than unity; and if it be equal to unity,* we shall have, from [10171],

[10180] $$Z=1-\alpha z^2;$$

consequently ‡

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* (4340) If Z be greater than unity, the expression of $\frac{dy}{dz}$, deduced from [10172], will
[10177a] become imaginary. The same also follows from the expression $Z=\sin.\varpi_{,}$ [10168f], which gives an imaginary value of $\varpi_{,}$, or ϖ' [10173], when $Z>1$.

† (4341) This is proved in [10161i], where it is shown that, at the point A, fig. 162, page 910, $EA=FG$; or, in other words, $z=0$. Substituting this in [10171], it becomes,
[10178a] for that point, $Z=\sin.\varpi+\alpha q^2$; and as Z is equal to or less than 1 [10178], it follows that
[10178b] $\sin.\varpi+\alpha q^2$ is equal to or less than 1, as in [10179]. If $\sin.\varpi+\alpha q^2=1$, the equation [10171] will become as in [10180].

‡ (4342) Substituting the value of Z [10180] in [10172], it becomes by reduction as in [10181], observing that the radicals are considered as positive; and we have inserted the
[10181a] sign $-$ in the second member in order to conform to the supposition in [10158]; from which it appears that near the first plane *the value of z is negative;* that dy, dz, have the same
[10181a'] signs, and that Z [10177a, &c.] is positive. Now putting, for brevity, $v=\sqrt{1-\frac{1}{2}\alpha z^2}$, we shall find that the integral of [10181] is

[10181b] $$y=\frac{1}{2\sqrt{2\alpha}}.\log.\left(\frac{1+v}{1-v}\right)-\frac{2v}{\sqrt{2\alpha}}+\text{constant};$$

as we can easily prove by taking its differential, which gives

[10181c] $$dy=\frac{1}{2\sqrt{2\alpha}}.\left(\frac{2dv}{1-v^2}\right)-\frac{2dv}{\sqrt{2\alpha}}=\frac{-dv}{\sqrt{2\alpha}.(1-v^2)}.\{-1+2(1-v^2)\}=\frac{-dv}{\sqrt{2\alpha}.(1-v^2)}.\{1-2v^2\}.$$

Now the differential of v [10181a'] gives

[10181d] $$dv=-\frac{\frac{1}{2}\alpha z dz}{\sqrt{1-\frac{1}{2}\alpha z^2}};$$

and as $1-v^2=\frac{1}{2}\alpha z^2$, we have

[10181d'] $$\frac{-dv}{1-v^2}=\frac{dz}{z\sqrt{1-\frac{1}{2}\alpha z^2}};$$

$$dy = \frac{-(1-\alpha z^2).dz}{z\sqrt{\alpha}.\sqrt{2-\alpha z^2}}.$$ [10181]

The integral of this equation being taken so that z may vanish between the limits of y, gives for y, and of course for the mutual distance of the two planes, an infinite value, [10181i]; therefore, when this distance is finite, and there is a point of inflexion in the section of the surface of the interior fluid, $\sin.\varpi + \alpha q^2$ will be less than unity [10181l]; consequently $\sin.\varpi' + \alpha q'^2$ will likewise be less than unity [10181m], in virtue of the equation [10177]. [10182] [10183]

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multiplying this by $1-2v^2 = -(1-\alpha z^2)$, we get

$$\frac{-dv}{(1-v^2)}.(1-2v^2) = \frac{-(1-\alpha z^2).dz}{z\sqrt{1-\frac{1}{2}\alpha z^2}}.$$ [10181e]

Dividing this last expression by $\sqrt{2\alpha}$, we get the last value of dy [10181c], under the form

$$dy = \frac{-dv}{\sqrt{2\alpha}.(1-v^2)}.(1-2v^2) = \frac{-(1-\alpha z^2).dz}{z\sqrt{\alpha}.\sqrt{2-\alpha z^2}},$$ [10181f]

being the same as in [10181]; therefore the expression [10181b] is the integral of [10181]. If we take the constant quantity, so that, when $y=0$, we shall have $z=-q$ [10168], $v=\sqrt{1-\frac{1}{2}\alpha q^2}$ [10181a']; and then we shall get for the complete value of y [10181b], the following expression; [10181g]

$$y = \frac{1}{2\sqrt{2\alpha}}.\left\{\log.\left(\frac{1+\sqrt{1-\frac{1}{2}\alpha z^2}}{1-\sqrt{1-\frac{1}{2}\alpha z^2}}\right) - \log.\left(\frac{1+\sqrt{1-\frac{1}{2}\alpha q^2}}{1-\sqrt{1-\frac{1}{2}\alpha q^2}}\right)\right\} - \frac{2\sqrt{1-\frac{1}{2}\alpha z^2}}{\sqrt{2\alpha}} + \frac{2\sqrt{1-\frac{1}{2}\alpha q^2}}{\sqrt{2\alpha}}.$$ [10181h]

At the first plane, where $z=-q$, this gives $y=0$; and when $z=0$, the logarithmic part of this expression becomes infinite, and we have $y=\infty$. Hence it appears that *we cannot obtain a value* $z=0$, *corresponding to a point of contrary flexure* [10178], *and to the assumed value* $\sin.\varpi.+\alpha q^2=1$ [10179], *without supposing the planes to be at an infinite distance from each other.* Therefore, if there is a point of contrary flexure, and the planes are at a finite distance from each other, we must have $\sin.\varpi+\alpha q^2<1$ [10179]; and as $\sin.\varpi+\alpha q^2 = \sin.\varpi'+\alpha q'^2$ [10177], we shall also have, as in [10183], $\sin.\varpi'+\alpha q'^2<1$. [10181$i$] [10181$k$] [10181$l$] [10181$m$]

The logarithmic part of the expressions [10181b, h] may be changed by observing that $\frac{1+v}{1-v} = \frac{1-v^2}{(1-v)^2} = \frac{\frac{1}{2}\alpha z^2}{(1-v)^2}$ [10181a']; and if we put v for the value of v, when $z=-q$, we shall, in like manner, obtain $\frac{1+\text{v}}{1-\text{v}} = \frac{\frac{1}{2}\alpha q^2}{(1-\text{v})^2}$. The difference of the logarithms of these two quantities is [10181n] [10181o]

$$\log\frac{1+v}{1-v} - \log.\frac{1+\text{v}}{1-\text{v}} = \log.\frac{(1-\text{v})^2.z^2}{(1-v)^2.q^2} = \log.\left(\frac{(\text{v}-1).z}{(1-v).q}\right)^2 = 2\log.\frac{(\text{v}-1).z}{(1-v).q};$$ [10181p]

observing that we have changed $(1-\text{v})^2$ into $(\text{v}-1)^2$, so as to render the number

[10184] When the planes are at an infinite distance from each other, y must be infinite,
$q_{,}$ which requires that Z should be equal to 1, when* $z=0$; therefore, *putting* $q_{,}$
[10184′] *for the depression of the fluid in this case*, or, in other words, *the depression of the fluid outside of the first plane*, we shall have

[10185] $$\alpha q_{,}^2+\sin.\varpi=1, \quad \text{or} \quad \alpha q_{,}^2=1-\sin.\varpi;$$

[10186] q is therefore less than $q_{,}$. Now if we apply the method of computation which is used in [9580*a*—*d*], we shall find that the first plane is pressed *outwards* by

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[10181*q*] whose logarithm is to be found, positive when z is negative, which corresponds with one of the forms of the equation where we have used the integral [10181*v*]. Substituting [10181*p*] in [10181*h*], and then replacing the values of v, v, we get

[10181*r*] $$y=\frac{1}{2\sqrt{2\alpha}}\cdot\left\{2\log.\frac{(\sqrt{1-\frac{1}{2}\alpha q^2}-1)\cdot z}{(1-\sqrt{1-\frac{1}{2}\alpha z^2})\cdot q}\right\}-\frac{2\sqrt{1-\frac{1}{2}\alpha z^2}}{\sqrt{2\alpha}}+\frac{2\sqrt{1-\frac{1}{2}\alpha q^2}}{\sqrt{2\alpha}}+\text{constant } r,$$

for the integral of [10181]; the constant r being added upon the supposition that $y=r$,
[10181*s*] when $z=-q$, instead of supposing $y=0$, when $z=-q$, as in [10181*g*]. Substituting
[10181*t*] $\alpha=\alpha^{-2}$ [9328], and making a few slight reductions, we find that the differential equation [10181] becomes as in [10181*u*], and its integral [10181*r*] can be put under the form [10181*v*], by altering the arrangement of the terms;

[10181*u*] $$dy=\frac{(z^2-\alpha^2)\cdot dz}{z\sqrt{2\alpha^2-z^2}};$$

[10181*v*] $$y=r+\sqrt{2\alpha^2-q^2}-\sqrt{2\alpha^2-z^2}+\frac{\alpha}{\sqrt{2}}\cdot\log.\frac{(\sqrt{2\alpha^2-q^2}-\alpha\sqrt{2})\cdot z}{(\alpha\sqrt{2}-\sqrt{2\alpha^2-z^2})\cdot q};$$

which will be of use hereafter.

* (4343) The value of y, corresponding to the distance of the second plane from the first, becomes infinite, when the distance of the planes is infinite, which happens
[10184*a*] when the denominator $1-\sqrt{1-\frac{1}{2}\alpha z^2}$ of the first term of [10181*h*] vanishes; and
[10184*b*] then we evidently have $z=0$. This value $z=0$ corresponds to the point of inflexion [10178], which is at an infinite distance from the planes, where the fluid must evidently be horizontal, or $\varpi_{,}=90^{d.}$ [10168*d*]; consequently $Z=\sin.\varpi_{,}=1$ [10168*f*];
[10184*c*] substituting these values in [10171], we get $1=\sin.\varpi+\alpha q^2$, *for the relation between* ϖ *and* q, *when the second plane is at an infinite distance from the first.* Now this may
[10184*d*] be considered as the actual situation of the fluid outside of the first plane, because we may suppose a plane parallel to IK, fig. 162, page 910, to be situated at an infinite distance in
[10184*e*] the direction IVG', without altering the curvature of the surface $G'HV$ near the first plane, and the equation [10184*c*] will correspond to this surface by merely changing q into $q_{,}$, as
[10184*f*] in [10184′], which makes it become $1=\sin.\varpi+\alpha q_{,}^2$, as in [10185], or, as it may be
[10184*g*] written, $\alpha q_{,}^2=1-\sin.\varpi$. But from [10183], we have $\sin.\varpi+\alpha q^2<1$, or $\alpha q^2<1-\sin.\varpi$; hence $\alpha q^2<\alpha q_{,}^2$, or $q<q_{,}$, as in [10186].

a force which is equal to the weight of a fluid prism, whose height is $\frac{1}{2}.(q_{,}+q)$, depth $(q_{,}-q)$, and width the same as that of the plane.* [10187]

The equation [10177] gives, generally, $\sin.\varpi'+\alpha q'^2$ less than unity [10181m]; [10188]
but when the planes are at an infinite distance, it gives this function equal to unity [10181i,k,l]. Let $q'_{,}$ *be what* q' *then becomes;* or, in other words, *let* $q'_{,}$ [10188']
be the elevation of the fluid outside of the second plane, $q'_{,}$ *will be greater* [10189]

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* (4344) If we continue the horizontal canal DB, fig. 162, page 910, to meet the first plane at B', below the point U, and above the point V', VV' being horizontal, it is evident that, if we neglect the pressure of the atmosphere, which acts equally on opposite sides of the first plane, we shall have for the capillary action of the fluid at A, the expression K [9259]; therefore the pressure at B, and in the canal BB', is equal to $K+gD\times AB$, g being the gravity, and D the density of the fluid. Subtracting from this the capillary action of the fluid at the first plane at B', namely K [9259], we obtain $gD\times AB$, for the pressure at the point B', *in the direction* BB'; and by putting $AB=RB'=w$, this pressure becomes gDw; which is similar to the expression gw [9580b], changing g into gD. Hence we find, by the method used in [9580a—d], that the whole pressure on the part UB', will be, as in [9580b], equal to $\frac{1}{2}gD.(w^2-RU^2)$, and in the whole space UV' will be $\frac{1}{2}gD.(RV'^2-RU^2)=\frac{1}{2}gD.(q_{,}^2-q^2)=\frac{1}{2}gD.(q_{,}+q).(q_{,}-q)$, which is evidently equal to the expression [10187]. Moreover, this pressure is outwards, *or in the repulsive direction* BB'. The pressures below VV', on opposite sides of this plane, are equal and opposite, and therefore destroy each other. Multiplying [10185] by α^2, and using $a\alpha^2=1$ [9323p], we get $q_{,}^2=\alpha^2.(1-\sin.\varpi)$; substituting this in [10187g], we get the expression of the repulsive force at the first plane [10187k]. In like manner we obtain from [10190n, 10189b, &c.] the similar expression [10187l], representing the repulsive force at the second plane; [10187a] [10187b] [10187c] [10187c'] [10187d] [10187e] [10187f] [10187g] [10187h] [10187i]

$$\tfrac{1}{2}gD.\{\alpha^2.(1-\sin.\varpi)-q^2\}=\text{repulsive force at the first plane;} \qquad [10187k]$$

$$\tfrac{1}{2}gD.\{\alpha^2.(1-\sin.\varpi')-q'^2\}=\text{repulsive force at the second plane.} \qquad [10187l]$$

If we substitute $q^2=\alpha^2.(\cos.i-\sin.\varpi)$ [10170b'] and $q'^2=\alpha^2.(\cos.i-\sin.\varpi')$ [10170a] in the expressions of the repulsive forces [10187k, l], they will become equal to each other, and will be expressed by [10187m]

$$\tfrac{1}{2}gD.\alpha^2.\{1-\cos.i\}=\text{repulsive force at the first or second plane.} \qquad [10187n]$$

Moreover, if the curve is symmetrical above and below the level of the fluid in the vase, or $\varpi=\varpi'$ [10170p], and the distance of the planes $2l$ is small in comparison with α, we may put, as in [10171d], $i=\mu'$ nearly, and then [10187n] becomes, by using $\mu'=\frac{1}{2}\pi-\varpi'$ [10170z]. [10187o]

$$\tfrac{1}{2}gD.\alpha^2.(1-\cos.\mu')=\tfrac{1}{2}gD.\alpha^2.(1-\sin.\varpi')=\text{the repulsive force of the planes;} \qquad [10187p]$$

which is independent of the distance of the two planes, and if we suppose $\varpi=\varpi'=0$, the repulsive force becomes $\frac{1}{2}gD.\alpha^2$. In all these calculations, we have supposed the angle ϖ to be the same on both sides of the first plane, and ϖ' to be the same on both sides of the second plane, so that the correction [9983x] vanishes. [10187q] [10187r]

*than** q'; and it follows also from the above theory [9580a, &c.], that the second plane will be pressed *outwards* by a force which is equal to the weight
[10190] of a fluid prism whose height is $\frac{1}{2}(q'_{,}+q')$, depth $(q'_{,}-q')$, and width that of
[10190'] the second plane,† *which we shall here suppose to be the same as that of the*

[10189a] * (4345) When the two planes are at an infinite distance from each other, q will change into $q_{,}$ [10184'], also q' into $q'_{,}$; and the equation [10181l] will become
[10189b] $\sin.\varpi+\alpha q_{,}^{2}=\sin.\varpi'+\alpha q'^{2}_{,}=1$ [10185]; or $\alpha q'^{2}_{,}=1-\sin.\varpi'$. But by [10181m], $\alpha q'^{2}<1-\sin.\varpi'$; therefore $\alpha q'^{2}_{,}>\alpha q'^{2}$, or $q'_{,}>q'$, as in [10189].

† (4346) To find the pressure on the second plane LM, we shall refer to fig. 163, which is similar to fig. 162, but drawn upon a larger scale, for the sake of distinctness. Through any point r of the surface of
[10190a] the fluid, a canal $rstu$ is drawn, whose branches sr, tu, are vertical, and st horizontal; moreover the line rx is drawn perpendicular to the plane OX. Then it is evident that the equal and opposite pressures on the parts of this plane below
[10190b] the horizontal line TT' mutually destroy each other; so that it is only necessary to notice the pressure in the part
[10190c] corresponding to $T'O$; therefore, if we put $Xx=w$, its extreme values will be $XT'=ST=q'$, and $XO=q'_{,}$. Now neglecting the atmospherical pressure, as in [10187b],

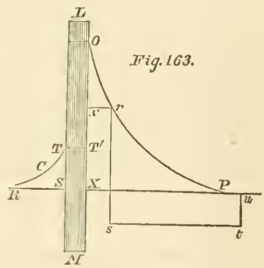

Fig. 163.

[10190d] we shall have, for the capillary action at r, the expression $K-\frac{H}{2b}$ [9558b]. Adding this to the gravity of the column rs, which is equal to $gD\times rs$, we get the pressure in the canal
[10190e] at s, equal to $K-\frac{H}{2b}+gD\times rs$. In like manner the capillary action at u is K [9259]; adding this to the gravity of the column tu, namely $gD\times tu$, we get the pressure at t, in the
[10190f] canal tu equal to $K+gD\times tu$; and this must be equal to the pressure at s; hence

[10190g] $$K-\frac{H}{2b}+gD\times rs=K+gD\times tu;$$

consequently

[10190g'] $$\frac{H}{2b}=gD.(rs-tu)=gD.Xx=gD.w \quad [10190c].$$

[10190h] Substituting this in the expression of the capillary action [10190d], it becomes $K-gD.w$. This being the action at any sensible distance from the surface, it must, from the nature of fluids, act in every direction, consequently in the direction rx. But the action of the fluid at x, in the direction xr, being K [9259], there must be an excess of pressure in the direction
[10190i] xr, equal to $gD.w$; that is, the pressure at x is *outwards*, and equal to $gD.w$. This is of
[10190k] the same form as that for the first plane in [10187e]; and by making the calculation as in [10187d—h], we shall find that the whole pressure on the part $T'O$, in the repulsive

first. We may thence conclude that the pressures suffered by the planes, and tending to repel them from each other, are equal. For the products [10191]

$$\tfrac{1}{2}(q'_{,}+q')\cdot(q'_{,}-q')=\tfrac{1}{2}(q'^{2}_{,}-q'^{2}),\quad\text{and}\quad \tfrac{1}{2}(q_{,}+q)\cdot(q_{,}-q)=\tfrac{1}{2}(q^{2}_{,}-q^{2}),$$ [10192]

are respectively equal to the following expressions,*

$$\frac{1}{2\alpha}\cdot(1-\sin.\varpi'-\alpha q'^{2}),\qquad\qquad \frac{1}{2\alpha}\cdot(1-\sin.\varpi-\alpha q^{2});$$ [10193]

and these two last quantities are equal, in virtue of the equation [10189*b*]. [10194]

There will always be an inflexion in the middle of the surface of the fluid included between the planes, if we have $\varpi=\varpi'$, however near these planes may be to each other; therefore the planes will repel each other at all distances.† [10195]

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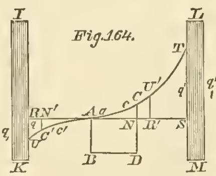
direction RS, is, as in [10187*g*], equal to

$$\tfrac{1}{2}gD.(XO^{2}-XT'^{2})=\tfrac{1}{2}gD.(q'^{2}_{,}-q'^{2})=\tfrac{1}{2}gD.(q'_{,}+q')\cdot(q'_{,}-q'),$$ [10190*l*]

which is evidently the same as in [10190], and in the *repulsive* direction RS. If we change [10190*m*] the signs of these expressions, we shall get $\tfrac{1}{2}gD.(q'^{2}-q'^{2}_{,})$ for the *pressure of the planes towards each other.* [10190*n*]

* (4347) We have, in [10189*b*], $q'^{2}_{,}=\frac{1-\sin.\varpi'}{\alpha}$, $q^{2}_{,}=\frac{1-\sin.\varpi}{\alpha}$; substituting these [10193*a*] values in the second members of the expressions [10192], they become respectively as in [10193]; and they are evidently equal to each other, as in [10181*m*]. Now the pressure on the first plane is $\tfrac{1}{2}gD.(q^{2}_{,}-q^{2})$ [10187*g*], and that on the second plane is [10193*b*] $\tfrac{1}{2}gD.(q'^{2}_{,}-q^{2})$ [10190*l*], neglecting the consideration of the widths of the planes, which are equal to each other [10190']; and as the factors $q^{2}_{,}-q^{2}$, $q'^{2}_{,}-q^{2}$, are equal [10192, 10194], [10193*c*] these pressures must be equal, as in [10191]. This is also proved in another way in [10187*n*].

† (4348) When there is a point of inflexion A, fig. 164, we shall have at that point $z=0$ [10161*i*]; or, in other words, this [10195*a*] point must fall on the horizontal line RS, on the level of the surface of the external fluid; hence it is evident that, if the curved surface of the fluid UAT falls wholly above this horizontal line RS, there will be no inflexion. This is also [10195*b*] evident from the equation $\frac{1}{R}=2\alpha z$ [10161*c*], which makes R



finite when z has a positive value, and this is incompatible with a point of inflexion. We shall now suppose that there is a point of inflexion A, which may be considered as the origin of the coordinates; we shall have for the general equation of the surface, the same expression as in [10168*g*], which, by putting, for brevity, a equal to the constant quantity $\sin.\varpi+\alpha q^{2}$, [10195*c*] becomes

$$\sin.\varpi_{,}=a-\alpha z^{2};$$ [10195*d*]

But if ϖ differ from ϖ', the line of inflexion of the surface will approach
[10195'] nearer to the first plane* if $\varpi > \varpi'$; but the point of inflexion will approach
[10196] nearer to the second plane if $\varpi < \varpi'$. If we now suppose that $\varpi > \varpi'$, we shall have $q_,$ less than† $q'_,$; or, in other words, the fluid will be less depressed without the first plane, than it will be elevated without the second. If we bring the planes gradually nearer to each other, the line of inflexion of
[10197] the surface will finally coincide with the first plane. For the equation [10177],

[10198] $$\sin.\varpi - \sin.\varpi' = \alpha q'^2 - \alpha q^2,$$

[10195e] $\varpi_,$ being the acute angle which the arc Cc of the curve makes with the ordinate $NC=z$ [10168d]. Hence it appears that, if we take, on opposite sides of the point A, two points of the curve C, C', so situated that their corresponding ordinates CN, $C'N'$, may be equal,
[10195f] we shall have, from [10195d], the values of $\sin.\varpi_,$ equal to each other, at the points C, C'; therefore the values of $\varpi_,$ will also be equal. Hence we easily perceive that the curves
[10195g] AC, AC', are perfectly similar; so that, if we put $AN=x$, $NC=z$, we shall have $AN'=-x$, $N'C'=-z$, angle $NCc=$ angle $NC'c'$. From this it evidently follows that
[10195h] if the point A is situated in the middle of the line RS, we shall have $AS=AR$, $RU=ST$, and the angle $\varpi=$ the angle ϖ', whatever be the distance RS of the planes,
[10195i] which will then always repel each other, as in [10191]. This agrees with what is stated in [10195].

* (4349) If the point A, fig. 164, page 925, be nearer to the first plane IK than to the
[10196a] second plane LM, and we make $AR'=AR$, then draw the ordinate $R'U'$, we shall have,
[10196b] as in [10195h], the angle $\varpi=$ the angle $\varpi_,$; $\varpi_,$ being the acute angle formed by the ordinate
[10196c] $R'U'$ and the curve at U'. Now by putting successively in [10195d] $z=R'U'$, $z=ST$,
[10196d] we easily perceive that the acute angle $\varpi_,$ is less at the point T than at the point U'; but the
[10196e] angle at the point T is represented by ϖ' [10173], and at U' it is equal to ϖ [10196b].
[10196f] Hence we have $\varpi > \varpi'$, when the point A falls nearest to the first plane, as in [10195'].
[10196g] By a similar process we can easily prove that, when the point A falls nearer to the second
[10196h] plane LM than to the first plane IK, we shall have $\varpi < \varpi'$, as in [10196]; the only change required in the demonstration and in the figure, is that the ordinate $R'U'$ should be drawn on
[10196i] the other side of the point A, between A and R, at a point N' where $AN'=AS$, AS being supposed less than AR.

† (4350) We have, in [10189b],

[10197a] $$\sin.\varpi + \alpha q_,^2 = \sin.\varpi' + \alpha q'^2_,;$$

hence

[10197b] $$\alpha.(q'^2_, - q_,^2) = \sin.\varpi - \sin.\varpi',$$

[10197c] ϖ, ϖ', being acute [10166, 10173]; hence it is evident that, when $\varpi > \varpi'$, we shall have $q'_, > q_,$, or $q_, < q'_,$, as in [10196].

shows that $\alpha q'^2$ always exceeds $\sin.\varpi - \sin.\varpi'$; and yet it is evident by the [10199]
equation [10172], that, if there be an inflexion in the surface of the interior fluid, q' will be of the same order as the distance of the planes from each other,* which, by approaching towards each other, may become less than any [10200]
assignable magnitude. There is, therefore, a limit of approximation, where this inflexion ceases, and where, of consequence, the line of inflexion must coincide with this first plane.† *Beyond this limit, while we gradually draw the* [10201]

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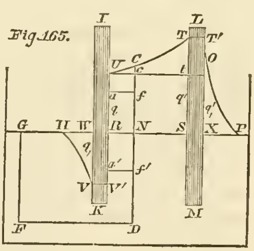
* (4351) If we put, as in [10168d], $\varpi_{,}$ equal to the acute angle which is formed by the vertical ordinate z, and the corresponding part of the curve surface, we shall have $Z = \sin.\varpi_{,}$ [10199a]
[10168f]; consequently $\frac{Z}{\sqrt{1-Z^2}} = \frac{\sin.\varpi_{,}}{\cos.\varpi_{,}} = \text{tang}.\varpi_{,}$. Substituting this in the equation [10199b]
[10172], we get $dy = dz.\text{tang}.\varpi_{,}$; whose integral is $y - \text{constant} = \int dz.\text{tang}.\varpi_{,}$. Now [10199$c$]
each of the elements $dz.\text{tang}.\varpi_{,}$, in proceeding from the point A to T, fig. 164, page 925, may be considered as of the same order as dz; and if we commence the integration [10199d]
at the point A, where $y = RA$ [10163], we shall have $y - RA = \int dz.\text{tang}.\varpi_{,}$; therefore $y - RA$ may be considered as of the same order as z; or, in other words, AS is of the [10199e]
same order as q'. In like manner, AR is of the same order as q. Hence $AS + AR$ [10199f]
or RS is of the same order as $q + q'$; so that we may consider $q + q'$ as being of the [10199f']
same order as the distance of the planes RS, which corresponds with the remarks of the author in [10200]. All that we have here stated is founded upon the supposition that there [10199g]
is a point of inflexion A, where $z = 0$. For if the point A fall above the line RS, by a quantity c, the integral $y - RA = \int dz.\text{tang}.\varpi_{,}$ [10199e] will not prove that AS is of the [10199h]
same order as q' [10199f], but that AS is of the same order as $q' - c$; therefore q', instead of being of the order AS [10199f], will be of the order $AS + c$. [10199h']

† (4352) In the case treated of in [10196, &c.], where ϖ, ϖ', are supposed to be given, and $\varpi > \varpi'$, we shall have $\sin.\varpi - \sin.\varpi'$ equal to a given finite and positive quantity, [10199i]
which we shall represent by C; substituting this in [10198], we shall get $\alpha.(q'^2 - q^2) = C$. [10199k]
Now we have seen, in [10199f, &c.], that q, q', are of the order RS; therefore $\alpha.(q'^2 - q^2)$, or its equal C, must not exceed a certain given quantity of the order $\alpha.RS^2$, which cannot be the case when RS is infinitely small; hence it follows that the supposition of there being a point of inflexion must be [10199k']
restricted to greater values of RS. We shall now suppose that the planes are placed, in the first instance, at a sufficient distance from each other to have a point of inflexion in the surface of the fluid between them, as at A, in fig. 164, page 925. [10199l] Then, moving the planes gradually towards each other, the point of inflexion A will approach towards the first plane, till it coincides with it at R; and the fluid will continue to rise between the planes as the distance is decreased, so that the surface will finally be wholly above the horizontal level of the fluid in the vase RS, as in fig. 165, [10199m]

Fig. 165.



[10201′] *planes towards each other, they will continue to repel each other until*

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where the surface is represented by UCT. Through any point C of this surface suppose the infinitely slender and uniform canal $CDFG$ to be drawn, with the vertical legs
[10199n] DC; FG, and the horizontal leg DF; then the capillary action at C is $K-\frac{H}{2b}$ [9558b], b being the radius of curvature at C; and the pressure at the bottom of the canal, being found, as in [10187c], is $K-\frac{H}{2b}+gD\times CD$, and at the bottom of the canal FG, is $K+gD\times FG$. Putting these expressions equal to each other, on account of the equilibrium
[10199o] in the canal FD, we get $\frac{H}{2b}=gD.(CD-FG)=gD\times NC$; consequently the capillary action at C [10199n] becomes $K-\frac{H}{2b}=K-gD\times NC$. Through any point f of the canal CD, above the horizontal line RS, draw the horizontal canal fa; then the pressure at
[10199p] f, in the canal Cf, will be equal to the weight of the column Cf, namely, $gD\times Cf$, increased by the capillary action at C, which we have just found to be equal to $K-gD\times NC$. Hence the pressure at f, in the direction fa, is equal to

[10199q] $$K-gD\times NC+gD\times Cf=K-gD.(NC-Cf)=K-gD\times Nf.$$

Now the capillary action of the fluid at a, in the opposite direction af, being equal to K [9259], the whole pressure at a is equivalent to the difference of these quantities, namely, $gD\times Nf$, in the direction af, or $-gD\times Nf$, in the *outward* or *repulsive* direction fa. Putting now $Nf=Ra=w$, we find that this repulsive action upon the space dw is $-gD.wdw$; and its integral, commencing at the point R, where $w=0$, is $-\frac{1}{2}gD.w^2$; so
[10199r] that the whole *repulsive* action upon the part RU is equal to $-\frac{1}{2}gD.RU^2$.

Instead of supposing the point f to be above the line RS, we shall now suppose it to fall below the line RS, as at f'; and we shall draw the horizontal canal $f'a'$; then we shall
[10199s] have, as in [10199q], $K-gD.(NC-Cf')$, or $K+gD.Nf'$, for the pressure in the point a', in the direction $f'a'$. Subtracting from this the capillary action in a', namely, K [9259], we get the whole action at a', equal to $gD\times Nf'$, in the repulsive direction $f'a'$.
[10199t] Putting now, as above, $Nf'=Ra'=w$, we find that the force on the space dw is equal to
[10199u] $gD.wdw$; and its integral, taken from $w=0$ to $w=RV'$, is $\frac{1}{2}gD.RV'^2$, which represents the action on the part RV', in the *repulsive* direction $f'a'$. The sum of the two forces [10199r, u], gives the whole force acting upon the plane UV', in the *repulsive* direction
[10199v] SR, equal to $\frac{1}{2}gD.\{RV'^2-RU^2\}$. From this it follows that, when $RU<RV'$, the
[10199w] factor RV'^2-RU^2 is *positive*, and this force will be *repulsive*. When $RU=RV'$, this factor vanishes, and the capillary action becomes equal to nothing. Lastly, when
[10199x] $RU>RV'$, this factor becomes *negative*, and the capillary action becomes *attractive*, as is
[10199y] stated in [10205, &c.]. We shall conclude with the remark that the computation of these
[10199z] forces, and the form of the curve UCT, will be treated of more fully in the remaining part of this note, by a somewhat different method, depending on elliptical functions.

[10200a] As it is an object of considerable interest to ascertain the form of the surface of the fluid,

the fluid is as much elevated above the level within the first plane [10201″]

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and the other phenomena, when the distance of the planes is very small, we shall here enter [10200b]
into a particular examination of the subject, using elliptical functions in finding the integrals, in nearly the same manner as is done by M. Poisson. In the annexed figures 166, 167, IK, LM, [10200c]
represent the two vertical planes; $UuPptT$, the surface of the fluid [10200d]
between them. The surface, in fig. 166, is supposed to be concave; [10200e]
and in fig. 167, the first branch UA is convex, the second branch AT is concave, and there is a point of inflexion A. Taking now, [10200f]
in the plane of each figure, the points u, t, of the surface, infinitely near to U, T, respectively, we shall put the angle $IUu=w$; the [10200g]
angle $LTt=w'$; the lines UI, TL, being always drawn in a vertical direction towards the upper part of the planes; so that when the surface is concave in both branches, as in fig. 166, the [10200h]
angles w, w', will be obtuse; and by comparing these values with those of ϖ, ϖ' [10166, 10173], we shall have, in this case, [10200i]
$w=\pi-\varpi$, $w'=\pi-\varpi'$. In fig. 167, where the surface is convex in the first branch UA, and concave in the second branch [10200k]
AT, we shall have $w=\varpi$, $w'=\pi-\varpi'$. Lastly, when the surface is convex in both branches, as in fig. 128, page 777, we shall have $w=\varpi$, $w=\varpi'$. This last case can be treated of in the [10200l]
same manner as the first; we shall not, therefore, notice it separately, as the calculation is not attended with any difficulty.

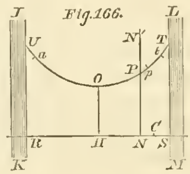
Fig. 166.

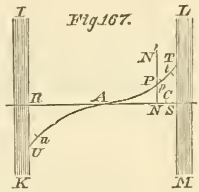
Fig. 167.

Case of a concave surface.

We shall first take into consideration the case corresponding to fig. 166, *where the surface is concave*, and the angles w, w', obtuse; and shall suppose HS to be the axis of y; HO, [10200m]
the axis of z; H, the origin of these coordinates; we shall then have, for the differential equation of the surface of the fluid, as in [9415f, or 10161],

$$\frac{\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}}=2\alpha z.$$ [Differential equation of the surface.] [10200n]

The form of this differential equation will not be altered, if we change the origin, from the point H to any point C of the line HS, near the second plane LM; since the only effect will be to change y into $y-HC$, which does not alter the differential dy in [10200o]
[10200n]. Therefore, if we suppose a vertical line NPN' to be drawn parallel to the second plane LM, and take the point C, situated between N and M, for the new origin of the rectangular coordinates y, z, the differential equation of the surface $PptT$, included between the planes NN', LM, will be represented by the equation [10200n]; the positive values of y being on the line CS, and the negative values on the line CN. Then [10200p]

[10201‴] *as it is depressed without it,* as we can easily satisfy ourselves by

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[10200p′] the obtuse angle $LTt = w'$, being equal to $\pi - \varpi'$ [10200i], we shall have, as in [10174, &c.],

[10200q] $$\frac{1}{\sqrt{1+\frac{dz^2}{dy^2}}} = \sin.w', \qquad \frac{\frac{dz}{dy}}{\sqrt{1+\frac{dz^2}{dy^2}}} = -\cos.w',$$

considering the radical as a positive quantity. Taking now any part of the surface TP,
[10200r] corresponding to positive values of y, and putting φ for the *obtuse* angle formed by the vertical ordinate z and the curve, we shall evidently have the same differential expression of $-\cos.\varphi$, as that of $-\cos.w'$ [10200q]; so that, for any point of the surface PT, corresponding to a positive value of y, we shall have

[10200s] $$\frac{\frac{dz}{dy}}{\sqrt{1+\frac{dz^2}{dy^2}}} = -\cos.\varphi, \qquad \text{[Equation of the surface.]}$$

using the values of y, z, corresponding to that point. If the point of the surface correspond to negative values of y, we may put $y = -y'$; and then [10200s] will become

[10200t] $$-\frac{\frac{dz}{dy'}}{\sqrt{1+\frac{dz^2}{dy'^2}}} = -\cos.\varphi.$$

[10200t′] If we put $\varphi = \pi - w_{,}$, so that $w_{,}$ may represent the *acute* angle formed by the vertical ordinate z and the curve, we shall have, by substitution, and changing the signs of all the terms,

[10200u] $$\frac{\frac{dz}{dy'}}{\sqrt{1+\frac{dz^2}{dy'^2}}} = -\cos.w_{,}. \qquad \text{[Equation of the surface.]}$$

If this expression be now supposed to correspond to the point P, where we shall put the acute
[10200u′] angle $N'Pp = w_{,}$, we shall have

[10200v] $$\frac{\frac{dz}{dy'}}{\sqrt{1+\frac{dz^2}{dy'^2}}} = -\cos.w_{,},$$

[10200w] supposing $y' = CN$ in the resulting expression. We shall now suppose that there is placed in the fluid a vertical plane NPN', parallel to LM, so that the surface which faces the plane LM may pass through the point P. We shall also suppose that the plane NPN' is of such a nature that $w_{,}$ may be the angle which specially agrees with the matter of the
[10200x] plane and of the fluid. Then it is evident that the equilibrium of the fluid will not be troubled between NN', LM; for the curve PT satisfies, by hypothesis, the general

the theory explained in [10199*i*—*x*]. In this case, *q being the* [10202]

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equation of the capillary surface [10200*n*]; moreover the equations [10200*q*, *v*], which correspond to the extreme points T, P, are also satisfied, when we consider the equilibrium of the fluid between the two planes NN', LM; $w_{,}$, w', being given angles. Thus we see [10200*y*]
that there is a state of equilibrium, in which the fluid is elevated along the sides of two parallel planes NN', LM, although the cosines of the angles $w_{,}$, w', corresponding to them, [10200*z*]
have contrary signs. The fluid in the case we have considered is elevated above the level in the vase, and its upper surface is concave, because we have supposed that the obtuse angle $w' = LTt$, corresponding to the second plane LM, is greater than the supplement of the acute angle $N'Pp = w_{,}$, corresponding to the plane NN'. [10201*a*] On the contrary, the fluid will be depressed, and the surface convex, if the obtuse angle is less than the supplement of the acute angle.

The curve surface PT can be obtained by means of the equation [9417*m*]; and if we change φ into Θ, so that it may refer to the plane NN', the corresponding absciss y' will [10201*b*]
represent the distance of the two planes $SN = 2l$ [9417*a*—*b*], for the case relative to fig. 166, page 929; and we shall have, by putting $y' = 2l$, and $\varphi = \Theta$, in [9417*m*],

To find l in terms of Θ, Θ'.

$$\pm\frac{2l.\sqrt{2}}{\alpha} = \frac{2-c^2}{c}.[\mathrm{F}(c,\Theta') - \mathrm{F}(c,\Theta)] - \frac{2}{c}.[\mathrm{E}(c,\Theta') - \mathrm{E}(c,\Theta)] + \frac{2c.\sin.\Theta'.\cos.\Theta'}{\sqrt{1-c^2.\sin.^2\Theta'}} - \frac{2c.\sin.\Theta.\cos.\Theta}{\sqrt{1-c^2.\sin.^2\Theta}}, \qquad [10201c]$$

the double sign being prefixed on account of the radical $\pm\sqrt{2}$; the upper sign is to be used [10201*d*]
when the second member of [10201*c*] is positive; the lower sign, when it is negative. The angles Θ, Θ', which correspond to the planes NN', LM, respectively, may be obtained [10201*d'*]
from [9416*w*], which gives, by using $1 - c^2 = b^2$ [10168*w*],

$$\tan g.^2\Theta = \frac{1-\sin.w_{,}}{(1-c^2).(1+\sin.w_{,})} = \frac{1-\sin.w_{,}}{b^2.(1+\sin.w_{,})}; \qquad \text{[To find } \Theta \text{ at the plane } NN'.] \qquad [10201e]$$

$$\tan g.^2\Theta' = \frac{1-\sin.w'}{(1-c^2).(1+\sin.w')} = \frac{1-\sin.w'}{b^2.(1+\sin.w')}. \qquad \text{[To find } \Theta' \text{ at the plane } LM.] \qquad [10201f]$$

The elevations q, q', of the fluid, near the points P, T, above the level of the fluid in the [10201*g*]
vase, are given by the formula [9416*s*]; from which we get, in the present notation,

[Internal elevations near the planes.]

$$q^2 = h^2 + \alpha^2.(1-\sin.w_{,}), \qquad [10201h]$$

$$q'^2 = h^2 + \alpha^2.(1-\sin.w'); \qquad [10201i]$$

h being the elevation of the point O, situated on the prolongation of TP, where the tangent [10201*k*]
to the surface would be parallel to the horizon [9415*k*], its value being given, as in [9416*c*], namely,

$$h = \frac{\alpha\sqrt{2}}{c}.\sqrt{1-c^2} = \frac{\alpha b\sqrt{2}}{c}. \qquad \text{[Elevation of the lowest point of the surface.]} \qquad [10201l]$$

These ordinates have the same signs, which are positive when $w_{,} + w' > \pi$, *and negative* [10201*m*]

[10202′] *elevation of the fluid between the two planes, and near the first*

[10201m′] *when* $w_{,}+w'<\pi$. In both cases, the planes are pressed towards each other, as we have seen in [9580, 9584].

Case where there is a point of inflexion.

We may proceed in the same manner when the surface has a point of inflexion A, as in fig. 167, page 929; and we may take a portion, situated between the two parallel planes NN', LM, on the same side of the point A, for a position of equilibrium; the cosines of the

[10201n] special angles $w_{,}$, w', corresponding to these two planes, having contrary signs. In this new state of equilibrium, as in that we have already treated of, the fluid will be elevated on both

[10201n′] planes, or depressed on both planes; but the figure of the included surface PT will be different in the two cases. In the present case, where there is a point of inflexion, we may determine the angles Θ, Θ', corresponding to the planes NN', LM, respectively, by means of the equations [10201p, p′], derived from [10170c′]; the vertical coordinate z, by means of the equation [10168x], which is the same as [10201q]; the ordinate y', whose origin is at the point S, by means of the equation [10170m], which is repeated in [10201r]; lastly, the

[10201o] modulus c may be derived from the equation [10201s], which is derived from [10201r], by putting $\varphi=\Theta$, and $y'=2l=SN$, prefixing the sign $\pm$ on account of the radical $\sqrt{2}$.

[10201p]
$$\text{tang.}^2\Theta=\frac{2c^2-1-\sin.w_{,}}{(1-c^2).(1+\sin.w_{,})}=\frac{\cos.i-\sin.w_{,}}{\sin.^2\tfrac{1}{2}i.(1+\sin.w_{,})};$$

[To find Θ, Θ', near the planes NN', LM.]

[10201p′]
$$\text{tang.}^2\Theta'=\frac{2c^2-1-\sin.w'}{(1-c^2).(1+\sin.w')}=\frac{\cos.i-\sin.w'}{\sin.^2\tfrac{1}{2}i.(1+\sin.w')}.$$

[10201q]
$$z^2=\frac{2\alpha^2c^2.(1-c^2).\sin.^2\varphi}{1-c^2.\sin.^2\varphi};$$

[10201r]
$$\frac{y'\sqrt{2}}{\alpha}=\text{F}(c,\Theta')-\text{F}(c,\varphi)-2\,\text{E}(c,\Theta')+2\,\text{E}(c,\varphi)+\frac{2c^2.\sin.\Theta'.\cos.\Theta'}{\sqrt{1-c^2.\sin.\Theta'.\cos.\Theta'}}-\frac{2c^2.\sin.\varphi.\cos.\varphi}{\sqrt{1-c^2.\sin.^2\varphi}};$$

Equations of the surface when there is a point of inflexion.

[10201s]
$$\pm\frac{2l\sqrt{2}}{\alpha}=\text{F}(c,\Theta')-\text{F}(c,\Theta)-2\,\text{E}(c,\Theta')+2\,\text{E}(c,\Theta)+\frac{2c^2.\sin.\Theta'.\cos.\Theta'}{\sqrt{1-c^2.\sin.^2\Theta'}}-\frac{2c^2.\sin.\Theta.\cos.\Theta}{\sqrt{1-c^2.\sin.^2\Theta}}.$$

This last equation is similar to [10201c], and the double sign $\pm$ is to be used in the same manner as in [10201d]; that is, the upper sign is to be used, if the second member of [10201s] is positive, otherwise the lower sign. The elevation of the fluid, near the second

[10201t] plane, is $ST=q'$; and this is given by the equation [10170a], which is the same as

[10201u] [10201w]; and by changing in it w' into $w_{,}$, and q' into $q=NP$, we get the value of q^2 [10201v]; observing that i is determined by the equation $\cos.\tfrac{1}{2}i=c$ [10168w];

Elevations q, q', near the planes NN', LM, and between them.

[10201v]
$$q^2=\alpha^2.(\cos.i-\sin.w_{,});$$

[10201w]
$$q'^2=\alpha^2.(\cos.i-\sin.w').$$

[10201x] *The values of* q, q', *must have the same signs, and we must observe the same rules as in*

plane, we shall have, from [10202, 10185], [10202″]

[10201m, m'], *namely, that* q, q', *must be positive when* $w_{,}+w'>\pi$, *and negative when* $w_{,}+w'<\pi$. In both cases, the planes will be pressed towards each other, as in [9580, 9584]. [10201y]

Hence it appears that, if there be two planes, of such a nature that, when dipped separately into the same fluid, it will rise near one of them, and sink near the other, we can have three different situations of equilibrium of the fluid included between these two planes, supposing them to be dipped vertically into the same fluid in parallel positions to each other. *The first of these cases* corresponds to fig. 167, page 929, where the fluid is elevated near the second plane LM, at T, and depressed near the first plane IK, at U, with a point of inflexion at A; then, if we suppose the angles $IUu=w$, $LTt=w'$, to be given, together with the distance $RS=2l$ of the two planes, we can determine c by the formula [10170f]; and the coordinates z, y, as in [10168x, 10169x]. *The second case* corresponds to fig. 166, page 929, supposing the planes to be NN', LM, and that we have given the angle $N'Pp=w_{,}$, the angle $LTt=w'$, and the distance of the two planes $NS=2l$. In this case, the modulus c is found by the formula [10201c]; the absciss y', by the formula [9417m]; and the ordinate z, as in [9416o]; then it is evident that if we continue the curve TP towards U, with coordinates having these relations, we can obtain a point O, where the tangent to the surface is parallel to the horizon. *The third case* corresponds to fig. 167, page 929, where the planes are NN', LM, and, as in [10201n], the angle $N'Pp=w_{,}$, the angle $LTt=w'$, $NS=2l$. Here the modulus is found by the formula [10201s]; the absciss y', by the formula [10201r]; the ordinate z, by the formula [10201q]; and by continuing the curve TP towards U, with coordinates having these relations, we can easily obtain a point A, where the surface passes through a point of inflexion. [10201z] [10202a] [10202b] [10202c] [10202d] [10202e] [10202f] [10202g] [10202h]

We shall now illustrate the two last of these three cases by examples, corresponding to the planes NN', LM, fig. 166, 167, page 929; and for greater simplicity we shall suppose the angle $LTt=w'=\pi$, and the angle $N'Pp=2v$, $2v$ being a very small quantity. *We shall, in the first place, notice the case which corresponds to fig.* 166, *where the continuation of the curve* TP *gives a point* O, *with a tangent parallel to the horizon.* Substituting the value of $w'=\pi$, in the second value of $\text{tang.}^2\Theta'$ [10201f], we get $\text{tang.}^2\Theta'=\frac{1}{b^2}$; whence we easily deduce [10202i]

$$\text{tang.}\Theta'=\frac{1}{b},\qquad \sin.\Theta'=\frac{1}{\sqrt{1+b^2}},\qquad \cos.\Theta'=\frac{b}{\sqrt{1+b^2}},\qquad c^2.\sin.\Theta'.\cos.\Theta'=\frac{bc^2}{1+b^2} \qquad [10202k]$$

In like manner, by substituting $w_{,}=2v$, in the second value of $\text{tang.}^2\Theta$ [10201e], and neglecting the second and higher powers of v, we get

$$\text{tang.}^2\Theta=\frac{1}{b^2}.\frac{1-2v}{1+2v}=\frac{1}{b^2}.(1-4v); \qquad [10202l]$$

whence

$$\text{tang.}\Theta=\frac{1}{b}.(1-2v)=(1-2v).\text{tang.}\Theta',\quad\text{or}\quad \text{tang.}\Theta'-\text{tang.}\Theta=2v.\text{tang.}\Theta'; \qquad [10202l']$$

[10203] $$\alpha q^2 = \alpha q_{\prime}^2 = 1 - \sin.\varpi.$$ [Case when the elevation $UR = q$, is equal to the depression $WV = q_{\prime}$, fig. 165.]

[10202m] and as the first member of this expression is equal to $\mathrm{tang.}(\Theta' - \Theta).(1 + \mathrm{tang.}\Theta.\mathrm{tang.}\Theta')$, by [30], Int., we shall have very nearly, by putting for brevity $\Theta' - \Theta = x$, x being a very small quantity,

[10202m'] $$x.(1 + \mathrm{tang.}^2\Theta') = 2v.\mathrm{tang.}\Theta', \quad \text{or} \quad x.\left(1 + \frac{1}{b^2}\right) = 2v.\frac{1}{b};$$

whence we finally obtain

[10202m''] $$x = \frac{2bv}{1 + b^2}.$$

If we now, for abridgment, put the second number of [10202n] equal to $\mathrm{G}(\Theta')$, G being the symbol of a function, we shall have [10202o], by changing Θ' into Θ,

[10202n] $$\mathrm{G}(\Theta') = \frac{2 - c^2}{c}.\mathrm{F}(c, \Theta') - \frac{2}{c}.\mathrm{E}(c, \Theta') + \frac{2c.\sin.\Theta'.\cos.\Theta'}{\sqrt{1 - c^2.\sin.^2\Theta'}},$$

[10202o] $$\mathrm{G}(\Theta) = \mathrm{G}(\Theta' - x) = \frac{2 - c^2}{c}.\mathrm{F}(c, \Theta) - \frac{2}{c}.\mathrm{E}(c, \Theta) + \frac{2c.\sin.\Theta.\cos.\Theta}{\sqrt{1 - c^2.\sin.^2\Theta}}.$$

Substituting these in [10201c], we get [10202p], which, being developed, by Taylor's theorem, according to the powers of x, neglecting x^3, x^4, &c., becomes, as in [10202q],

[10202p] $$\pm \frac{2l\sqrt{2}}{\alpha} = \mathrm{G}(\Theta') - \mathrm{G}(\Theta' - x)$$

[10202q] $$= x.\left(\frac{d.\mathrm{G}(\Theta')}{d\Theta'}\right) - \tfrac{1}{2}x^2.\left(\frac{d^2.\mathrm{G}(\Theta')}{d\Theta'^2}\right).$$

If we put, for brevity, in like manner as in [9416f],

[10202r] $$\Delta = \sqrt{1 - c^2.\sin.^2\Theta'}, \qquad X = \frac{\sin.\Theta'.\cos.\Theta'}{\Delta},$$

we shall get, as in [9416i],

[10202r'] $$\left(\frac{dX}{d\Theta'}\right) = \left(\frac{d.\frac{\sin.\Theta'.\cos.\Theta'}{\Delta}}{d\Theta'}\right) = \frac{\Delta}{c^2} - \frac{(1 - c^2)}{c^2\Delta^3} = \frac{\Delta}{c^2} - \frac{b^2}{c^2\Delta^3}.$$

Substituting this, and the values of the elliptical symbols [8910k, l], in the differential of
[10202s] [10202n], divided by $d\Theta'$, we get the first of the expressions in [10202t]; reducing this, and substituting $2 - c^2 = 1 + b^2$ [10168w], we finally obtain the last of the forms [10202t]. The differential of this last form being divided by $d\Theta'$, gives [10202u],

[10202t] $$\left(\frac{d.\mathrm{G}(\Theta')}{d\Theta'}\right) = \frac{2 - c^2}{c}.\frac{1}{\Delta} - \frac{2}{c}.\Delta + 2c.\left\{\frac{\Delta}{c^2} - \frac{b^2}{c^2\Delta^3}\right\} = \frac{2 - c^2}{c}.\frac{1}{\Delta} - \frac{2b^2}{c\Delta^3}$$
$$= \frac{1}{c\Delta}.\left\{(2 - c^2) - \frac{2b^2}{\Delta^2}\right\} = \frac{1}{c\Delta}.\left\{1 + b^2 - \frac{2b^2}{\Delta^2}\right\};$$

[10202u] $$\left(\frac{d^2.\mathrm{G}(\Theta')}{d\Theta'^2}\right) = \frac{1}{c\Delta^2}.\left\{-(1 + b^2) + 6.\frac{b^2}{\Delta^2}\right\}.\left(\frac{d\Delta}{d\Theta'}\right).$$

Now, from [10202k], we get

[10202u'] $$\Delta^2 = 1 - c^2.\sin.^2\Theta' = 1 - \frac{c^2}{1 + b^2} = \frac{(1 - c^2) + b^2}{1 + b^2} = \frac{2b^2}{1 + b^2};$$

The equation [10177], which always holds good, then gives, [10203′]

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whence

$$\frac{2b^2}{\Delta^2}=1+b^2, \quad \text{or} \quad \frac{6b^2}{\Delta^2}=3(1+b^2);$$ [10202v]

and the differential of the first of these expressions of Δ^2 [10202u′], being divided by $2d\Theta'$, gives

$$\Delta.\left(\frac{d\Delta}{d\Theta'}\right)=-c^2.\sin.\Theta'.\cos.\Theta'=-\frac{bc^2}{1+b^2} \quad [10202k].$$ [10202w]

Dividing this by $\Delta=\frac{b\sqrt{2}}{\sqrt{1+b^2}}$ [10202u′], we get [10202x]

$$\left(\frac{d\Delta}{d\Theta'}\right)=-\frac{bc^2}{1+b^2}.\frac{\sqrt{1+b^2}}{b\sqrt{2}}=-\frac{c^2}{\sqrt{2}.\sqrt{1+b^2}}.$$ [10202y]

Substituting the values [10202v, y], in the last expressions of [10202t, u], we obtain

$$\left(\frac{d.\text{G}(\Theta')}{d\Theta'}\right)=\frac{1}{c\Delta}.\{1+b^2-(1+b^2)\}=0;$$ [10202z]

$$\left(\frac{d^2.\text{G}(\Theta')}{d\Theta'}\right)=\frac{1}{c\Delta^2}.\left\{-(1+b^2)+3(1+b^2)\right\}.\left(\frac{d\Delta}{d\Theta'}\right)=\frac{2(1+b^2)}{c\Delta^2}.\left(\frac{d\Delta}{d\Theta'}\right)$$
$$=\frac{(1+b^2)^2}{b^2c}.\left(\frac{d\Delta}{d\Theta'}\right)=-\frac{(1+b^2)^{\frac{3}{2}}.c}{b^2\sqrt{2}}.$$ [10203a]

Substituting the values [10202z, 10203a], in [10202q], then dividing by $\sqrt{2}$, and neglecting the double sign $\mp$, we get

$$\frac{2l}{\alpha}=\frac{(1+b^2)^{\frac{3}{2}}.c}{4b^2}.x^2;$$ [10203b]

finally, substituting the value of x [10202m″], and putting $c=\sqrt{1-b^2}$, we get

$$\frac{2l}{\alpha}=v^2.\sqrt{\frac{1-b^2}{1+b^2}}.$$ [10203c]

From this we easily deduce the following value of b, and then $c=\sqrt{1-b^2}$;

$$b=\sqrt{\frac{\alpha^2v^4-4l^2}{\alpha^2v^4+4l^2}}; \qquad c=\frac{2l\sqrt{2}}{\sqrt{\alpha^2v^4+4l^2}}.$$ [10203d]

Hence it appears that nothing more is necessary, to obtain real values of b, c, less than unity, *than to have the distance of the planes* $2l=NS$, fig. 166, page 929, *very small, and less than* αv^2. When this condition is satisfied, we shall have, by substituting the values of b, c, [10203d] in the second value of h [10201l], [10203e]

$$h=\frac{a}{2l}.\sqrt{\alpha^2v^4-4l^2}.$$ [10203f]

Substituting this value of h, and $\sin.w'=0$ [10202i] in [10201i], we get

$$q'^2=\frac{\alpha^4v^4}{4l^2}, \quad \text{or} \quad q'=\frac{\alpha^2v^2}{2l};$$ [10203g]

and as the elevation of the fluid, on the outer side of the plane, is $q'_{\prime}=\alpha$ [9425k, 10188′], we shall have, for the pressure of *the planes towards each other*, as in [10190n], [10203h]

[10203''] [as in 10205a, &c.],

Attractive force where there is not a point of inflexion.

[10203i] $$\tfrac{1}{2}gD.(q'^2-q_{\prime}'^2)=\tfrac{1}{2}gD.\left(\frac{\alpha^4v^4}{4l^2}-\alpha^2\right)=\tfrac{1}{2}gD.\frac{\alpha^2}{4l^2}.(\alpha^2v^4-4l^2);$$

therefore, for the same value of v, the elevation of the fluid q' [10203g] will be inversely

[10203k] as the distance $2l$ of the two planes from each other; and if this distance is very small in

[10203l] comparison with αv^2, the force [10203i], which presses the planes towards each other, *will be inversely as the square of the distance $2l$ of the planes.*

[10203m] *We shall now take into consideration the case which corresponds to* fig. 167, page 929, *where the continuation of the curve TP gives a point of inflexion A;* and we shall use the

[10203m'] same values of w, w', as in the preceding example, namely, $w_{\prime}=2v$, $w'=\pi$ [10202i]. Substituting these in the second forms of [10201p, p'], we get

[10203n] $$\text{tang.}^2\theta=\frac{\cos.i-\sin.2v}{\sin.^2\tfrac{1}{2}i.(1+\sin.2v)},$$

[10203o] $$\text{tang.}^2\theta'=\frac{\cos.i}{\sin.^2\tfrac{1}{2}i},\quad\text{or}\quad \text{tang.}\theta'=\frac{\sqrt{\cos.i}}{\sin.\tfrac{1}{2}i}.$$

[10203p] If we neglect the square and higher powers of v, we may put $\sin.2v=2v$; and if we suppose $\cos.i$ to be much greater than $2v$, we may develop the expression [10203n] in the following manner;

[10203q] $$\text{tang.}^2\theta=\frac{\cos.i-2v}{\sin.^2\tfrac{1}{2}i.(1+2v)}=\frac{\cos.i}{\sin.^2\tfrac{1}{2}i}.\left(1-2v.\frac{(1+\cos.i)}{\cos.i}\right)=\frac{\cos.i}{\sin.^2\tfrac{1}{2}i}.\left(1-4v.\frac{\cos.^2\tfrac{1}{2}i}{\cos.i}\right),$$

whose square root gives

[10203r] $$\text{tang.}\theta=\frac{\sqrt{\cos.i}}{\sin.\tfrac{1}{2}i}.\left(1-2v.\frac{\cos.^2\tfrac{1}{2}i}{\cos.i}\right)=\frac{\sqrt{\cos.i}}{\sin.\tfrac{1}{2}i}-2v.\frac{\cos.^2\tfrac{1}{2}i}{\sin.\tfrac{1}{2}i\sqrt{\cos.i}}$$

[10203s] Subtracting this value of tang.θ from that of tang.θ' [10203o], and putting $\theta'-\theta=x$, we get, as in [10202m],

[10203t] $$2v.\frac{\cos.^2\tfrac{1}{2}i}{\sin.\tfrac{1}{2}i\sqrt{\cos.i}}=\text{tang.}\theta'-\text{tang.}\theta=\text{tang.}(\theta'-\theta).(1+\text{tang.}\theta.\text{tang.}\theta')=x.(1+\text{tang.}^2\theta'),$$

nearly. Now, by substituting $\cos.i=1-2\sin.^2\tfrac{1}{2}i$ [1], Int., in tang.$^2\theta'$ [10203o], we get

[10203u] $$1+\text{tang.}^2\theta'=1+\frac{1-2\sin.^2\tfrac{1}{2}i}{\sin.^2\tfrac{1}{2}i}=\frac{1-\sin.^2\tfrac{1}{2}i}{\sin.^2\tfrac{1}{2}i}=\frac{\cos.^2\tfrac{1}{2}i}{\sin.^2\tfrac{1}{2}i};$$

and if we divide [10203t] by this value, we shall get

[10203v] $$x=2v.\frac{\sin.\tfrac{1}{2}i}{\sqrt{\cos.i}}.$$

Instead of the functions G(θ'), G(θ) [10202n, o], we shall now assume the following values;

[10203w] $$G(\theta')=F(c,\theta')-2E(c,\theta')+\frac{2c^2.\sin.\theta'.\cos.\theta'}{\sqrt{1-c^2.\sin.^2\theta'}};$$

[10203x] $$G(\theta)=G(\theta'-x)=F(c,\theta)-2E(c,\theta)+\frac{2c^2.\sin.\theta.\cos.\theta}{\sqrt{1-c^2.\sin.^2\theta}};$$

$$\alpha q'^2 = \alpha q_{\prime}'^2 = 1 - \sin.\varpi' ; \quad \left[\begin{array}{l}\text{Case where the elevation } ST = q', \text{ fig. 162,} \\ \text{page 910, is equal to the elevation } XO = q_{\prime}'.\end{array}\right]$$ [10204]

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by substituting them in [10201s], and then developing the resulting expression, we get, as in [10202p, q], neglecting x^3, x^4, &c. [10203x']

$$\pm \frac{2l\sqrt{2}}{\alpha} = \mathrm{G}(\Theta') - \mathrm{G}(\Theta' - x)$$ [10203y]

$$= x.\left(\frac{d.\mathrm{G}(\Theta')}{d\Theta'}\right) - \tfrac{1}{2}x^2.\left(\frac{d^2.\mathrm{G}(\Theta')}{d\Theta'^2}\right).$$ [10203z]

Substituting the values of the elliptical functions [8910k, l], and using the abridged symbols X, Δ [10202r], we get the following differential of $\mathrm{G}(\Theta')$ [10203w] divided by $d\Theta'$, which is reduced by substituting the value of $\left(\frac{dX}{d\Theta'}\right)$ [10202r']; [10204a]

$$\left(\frac{d.\mathrm{G}(\Theta')}{d\Theta'}\right) = \frac{1}{\Delta} - 2\Delta + 2c^2.\left(\frac{\Delta}{c^2} - \frac{b^2}{c^2\Delta^3}\right) = \frac{1}{\Delta} - \frac{2b^2}{\Delta^3}.$$ [10204b]

Taking the differential of this last expression, dividing it by $d\Theta'$, and using

$$\left(\frac{d\Delta}{d\Theta'}\right) = -\frac{c^2.\sin.\Theta'.\cos.\Theta'}{\Delta} \quad [10202w],$$ [10204b']

we get

$$\left(\frac{d^2.\mathrm{G}(\Theta')}{d\Theta'^2}\right) = \left(-\frac{1}{\Delta^2} + \frac{6b^2}{\Delta^4}\right).\left(\frac{d\Delta}{d\Theta'}\right) = \left(\frac{1}{\Delta^2} - \frac{6b^2}{\Delta^4}\right).\frac{c^2.\sin.\Theta'.\cos.\Theta'}{\Delta}.$$ [10204c]

We have, in [10168w],

$$b = \sin.\tfrac{1}{2}i, \qquad c = \cos.\tfrac{1}{2}i, \qquad \cos.i = 1 - 2bb;$$ [10204c']

hence the expression of $\mathrm{tang}.\Theta'$ [10203o] becomes

$$\mathrm{tang}.\Theta' = \frac{\sqrt{1-2b^2}}{b},$$ [10204c'']

consequently

$$1 + \mathrm{tang}.^2\Theta' = \frac{1-b^2}{b^2} = \frac{c^2}{b^2}.$$ [10204d]

Dividing 1 by the square root of this expression, we get $\cos.\Theta' = \frac{b}{c}$; and then [10204$d'$]

$$\sin.\Theta' = \cos.\Theta'.\mathrm{tang}.\Theta' = \frac{\sqrt{1-2b^2}}{c}.$$ [10204d'']

The square of this value of $\sin.\Theta'$, being multiplied by c^2, gives

$$c^2.\sin.^2\Theta' = 1 - 2b^2;$$ [10204e]

whence

$$\Delta^2 = 1 - c^2.\sin.^2\Theta' = 2b^2, \quad \text{or} \quad \Delta = b\sqrt{2}.$$ [10204e']

Substituting this value of Δ in [10204b, c], we get

$$\left(\frac{d.\mathrm{G}(\Theta')}{d\Theta'}\right) = 0, \qquad \left(\frac{d^2.\mathrm{G}(\Theta')}{d\Theta'^2}\right) = -\frac{c^2.\sin.\Theta'.\cos.\Theta'}{\sqrt{2}.b^3}.$$ [10204f]

Substituting the values [10204f] in [10203z], and putting, as in [10204d', d''],

$$c^2.\sin.\Theta'.\cos.\Theta' = b\sqrt{1-2b^2},$$ [10204f']

[10204'] and it follows also, from [10205a—e], *that the second plane then*

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we get, by using [10204c'],

[10204g] $$\pm\frac{2l\sqrt{2}}{\alpha}=\frac{c^2.\sin.\Theta'.\cos.\Theta'}{2\sqrt{2}.b^3}.x^2=\frac{\sqrt{1-2b^2}}{2\sqrt{2}.b^2}.x^2=\frac{\sqrt{\cos.i}}{2\sqrt{2}.\sin.^2\frac{1}{2}i}.x^2.$$

Substituting the value of x [10203v], dividing by $\sqrt{2}$, and neglecting the double sign, we finally get

[10204h] $$\frac{2l}{\alpha}=v^2.\frac{1}{\sqrt{\cos.i}};$$

and from this we obtain

[10204i] $$\cos.i=\frac{\alpha^2v^4}{4l^2}.$$

[10204i'] *In order that* i *may be a real angle, we must therefore have* $2l$, or NS, fig. 167, page 929, greater than αv^2. We have also supposed that x or $\Theta'-\Theta$ is small, which requires that

[10204k] $\frac{v}{\sqrt{\cos.i}}$ [10203v] should be small; and as $\frac{v}{\sqrt{\cos.i}}=\frac{2l}{\alpha v}$ [10204h], it is necessary that $\frac{2l}{\alpha v}$

[10204l] should be small. *If we suppose, therefore, that the distance* $2l$ *exceeds* αv^2, *and also that it is very small relative to* αv, *the equilibrium here treated of can take place.* In this case, the

[10204l'] expression of the elevation of the fluid q' near the plane LM, can be obtained from [10201w], by substituting the value of $w'=\pi$ [10203m'], and that of $\cos.i$ [10201i]; hence we get

[10204m] $$q'=\frac{\alpha^2v^2}{2l}.$$

[10204n] With this value of q', and that of $q'_{,}=\alpha$ [10203h], we get the repulsive force of the planes from each other, in the case of fig. 167, page 929, as in [10190l],

Repulsive force where there is a point of inflexion.

[10204o] $$\tfrac{1}{2}gD.(q'^2_{,}-q'^2)=\tfrac{1}{2}gD.\left(\alpha^2-\frac{\alpha^4v^4}{4l^2}\right)=\tfrac{1}{2}gD\alpha^2.\left(1-\frac{\alpha^2v^4}{4l^2}\right).$$

[10204p] When $2l=\alpha v^2$, this repulsive force [10204o] vanishes, and the same is to be observed relative to the attractive force, in the case of fig. 166, page 929, computed in [10203i].

[10204p'] Multiplying this value of $2l$ by α, we get $\alpha^2v^2=2l\alpha$; substituting this in [10204m], we get

[10204q] $q'=\alpha$; and the same result is obtained with the value of q' corresponding to the case of fig. 166, page 929 [10203g]. If the distance $2l$ be supposed, in the first place, to exceed αv^2, the form of equilibrium corresponding to fig. 167, page 929, will be established, as in

[10204r] [10204l], with a *repulsive force* of the planes [10204o]. By bringing the planes nearer

[10204r'] together, we find that this force *vanishes*, when $2l=\alpha v^2$; and by still decreasing the distance $2l$, so that $2l$ may be less than αv^2 [10203e], we fall on the case first treated of,

[10204s] corresponding to fig. 166, page 929, where the force becomes *attractive* [10203i], and is expressed by the function [10203i].

By comparing together what we have found in the case where there is a point of inflexion between two planes, we see that the fluid can have two different figures of equilibrium for

[10204t] the same distance $2l$ of the planes, supposing this distance to be very small in comparison with α. One of these figures is very nearly represented by a cubic parabola [10171o]; and

*ceases to be repelled by the first, so that the repulsion changes into attraction at the same instant in both the planes.** [10205]

It is easy to determine the distance of the planes from each other when this change takes place. For αq^2 being then equal to $1-\sin.\varpi$ [10203], we shall have, from [10171], [10206]

$$Z=1-\alpha z^2;$$ [10206']

and the differential equation [10172] becomes†

$$dy=\frac{(1-\alpha z^2).dz}{z\sqrt{\alpha}.\sqrt{2-\alpha z^2}};$$ [10207]

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in the case of $\varpi=\varpi'=0$, the repulsive force of the planes is equal to the constant quantity $\frac{1}{2}gD.\alpha^2$ [10187q], which is wholly independent of the distance $2l$ of the two planes, and this distance may be extremely small. In the other case, the curve will be of two different natures, according as we shall have NS or $2l>\alpha v^2$, or $2l<\alpha v^2$; and the surface of the fluid between the planes will be wholly above the level of the surface of the fluid in the vase, and there will be neither an inflexion, nor a point where the surface is horizontal, between the planes. When NS or $2l>\alpha v^2$, the surface will correspond to fig. 167, page 929 [10204l], and will be part of a cubic parabola [10171p]; and when $2l<\alpha v^2$, the surface will correspond to fig. 166, page 929 [10203e], which is nearly that of an arc of a circle [9336g, &c.]. In this change of curvature, the *repulsive* force [10204o] changes into an attractive force [10203i]. [10204u] [10204v] [10204w] [10204x]

(4353) This note is referred to in [10203''] for the purpose of investigating the equation [10204]. The equation [10177] gives $\alpha q^2=\alpha q'^2-\sin.\varpi+\sin.\varpi'$; putting this equal to the expression of αq^2 [10203], we get $\alpha q'^2-\sin.\varpi+\sin.\varpi'=1-\sin.\varpi$; whence we have $\alpha q'^2=1-\sin.\varpi'=\alpha q_{\prime}'^2$ [10189b], agreeing with [10204]. From this we get $q'=q_{\prime}'$; or, in other words, $ST=XO=XT'$, fig. 165, page 927; hence the pressure on the second plane $\frac{1}{2}gD.(XO^2-XT'^2)$ [10190l] vanishes, as is remarked in [10205]. We may also observe, that this general expression of the pressure $\frac{1}{2}gD.(XO^2-XT'^2)$, is supposed to be in the direction RS [10190l, m], or *repulsive*, the factor $XO^2-XT'^2$ being positive, or $XO>XT'$; and it changes into an attractive force when $XO<XT'$, agreeing with [10205]. [10205a] [10205b] [10205c] [10205d] [10205e]

* (4354) When we have $q^2=q_{\prime}^2$, as in [10203], we also have simultaneously $q'=q_{\prime}'$ [10204]; hence we get $q_{\prime}^2-q^2=0$, $q_{\prime}'^2-q'^2=0$; and by substituting these values in the pressures on the first and second planes [10187g, 19190l], we find that both of them become equal to nothing, as in [10205]. [10206a] [10206b] [10206c]

† (4355) Substituting the value of Z [10206'] in [10172], it becomes as in [10207], which is of the same form as [10181], and differs only in the sign of the second member; [10208a]

whence we obtain by integration,

[10208] $$y=\frac{1}{2\sqrt{2\alpha}}\cdot\log.\left\{\frac{1-\sqrt{1-\frac{1}{2}\alpha z^2}}{1+\sqrt{1+\frac{1}{2}\alpha z^2}}\right\}+\frac{2\sqrt{1-\frac{1}{2}\alpha z^2}}{\sqrt{2\alpha}}+\text{constant}.$$ [Equation of the surface when the repulsion changes into attraction.]

[10208′] To determine the constant quantity, we shall observe that, when $y=0$ [10163], we have $z^2=q^2$ [10162, 10168], and then, from [10203], we obtain

[10209] $$\alpha z^2=1-\sin.\varpi.$$

$2l$ [10210] [10211] *Now, if we put $2l$ for the distance of the planes from each other*, we shall have $z^2=q'^2$ [10162, 10175] when $y=2l$ [10163]; therefore we shall then have, from [10210, 10204],

[10212] $$\alpha z^2=1-\sin.\varpi'.$$

We shall now put

[10213] $$\varpi=\tfrac{1}{2}\pi-\theta,\qquad \varpi'=\tfrac{1}{2}\pi-\theta';$$

[10214] and then θ,θ', will denote the inclinations of the two extreme sides of the section to the horizon [10166, 10173]; therefore *we shall have for the distance of the two planes, when the repulsive force changes into an attractive force*,*

[10215] $$2l=\frac{1}{\sqrt{2\alpha}}\cdot\log.\left\{\frac{\text{tang.}\frac{1}{4}\theta'}{\text{tang.}\frac{1}{4}\theta}\right\}-\frac{2}{\sqrt{2\alpha}}\cdot\{\cos.\tfrac{1}{2}\theta-\cos.\tfrac{1}{2}\theta'\};\quad (t)$$ [Distance $2l$ of the planes when the repulsion changes into attraction.]

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observing that, in this case, the quantities dy, dz, have the same signs [10181a]; but that [10208b] the quantity z, which occurs in the denominator of the expression of dy, has a positive value when $y=0$, in the present example [10201, &c.], instead of the negative value used in [10181a]. Hence it is evident that the integral of [10207] can be deduced from that of [10208c] [10181], which is given in [10181b], by merely changing the signs of the terms in the second member of [10181b]; therefore we get for the integral of [10207] the following expression;

[10208d] $$y=\frac{1}{2\sqrt{2\alpha}}\cdot\log.\left(\frac{1-v}{1+v}\right)+\frac{2v}{\sqrt{2\alpha}}+\text{constant};$$

[10208e] and by substituting $v=\sqrt{1-\frac{1}{2}\alpha z^2}$ [10181a'], it becomes as in [10208].

[10214a] * (4356) Substituting $y=0$ [10208′], and $\alpha z^2=1-\sin.\varpi=1-\cos.\theta$ [10209, 10213], in the equation [10208], and then subtracting the resulting expression from [10208], we obtain *the general value of y, under the following form;*

[10214b] $$y=\frac{1}{2\sqrt{2\alpha}}\cdot\left\{\log.\left(\frac{1-\sqrt{1-\frac{1}{2}\alpha z^2}}{1+\sqrt{1-\frac{1}{2}\alpha z^2}}\right)-\log.\left(\frac{1-\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta}}{1+\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta}}\right)\right\}+\frac{2\sqrt{1-\frac{1}{2}\alpha z^2}}{\sqrt{2\alpha}}-\frac{2\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta}}{\sqrt{2\alpha}}.$$

Substituting the values $y=2l$ [10211], $\alpha z^2=1-\sin.\varpi'=1-\cos.\theta'$ [10212, 10213], we get

[10214c] $$2l=\frac{1}{2\sqrt{2\alpha}}\cdot\left\{\log.\left(\frac{1-\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta'}}{1+\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta'}}\right)-\log.\left(\frac{1-\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta}}{1+\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta}}\right)\right\}+\frac{2\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta'}}{\sqrt{2\alpha}}-\frac{2\sqrt{\frac{1}{2}+\frac{1}{2}\cos.\theta}}{\sqrt{2\alpha}}.$$

we must here observe that the logarithm is hyperbolic. If θ be infinitely small, [10216]
the fluid will descend but an infinitely small quantity without the first plane; and in this case the preceding expression of $2l$ becomes infinite;* consequently [10216′]
the two planes attract each other at all distances [10216a]. Thus the supposition of θ equal to nothing, is the limit in which the two planes begin to have power to repel each other. When θ increases, and becomes equal to θ', [10217]
$2l$ becomes nothing; and in this case, the two planes repel each other at all distances. Between these two limits, the planes, after being repelled, attract each other when the preceding expression is less than $2l$ [10214m]. We may [10218]

This may be much reduced by means of [6], Int., which gives

$$\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos.\theta}=\sqrt{\cos.^2\tfrac{1}{2}\theta}=\cos.\tfrac{1}{2}\theta; \qquad [10214c']$$

and then

$$\frac{1-\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos.\theta}}{1+\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos.\theta}}=\frac{1-\cos.\tfrac{1}{2}\theta}{1+\cos.\tfrac{1}{2}\theta}=\text{tang.}^2\tfrac{1}{4}\theta \quad [40],\ \text{Int.} \qquad [10214d]$$

Substituting these and the similar reductions in the terms depending on θ', we get

$$2l=\frac{1}{2\sqrt{2\alpha}}\cdot\{2\log.\text{tang.}\tfrac{1}{4}\theta'-2\log.\text{tang.}\tfrac{1}{4}\theta\}+\frac{2\cos.\tfrac{1}{2}\theta'}{\sqrt{2\alpha}}-\frac{2\cos.\tfrac{1}{2}\theta}{\sqrt{2\alpha}}, \qquad [10214e]$$

which is easily reduced to the form [10215].

We may here remark that we have seen, in [10195], that, when $\varpi=\varpi'$, there is a point [10214f]
of inflexion A, fig. 152, page 910, midway between the two planes; and then these two planes *repel* each other. If $\varpi>\varpi'$, as in [10196], and we begin to draw the planes nearer to each other, we shall find, as in [10195′], that this point of inflexion will approach nearer and [10214g]
nearer to the first plane, and will finally touch it, as in [10197]. If we continue still to draw the planes nearer to each other, the fluid will continue to rise between the planes until it shall [10214h]
attain to the height q, near the first plane, the *repulsion* of the planes continuing during all this time. When the fluid has arrived at the height q, and near to the first plane, the *repulsion* [10214i]
will cease, as in [10205]; and by continuing to bring the planes still nearer to each other, the repulsion will change into attraction, as in [10205]; and the distance of the planes where [10214k]
this change takes place being represented by $2l$, its value is given by the formula [10215]; [10214l]
so that, if the planes are at a less distance from each other than the value of $2l$ [10215], *they will attract each other, and if they are at a greater distance than that value of $2l$, they will repel each other.* [10214m]

* (4357) When $\theta=0$, we evidently have $2l=\infty$ [10215]; and then, from [10214m], [10216a]
it follows that, whatever be the distance of the planes, they will attract each other; and this infinite distance is evidently the extreme limit at which the attraction can commence, as in [10216′]. On the other hand, when $\theta=\theta'$, and $\log.\frac{\text{tang.}\frac{1}{4}\theta'}{\text{tang.}\frac{1}{4}\theta}=\log.1=0$, the expression [10216$b$]
of $2l$ is equal to nothing; and this may be considered as the other limit, or point where the [10216c]
planes begin to repel each other, as in [10214m, 10217].

determine their attraction or repulsion, by means of the following theorem, which may be easily deduced from the preceding theory [9552—9586].

[10219] Theorem relative to the attractive or repulsive forces of planes. "Whatever be the substances of which the planes are formed, *the tendency of each of them towards the other*, is equal to the weight of a fluid prism, whose height is the elevation above the level, of the extreme points of contact of the *interior* fluid with the plane, *minus* that elevation without the plane; whose
[10220] length is the half sum of these elevations, and whose width is that of the planes in a horizontal direction. *We must suppose the elevation to be negative, when it*
[10221] *changes into a depression below the level.* If the product of these three dimensions is negative, the tendency becomes repulsive."*

We shall here observe that the tendency is the same, and has the same sign
[10222] for both planes. For the two first factors being $q-q_,$, and $\frac{1}{2}(q+q_,)$, for the
[10223] first plane; their product is $\frac{1}{2}(q^2-q_,^2)$ [10220*a*]. The analogous product for the second plane is $\frac{1}{2}(q'^2-q_,'^2)$ [10190*m*, &c.]; thus, the width of the two planes being supposed to be the same, the two fluid prisms whose weights are equal to the tendencies of the one towards the other, are equal, provided
[10224] $q^2-q_,^2$ is equal to $q'^2-q_,'^2$; now this equality takes place in virtue of the
[10225] equation [10176], which, by substituting for $\sin.\varpi$, $\sin.\varpi'$, their values $1-\alpha q_,^2$, $1-\alpha q_,'^2$ [10203,10204], becomes

[10226]
$$\alpha.(q'^2-q_,'^2)=\alpha.(q^2-q_,^2).$$

The action and reaction of the planes are equal. [10227] *Thus, although the two planes act upon each other by the capillary force of an intermediate fluid, yet this reciprocal action is such that the action and the reaction are equal to each other.*

When the two planes are very near to each other, z will differ but very little from q; so that, if we put†

[10228]
$$z-q=z', \quad \text{or} \quad z=q+z',$$

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* (4358) Considering q, $q_,$ [10168, 10184′], as positive quantities, the height of the
[10220*a*] prism [10219] will be $q-q_,$, its length $\frac{1}{2}(q+q_,)$ [10220]; and if we put its width equal to unity [10220], its volume will be $\frac{1}{2}(q^2-q_,^2)$, and its mass $\frac{1}{2}gD.(q^2-q_,^2)$. This represents the *attractive* force [10219]; and by changing its sign, we may consider it as a
[10220*b*] repelling force represented by $\frac{1}{2}gD.(q_,^2-q^2)$, being the same as that in [10187*g*].
[10220*c*] Moreover the pressures on the two planes are equal, as is seen in [10191, &c., 10222, &c.]; hence we easily perceive the correctness of the theorem in [10219—10221].

† (4359) Through the point U, fig. 165, page 927, draw the horizontal line Uct, cutting
[10228*a*] the ordinate NC in c; then $RN=y$ [10163], $NC=z$ [10162], and $RU=Nc=St$ is equal to q [10202]; so that, if we put $z=NC=RU+Cc=q+z'$, as in [10228], we

z' will be a very small quantity, whose square may be neglected. Then we shall have [10229]

$$Z = \sin.\varpi - 2\alpha q z'; \quad [10230]$$

consequently *

$$dz = dz' = -\frac{dZ}{2\alpha q}; \quad [10231]$$

hence the equation [10172] will become

$$dy = \frac{-ZdZ}{2\alpha q.\sqrt{1-Z^2}}; \quad [10232]$$

and by integration,

$$y = \text{constant} + \frac{\sqrt{1-Z^2}}{2\alpha q}. \quad [10233]$$

To determine the constant quantity, we shall observe that, when $y=0$, we also have $z'=0$ [10228a], and then † $Z=\sin.\varpi$; therefore we shall have [10234]

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shall have Cc of the order z'; and its greatest value $z'=Tt$ may be considered as of the same order as the distance of the planes $Ut=2l$; and when $2l$ is very small, this quantity [10228b] z' will be very small in comparison with q, as in [10229]. Now, substituting the value of $z=q+z'$ [10228], in [10171], we get [10230], neglecting the second and higher powers [10228c] of z'.

* (4360) The differential of [10228] gives $dz=dz'$; substituting this in the differential of [10230], and then dividing by $-2\alpha q$, we get [10231]. Substituting this value of dz [10231a] in [10172], we obtain [10232], whose integral gives [10233].

† (4361) We have generally, as in [10168f], $Z=\sin.\varpi_{,}$; $\varpi_{,}$ being the acute angle [10235a] formed by the ordinate Cc, fig. 165, page 927, and the arc of the curve at C. At the point U, where $y=0$, we have $\varpi_{,}=\varpi$ [10166], and then the preceding value of Z becomes [10235a'] $Z=\sin.\varpi$, as in [10234]; which is the same as is given by the equation [10230], putting $z'=0$. This last value of Z gives $\sqrt{1-Z^2}=\cos.\varpi$; and by substituting it in [10233], [10235b] we get at the point U, $0=\text{constant}+\frac{\cos.\varpi}{2\alpha q}$, as in [10235]. Substituting this in [10233], [10235c] we obtain, for the general value of y, $y=\frac{\sqrt{1-Z^2}}{2\alpha q}-\frac{\cos.\varpi}{2\alpha q}$; and at the point T, where [10235d] $\varpi_{,}=\varpi'$ [10173], $Z=\sin.\varpi'$ [10235a], $\sqrt{1-Z^2}=\cos.\varpi'$, and $y=2l$ [10210], it becomes [10235e] as in [10238], from which we easily deduce the value of q [10239]. If we put $\varpi'=0$, and $\varpi=2v$, as in [10202i, &c.], the value of q [10239] will become $q=\frac{1-\cos.2v}{4\alpha l}$. [10235$f$] Substituting $\frac{1}{\alpha}=\alpha^2$ [9323p], $\cos.2v=1-2v^2$ [44], Int., neglecting higher powers of v^2, it becomes $q=\frac{\alpha^2v^2}{2l}$, agreeing with the elevation q' computed in [10203g, 10201m], by [10235g] means of elliptical functions.

[10235]
$$\text{constant} = -\frac{\cos.\varpi}{2\alpha q};$$

[10236] moreover, $2l$ being the distance of the planes from each other [10210],
[10237] we shall have, when $y=2l$, $z=q'$ [10175], and $Z=\sin.\varpi'$ [10235e]; consequently

[10238]
$$2l = \frac{\cos.\varpi' - \cos.\varpi}{2\alpha q};$$

hence we get

[10239]
$$q = \frac{\cos.\varpi' - \cos.\varpi}{4\alpha l}.$$

[Elevation q of a fluid between two very near planes whose distance is $2l$.]

The height of the fluid between the planes is then in the inverse ratio of their
[10240] *distance from each other. We may therefore, from this analysis, deduce the following theorem.*

"When the planes are very near to each other, the elevation of the fluid
[10241] between them is in the inverse ratio of their distance. This elevation is equal to the half sum of the elevations which will occur if we suppose, in the first place, that the first plane is of the same substance as the second, and, in the
[10242] next place, that the second plane is of the same substance as the first; observing to prefix the negative sign to the elevation, when it changes into a depression."* This theorem is a corollary to that which we have before given upon the elevation of a fluid between two prismatic surfaces of different substances,
[10242′] of which the one is included within the other.†

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* (4362) It appears, from [9453, 9454], that the elevation of a fluid between two parallel planes, made of the same substance as the second plane, and whose distance from each other
[10241a] is $2l$, is very nearly equal to $\frac{H}{g}\cdot\frac{\sin.\theta'}{2l} = \frac{\sin.\theta'}{2\alpha l}$ [9328]; and since $\theta' = \frac{1}{2}\pi - \varpi'$ [9346, 10213],
[10241b] it becomes $\frac{\cos.\varpi'}{2\alpha l}$. The depression of the fluid between two similar planes of the same
[10241c] nature as that of the first plane, would in like manner be represented by $-\frac{\cos.\varpi}{2\alpha l}$, as is evident from [9454, 10166], its sign being changed as in [10242]. The half sum of these
[10241d] two expressions is $\frac{\cos.\varpi' - \cos.\varpi}{4\alpha l}$, which is equal to the expression of q [10239], as in [10241, &c.]. This expression of q [10239] agrees with that given by M. Poisson in page 182 of his *Nouvelle Théorie, &c.*, changing $\frac{1}{\alpha}$ into a^2, $2l$ into δ, q into h, $\cos.\varpi'$ into $\sin.\mu$, and $-\cos.\varpi$ into $\sin.\mu'$, to conform to his notation.

† (4363) If we suppose the prisms treated of in [10102—10105] to be cylinders
[10242a] whose radii are infinite, we shall have $\frac{c'}{c} = 1$ [10095]; and since the expression [10105′]

We see by this theorem, and by that we have mentioned before [10219—10221], that the repulsive force of the planes is much weaker than the attractive force,* which commences when the planes are very near to each other, and draws them towards each other with an accelerated motion. [10243] In this case, the interior elevation of the fluid, or that between the planes, is very great in comparison with the exterior elevation near the same planes. Therefore, by neglecting the square of this last elevation in comparison with the square of the first, we shall find that the fluid prism whose weight expresses the tendency of one of the planes towards the other, [10244] in virtue of the first of the two preceding theorems [10219—10221], will be equal to the product of the square of the elevation of the interior fluid by half the width of the planes in a horizontal direction.† This elevation being, by the second of these theorems [10245]

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may be put under the form

$$h=\frac{q+q_{,}\cdot\frac{c'}{c}}{1+\frac{c'}{c}}, \quad \text{it becomes} \quad h=\frac{q+q_{,}}{2}, \qquad [10242b]$$

in which q, $q_{,}$ [10104], represent the elevations of the fluid in cylinders having the same *radius* l, but composed successively of the two different substances. These quantities q, $q_{,}$, [10242c] also represent the similar elevations between parallel planes of the like substances, whose distance is l, as appears from [9998c]. Lastly, h represents, as in [10097'], the elevation of the fluid between two of these planes of different substances, whose distance is l. These [10242d] values agree with [10241].

* (4364) It follows from [10220a], that the *attractive* force of two planes is $\frac{1}{2}gD.(q^2-q_{,}^2)$, and that it becomes *repulsive* when the factor $q^2-q_{,}^2$ is negative; so [10243a] that its greatest *repulsive* force corresponds to $q=0$, and it then becomes, by neglecting its sign, $\frac{1}{2}gD.q_{,}^2$. Therefore the *attractive* force is to the greatest *repulsive* force as $q^2-q_{,}^2$ [10243b] to $q_{,}^2$. Now $q_{,}$, which represents the exterior elevation or depression of the fluid near the planes [10184'], must always be very small; but q [10239] may become very large, when [10243c] l is small; therefore the *repulsive* force is much smaller than the attractive force, and the ratio of these forces is nearly as $q_{,}^2$ to q^2. [10243d]

† (4365) The *attractive* force $\frac{1}{2}gD.(q^2-q_{,}^2)$ [10243a], when the planes are very [10245a] near to each other, and $q_{,}$ is much smaller than q, becomes nearly equal to $\frac{1}{2}gD.q^2$ [10243c]; and if we substitute the value of q [10239], it will become

$$\tfrac{1}{8}gD.\left(\frac{\cos.\varpi'-\cos.\varpi}{\alpha}\right)^2.\left(\frac{1}{2l}\right)^2, \qquad [10245b]$$

which is inversely proportional to the square of the distance $2l$ of the two planes from each other, as in [10246].

[10241, 10242], inversely proportional to their distance from each other, the prism will be proportional to their horizontal width, divided by the square of that distance; *the tendency of the two planes towards each other will therefore be*
[10246] *in the inverse ratio of the square of their distance* [10245b]; *consequently it will follow the law of universal attraction, a law which seems to be followed by all*
[10247] *attractions and repulsions exerted at sensible distances, like electricity and magnetism.*

Wishing to determine by experiment the singular phenomenon of the repulsion of planes, which changes into attraction by drawing the planes nearer
[10248] together, I requested M. Haüy to make some experiments relative to this curious result of the theory of capillary action. For this purpose he made some experiments with planes of ivory, which, as is well known, may be moistened
[10249] by water, and with some talc laminæ, having to the touch a sort of greasiness, which prevents them from being moistened. These experiments have fully confirmed the results of the theory, as may be seen in the following account which he communicated.

"There was suspended, by a very delicate thread, a small square leaf of a
[10250] talc lamina, so that its lower end was dipped into the water. There was also dipped into the same water, at the distance of a few centimetres, the lower part of a parallelopiped or plate of ivory, so that one of its faces was parallel to the leaf of talc; and it was always kept in this parallel position, stopping the plate of ivory occasionally, in order to be sure that the effect of the motion,
[10251] which it might impress upon the fluid, was insensible in the experiment; then the leaf of talc was repelled by the ivory.* Afterwards the ivory was moved in a very slow manner towards the talc, until the distance between them was very small, when suddenly the talc approached towards the ivory, and came in
[10252] contact with it. Upon separating the two bodies, it was found that the ivory plate was moistened to a certain height above the level of the water; and by
[10253] repeating the experiment before it was wiped, the attraction commenced sooner, and sometimes it took place at the first moment, without being preceded by any sensible repulsion. These experiments, repeated several times with care, have always furnished the same results."

When the ivory plane is very moist, the water covers its surface, and forms

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* (4366) In this experiment, the talc is what we have heretofore called the first plane
[10251a] [10158], to which the quantities q, $q_{,}$, ϖ, θ, correspond; and the ivory is the second plane, to which the quantities q', $q'_{,}$, ϖ', θ', correspond.

a new plane, which attracts the lamina of talc; and in this case, the corresponding angle θ' of the formula [10215] is the greatest possible, being by the theory [10254]
equal to a right angle [9372f]. The value of $2l$, given by the formula [10215], which expresses the distance of the planes where the attraction begins, then becomes greater* than when the ivory was not moistened, agreeing with the [10255]
experiment. Moreover, it may happen, by the effect of the friction of the fluid against the talc lamina, when it sinks down, after it has been raised up between the planes, that the angle θ near the place of contact of the fluid with the talc [10256]
may become nothing or insensible, in the same manner as is observed relative to the similar angle with mercury in a barometer, which decreases when the fluid sinks; then the preceding expression of $2l$ becomes infinite,† and the attraction [10257]
is not preceded by any sensible repulsion.

ON THE ADHESION OF A PLATE TO THE SURFACE OF A FLUID.

When we lay a plate or disk upon the surface of a stagnant fluid, in a vessel of great extent, we find that, to detach it, even in a vacuum, we experience a [10258]
resistance which increases with the size of the disk. The disk, as it rises, lifts up with it a column of the fluid, which follows it till it gets to a certain limit,

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* (4367) The expression of the distance $2l$ between the planes, where the attraction changes into repulsion, is given in [10215]; and it may be put under the form

$$2l=\frac{1}{\sqrt{2\alpha}}.\{\text{log.tang.}\tfrac{1}{4}\theta'+2\cos.\tfrac{1}{2}\theta'\}-\frac{1}{\sqrt{2\alpha}}.\{\text{log.tang.}\tfrac{1}{4}\theta+2\cos.\tfrac{1}{2}\theta\}; \qquad [10255a]$$

from which it is easy to show, that, while θ' increases, this value of $2l$ will also increase; for, by taking its differential, considering l, θ', only to vary, we get the first expression in [10255b]
[10255d]; and by making successive reductions, using [34', 31, 1], Int., which give

$$\cos.^2\tfrac{1}{4}\theta'.\text{tang.}\tfrac{1}{4}\theta'=\cos.\tfrac{1}{4}\theta'.\sin.\tfrac{1}{4}\theta'=\tfrac{1}{2}\sin.\tfrac{1}{2}\theta', \qquad 1-2\sin.^2\tfrac{1}{2}\theta'=\cos.\theta', \qquad [10255c]$$

we finally obtain the expression of $2dl$ [10255e],

$$2dl=\frac{d\theta'}{\sqrt{2\alpha}}.\left\{\frac{\frac{1}{4}}{\cos.^2\frac{1}{4}\theta'.\text{tang.}\frac{1}{4}\theta'}-\sin.\tfrac{1}{2}\theta'\right\}=\frac{d\theta'}{\sqrt{2\alpha}}.\left\{\frac{\frac{1}{4}}{\frac{1}{2}\sin.\frac{1}{2}\theta'}-\sin.\tfrac{1}{2}\theta'\right\} \qquad [10255d]$$

$$=\frac{d\theta'}{\sqrt{2\alpha}}.\left\{\frac{1-2\sin.^2\frac{1}{2}\theta'}{2\sin.\frac{1}{2}\theta'}\right\}=\frac{d\theta'}{\sqrt{2\alpha}}.\frac{\cos.\theta'}{2\sin.\frac{1}{2}\theta'}; \qquad [10255e]$$

and as the factor $\frac{\cos.\theta'}{\sin.\frac{1}{2}\theta'}$ is always positive, when θ' increases from 0 to a right angle, it [10255f]
follows that $2l$ will increase with θ', while it passes through those limits, agreeably to what is stated in [10255].

† (4368) When $\theta=0$, the expression $-\text{log.tang.}\frac{1}{4}\theta$ becomes infinite; and the value of [10257a]
$2l$ [10255a] then becomes infinite, as in [10257].

when it falls back again into the vessel. At this limit, the column will be sustained in equilibrium, if the force which raises the disk be exactly that which corresponds to the state of equilibrium; and it is evident that, for this to take place, the force ought to be equal to the weight of the disk and that of the
[10259] column which is raised up. Thus the adhesion of the disk to the fluid is a capillary phenomenon; but to prove it incontestably, we shall compute this force by analysis, and shall compare the results with experiments.

We shall consider a section of the surface of the column, by a vertical plane
[10260] passing through the centre of the disk, supposing it to be circular. This section will be the generating curve, which, by its revolution about the vertical line or axis passing through the centre of the disk, generates the external surface of the column. Then we shall put

[10261] $l =$ the radius of the disk;

[10262] $l+y =$ the distance from the vertical axis to any point of the generating curve;

[10263] $z =$ the height of the same point above the level of the fluid in the vessel.

The equilibrium of the columns will give, as in [9324], by observing
[10263′] that* $\frac{1}{b}=0$,

[10264] $$\frac{\frac{ddz}{dy^2}}{\left(1+\frac{dz^2}{dy^2}\right)^{\frac{3}{2}}}+\frac{1}{l+y}\cdot\frac{\frac{dz}{dy}}{\sqrt{1+\frac{dz^2}{dy^2}}}=2\alpha z.$$ [Differential equation of the surface.]

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* (4369) In the annexed figures 168, 169, $LECMH$ is the level surface of the fluid in
[10264a] the vessel; FAR the diameter of the disk, which is supposed to be parallel to HL; AE the vertical line or axis perpendicular to HL;
[10264b] $MmKkf'F$ the curve whose revolution about the axis AE generates the external surface of the fluid; K, k, are two points of this curve which are infinitely near to each other; KD, kd, the corresponding
[10264c] ordinates parallel to AF; and the point m is infinitely near to M, the point f' infinitely near to F; moreover the line Ff is on the continuation of $f'F$. Then, in the present notation, we shall have

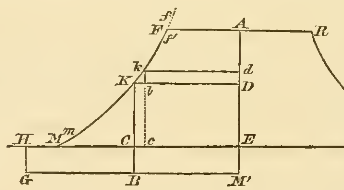

Fig. 168.

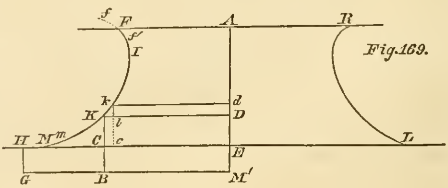

Fig. 169.

To integrate this equation, we shall put [10264']

$$AF=l, \qquad ED=z, \qquad DK=l+y, \qquad \text{angle } DKk=\varpi \ [10265].$$ [10264d]

At the point M we have the angle $\varpi=0$; and at the point F, it becomes

$$AFf=\pi-AFf'=\pi-\varpi' \ [10279].$$ [10264e]

When the fluid is mercury, and the disk of glass, we shall have the angle $AFf=48^\circ$ [10353], and the surface of the fluid will have a gradual slope from F to M, as in fig. 168. But if the fluid be water, and the disk of glass, well moistened with water, we shall have the angle $AFf'=0^\circ$ [9359k], as in fig. 169; and the angle ϖ will gradually increase from the point M, where $\varpi=0$, to the point I, where $\varpi=\frac{1}{2}\pi$, and the surface of the fluid is vertical, as is evident from the equation [10266]; lastly, from I to F, the angle ϖ will continue to increase till it becomes $\varpi=\pi$, at the point F; and then the curve Ff' will be horizontal, as appears from the same equation [10266]. Hence it is evident that the surface which is formed by the revolution of this curve about its axis will be grooved like a pulley. To find the equation of this surface, we shall suppose a canal $KCBGH$ to be drawn, with the vertical branches BCK, GH, and the horizontal branch BG, the point H being situated upon the level surface of the fluid in the vessel. Then putting, as in [9310], R, R', for the greatest and the least radii of curvature at the point K, the capillary action at K will be equal to $K-\frac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)$ [9315b]; to which we must add the pressure of the atmosphere P, and the weight of the column BK, namely $gD\times BK$ [10187c], to obtain the pressure at the point B, in the canal BK, equal to $K+P-\frac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)+gD.BK.$ Again, the capillary action at the point H, which is represented by K [9259], being added to the pressure of the atmosphere P, and to the weight $gD\times GH$ of the column GH, gives the pressure at G, at the bottom of the canal HG, equal to $K+P+g\times GH$. Putting the expressions [10264m, o] equal to each other, on account of the equilibrium of the canal BG, and neglecting $K+P$, which occurs in both members of the equation, we get [10264f] [10264g] [10264h] [10264i] [10264k] [10264l] [10264m] [10264n] [10264o]

$$\tfrac{1}{2}H.\left(\frac{1}{R}+\frac{1}{R'}\right)=gD.(BK-GH)=gD.CK=gDz \ [10263].$$ [10264p]

Substituting $gD=H.\alpha$ [9372z'], and dividing by $\frac{1}{2}H$, we get $\frac{1}{R}+\frac{1}{R'}=2\alpha z$; and by using the values of R, R', [9326, 9326'], it becomes [10264p']

$$\frac{\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}=2\alpha z;$$ [10264q]

observing that the quantity u [9320] represents the ordinate $DK=l+y$ [10264d], so that $du=dy$. Substituting these values of u, du, in the preceding equation [10264q], it becomes as in [10264], where du or *dy is considered as constant* [9327c]. The equation [10264q] is equivalent to the equation (2), in page 228 of M. Poisson's *Nouvelle Théorie, &c.*, changing u into t, and α into a^{-2} [9323p], to conform to his notation. [10264r] [10264s]

[10265] $\varpi =$ the angle which an infinitely small arc of the generating section makes with the horizontal line drawn from the lower end of this small arc to the vertical line which passes through the centre of the disk. We shall have *

[10266] $$\frac{dz}{dy} = -\text{tang.}\varpi;$$

and then the preceding equation will become †

[10267] $$-\frac{d\varpi}{dy}.\cos.\varpi - \frac{\sin.\varpi}{l+y} = 2\alpha z; \qquad (s)$$

Multiplying this by $dy.\text{tang.}\varpi$, or $-dz$ [10266], we get, by integration,

[10268] $$\cos.\varpi + \int \frac{dz.\sin.\varpi}{l+y} = \text{constant} - \alpha z^2.$$

[10269] *We shall suppose that the integral commences with z;* observing that, when $z=0$, the arc of the surface will coincide with the level surface, which renders

[10270] ϖ nothing; consequently $\cos.\varpi = 1$, and we shall have, constant $= 1$; therefore

[10271] $$\alpha z^2 = 1 - \cos.\varpi - \int_0^z \frac{dz.\sin.\varpi}{l+y}. \qquad (t)$$ [Equation of the surface.]

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* (4370) Draw kl, fig. 168, 169, page 948, parallel and equal to dD; then it is evident

[10266a] that $\frac{kl}{Kl} = \text{tang.}kKl$, or by [10264d], $\frac{dz}{-dy} = \text{tang.}\varpi$, whence we easily deduce [10266],

[10266b] or $dy = -\frac{dz}{\text{tang.}\varpi}$.

† (4371) The equation [10266] gives

[10268a] $$\sqrt{1+\frac{dz^2}{dy^2}} = \sqrt{1+\text{tang.}^2\varpi} = \frac{1}{\cos.\varpi};$$

moreover, the differential of [10266] gives, by observing that dy is constant [10264r],

[10268b] $$\frac{ddz}{dy} = -\frac{d\varpi}{\cos.^2\varpi};$$

substituting these in [10264], we get [10267]. Multiplying the first term of [10267] by $dy.\text{tang.}\varpi$, and the other terms of that equation by the equivalent expression $-dz$ [10266], we get

[10268c] $$-d\varpi.\sin.\varpi + \frac{dz.\sin.\varpi}{l+y} = -2\alpha z dz,$$

[10268d] whose integral gives [10268]. When $z=0$, the angle ϖ corresponding to the point M, fig. 168, 169, page 948, will be $\varpi=0$, and the integral $\int \frac{dz.\sin.\varpi}{l+y}$ then commences

[10268e] [10269]; so that the equation [10268] becomes, at that point, $1 =$ constant. Substituting this in [10268], we easily obtain the value of αz^2 [10271].

When the disk is very large, l will be great in comparison with* $\frac{1}{\sqrt{a}}$; thus we [10272]
shall have a first approximate value of z, by neglecting the integral $-\int_0^z \frac{dz.\sin.\varpi}{l+y}$, in the equation [10271], which gives

$$z = \frac{\sqrt{2}}{\sqrt{a}}.\sin.\tfrac{1}{2}\varpi. \qquad \text{[Approximate value of } z.] \qquad [10273]$$

Then we may substitute this value of z, in the integral $-\int \frac{dz.\sin.\varpi}{l+y}$, which [10273']
by this means becomes†

$$-\int_0^z \frac{dz.\sin.\varpi}{l+y} = -\sqrt{\frac{2}{a}}.\int_0^\varpi \frac{d\varpi.\sin.\frac{1}{2}\varpi.\cos.^2\frac{1}{2}\varpi}{l+y}; \qquad [10274]$$

this integral is easily reduced to the following form,

$$-\int_0^z \frac{dz.\sin.\varpi}{l+y} = -\frac{2\sqrt{\frac{2}{a}}}{3(l+y)}.\{1-\cos.^3\tfrac{1}{2}\varpi\} - \tfrac{2}{3}.\sqrt{\frac{2}{a}}.\int_\theta^\varpi \frac{dy.(1-\cos.^3\frac{1}{2}\varpi)}{(l+y)^2}. \qquad [10275]$$

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* (4372) While ϖ varies from 0° to 48° [10264f], if the fluid be mercury; or from $\varpi=0^\circ$ to $\varpi=200^\circ$ [10264h], if the fluid be water; the ordinate z will vary from $z=0$, to $z=EA$, its greatest value; and if l *be very large*, we may consider the integral [10272a]
$\int \frac{dz.\sin.\varpi}{l+y}$, as not exceeding a quantity of the order $\frac{AE}{l}$, or $\frac{AE}{AF}$, which must evidently [10272b]
be much less than unity, from the nature of the capillary action; so that this term must, in general, be much less than either of the terms $1, \cos.\varpi$, which occur in [10271]; and if we retain only these two terms, we shall have nearly $az^2 = 1-\cos.\varpi = 2\sin.^2\frac{1}{2}\varpi$ [1], Int. Dividing [10272c]
this by a, and extracting the square root, we get z [10273], which is of the order $\frac{1}{\sqrt{a}}$, mentioned in [10272], and is much less than l, as is observed in [10272], and is also [10272c']
evident from the experiments which are mentioned in [10317, 10333, &c.]. If we substitute $a^{-\frac{1}{2}} = \alpha$ [9323p] in [10273], it will become

$$z = \sqrt{2}.\alpha.\sin.\tfrac{1}{2}\varpi. \qquad \text{[Approximate value of } z.] \qquad [10272d]$$

† (4373) The differential of [10273] gives $dz = \frac{\sqrt{2}}{\sqrt{a}}.\frac{1}{2}d\varpi.\cos.\frac{1}{2}\varpi$; substituting this in [10274a]
[10273'], and then putting $\sin.\varpi = 2\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi$ [31], Int., we get [10274]. The [10274b]
integral of [10274] can easily be reduced to the form [10275]; their differentials being equal to each other, as is easily perceived by observing that the differential of the last term of [10274c]
[10275] is destroyed by the part of the differential of the first term depending on the differential of $l+y$, as is evident by inspection; lastly, the differential of the first term [10274d]
depending on $\cos.^3\frac{1}{2}\varpi$, produces the differential of [10274].

The element of this last integral is never infinite; for, although $\frac{dy}{d\varpi}$ becomes infinite, when ϖ is nothing, because we have*

[10276] $$\frac{dy}{d\varpi}=-\frac{dz.\cos.\varpi}{d\varpi.\sin.\varpi}=-\frac{1}{2\sqrt{2\alpha}}.\frac{\cos.\varpi}{\sin.\frac{1}{2}\varpi},$$

yet, as it is multiplied, in the preceding integral [10275], by $-d\varpi.(1-\cos.^3\frac{1}{2}\varpi)$, the
[10277] coefficient of $d\varpi$ in this product is never infinite. If we neglect the term divided by $(l+y)^2$, in comparison with that which is divided by $l+y$, we shall have

[10278] $$-\int_0^z\frac{dz.\sin.\varpi}{l+y}=-\frac{2\sqrt{2}.(1-\cos.^3\frac{1}{2}\varpi)}{3(l+y).\sqrt{\alpha}}.$$

[10278′] The integral must be taken from $\varpi=0$ to $\varpi=\pi-\varpi'$, supposing that†

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* (4374) We have, in [10266],

[10276a] $$dy=-dz.\frac{1}{\text{tang}.\varpi}=-dz.\frac{\cos.\varpi}{\sin.\varpi}=-\left(\frac{\sqrt{2}}{\sqrt{\alpha}}.\tfrac{1}{2}d\varpi.\cos.\tfrac{1}{2}\varpi\right).\frac{\cos.\varpi}{\sin.\varpi}\quad[10274a];$$

now, substituting $\sin.\varpi=2\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi$, and dividing by $d\varpi$, we get

[10276a′] $$\frac{dy}{d\varpi}=-\frac{1}{2\sqrt{2\alpha}}.\frac{\cos.\varpi}{\sin.\frac{1}{2}\varpi},$$

as in [10276]. The value of dy, deduced from this expression, being substituted in the part of the second member of [10275], under the sign $\int$, gives

[10276b] $$\int\frac{dy.(1-\cos.^3\frac{1}{2}\varpi)}{(l+y)^2}=-\frac{1}{2\sqrt{2\alpha}}.\int\frac{d\varpi.\cos.\varpi.(1-\cos.^3\frac{1}{2}\varpi)}{(l+y)^2.\sin.\frac{1}{2}\varpi}.$$

Now the factor $(1-\cos.^3\frac{1}{2}\varpi)$ is equal to

[10276c] $$(1-\cos.\tfrac{1}{2}\varpi).(1+\cos.\tfrac{1}{2}\varpi+\cos.^2\tfrac{1}{2}\varpi)=2\sin.^2\tfrac{1}{4}\varpi.(1+\cos.\tfrac{1}{2}\varpi+\cos.^2\tfrac{1}{2}\varpi),$$

and

[10276c′] $$\sin.\tfrac{1}{2}\varpi=2\sin.\tfrac{1}{4}\varpi.\cos.\tfrac{1}{4}\varpi\quad[1, 31], \text{Int.};$$

hence, by substitution in [10276b], it becomes

[10276d] $$-\frac{1}{2\sqrt{2\alpha}}.\int\frac{d\varpi}{(l+y)^2}.\cos.\varpi.(1+\cos.\tfrac{1}{2}\varpi+\cos.^2\tfrac{1}{2}\varpi).\text{tang}.\tfrac{1}{4}\varpi;$$

[10276e] and as ϖ never exceeds 200° [10272a], the coefficient of $d\varpi$ must always be finite.
[10276f] Moreover, as the elements of this integral are divided by the great quantity $(l+y)^2$, it is evident that the whole integral is very small; and by neglecting it, we obtain [10278].

† (4375) Substituting [10278] in [10271], then changing z into z' [10279′] and
[10280a] ϖ into $\pi-\varpi'$ [10278′], so as to correspond to the upper point F of the curve, fig. 168, 169, page 948, where we have the angle $AFf'=\varpi'$, it becomes as in
[10280b] [10280]; observing that, at this point, the ordinate $l+y$ becomes simply equal to l, because the fluid is supposed to terminate at the circular border of the lower surface of
[10280b′] the disk. Now, dividing [10280] by α, substituting $1+\cos.\varpi'=2\cos.^2\frac{1}{2}\varpi'$ [6], Int.,

$\varpi'=$ the angle which is formed, in the plane of the generating section of the fluid, by two lines drawn through the upper point of contact of the fluid with the plane, in directions respectively parallel to the tangents of the surface of the fluid, and of the surface of the attracting solid, at an insensible distance from that point of contact; both these tangents being drawn in a downward direction, or towards the interior part of the fluid; [10279]

$z'=$ the extreme value of z, or the whole height of the column EA, which is raised by the disk. [10279']

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and then extracting the square root, we get z', as in [10281], neglecting the third and higher powers of $\alpha^{-\frac{1}{2}}$.

We must not use the value $\varpi=\pi-\varpi'$ [10280a], for the upper limit of the surface, if the fluid rises ever so little on the outer rim $FOPQ$ of the disk, fig. 170, 171. This will [10280c]
be evident *by considering the simple case, where a glass disk, whose figure is that of a right cylinder with a circular base, is dipped* [10280d] *into a vessel of water, to a considerable depth, and then gradually raised up in a vertical direction;* taking care that the operation is performed with sufficient steadiness to enable the fluid on the surface to adjust itself to its successive states of equilibrium. In [10280d']
order to form a more distinct idea of the figure of the surface of the fluid, in the different situations of the cylinder, we shall suppose, as in [10012m—o], that the rim or intersection of the base of the cylinder with its curved surface, is rounded off in the form of [10280e]
a quadrant of a circle FOP, whose radius is extremely small. The cylinder being well

Fig. 170.

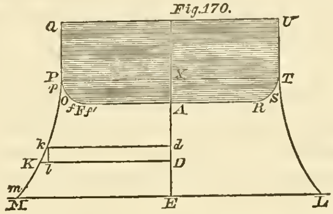
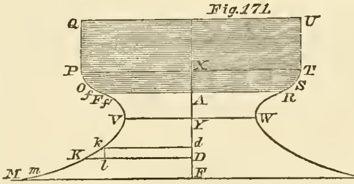


Fig. 171.



moistened, the fluid will ascend on the vertical sides PQ, TU, of the cylinder, fig. 170, and will adhere, so as to form with them, at the upper part of the fluid, the angle ϖ'. In this [10280f]
case, the general value of the angle $DKk=\varpi$ will become, at the side of the cylinder, [10280g]
$\varpi=\frac{1}{2}\pi$; and it will retain the same value, while the cylinder is gradually raised up in a [10280g']
vertical direction, until the upper part of the fluid has descended along the side QP to the point P. When the fluid has attained to the point P, the angle ϖ, corresponding to its [10280h]
summit, will increase by passing over the curved surface POF, as the cylinder is slowly elevated; so that, when this summit is at the point F, fig. 171, where the angle $AFf'=\varpi'=0$ [10280i]
[10279, 10280d, 9359k], we shall have $\varpi=AFf=\pi$. Hence it appears that, while the summit of the fluid descends through the quadrantal arc POF, the angle ϖ increases from

Then the equation [10271] will give

[10280] $$\alpha z'^2 = 1 + \cos.\varpi' - \frac{2\sqrt{2}.\{1-\sin.^3\tfrac{1}{2}\varpi'\}}{3l\sqrt{\alpha}};$$

[10280k] $\varpi = \frac{1}{2}\pi$ to $\varpi = \pi$, the whole increment being $\frac{1}{2}\pi$, or a quadrantal arc. During the whole of this descent along the arc POF, the angle ϖ', which is formed by the upper part of the section of the fluid and the section of the cylinder at the same point, remains invariable, and,
[10280k′] in the present hypothesis, is $\varpi' = 0$ [10280i]; so that, if we put i for the *acute* angle which
[10280l] a vertical line, drawn through any point O of the quadrantal arc POF, forms with the arc
[10280l′] at O, we may put generally $\varpi = \frac{1}{2}\pi + i$, for the value of ϖ, corresponding to the upper limit of the surface of the fluid, when it is at the point O. The values of the angle ϖ,
[10280m] corresponding to the points P, O, F, which we have found to be $\frac{1}{2}\pi$, $\frac{1}{2}\pi + i$, π [10280k, l′], respectively, have been computed upon the supposition that the angle $\varpi' = 0$ [10280i]; but if this angle has a finite value, the preceding expressions of ϖ [10280m] must be decreased by ϖ', and then they will become respectively $\frac{1}{2}\pi - \varpi'$, $\frac{1}{2}\pi + i - \varpi'$, $\pi - \varpi'$; so that we
[10280n] shall have for the extreme limit of ϖ [10280l′], corresponding to any point O of the arc POF, the following expression;

[10280o] $$\varpi = \tfrac{1}{2}\pi + i - \varpi';$$

in which i represents the angle corresponding to the circular arc PO, its least value being
[10280p] $i=0$, its greatest value $i = \frac{1}{2}\pi$ [10280m]. This extreme limit of the value ϖ is to be
[10280p′] used instead of $\varpi = \pi - \varpi'$, given by the author in [10278′]; and the effect of making this
[10280q] alteration is equivalent to that of changing his value of ϖ', in [10280p′], into $\varpi' + \frac{1}{2}\pi - i$; since, by this means, the expression $\varpi = \pi - \varpi'$ [10280p′] changes into

[10280q′] $$\varpi = \pi - (\varpi' + \tfrac{1}{2}\pi - i) = \tfrac{1}{2}\pi + i - \varpi',$$

as in [10280o]. Making the same change in the value of ϖ' in the formulas [10280, 10281], they become respectively

[10280r] $$\alpha z^2 = 1 + \sin.(i - \varpi') - \frac{2\sqrt{2}.\{1 - \sin.^3(\frac{1}{2}\varpi' + \frac{1}{4}\pi - \frac{1}{2}i)\}}{3l\sqrt{\alpha}},$$

[10280r′] $$z' = \sqrt{\frac{2}{\alpha}}.\cos.(\tfrac{1}{2}\varpi' + \tfrac{1}{4}\pi - \tfrac{1}{2}i) - \frac{\{1 - \sin.^3(\frac{1}{2}\varpi' + \frac{1}{4}\pi - \frac{1}{2}i)\}}{3l\alpha.\cos.(\frac{1}{2}\varpi' + \frac{1}{4}\pi - \frac{1}{2}i)}.$$

These expressions may be simplified by putting

[10280s] $$i = \varpi' - \tfrac{1}{2}\pi + 2v, \quad \text{or} \quad v = \tfrac{1}{4}\pi + \tfrac{1}{2}i - \tfrac{1}{2}\varpi',$$

from which we get $i - \varpi' = -(\frac{1}{2}\pi - 2v)$, $\frac{1}{2}\varpi' + \frac{1}{4}\pi - \frac{1}{2}i = \frac{1}{2}\pi - v$. Substituting these in
[10280t] [10280r], and then multiplying by $\alpha^{-1} = a^2$ [9323p], we get, by observing that $1 + \sin.(i - \varpi')$ becomes $1 - \sin.(\frac{1}{2}\pi - 2v) = 1 - \cos.2v = 2\sin.^2 v$ [1], Int.,

[10280u] $$z'^2 = 2a^2.\sin.^2 v - \frac{2\sqrt{2}.a^3}{3l}.(1 - \cos.^3 v).$$

This expression is the same as the equation (6) in page 232 of M. Poisson's *Nouvelle*
[10280v] *Théorie, &c.*, neglecting terms of the order a^4. If we neglect terms of the order a^3, in [10280u], and extract the square root, we shall get very nearly

whence we obtain very nearly

$$z'=\sqrt{\frac{2}{\alpha}}.\cos.\tfrac{1}{2}\varpi'-\frac{(1-\sin.^3\frac{1}{2}\varpi')}{3l\alpha.\cos.\frac{1}{2}\varpi'}. \quad \text{[Height of the elevated column.]} \qquad [10281]$$

$$z'=\sqrt{2}.\alpha.\sin.v=\sqrt{2}.\alpha.\sin.(\tfrac{1}{4}\pi+\tfrac{1}{2}i-\tfrac{1}{2}\varpi') \quad [10280s]. \qquad [10280w]$$

From this last expression of z', it is evident that, while i increases, from $i=0$ to its greatest [10280x] value $i=\frac{1}{2}\pi$ [10280p], the value of z' will increase, and that z' will attain its maximum [10280y] when $i=\frac{1}{2}\pi$, and then $z'=\sqrt{2}.\alpha.\sin.(\frac{1}{2}\pi-\frac{1}{2}\varpi')=\sqrt{2}.\alpha.\cos.\frac{1}{2}\varpi'$; which is the case [10280y'] treated of by La Place in [10280, &c]. This maximum value corresponds to the state where the summit of the fluid is situated on the lowest point F of the arc POF, as in fig. 171, page 953. If we put $\varpi'=0$, as in the case of water with a glass disk [10280i], this [10280z] maximum value of z' will become $z'=\sqrt{2}.\alpha$. In the same case of $\varpi'=0$, if we put $i=0$, we shall have, from [10280s], $v=\frac{1}{4}\pi$, whence $\sin.v=\sqrt{\frac{1}{2}}$, and then the expression of z' [10280w] becomes $z'=\alpha$; this corresponds to the case where the summit of the [10281a] fluid ascends to the point P, and agrees with the calculation in [9425k]. Hence it appears, that, when $\varpi'=0$, the value of z' increases with the angle i, from the upper point P, where $i=0$, and $z'=\alpha$ [10281a], to the lower point F, where $i=\frac{1}{2}\pi$, and $z'=\sqrt{2}.\alpha$ [10281b] [10280z], being its greatest value. This maximum value of z' must correspond very nearly to the situation of the disk when the greatest quantity of fluid is elevated, the radius of the [10281c] disk being supposed much greater than z' or α; and it being evident that this volume would be decreased, if the outer limit of the fluid were removed from F towards A, on the lower [10281d] surface of the disk.

In the case we have just considered, where water is elevated by a glass disk or cylinder, the external surface of the fluid MKP, fig. 170, 171, page 953, is always concave, and the lower part of the surface, near M or L, has the level line of the fluid in the vase MEL for [10281e] its *asymptote*. As the summit of the fluid descends along the arc POF, in consequence of the gradual elevation of the disk, the external surface of the fluid becomes grooved in fig. 171, at the point V, where the ordinate YV, or the distance of the surface from the axis AE, is a [10281f] minimum, and the *tangent to the curve is vertical*, or $\varpi=\frac{1}{2}\pi$. The figure of the surface [10281g] FVM may be very easily investigated by means of the equations [10271, 10266]; and if we neglect the last term of [10271], on account of its smallness, we shall have, as in [10281h] [10272d],

$$z=\sqrt{2}.\alpha.\sin.\tfrac{1}{2}\varpi. \qquad [10281i]$$

At the point F, where $\varpi=\pi-\varpi'=w$ [10280c, 9980m'], it becomes $EA=\sqrt{2}.\alpha.\sin.\frac{1}{2}w$; [10281k] and at the point V, where $\varpi=\frac{1}{2}\pi$ [10281g], or $\sqrt{2}.\sin.\frac{1}{2}\varpi=1$, we get $EY=\alpha$ [10281i]. [10281l] The difference of these two expressions is the depression of the point V below the lower surface of the disk, namely,

$$AY=\alpha.\{\sqrt{2}.\sin.\tfrac{1}{2}\varpi-1\}. \qquad [10281m]$$

We have, from [31', 31, 1], Int., $\frac{1}{\text{tang.}\varpi}=\frac{\cos.\varpi}{\sin.\varpi}=\frac{1-2\sin.^2\frac{1}{2}\varpi}{2\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi}$; substituting this, and the [10281n] differential of [10281i], $dz=\frac{\alpha}{\sqrt{2}}.d\varpi.\cos.\frac{1}{2}\varpi$, in [10266b], we get

To obtain the volume of the whole column raised up, we must first multiply
[10282] this value of z' by the lower surface of the disk, or by πl^2, and then add to

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[10281o] $$dy = -\frac{dz}{\text{tang.}\varpi} = -\left(\frac{\alpha}{\sqrt{2}}.d\varpi.\cos.\tfrac{1}{2}\varpi\right).\left(\frac{1-2\sin.^2\tfrac{1}{2}\varpi}{2\sin.\tfrac{1}{2}\varpi.\cos.\tfrac{1}{2}\varpi}\right) = -\frac{\alpha}{\sqrt{2}}.\frac{\tfrac{1}{2}d\varpi}{\sin.\tfrac{1}{2}\varpi} + \frac{\alpha}{\sqrt{2}}.d\varpi.\sin.\tfrac{1}{2}\varpi,$$

whose integral is

[10281p] $$y = -\frac{\alpha}{\sqrt{2}}.\int\frac{\tfrac{1}{2}d\varpi}{\sin.\tfrac{1}{2}\varpi} + \frac{\alpha}{\sqrt{2}}.\int d\varpi.\sin.\tfrac{1}{2}\varpi = -\frac{\alpha}{\sqrt{2}}.\text{hyp.log.tang.}\tfrac{1}{4}\varpi - \sqrt{2}.\alpha.\cos.\tfrac{1}{2}\varpi + \text{constant};$$

observing that, from [59, 54, 31], Int., we have

[10281q] $$d.(\text{hyp.log.tang.}\tfrac{1}{4}\varpi) = \frac{d.\text{tang.}\tfrac{1}{4}\varpi}{\text{tang.}\tfrac{1}{4}\varpi} = \frac{\tfrac{1}{4}d\varpi.(\cos.\tfrac{1}{4}\varpi)^{-2}}{\text{tang.}\tfrac{1}{4}\varpi} = \frac{\tfrac{1}{4}d\varpi}{2\sin.\tfrac{1}{4}\varpi.\cos.\tfrac{1}{4}\varpi} = \frac{\tfrac{1}{2}d\varpi}{\sin.\tfrac{1}{2}\varpi}.$$

The constant quantity in [10281p] must be so taken that, at the point F, fig. 171, page 953, where $\varpi = w$ [10281k], we may have $y = 0$; hence we get

[10281r] $$\text{constant} = \frac{\alpha}{\sqrt{2}}.\text{hyp.log.tang.}\tfrac{1}{4}w + \sqrt{2}.\alpha.\cos.\tfrac{1}{2}w;$$

substituting this in [10281p], we obtain the general value of y, in the following form;

[10281s] $$y = \frac{\alpha}{\sqrt{2}}.\text{hyp.log.}\frac{\text{tang.}\tfrac{1}{4}w}{\text{tang.}\tfrac{1}{4}\varpi} + \sqrt{2}.\alpha.(\cos.\tfrac{1}{2}w - \cos.\tfrac{1}{2}\varpi).$$

At the point V, where $\tfrac{1}{2}\varpi = \tfrac{1}{4}\pi$, we have $\sin.\tfrac{1}{2}\varpi = \cos.\tfrac{1}{2}\varpi = \sqrt{\tfrac{1}{2}}$; consequently
[10281t] $\text{tang.}\tfrac{1}{4}\varpi = \frac{\sin.\tfrac{1}{2}\varpi}{1+\cos.\tfrac{1}{2}\varpi}$ [41], Int., becomes $\text{tang.}\tfrac{1}{4}\varpi = \frac{\sqrt{\tfrac{1}{2}}}{1+\sqrt{\tfrac{1}{2}}} = \frac{1}{1+\sqrt{2}}$. Substituting these in [10281s], we get the value of y corresponding to the point V, namely,

[10281u] $$y = \frac{\alpha}{\sqrt{2}}.\text{hyp.log.}\{(1+\sqrt{2}).\text{tang.}\tfrac{1}{4}w\} + \alpha.\{\sqrt{2}.\cos.\tfrac{1}{2}w - 1\}.$$

In the case of water with a glass cylinder, we have $\varpi' = 0$ [10280i], and $w = \pi$ [10281k];
[10281v] hence $\sin.\tfrac{1}{2}w = 1$, $\cos.\tfrac{1}{2}w = 0$, $\text{tang.}\tfrac{1}{4}w = 1$; substituting these in [10281m, u], we get very nearly

[10281w] $$AY = \alpha.(\sqrt{2}-1) = \alpha \times 0{,}4142.$$

[10281x] $$y = \frac{\alpha}{\sqrt{2}}.\text{hyp.log.}(1+\sqrt{2}) - \alpha = -\alpha \times 0{,}3768.$$

[10281y] If we suppose, as in [9372o], that $\alpha = 3^{\text{mi.}},88972$, we shall get $AY = 1^{\text{mi.}},611$; $y = -1^{\text{mi.}},466$; *which correspond to the case where the radius of the disk or cylinder is excessively large in comparison with* α. The expressions [10281i, m, s, u, w, x] agree with those of M. Poisson, in page 230 of his *Nouvelle Théorie, &c.*, by making some slight reductions.

[10281z] The supposition that the lower edge of the surface of the cylinder is terminated by the small circular arc POF, may be applied when the external surface of the fluid is convex, as in the case of mercury with a wet glass cylinder. In this case, we shall suppose that the
[10282a] mercury forms the acute angle w with the vertical side PQ of the cylinder, when the upper surface of the fluid rests against the point P; and if we slowly raise up the cylinder in a

this product the volume which surrounds the fluid cylinder whose base is the lower surface of the disk. This last volume is equal to the integral* $-2\pi\int(l+y).zdy$, taken from $\varpi=0$ to $\varpi=\pi-\varpi'$; thus we shall have, for the expression of the whole volume V of the elevated column, [10283]

$$V=\pi l^2.\sqrt{\tfrac{2}{\alpha}.\cos.\tfrac{1}{2}\varpi'}-\frac{\pi l.(1-\sin.^3\tfrac{1}{2}\varpi')}{3\alpha.\cos.\tfrac{1}{2}\varpi'}-2\pi.\int(l+y).zdy. \quad \left[\begin{smallmatrix}\text{Volume of the}\\ \text{elevated fluid.}\end{smallmatrix}\right] \qquad [10284]$$

We may rigorously determine this last integral in the following manner.

The equation [10267], being multiplied by $-(l+y).dy$, gives, by integration,†

$$-2\alpha.\int(l+y).zdy=(l+y).\sin.\varpi+\text{constant}. \qquad [10285]$$

To determine the constant quantity, we shall observe that the integral must be taken between the limits $\varpi=0$ and $\varpi=\pi-\varpi'$ [10283]; and we shall now show that $(l+y).\sin.\varpi$ is nothing when $\varpi=0$. For $l+y$ becomes infinite when ϖ is nothing; therefore, by reducing its expression into a series ascending according to the powers of ϖ, the first term of this series will be of the form $A.\varpi^{-r}$. Moreover, z being nothing when $\varpi=0$, if we also reduce its value into a series ascending according to the powers of ϖ, the first term of it will be of the form $A'.\varpi^{r'}$, *r and r′ being positive.* Then the equation [10286] [10287] [10288]

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vertical direction, so as to bring the point O, which corresponds to the angle i, against the upper surface of the fluid, its inclination with the vertical will be $i+w$; and when this is equal to $\frac{1}{2}\pi$, the fluid will become horizontal, and upon a level with the surface of the fluid in the vessel. Continuing still to elevate the cylinder, so that the upper part of the fluid may fall upon a point between O and F, the angle $i+w$ will exceed $\frac{1}{2}\pi$, and the surface of the fluid near that point will become concave. [10282b] [10282c] [10282d]

* (4376) Continue the line kl, fig. 168 or 169, page 948, to meet ME in c; then the area $CKkc=CK\times Kl=-zdy$; multiplying this by the circumference described by the point K, in its revolution about the axis AE, it becomes $-2\pi.(l+y).zdy$, which represents the element of the volume of the fluid in question. Its integral is as in [10283]; its limits being from the point M, where $\varpi=0$, to the point F, where $\varpi=\pi-\varpi'$ [10280a]. Adding this to the quantity $\pi l^2z'$ [10282], we get the whole volume V of the fluid, namely, $V=\pi l^2z'-2\pi.\int(l+y).zdy$; and by substituting the value of z' [10281], it becomes as in [10284]. We may remark that, for convenience, we have inserted the symbol V in [10284], but it is not in the original work. [10283a] [10283b] [10283c] [10283d] [10283e] [10283f] [10283g]

† (4377) This product is $d\varpi.\cos.\varpi.(l+y)+dy.\sin.\varpi=-2\alpha.(l+y).zdy$, whose integral gives [10285]. [10285a]

[10289] $$\frac{dz}{dy} = -\text{tang.}\varpi \quad [10266]$$

[10290] will give, by noticing only these first terms, and observing that $\text{tang.}\varpi$ becomes nearly equal to ϖ when it is very small,*

[10291] $$\frac{r'.A'.\varpi^{r'}}{r.A.\varpi^{-r}} = \varpi;$$

whence we deduce, by the comparison of the exponents of ϖ,

[10292] $$1 - r = r';$$

[10293] $(l+y).\sin.\varpi$ will therefore become† $A.\varpi^{r'}$, by substituting $A.\varpi^{-r}$ for $l+y$, and ϖ for $\sin.\varpi$; so that $(l+y).\sin.\varpi$ is nothing when $\varpi=0$; consequently

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[10291a] * (4378) We have, in [10287, 10288], $l+y=A.\varpi^{-r}+\&c.$; $z=A'.\varpi^{r'}+\&c.$; $-\text{tang.}\varpi = -\varpi - \&c.$ [45], Int. Substituting these in [10289], it becomes

[10291b] $$\frac{r'.A'.\varpi^{r'-1}d\varpi + \&c.}{-r.A.\varpi^{-r-1}d\varpi + \&c.} = -\varpi - \&c.;$$

and, as the first term of this equation must be the same in both members of the equation, we shall have

[10291c] $$\frac{r'.A'.\varpi^{r'-1}d\varpi}{-r.A.\varpi^{-r-1}d\varpi} = -\varpi;$$

whence we easily deduce [10291]. This may also be put under the form $\frac{r'.A'}{r.A}.\varpi^{r'+r}=\varpi$;

[10291d] and, to make the exponents of ϖ equal to each other, we must put $r'+r=1$, as in [10292].

† (4379) The first term of $l+y$ [10291a] is $A.\varpi^{-r}$, and the first term of $\sin.\varpi$ is ϖ

[10293a] [43], Int.; therefore the first term of their product $(l+y).\sin.\varpi$ is $A.\varpi^{1-r}=A.\varpi^{r'}$ [10292]; and as r' is positive [10288], this will vanish when $\varpi=0$; consequently the second member of the expression [10285] must give the constant quantity equal to nothing,

[10293b] and we shall have $-2\alpha.\int(l+y).zdy=(l+y).\sin.\varpi$. At the second limit, where $\varpi=\pi-\varpi'$ [10286], and $l+y=l$ [10280b], corresponding to the point F, fig. 168, 169, page 948,

[10293c] the integral becomes $-2\alpha\int(l+y).zdy=l.\sin.\varpi'$. Multiplying this by $\frac{\pi}{\alpha}$, we get [10294],

[10293c′] and by putting $\sin.\varpi'=2\sin.\frac{1}{2}\varpi'.\cos.\frac{1}{2}\varpi'$ [31], Int., it becomes

[10293d] $$-2\pi.\int(l+y).zdy=\frac{2\pi}{\alpha}.l.\sin.\tfrac{1}{2}\varpi'.\cos.\tfrac{1}{2}\varpi'.$$

Instead of restricting the calculation to the particular value of $\varpi=\pi-\varpi'$, which is used by

[10293e] La Place in [10278′, &c.], we may use the more general expression $\varpi=\frac{1}{2}\pi+i-\varpi$ [10280o], which requires that we should change ϖ' into $\varpi'+\frac{1}{2}\pi-i$ [10280q], in La Place's formulas; and by this means the expression [10294] becomes

the constant term of the preceding integral is nothing. We shall therefore have, by observing that, at the other limit of the integral, ϖ becomes $\pi-\varpi'$, and $y=0$ [10280b],

$$-2\pi.\int_0^{\pi-\varpi'}(l+y).zdy=\frac{\pi}{\mathfrak{a}}.l.\sin.\varpi'; \qquad [10294]$$

consequently the whole volume of the elevated column will be*

$$-2\pi.\int(l+y).zdy=\frac{\pi}{\mathfrak{a}}.l.\cos.(i-\varpi'). \qquad [10293f]$$

Substituting this in the expression of the volume V [10283f], it becomes

$$V=\pi l^2z'+\frac{\pi}{\mathfrak{a}}.l.\cos.(i-\varpi'). \qquad [10293g]$$

Multiplying this by the gravity g, and the density D, we get the following expression of the mass M of the fluid, which is elevated by the disk; [10293h]

$$M=\pi gD.l^2z'+\frac{\pi}{\mathfrak{a}}.gD.l.\cos.(i-\varpi'). \qquad [10293i]$$

This may be reduced to the same form as the equation (1) in page 226 of M. Poisson's *Nouvelle Théorie, &c.*, by changing M, l, z', D, ϖ', $\mathfrak{a}$, into Δ, r, k, ρ, $\pi-w$, a^{-2}, [10293k]
respectively, to conform to his notation. In the case treated of by La Place, where $i=\frac{1}{2}\pi$ [10293l]
[10280y'], the value of V [10293g] becomes

$$V=\pi l^2z'+\frac{\pi}{\mathfrak{a}}.l.\sin.\varpi', \qquad [10293m]$$

which, by substituting the values of z', $\sin.\varpi'$ [10281, 10293c'], becomes as in [10295a]; or, [10293n]
by reduction, as in [10295], corresponding to the case where the summit of the fluid is situated at F, fig. 171, page 953, on the lowest point of the quadrantal arc POF. If $i=0$, and $\varpi'=0$, the expression [10293g] becomes $V=\pi l^2z'+\frac{\pi}{\mathfrak{a}}.l$; and in this case, we have [10293o]
$z'=\alpha$ [10281a], also $\mathfrak{a}=\alpha^{-2}$ [10280t]; therefore this value of V becomes

$$V=\pi l^2\alpha+\pi l\alpha^2. \qquad [10293p]$$

The first term of the second member of this expression $\pi l^2\alpha$ is equal to the volume of the fluid immediately under the disk, which is in the form of a cylinder whose altitude is $z'=\alpha$, [10293q]
and the radius of its base equal to l; the second term $\pi l\alpha^2$ is equal to the volume of the elevated fluid which is situated without this cylinder, being the same as that we have already [10293r]
computed in [10107i]; and these two expressions ought evidently to be equal to each other, because the action of the fluid cylinder [10293q] upon the external mass [10293r], is exactly [10293s]
the same as that of a glass tube of the same dimensions well moistened [9220].

* (4380) Substituting [10293d] in [10284], it becomes

$$V=\frac{\pi l^2.\sqrt{2}.\cos.\frac{1}{2}\varpi'}{\sqrt{\mathfrak{a}}}-\frac{\pi l}{3\mathfrak{a}.\cos.\frac{1}{2}\varpi'}.\{1-\sin.^3\tfrac{1}{2}\varpi'-6\sin.\tfrac{1}{2}\varpi'.\cos.^2\tfrac{1}{2}\varpi'\}; \qquad [10295a]$$

and, by putting $\cos.^2\frac{1}{2}\varpi'=1-\sin.^2\frac{1}{2}\varpi'$, in the last term, it becomes as in [10295].

[10295] $$V=\frac{\pi l^2.\sqrt{2}.\cos.\frac{1}{2}\varpi'}{\sqrt{\alpha}}-\frac{\pi l}{3\alpha.\cos.\frac{1}{2}\varpi'}.\{1-6\sin.\tfrac{1}{2}\varpi'+5\sin.^3\tfrac{1}{2}\varpi'\}.$$ [Volume of the elevated column.]

We shall have the value of $\frac{1}{\alpha}$ by means of the following equation, given in [9358],*

[10296] $$q=\frac{2\cos.\varpi'}{\alpha h}.\left\{1-\frac{h}{6q.\cos.^3\varpi'}.(1-\sin.\varpi')^2.(1+2\sin.\varpi')\right\};$$ [Elevation q in the axis of a tube whose diameter is h.]

[10297] *h being the interior diameter of the tube, q the height to which the lowest point of the included fluid is elevated above the level* [9998′, 10000]. *In case the*

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* (4381) In the calculation [10296, &c.], the author supposes that the tube whose
[10296a] diameter is $h=2l$ [10297, 9359], is formed of the same substance as the disk; and if this
[10296a′] tube is dipped into the same fluid as that which is used with the disk, the elevation of the fluid, in the axis of the tube, will be represented by q [10297, &c.], in conformity with the notation in [9353, &c.]. Then, if we compare [9346] with [10279, or 10280a], we get
[10296b] $\theta'=\frac{1}{2}\pi-\varpi'$; also $\frac{H}{g}=\frac{1}{\alpha}$ [9328]; hence [9358] becomes, by successive reductions,

[10296c] $$\begin{aligned}q&=\frac{\cos.\varpi'}{\alpha l}.\left\{1-\frac{l}{q.\cos.\varpi'}.\left(1-\tfrac{2}{3}.\frac{(1-\sin.^3\varpi')}{\cos.^2\varpi'}\right)\right\}\\&=\frac{\cos.\varpi'}{\alpha l}.\left\{1-\frac{l}{3q.\cos.^3\varpi'}.(3\cos.^2\varpi'-2+2\sin.^3\varpi')\right\}\\&=\frac{\cos.\varpi'}{\alpha l}.\left\{1-\frac{l}{3q.\cos.^3\varpi'}.(1-3\sin.^2\varpi'+2\sin.^3\varpi')\right\}\end{aligned}$$

[10296d] $$=\frac{\cos.\varpi'}{\alpha l}.\left\{1-\frac{l}{3q.\cos.^3\varpi'}.(1-\sin.\varpi')^2.(1+2\sin.\varpi')\right\};$$

and, as $l=\frac{1}{2}h$ [10296a], it may be changed into [10296]. Now, putting for brevity
[10296e] $\text{w}=\frac{h}{6q.\cos.^3\varpi'}.(1-\sin.\varpi')^2.(1+2\sin.\varpi')$, we find that the preceding expression [10296]
[10296f] becomes $q=\frac{2\cos.\varpi'}{\alpha h}.(1-\text{w})$. Dividing this by the coefficient of $\frac{1}{\alpha}$, and neglecting w^2,
[10296g] we obtain $\frac{1}{\alpha}=\frac{h}{2\cos.\varpi'}.(q+q\text{w})$, as in [10298]; so that the correction of the altitude q is
[10296h] $q\text{w}$; and if we put the corrected altitude $q+q\text{w}=q'$, we shall have $\frac{1}{\alpha}=\frac{1}{2\cos.\varpi'}.hq'$; hence, as α, ϖ', are given, we shall have q' inversely as h, as in [10299]. When $\varpi'=0$,
[10296i] $q\text{w}=\frac{1}{6}h=\frac{1}{3}l$ [10296e, a], as in [10299]; hence it appears that, when the fluid perfectly moistens the sides of the tube, and we have $\varpi'=0$, we must increase the observed height of the lowest point of the elevated fluid q, by one sixth part of the diameter of the tube, to
[10296k] obtain the corrected height $q'=q+\frac{1}{6}h$; and then we shall have, from [10296h], $\frac{2}{\alpha}=hq'$ This agrees with what we have already found in [9372x, 10003].

fluid is depressed below the level, q becomes negative, and denotes the depression of the most elevated point of the interior fluid. This equation gives, very nearly, [10297′]

$$\frac{1}{\alpha}=\frac{h}{2\cos.\varpi'}\cdot\left\{q+\frac{h}{6\cos.^3\varpi'}\cdot(1-\sin.\varpi')^2.(1+2\sin.\varpi')\right\}. \qquad [10298]$$

Thus, to obtain the elevations inversely proportional to the interior diameters of the tubes, we must add to the observed elevation q, the sixth part of the diameter multiplied by the factor [10299]

$$\frac{(1-\sin.^2\varpi').(1+2\sin.\varpi')}{\cos.^3\varpi'}; \qquad [10300]$$

and this factor is reduced to unity when ϖ' is nothing. This correction is necessary in experiments made with great care, like those we shall now give an account of. M. Gay-Lussac has undertaken them, at my request, and has contrived a method of measuring the elevations and depressions of the fluids in transparent capillary tubes, by a process which gives to his experiments the correctness of astronomical observations; so that we may with confidence adopt his results. Tubes were selected of a uniform calibre, and their interior diameters were ascertained by means of the weights of the columns of mercury which filled them; this being the most accurate method of determining the diameters. [10301] [10302]

Experiments on water.

Observers do not agree in their experiments on the ascent of water in a glass capillary tube of a given diameter; their results differ from each other at least one half [10322*a*—*g*]. These differences depend, in a great measure, upon the greater or less degree of moisture on the sides of the tube. When the tubes are very wet, as they were in the following experiments, the water always rises very nearly to the same height, in the same tube. The *first tube* used by M. Gay-Lussac was of white glass, and its interior diameter was $1^{mi.},29441$. He measured the elevation of the lowest point of the inner surface of the water above the level of this fluid, in a very large vessel, into which this tube was dipped at its lower end; and found, by several experiments, which agree with each other, that it was equal to $23^{mi.},1634$, the temperature being about $8^{\circ},5$ of the centigrade thermometer. *Here the angle ϖ' was nothing* [10280*k*′], *as the water perfectly moistened the sides of the tube.* Augmenting this elevation by a sixth part of the diameter of the tube [10296*k*], we get $23^{mi.},3791$; multiplying this quantity by the diameter of the tube, we obtain, by what has been said in [10296*k*], the value of $\frac{2}{\alpha}$; hence we deduce [10303] [10304] [10305] [10306] [10306′] [10307]

[10308] $\frac{2}{\alpha} = 2a^2 = 30^{mi.mi.},2621$, or $\frac{1}{\alpha} = a^2 = 15^{mi.mi.},131$, $a = 3^{mi.}890$. [Values of α, a, for water.]

[10309] In a *second glass tube*, whose interior diameter was $1^{mi.},90381$, M. Gay-Lussac observed, at the same temperature, the elevation of the lowest point of the interior surface above the level to be $15^{mi.},5861$, which gives $15^{mi.},9034$, by
[10310] adding to it the sixth part of the diameter of the tube [10296*k*]. The elevation of the fluid in the first tube, being augmented in like manner, gives,
[10310′] by the formula [10296*k*], the corrected elevation of the second tube,* $15^{mi.},896$, which differs but very little from the result of observation; and this proves,
[10311] *First, that the corrected elevations are very nearly in the inverse ratio of the diameters of the tubes; Second, that, in very accurate experiments, the correction*
[10312] *made by the addition of the sixth part of the diameter of the tubes is indispensable.*

We may also determine the value of $\frac{2}{\alpha}$, by means of the elevation of the lowest point of the surface of the water which rises between two glass plates, placed very near to each other, in vertical and parallel positions, and dipped into
[10313] a vessel filled with that fluid. M. Gay-Lussac has found, by the mean of five experiments which differ but little from each other, that this elevation is
[10314] equal to $13^{mi.},574$, the distance of the plates from each other being $1^{mi.},069$. This distance is exactly equal to the diameter of a cylinder of iron wire; and to measure this diameter, several pieces of the same wire were placed at the side

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* (4382) This quantity $15^{mi.},896$ is equal to $30^{mi.},2621$ [10308] divided by $1^{mi.},90381$ [10309]. The value of $\frac{2}{\alpha}$, deduced from this experiment, is

[10310*a*] $$1^{mi.},90381 \times 15^{mi.},9034 = 30^{mi.mi.},277,$$

which differs from the value [10308], namely, $30^{mi.mi.},262$, by the quantity $0^{mi.mi.},015$. If
[10310*a*′] we had not corrected the altitudes for the sixth part of the diameter, the value of $\frac{2}{\alpha}$, deduced from the first experiment [10305, 10306], would be

[10310*b*] $$1^{mi.},29441 \times 23^{mi.},1634 = 29^{mi.mi.},983;$$

and that deduced from the second [10309, 10310], would be

[10310*c*] $$1^{mi.},90381 \times 15^{mi.},5861 = 29^{mi.mi.},673,$$

which differs from the preceding value $29^{mi.mi.},983$ by the quantity $0^{mi.mi.},310$, instead of
[10310*d*] $0^{mi.mi.},015$; so that the error 0,310 is reduced to 0,015, or $\frac{19}{20}$ths of it is taken away, by correcting the altitudes for the sixth part of the diameter of the tube. This agrees with [10312].

of each other, which, by the sum of their diameters, formed a considerable width; this was carefully measured, and then divided by the number of these diameters. The plates were perfectly plane, and had been well moistened; the temperature during the experiments was about 16°. If we add to the observed elevation the product of the half distance of the plates by $1-\frac{1}{4}\pi$, π being the [10315]
semi-circumference whose radius is unity, and then multiply the sum by the distance $1^{mi.},069$, we shall obtain the value of $\frac{1}{\alpha}$, as in [10317g]. Hence [10316]
we find *

$$\frac{1}{\alpha}=14^{mi.},524. \qquad [10317]$$

This result must be increased a little, to reduce it to the temperature of 8°,5; for [10318]
we have seen before that the attraction increases with the density of the fluid. It differs but little from the result $15^{mi.},131$ [10308], given by the elevation of [10319]
water in a glass tube, and this furnishes a new confirmation of the theory; according to which, the elevation between parallel planes must be about half the elevation in capillary tubes of a diameter equal to the distance of the planes. We shall here adopt, in preference, the value of $\frac{2}{\alpha}$, deduced from the [10320]

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* (4383) The plates being perfectly moistened, we have, as in [10306'], $\varpi'=0$, and then $\theta'=\frac{1}{2}\pi$ [10296b]; substituting this in [9153], also $\frac{H}{g}=\frac{1}{\alpha}$ [9328], it becomes [10317a]

$$q=\frac{1}{2\alpha l}\cdot\left\{1-\frac{2l}{q}\cdot(1-\tfrac{1}{4}\pi)\right\}, \qquad [10317b]$$

in which q' [9152] represents the elevation of the fluid between two vertical and parallel planes whose *distance* is $2l$, and q [9130] the elevation of the fluid in a vertical tube [10317c]
whose *radius* is l; so that $\frac{1}{2}q$ is nearly equal to the elevation of the fluid in a tube whose radius is $2l$; therefore, by [9110], $\frac{1}{2}q$ is nearly equal to the elevation of the fluid [10317d]
between two vertical, parallel planes, whose distance is $2l$; consequently $\frac{1}{2}q$ is nearly equal to q', or $q=2q'$ nearly. Substituting this in [10317b], it becomes [10317e]

$$q'=\frac{1}{2\alpha l}\cdot\left\{1-\frac{l}{q'}\cdot(1-\tfrac{1}{4}\pi)\right\}; \qquad [10317f]$$

then dividing by the coefficient of $\frac{1}{\alpha}$ and neglecting the square and higher powers of the small fraction $\frac{l}{q'}$, we get $\frac{1}{\alpha}=2l.\{q'+l.(1-\frac{1}{4}\pi)\}$. Now, substituting the values [10314], [10317g]
namely, $2l=1^{mi.},069$, $q'=13^{mi.},574$, $\pi=3,1416$, we get $\frac{1}{\alpha}=14^{mi.},633$, which differs [10317h]
a little from the value in [10317]. The value of $\frac{1}{\alpha}$ [10308] is $15^{mi.},131$, which differs but [10317i]
little from the preceding value $14^{mi.},633$ [10317h].

experiment upon the narrowest tube [10305], and we shall therefore suppose,
[10321] as in [10308], at the temperature of $8^{\circ},5$,*

[10322] $$\frac{2}{\alpha} = 30^{\text{mi.mi.}},2621, \quad \text{or} \quad \frac{1}{\alpha} = 15^{\text{mi.mi.}},131 = \alpha^2.$$ [Values of α, α, for water.]

This being premised, we find, by using both terms of the formula [10295],
[10323] which expresses the volume of the elevated fluid, and taking for unity the cubic centimetre, that the volume of the fluid elevated by a circular disk of white
[10324] glass, *whose diameter is* $118^{\text{mi.}},366$, will be expressed by†

[10325] $$V = 60,5327 - 0,9378.$$

[10326] The weight of a cubic centimetre of water, at its maximum of density, is a gramme. But, the preceding experiments having been made at the temperature
[10327] of about eight degrees and a half, the cubic centimetre of water will weigh
[10328] rather less than a gramme. Noticing this correction, we find that the weight of the elevated column of water, at the moment when it is ready to detach itself,
[10329] is $59^{\text{gram.}},5873$. M. Gay-Lussac found by several experiments, which differ but

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* (4384) The value of $\frac{2}{\alpha}$ adopted in [10322], differs considerably from that in [9670],
[10322a] namely, $13^{\text{mi.}},569$, agreeably to the remark in [10303]. A paper, by Dr. Young, in the
[10322b] Transactions of the Royal Society, for 1805, page 71, contains the product of the distance of two planes, by the height of the water elevated by the capillary action, which represents
[10322c] nearly the value of $\frac{1}{\alpha}$ [10317g], as given by several authors in square inches of English
[10322d] measure. That of Newton is the same as in [9678], $\frac{1}{\alpha} = 0^{\text{i.i.}},01 = \frac{(25^{\text{mi.}},3918)^2}{100} = 6^{\text{mi.mi.}},45$,
[10322e] which is much too small. Musschenbroek makes it $0^{\text{i.i.}},196$, or nearly $13^{\text{mi.mi.}}$; Weitbrecht, with considerable accuracy, $0^{\text{i.i.}},0214 = 14^{\text{mi.mi.}}$ nearly; Monge, $13^{\text{mi.mi.}}$ nearly;
[10322f] Atwood, $0^{\text{i.i.}},265 = 17^{\text{mi.mi.}}$ nearly. Doctor Young finally assumes $\frac{1}{\alpha} = 0^{\text{i.i.}},2 = 12^{\text{mi.mi.}},9$;
[10322g] or $\frac{2}{\alpha} = 25^{\text{mi.mi.}},8$, instead of $30^{\text{mi.mi.}},261$, which is used by La Place in [10322].

† (4385) When the fluid is water or alcohol, we have, as in [10306′], $\varpi' = 0$, and then
[10325a] the expression of the elevated volume of the fluid [10295] becomes $\pi l^2 . \sqrt{\frac{2}{\alpha}} - \frac{\pi l}{6} . \frac{2}{\alpha}$.
[10325b] Now, by making the calculation in centimetres, we have, in [10322], $\frac{2}{\alpha} = 0^{\text{c.c.}},302621$,
[10325c] $l = \frac{1}{2} \times 118^{\text{mi.}},366 = 5^{\text{c.}},9183$ [10324], $\pi = 3,1416$. Hence $\pi l^2 . \sqrt{\frac{2}{\alpha}} = 60,533$, and $\frac{\pi l}{6} \times \frac{2}{\alpha} = 0,9378$, as in [10325]; their difference [10325a] is the volume of the elevated
[10325d] fluid, 59,595, which is to be corrected for the temperature, as in [10327].

little from each other, that this weight was $59^{\text{gram.}},40$, agreeing, as well as could be expected, with the result of analysis. [10330]

Experiments with alcohol.

Some alcohol, whose specific gravity at the temperature of eight degrees, compared with that of water at the same temperature, was 0,81961, ascended in the first tube to the height of $9^{\text{mi.}},18235$, the temperature being always [10331] eight degrees. The alcohol moistened the glass perfectly, so that we must add to this height the sixth part of the diameter of the tube [10296k, 10305], and it then becomes equal to $9^{\text{mi.}},39808$. Multiplying this by the diameter of the [10332] tube [10305], we get, as in [10307, &c.], the value of $\frac{2}{\alpha}$, relative to this alcohol, namely,

$$\frac{2}{\alpha}=12^{\text{mi.mi.}},1649, \quad \frac{1}{\alpha}=\alpha^2=6^{\text{mi.mi.}},0824, \quad \alpha=2^{\text{mi.}},4662. \quad \left[\text{Values of } \alpha, \alpha, \text{ for alcohol.}\right]$$ [10333]

With this value we can compute the elevation of the alcohol in the *second tube* [10309], corrected by the addition of the sixth of the diameter of the tube, by dividing $\frac{2}{\alpha}$ by the diameter of that tube [10309]. This gives $6^{\text{mi.}},38976$, [10334] for the elevation which M. Gay-Lussac has found by experiment to be $6^{\text{mi.}},40127$. As these values differ so very little, it proves that the corrected [10335] elevations of the alcohol in different narrow capillary tubes, are inversely as the diameters of these tubes. If we use the preceding value of $\frac{2}{\alpha}$ [10333], we find that the volume of the alcohol, elevated by the glass disk, which is used in the first experiment [10324], is equal to *

$$V=38,3792-0,3770;$$ [10336]

the cubic centimetre being taken for unity. Multiplying this by the specific gravity of the alcohol, 0,81961 [10331], gives the weight of this mass of [10337] alcohol, equal to 31,1469 cubic centimetres of water, at the temperature of eight degrees; and this last weight is equal to $31^{\text{gram.}},1435$; which is therefore [10338] the weight necessary to detach the disk [10324] from the alcohol, when the temperature is eight degrees. M. Gay-Lussac found by experiment this

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* (4336) We have $\frac{2}{\alpha}=0^{\text{c.c.}},121649$ [10333], also $l=5^{\text{c.}},9183$ [10324], $\pi=3,1416$; [10336a] hence $\pi l^2.\sqrt{\frac{2}{\alpha}}=38,379$, and $\frac{\pi l}{6}.\frac{2}{\alpha}=0,377$; and by substituting them in the expression of the elevated fluid [10325a], it becomes as in [10336]. This requires a [10336b] reduction for the temperature, as in the case [10327—10329].

[10339] weight to be equal to $31^{gram.},08$, at the same temperature, which differs but very little from the result of the analysis.

Some alcohol, whose specific gravity at the temperature of ten degrees,
[10340] compared with that of water at the same temperature, was 0,8595, was
[10341] elevated in the first tube [10304], $9^{mi.},30079$, which gives $9^{mi.},51649$* for its
[10342] corrected elevation. Whence we find, relative to this alcohol, $\frac{2}{\alpha}=12^{mi.mi.},31905$.
This value of $\frac{2}{\alpha}$ gives the weight necessary to detach the preceding disk from
[10343] the surface of the alcohol, equal to $32^{gram.},86$; and M. Gay-Lussac has found by experiment $32^{gram.},87$, agreeing exactly with the calculation.

Lastly, some alcohol, whose density was 0,94153, at the temperature of eight
[10344] degrees, was elevated in the first tube [10304], $9^{mi.}99727$, which gives† $\frac{2}{\alpha}=13^{mi.mi.},2198$; consequently the adhesion of the preceding disk was equal
[10345] to $37^{gram.},283$. M. Gay-Lussac found by experiment, at the same temperature, this adhesion equal to $37^{gram.},152$.

Experiments with oil of turpentine.

Some oil of turpentine, whose specific gravity, at the temperature of eight
[10346] degrees, compared with that of water at the same temperature, was 0,869458, ascended in the first tube [10304] to the height $9^{mi.},95159$, which gives‡

[10341a] * (4387) One sixth part of the diameter of the tube [10305] is $0^{mi.},21573$; adding this to $9^{mi.},30079$ [10341], gives $9^{mi.},51652$, being nearly as in [10341]. Multiplying this by the diameter $1^{mi.},29441$ [10305], we get $\frac{2}{\alpha}=12^{mi.mi.},318$, agreeing nearly with [10342].
[10341b] Reducing this to centimetres, it becomes $\frac{2}{\alpha}=0^{c.c.},12318$; using this and l, π [10336a], we
[10341c] find that the expression [10325a] becomes $38,621-0,382=38,239$; multiplying this by the density 0,8595, we obtain the weight $32^{gram.},86$, as in [10343].

† (4388) Correcting the elevation $9^{mi.},99727$ [10344], by adding $0^{mi.},21573$ [10341a],
[10344a] it becomes $10^{mi.},213$; multiplying this by the diameter of the tube $1^{mi.},29441$ [10305], we get, as in [10341b], $\frac{2}{\alpha}=13^{mi.},2198$. Using this with the values of l, π [10336a], we find
[10344b] that the expression [10325a] becomes $40,0089-0,4097=39,5992$; multiplying this by the density 0,94153 [10344], it becomes $37^{gram.},283$ nearly, as in [10345].

[10346a] ‡ (4389) The corrected altitude is $9^{mi.},95159+0^{mi.},21573=10^{mi.},16732$, as is evident from [10346, 10341a], agreeing nearly with [10347]. This gives, by proceeding as in the
[10346b] last note, $\frac{2}{\alpha}=10^{mi.},16732\times 1^{mi.},29441=13^{mi.mi.},1606$, as in [10347]. Using this and l, π

$10^{mi.},16729$ for its corrected elevation, and $\frac{2}{\alpha}$, equal to $13^{mi.mi.},1606$. Hence [10347] we conclude that the adhesion of the preceding disk to the surface of this fluid is equal to $34^{gram.},35$. M. Gay-Lussac has found, at the same temperature of [10348] eight degrees, this adhesion equal to $34^{gram.},104$, which differs but very little from the preceding result

Experiments with mercury.

M. Gay-Lussac has made several experiments upon the adhesion of the preceding disk to mercury. But to compare them with the theory, we must know, *first*, the elevation of the mercury in a glass tube of a given diameter; *second*, the angle which the surface of the mercury forms with the glass, at the [10349] point of contact; and it is very difficult to ascertain these two points by experiment, on account of the friction of the mercury against the sides of the glass. This cause operates as an obstacle to the elevation or to the depression [10350] of the mercury in capillary tubes, and it has the effect to change considerably the angle of inclination of the surface of the mercury to that of the glass. The comparison of several observed capillary phenomena with the theory, has given for the mean value of $\frac{2}{\alpha}$, relative to mercury, at the temperature of ten degrees, [10351] the values *

$$\frac{2}{\alpha} = 2a^2 = 13^{mi.mi.}, \qquad \frac{1}{\alpha} = a^2 = 6^{mi.mi.},5, \qquad a = 2^{mi.},5495; \qquad \left[\begin{smallmatrix}\text{Values of } \alpha, a,\\ \text{for mercury.}\end{smallmatrix}\right] \qquad [10351']$$

and the acute angle formed by the sides of the glass, and by a plane which is a [10352] *tangent to the surface of the mercury, at the extremity of the sensible sphere of activity of its sides, equal to* † 48°; we shall therefore make use of [10353] these data, which may be rectified by further experiments. They make $\varpi' = 152°$, and $\frac{1}{2}\varpi' = 76°$. We may then find, by the preceding formula, that [10354] the weight of the column of mercury raised up by the above-mentioned glass

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[10336*a*], we find that the expression of the volume of the fluid [10325*a*] becomes $39{,}919 - 0{,}408 = 39{,}511$; multiplying it by the density 0,869158 [10346], we get the [10346*c*] weight $34^{gram.},35$, as in [10348].

* (4390) Dr. Young, in the paper mentioned in [10322*b*], supposes that the experiments on the depression of mercury in a tube, give for the value of $\frac{2}{\alpha}$ in English inches $0^{in.},015$; [10352*a*] which in French measure is $\frac{2}{\alpha} = 10^{mi.mi.}$ nearly, which differs $3^{mi.}$ from that in [10351'].

† (4391) This angle is represented in fig. 168, page 948, by $AFf = 48°$; hence $AFf' = 200° - 48° = 152° = \varpi'$ [10264*f*]. Dr. Young, in the paper mentioned in [10353*a*] [10322*b*], makes $\varpi' = 140^{d.} = 155°,5$, which differs a little from the preceding estimate.

[10355] disk, is* $207^{gram.},0$. M. Gay-Lussac has found very great differences in the results of his experiments upon this subject. In his observations upon the adhesion of a glass disk to the surface of a fluid, he suspended the disk from the end of the arm of a very accurate balance, which lifted it up vertically, by means of very small weights successively and slowly added to the scale at the end of the other arm. The sum of these small weights, at the moment when the disk was separated from the fluid, indicated the weight of the whole
[10356] elevated column. Using this method with mercury, he found that the sum was greater or less, according to the intervals of time in which he added the successive weights; and by adding them at very great intervals, he was enabled
[10356′] to increase the sum from $158^{gram.}$, to $296^{gram.}$. It depends, as we see by the preceding formula, upon the acute angle which the surface of the mercury forms with that of the glass, and it is very nearly proportional to the sine of the
[10357] half of that angle;† now we know, by daily experience with a barometer, that this angle may be considerably increased when the mercury descends very
[10358] slowly; the friction of the fluid against the sides of the tube impedes the descent of the particles of the fluid contiguous to those sides. The friction also prevents the mercurial column from separating from the disk. When it is
[10359] separating, it begins to quit the borders of the disk; then it contracts more and more near the disk, until it wholly separates. The friction of the mercury

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[10355a] * (4392) Substituting $\frac{1}{2}\varpi'=76^{\circ}$ [10354], and $\frac{2}{\alpha}=0^{c.c.},13$ [10351′], in the expression of the elevated volume [10295], it becomes $\pi l^2.\sqrt{0^{c.c.},13}.\cos.76^{\circ}+\frac{1}{6}\pi l.0,19768$; substituting the values of π, l [10325c], and then multiplying by the density of mercury
[10355b] 13,6, it becomes $198^{gram.},6+8^{gram.},3=206^{gram.},9$, as in [10355], nearly. This will be
[10355c] increased to about $218^{gram.}$, if we suppose $\frac{1}{2}\varpi'=67^{d.},15^{m.}$, and $\frac{2}{\alpha}=6^{mi.},5262$, which are the values assumed by M. Poisson, in page 235 of his *Nouvelle Théorie, &c.;* the result of
[10355d] his calculation, given in page 236 of the same work, is $222^{gram.},464$, differing a little from the above.

† (4393) We see, as in the last note, that the first term of [10295] is by far the greatest part of that expression of the volume of the elevated fluid; so that the whole expression is very nearly equal to that term,

[10357a] $$\pi l^2.\sqrt{\frac{2}{\alpha}}.\cos.\tfrac{1}{2}\varpi',$$

which is proportional to

$$\cos.\tfrac{1}{2}\varpi'=\sin.\tfrac{1}{2}(\pi-\varpi')=\sin.\tfrac{1}{2}AFf,$$

fig. 168, page 948, as in [10357].

against the lower surface of the disk ought therefore to prevent this effect, and to diminish, as in the descent of the barometer, the acute angle of contact of the surface of the disk with that of the mercury; and if all the particles of the fluid column have the time necessary to accommodate themselves to the new [10360]
state of equilibrium which results from it, we may easily conceive that we may increase considerably the whole weight necessary to detach the disk from the surface of the mercury. This weight would increase to nearly four hundred grammes, if the angle of contact was a right angle.* [10361]

Disks of different substances, which are perfectly moistened by a fluid, ought to produce the same resistance to their separation from the fluid, if their diameters are equal; for then this resistance is produced by the adhesion of the [10362]
fluid to its own particles, or to the stratum of the fluid which covers the lower surface of the disk. To verify this result, M. Gay-Lussac has placed in contact with water, a copper disk whose diameter was $116^{mi.},604$; and he found, at the [10363]
temperature of $18^{\circ},5$, the weight necessary to detach it from the fluid was $57^{gram.},945$. If we suppose that the value of $\frac{2}{\alpha}$ relative to copper, is the same as for glass, which is $\frac{2}{\alpha}=30^{mi.mi.},2621$ [10322], we shall find by the [10364]
preceding formula [10325a], that the weight of the water which is raised up by the disk is† $57^{gram.},757$; differing but very little from the results of the [10365]
experiment.

Experiments upon the adhesion of disks of different substances, at the surface of the same fluid, may serve to determine the ratios of their attractive [10366]
forces upon this fluid [or rather the values of the angles ϖ']. For, if we use

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* (4394) If we put $\varpi'=100^{\circ}$, we shall have $\cos.\frac{1}{2}\varpi=\sin.\frac{1}{2}\varpi'=\sqrt{\frac{1}{2}}$; hence $1-6\sin.\frac{1}{2}\varpi'+5\sin.^{3}\frac{1}{2}\varpi'=1-\frac{7}{2}.\sqrt{\frac{1}{2}}$; dividing this quantity by $\cos.\frac{1}{2}\varpi'=\sqrt{\frac{1}{2}}$, it becomes [10361$a$]
$\sqrt{2}-3,5=-2,086$; and the function [10295], by putting $\frac{2}{\alpha}=0^{c.c.},13$, is reduced to the form $\pi l^{2}.\sqrt{0^{c.c.},13}.\sqrt{\frac{1}{2}}+\frac{1}{6}\pi l\times0,13\times2,086$, which expresses the volume of the elevated [10361b]
fluid. Multiplying this by the density of the mercury 13,6, and then substituting the values of π, l [10325c], we get the weight of the elevated column $382^{gram.}+11^{gram.}=393^{gram.}$, [10361$c$]
being nearly equal to $400^{gram.}$, the quantity mentioned in [10361].

† (4395) Substituting in [10325a] the value of $\frac{2}{\alpha}$ [10322], and that of $2l=116^{mi.},604$ [10363], it becomes $57^{gram.},82$ nearly, as in [10365]. [10365a]

circular disks of a very large diameter, this adhesion, as we have already seen, will be very nearly equal to *

[10367] $$\frac{\pi l^2.\sqrt{2}.\cos.\frac{1}{2}\varpi'.D'}{\sqrt{\alpha}},$$ [Approximate value of the elevated mass p.]

D' being the density of the fluid; therefore by putting p for the weight necessary to separate the disk from the surface of the fluid, the preceding
[10368] quantity will be equal to p. The quantities D' and α [9328] depend only upon the fluid; the values of $\cos.\frac{1}{2}\varpi'$, relative to disks of the same diameter and of different substances, are therefore proportional to the weight p; consequently $\cos.^2\frac{1}{2}\varpi'$ is proportional to p^2; but we have shown in [9935] that, [in the hypothesis of a perfectly homogeneous fluid, we shall have]
[10369] $\rho=\rho'.\cos.^2\frac{1}{2}\varpi'$; and as ρ' is relative to the fluid, the values of ρ corresponding to the different disks [in this hypothesis], will be proportional to the squares of the corresponding weights p. These values of ρ, as we have seen in [10141],
[10370] correspond to equal volumes; to obtain the values corresponding to equal masses, we must divide them by the respective densities of the substances. They would be proportional to the attractive forces, if the law of attraction
[10371] were the same for different substances. In this case, the respective attractions of these substances upon the fluid, are, with equal volumes, as the squares of the weights necessary to detach the disk from the surface of this [homogeneous] fluid.

When a fluid perfectly moistens the disk, the experiments upon the adhesion of its surface indicate only the attraction of the fluid upon its own particles. But when it does not perfectly moisten the disk, its friction against the lower
[10372] surface of the disk produces great varieties in the results of the experiments upon adhesion, as we have seen relative to glass disks, applied to the surface of mercury [10350, &c.]. It then becomes difficult to distinguish the result which would take place independent of this cause of anomaly, or to obtain correctly the attraction of the disk upon the fluid.

We have seen, in [10056], that *the angle of contact of mercury with glass*
[10373] *vanishes in water;* so that a surface of mercury, covered with water, in a glass capillary tube, would form a convex hemisphere. Hence it follows that, if we

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* (4396) This is the same as the first and most important of the terms of the expression
[10367a] of the elevated volume of the fluid [10295], multiplied by the density D'; which must therefore represent very nearly the mass of that fluid

apply a glass disk to the surface of mercury, and then cover both the disk and the mercury in the vessel with a stratum of water, we shall have* $\varpi'=\pi$; [10374] which renders the preceding expression of the column of mercury elevated by the disk equal to nothing; *therefore it ought not to oppose any resistance to its separation from the mercury, which has in fact been found to be the case by the* [10375] *experiments of M. Gay-Lussac.*

ON THE FIGURE OF A LARGE DROP OF MERCURY, AND ON THE DEPRESSION OF MERCURY IN A GLASS TUBE OF A GREAT DIAMETER.

We shall suppose that a large circular drop of mercury is placed upon a glass horizontal plane; then the section of its surface, by a vertical plane [10376] passing through its centre, will be very little curved at its summit; but in proceeding from that point towards the borders of the drop, the curvature will increase more and more, until the tangent of the curve is vertical. At this point, the curvature and the width of the section will be at their maximum. Below this point, the curve will approach towards the axis, and will finally [10376′] terminate at the glass plane, forming with it an acute angle. We shall now determine the equation of this curve, putting †

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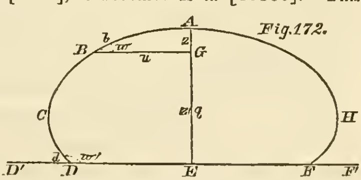
* (4397) We have the angle $AFf=\pi-\varpi'$ [10264e], and this vanishes in the case [10374a] treated of in [10373]; hence we have $\varpi'=\pi$; therefore $\sin.\frac{1}{2}\varpi'=1$, $\cos.\frac{1}{2}\varpi'=0$; substituting these in the expression of the volume of the elevated mass [10295], we find that its first term vanishes, and that the factor $\frac{1-6\sin.\frac{1}{2}\varpi'+5\sin.^3\frac{1}{2}\varpi'}{\cos.\frac{1}{2}\varpi'}$ of the second term becomes [10374b] of the form $\frac{0}{0}$; so that, to obtain its value, we must, according to the usual rule, take the differential of the numerator, and divide it by the differential of the denominator, and it then becomes

$$\frac{-3\cos.\frac{1}{2}\varpi'+\frac{15}{2}\cos.\frac{1}{2}\varpi'.\sin.^2\frac{1}{2}\varpi'}{-\frac{1}{2}\sin.\frac{1}{2}\varpi'},\qquad [10374c]$$

which vanishes, because it contains the factor $\cos.\frac{1}{2}\varpi'=0$; therefore the whole expression of the elevated column must vanish, as in [10375].

† (4398) Substituting $g=H\alpha$ [9328], in [9324], it becomes as in [10380]. This represents the equation of the section of the [10380a] drop $DdCBbAHF$, fig. 172, resting upon the horizontal plane $D'DEFF'$; AE is the vertical axis passing through the centre of its base E. Then the rectangular coordinates of [10380b] any point B of the surface are $AG=z$, $BG=u$ [10378, 10379]; also $EG=z'$ [10380c] [10378′], $AE=q$ [10403]; the angle $GBb=\varpi$, $EDd=\varpi'$ [10264e]; and it is evident



Symbols.

[10377] $b =$ the radius of curvature at the summit or vertex A of the drop, fig. 172;

[10377′] $q =$ the greatest height of the drop AE [10403];

[10378] $z =$ the vertical ordinate AG of any point B of the surface of the drop, the origin of z being at the vertex of the drop A; and the axis of z the vertical line AE drawn through this vertex A, in the direction of gravity;

[10378′] $z' = q - z =$ the vertical ordinate EG, or height above the glass plane [10403];

[10379] $u =$ the horizontal distance BG of the same point B of the surface from the axis of z;

[10379′] $\varpi =$ the angle GBb, which the tangent of the curve Bb makes with the ordinate BG [10405];

[10379″] $\varpi' =$ the angle EDd, which the tangent of the curve Dd makes with the ordinate DE [10408′].

Then we shall have, as in [9324],

[10380] $$\frac{\frac{ddz}{du^2}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}} + \frac{\frac{1}{u}\cdot\frac{dz}{du}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{1}{2}}} - 2\alpha z = \frac{2}{b}. \qquad (r)$$ [General equation of the surface.]

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that the top part of the curve must be very nearly horizontal, so that $\frac{dz}{du}$ will be very small.

[10380d] Thus, in the experiments mentioned in [10446], we have $DF = 100^{\text{mi.}}$, and $EA = 3^{\text{mi.}}$ nearly [10447]; hence it will evidently follow, that the part AB must be nearly level; so that,

[10380e] if we neglect $\frac{dz^3}{du^3}$, $\frac{dz^2}{du^2} \times \frac{ddz}{du^2}$, in [10380], and multiply the expression by u, it will become as in [10382]; which will represent very nearly the equation of the curve, in all parts of it, excepting near the extremities DC, FH. For convenience in referring to the symbols, we have collected them together in the table [10377—10379″]; observing, however, that

[10380f] q, z', ϖ, ϖ', in the original work, were defined in [10403, 10405, 10408′]; we have also, in these definitions, made reference to fig. 172, which is not used by La Place. The equation of the surface [10380], changing its signs, is equivalent to that marked (b) in page 202 of

[10380g] M. Poisson's *Nouvelle Théorie, &c.*, changing z into $q - z'$ [10378′], which reduces the

[10380h] equation [10380] to the form [10404]; then, putting $2\alpha q + \frac{2}{b} = 2\beta$, $\alpha = a^{-2}$, and $u = t$; remarking that M. Poisson omits the accent on z, so that his z, in the equation (b), is

[10380i] equivalent to z' in [10404]. The process of integration to obtain z [10384] is used by M. Poisson, and he gives an equivalent equation (l), in page 213 of his work. The equation

[10380k] [10402], neglecting the small term with the divisor $8u\sqrt{2\alpha}$, is equivalent to his equation (m), page 214. Lastly, the equation [10426], is the same as his equation (o), page 217.

When the drop is very large, we can, in a great part of its surface, neglect the third power of $\frac{dz}{du}$, and then the preceding equation is reduced to [10381]

$$u.\frac{ddz}{du^2}+\frac{dz}{du}-2\alpha uz-\frac{2u}{b}=0. \qquad (s)$$ [Equation of the upper part of the drop.] [10382]

Although this differential equation is much more simple than the equation [10380], yet it does not appear to be integrable by known methods; but we can satisfy it, by putting [10383]

$$z=\frac{1}{\alpha b\pi}.\int_0^\pi d\varphi.\{c^{u\sqrt{2\alpha}.\cos.\varphi}-1\}$$ [10384]

$$=\frac{1}{\alpha b\pi}.\int_0^\pi d\varphi.c^{u\sqrt{2\alpha}.\cos.\varphi}-\frac{1}{\alpha b};$$ [10384']

[Value of z on the upper surface of the drop, from the vertex towards the borders.]

the integral being taken from $\varphi=0$ to $\varphi=\pi$; for then we have * [10385]

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* (4399) Taking the partial differential of z [10384], relative to u, we get [10386, 10387]; substituting them in [10382], we obtain [10388], observing that the last term of z [10384] is easily reduced to

$$-\frac{1}{\alpha b\pi}.\int_0^\pi d\varphi=-\frac{1}{\alpha b\pi}.\pi=-\frac{1}{\alpha b},$$ [10384a]

as in [10384']; and this produces in the term $-2\alpha.uz$ [10382] the quantity $+\frac{2u}{b}$, which is [10384b] destroyed by the last term in [10382]. The general integral of [10388] is given in [10389], as is easily proved by taking its differential relative to φ, and substituting $-\sin.^2\varphi=\cos.^2\varphi-1$. This integral vanishes when $\varphi=0$, and when $\varphi=\pi$, which are [10384c] the two assumed limits [10385]; therefore the proposed value of z [10384] will satisfy the differential equation [10382]; though it must not be considered as its general integral, with two perfectly arbitrary constant quantities. It is, however, sufficient for the present problem, because at the vertex of the drop, where $u=0$, $z=0$, the tangent of the curve is horizontal, [10384d] which gives $\frac{dz}{du}=0$, and the radius of curvature is equal to b [10377]; and we shall soon [10384d'] show that these two conditions, which may be used for finding the values of the two arbitrary constant quantities in the present problem, are satisfied by the expression of z [10384e] [10384]. For, in the first place, by putting $u=0$, in [10384], it becomes $z=0$. Moreover, by putting $u=0$, in [10386], we get

$$\frac{dz}{du}=\frac{1}{\alpha b\pi}.\sqrt{2\alpha}.\int_0^\pi d\varphi.\cos.\varphi;$$ [10384f]

and since $\int d\varphi.\cos.\varphi=\sin.\varphi$ vanishes at the limits $\varphi=0$, $\varphi=\pi$, it becomes, as in [10384d'], $\frac{dz}{du}=0$. Lastly, putting $u=0$, in [10387], we obtain

$$\frac{ddz}{du^2}=\frac{1}{b\pi}.\int_0^\pi 2d\varphi.\cos.^2\varphi=\frac{1}{b\pi}.\int_0^\pi d\varphi.(1+\cos.2\varphi)=\frac{1}{b\pi}.\int_0^\pi d\varphi+\frac{1}{b\pi}.\int_0^\pi d\varphi.\cos.2\varphi.$$ [10384g]

Now $\int d\varphi.\cos.2\varphi=\frac{1}{2}\sin.2\varphi$ vanishes at the limits $\varphi=0$, $\varphi=\pi$; and as $\int_0^\pi d\varphi=\pi$, the [10384h]

[10386] $$\frac{dz}{du}=\frac{1}{\alpha b\pi}\cdot\int_0^\pi d\varphi\cdot\sqrt{2\alpha}\cdot\cos.\varphi\cdot c^{u\sqrt{2\alpha}.\cos.\varphi};$$

[10387] $$\frac{ddz}{du^2}=\frac{1}{\alpha b\pi}\cdot\int_0^\pi d\varphi\,.\,2\alpha\,.\cos.^2\varphi\,.\,c^{u\sqrt{2\alpha}.\cos.\varphi};$$

[10384i] expression [10384g] becomes $\frac{ddz}{du^2}=\frac{1}{b}$; but from [9326] we have, by putting $\frac{dz}{du}=0$,
[10384k] $\frac{ddz}{du^2}=\frac{1}{R}$, R being the radius of curvature, which is here represented by b [10377]. This agrees with [10384d'], and proves the correctness of the assumed value of z.

We may remark that the particular form assumed by the author in [10384], for the value
[10384l] of z, may be derived from a more general expression containing arbitrary constant coefficients.
[10384m] Thus, if we put $z=\mathrm{z}-\frac{1}{\alpha b}$, in [10382], it becomes $u.\frac{dd\mathrm{z}}{du^2}+\frac{d\mathrm{z}}{du}-2\alpha u\mathrm{z}=0$; and if we
[10384n] assume for z the following expression, (for remarks upon which see note in page 1018.)

[10384o] $$\mathrm{z}=e.\int_0^\pi d\varphi.\,c^{mu.\cos.\varphi}+f\mathrm{z}'',$$

which contains the arbitrary constant quantities e, m, f, (z'' being a particular value of z not
[10384p] yet obtained,) we may dispose of these quantities so as to satisfy the differential equation in z. For if we substitute this assumed value of z [10384o], in the differential equation [10384m], we shall get an equation which is equivalent to [10388]. This may be put equal to
[10384q] $me.\sin.\varphi.c^{mu.\cos.\varphi}$; and if we compare its differential with that of the preceding equation, we shall find that they may be rendered identical by putting, as in [10384], $m=\sqrt{2\alpha}$, leaving e, f, indeterminate; finally, these two quantities may be used so as to make
[10384r] $z=0$, and $\frac{dz}{du}=0$, when $u=0$, as in [10384d']; so that we shall finally obtain, after making some reductions, the precise form assumed by the author in [10384]; but we have not thought it to be necessary to insert the detail of these calculations.

[10384s] We shall put, for brevity, $u\sqrt{2\alpha}=\mathrm{u}$, and shall then develop the value of z [10384], in a series ascending according to the powers of $\mathrm{u}.\cos.\varphi$, by means of the formula [55], Int., which gives

[10384t] $$z=\frac{1}{\alpha b\pi}\cdot\left\{\mathrm{u}.\int_0^\pi d\varphi.\cos.\varphi+\frac{\mathrm{u}^2}{1.2}.\int_0^\pi d\varphi.\cos.^2\varphi+\frac{\mathrm{u}^3}{1.2.3}.\int_0^\pi d\varphi.\cos.^3\varphi+\&c.\right\}.$$

Now if n be a positive integer, we shall have $\int d\varphi.\cos.n\varphi=\frac{1}{n}.\sin.n\varphi$, which vanishes when
[10384u] $\varphi=0$, and when $\varphi=\pi$; hence we have $\int_0^\pi d\varphi.\cos.n\varphi=0$. From this it appears that, if we substitute, in [10384t], the values of $\cos.^2\varphi$, $\cos.^3\varphi$, &c. [6, 7, &c.], Int., all the terms depending on the uneven powers of $\cos.\varphi$, or on the uneven powers of u, will
[10384v] vanish; and with the even powers of $\cos.\varphi$, it is only necessary to retain the constant part, or that which is independent of φ; so that we may substitute the following parts, deduced from [6—10, &c.], Int.,

[10384w] $$\cos.^2\varphi=\frac{1}{2^2}.\frac{2}{1};\qquad \cos.^4\varphi=\frac{1}{2^4}.\frac{4.3}{1.2};\qquad \cos.^6\varphi=\frac{1}{2^6}.\frac{6.5.4}{1.2.3},\ \&c.$$

hence the first member of the equation [10382] becomes

$$\frac{1}{\alpha b\pi}\cdot\int_0^\pi d\varphi\cdot\{2\alpha u\,.\cos.^2\varphi+\sqrt{2\alpha}\,.\cos.\varphi-2\alpha u\}\,.\,c^{u\sqrt{2\alpha}\,.\cos.\varphi};\qquad [10388]$$

These are to be multiplied by $d\varphi$, and then integrated from $\varphi=0$ to $\varphi=\pi$, which is the same as to multiply them directly by π; hence the expression [10381t] becomes, by successive reductions, and using $\mathrm{u}=u\sqrt{2\alpha}$ [10384s],

$$z=\frac{1}{\alpha b}\cdot\left\{\frac{\mathrm{u}^2}{1.2}\times\frac{1}{2^2}\times\frac{2}{1}+\frac{\mathrm{u}^4}{1.2.3.4}\times\frac{1}{2^4}\times\frac{4.3}{1.2}+\frac{\mathrm{u}^6}{1.2.3.4.5.6}\times\frac{1}{2^6}\times\frac{6.5.4}{1.2.3}+\&\text{c.}\right\}\qquad [10384x]$$

$$=\frac{1}{\alpha b}\cdot\left\{\frac{\mathrm{u}^2}{2^2}+\frac{\mathrm{u}^4}{2^2.4^2}+\frac{\mathrm{u}^6}{2^2.4^2.6^2}+\&\text{c.}\right\}\qquad [10384y]$$

$$=\frac{1}{\alpha b}\cdot\left\{\frac{(u\sqrt{2\alpha})^2}{2^2}+\frac{(u\sqrt{2\alpha})^4}{2^2.4^2}+\frac{(u\sqrt{2\alpha})^6}{2^2.4^2.6^2}+\&\text{c.}\right\}.\qquad [10384z]$$

If we suppose z, u, to be so small that we may neglect terms of the order u^4, the expression [10384z] will give $z=\frac{u^2}{2b}$; which is evidently the equation of a small osculatory arc, whose sine is u, versed sine z, radius b; agreeing with the notation in [10377—10379]. [10385a]

The angle ϖ [10379′], which the tangent to the curve makes with the radius u, may be found by substituting the value z [10384z] in the equation $\text{tang.}\varpi=\frac{dz}{du}$ [10406a], from which we get [10385b]

$$\text{tang.}\varpi=\frac{1}{\alpha b}\cdot\left\{2\alpha\,.\,\frac{2u}{2^2}+(2\alpha)^2.\frac{4u^3}{2^2.4^2}+(2\alpha)^3.\frac{6u^5}{2^2.4^2.6^2}+\&\text{c.}\right\}.\qquad [10385c]$$

If we substitute $\frac{1}{\alpha}=6^{\text{mi}},5$ [10351′], in the expressions of z and $\text{tang.}\varpi$ [10384z, 10385c], they will become as in [10385e, f], respectively; *the numbers included between the brackets being the logarithms of the coefficients*, the index being as usual increased by 10, and in the last term of [10385f] by 20, because these numbers are less than unity; [10385d]

$$z=\frac{1}{\alpha b}\cdot\left\{\begin{array}{l}[8.8860566]\,.\,u^2+[7.1700532]\,.\,u^4+[5.1018673]\,.\,u^6\\ \qquad+[2.7838040]\,.\,u^8+[0.2719206]\,.\,u^{10}+\&\text{c.}\end{array}\right\}\qquad [10385e]$$

$$\text{tang.}\varpi=\frac{1}{\alpha b}\cdot\left\{\begin{array}{l}[9.1870866]\,.\,u+[7.7721132]\,.\,u^3+[5.8800186]\,.\,u^5\\ \qquad+[7.6868940]\,.\,u^7+[9.2719206]\,.\,u^9+\&\text{c.}\end{array}\right\}.\qquad [10385f]$$

These formulas were given by La Place in the *Connaisance des Tems*, for the year 1812, several years after the publication of the Second Supplement to this book, and were applied by him to the determination of the capillary action on the mercury in a barometer, as we shall see in [10443t]. [10385g]

If we put $z=\mathrm{z}-\frac{1}{\alpha b}$, as in [10384$m$], the equation [10380] multiplied by $\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$, may be reduced to the form [10385h]

$$\frac{dd\mathrm{z}}{du^2}+\frac{1}{u}\cdot\frac{d\mathrm{z}}{du}\cdot\left(1+\frac{d\mathrm{z}^2}{du^2}\right)=2\alpha\mathrm{z}\,.\left(1+\frac{d\mathrm{z}^2}{du^2}\right)^{\frac{3}{2}};\qquad [10385i]$$

and its general integral is expressed by

[10389] $$\frac{1}{\alpha b\pi}\cdot\sqrt{2\alpha}\cdot\sin.\varphi\cdot c^{u\sqrt{2\alpha}\,.\,\cos.\varphi}+\text{constant}.$$

The first limit of this integral being $\varphi=0$ [10385], the constant quantity
[10390] must be equal to nothing; and at the second limit, where $\varphi=\pi$ [10385], it again vanishes; therefore the equation [10382] is satisfied. The preceding value of z is not the complete integral of the equation; but it suffices for the
[10391] present case, in which z and $\frac{dz}{du}$ vanish when $u=0$.

[10392] Putting $\cos.\varphi=1-2\sin.^2\frac{1}{2}\varphi$, in the preceding expression of z [10384'], we obtain

[10393] $$z=\frac{c^{u\sqrt{2\alpha}}}{\alpha b\pi}\cdot\int_0^{\pi}d\varphi\cdot c^{-2u\sqrt{2\alpha}\,.\,\sin.^2\frac{1}{2}\varphi}-\frac{1}{\alpha b}.$$

[Value of z on the upper surface of the drop, from the vertex towards the borders.]

When $2u\sqrt{2\alpha}$ is a large quantity, which is the case near the borders of a great
[10394] drop, the value of $c^{-2u\sqrt{2\alpha}\,.\,\sin.^2\frac{1}{2}\varphi}$ becomes very small, and almost insensible, in
[10395] case φ has a sensible value. Therefore, if we put the integral $\int d\varphi\cdot c^{-2u\sqrt{2\alpha}\,.\,\sin.^2\frac{1}{2}\varphi}$ under the following form,*

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and the assumed expression [10384'] will give the following approximate value of z, as the integral of [10385i];

[10385k] $$\text{z}=\frac{1}{\alpha b\pi}\cdot\int_0^{\pi}d\varphi\cdot c^{u\sqrt{2\alpha}\,.\,\cos.\varphi}.$$

[10385l] If we put, as in [9323p], $\alpha=\alpha^{-2}$, the equation [10385i] will become as in [10385m], and its integral [10385k], as in [10385n], which will be of use hereafter;

[10385m] $$\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)=\frac{2z}{\alpha^2}\cdot\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$$

[10385n] $$\text{z}=\frac{\alpha^2}{b\pi}\cdot\int_0^{\pi}d\varphi\cdot c^{\frac{u\sqrt{2}}{\alpha}\,.\,\cos.\varphi}.$$

[10385o] If we make the same substitutions of $z=\text{z}-\frac{1}{\alpha b}$, $\alpha=\alpha^{-2}$, in [10402], and neglect the term of the order $u^{-\frac{3}{2}}$, we shall get for the approximate value of z,

[10385p] $$\text{z}=\frac{c^{u\sqrt{2\alpha}}}{\alpha b\cdot\sqrt{2\pi\,.\,u\sqrt{2\alpha}}}=\frac{\alpha^2}{b\sqrt{2\pi\sqrt{2}}}\cdot\sqrt{\frac{\alpha}{u}}\cdot c^{\frac{u\sqrt{2}}{\alpha}}.$$

* (4400) The coefficient of $d\varphi\cdot c^{-2u\sqrt{2\alpha}\,.\,\sin.^2\frac{1}{2}\varphi}$, neglecting the sign $\int$, in the second member of [10396], is

[10396a] $$\cos.\tfrac{1}{2}\varphi\cdot(1+\tfrac{1}{2}\sin.^2\tfrac{1}{2}\varphi)+2\sin.^4\tfrac{1}{4}\varphi\cdot(1+2\cos.^2\tfrac{1}{4}\varphi);$$

$$\int d\varphi \cdot c^{-2u\sqrt{2\alpha}\cdot\sin.^2\frac{1}{2}\varphi} = \int d\varphi \cdot \cos.\tfrac{1}{2}\varphi \cdot \{1+\tfrac{1}{2}\sin.^2\tfrac{1}{2}\varphi\} \cdot c^{-2u\sqrt{2\alpha}\cdot\sin.^2\frac{1}{2}\varphi}$$ [10396]

$$+\int 2d\varphi \cdot \sin.^4\tfrac{1}{4}\varphi \cdot \{1+2\cos.^2\tfrac{1}{4}\varphi\} \cdot c^{-2u\sqrt{2\alpha}\cdot\sin.^2\frac{1}{2}\varphi},$$

we may, without any sensible error, neglect this last term;* then, putting [10397]

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but from [1, 6], Int., we have

$$2\sin.^4\tfrac{1}{4}\varphi = 2(\sin.^2\tfrac{1}{4}\varphi)^2 = 2(\tfrac{1}{2}-\tfrac{1}{2}\cos.\tfrac{1}{2}\varphi)^2 = \tfrac{1}{2}-\cos.\tfrac{1}{2}\varphi+\tfrac{1}{2}\cos.^2\tfrac{1}{2}\varphi,$$

and

$$1+2\cos.^2\tfrac{1}{4}\varphi = 2+\cos.\tfrac{1}{2}\varphi;$$

the product of these two expressions is

$$2\sin.^4\tfrac{1}{4}\varphi\cdot(1+2\cos.^2\tfrac{1}{4}\varphi) = 1-\tfrac{3}{2}\cos.\tfrac{1}{2}\varphi+\tfrac{1}{2}\cos.^3\tfrac{1}{2}\varphi;$$ [10396b]

substituting this and $1+\frac{1}{2}\sin.^2\frac{1}{2}\varphi = \frac{3}{2}-\frac{1}{2}\cos.^2\frac{1}{2}\varphi$, in the factor [10396a], it becomes

$$\cos.\tfrac{1}{2}\varphi\cdot(\tfrac{3}{2}-\tfrac{1}{2}\cos.^2\tfrac{1}{2}\varphi)+(1-\tfrac{3}{2}\cos.\varphi+\tfrac{1}{2}\cos.^3\tfrac{1}{2}\varphi),$$

which, by neglecting the terms destroying each other, becomes 1; so that the first member of [10396] is reduced to the integral of the expression $d\varphi.c^{-2u\sqrt{2\alpha}\cdot\sin.^2\frac{1}{2}\varphi}$ multiplied by [10396c] the factor [10396a], as in [10396].

* (4401) The limits of φ [10385] being $\varphi=0$ and $\varphi=\pi$, we have

$$1+\tfrac{1}{2}\sin.^2\tfrac{1}{2}\varphi > 1 < \tfrac{3}{2}, \quad \text{and} \quad 1+2\cos.^2\tfrac{1}{4}\varphi > 2 < 3;$$ [10397a]

therefore the second of these factors is nearly double that of the first; hence the element of the expression in [10396, line 1], is to that in [10396, line 2], nearly as $\cos.\frac{1}{2}\varphi.c^{-2u\sqrt{2\alpha}\cdot\sin.^2\frac{1}{2}\varphi}$ to $4\sin.^4\frac{1}{4}\varphi.c^{-2u\sqrt{2\alpha}\cdot\sin.^2\frac{1}{2}\varphi}$. Now, if we substitute $\frac{2}{\alpha}=13^{\text{mi.mi.}}$ [10351′], and suppose the greatest value of $2u$ to be $100^{\text{mi.}}$ [10446], we shall have [10397a′]

$$2u\sqrt{2\alpha} = 200\sqrt{\tfrac{4}{13}} = 111$$ [10397b]

nearly; therefore the two terms of [10396], when near the borders of the drop at C and H, fig. 172, page 971, will be to each other nearly as $\cos.\frac{1}{2}\varphi.c^{-111.\sin.^2\frac{1}{2}\varphi}$ to $4\sin.^4\frac{1}{4}\varphi.c^{-111.\sin.^2\frac{1}{2}\varphi}$, [10397c] or as $\cos.\frac{1}{2}\varphi$ to $4\sin.^4\frac{1}{4}\varphi$; so that when φ is small, this last term must be much less than the first, and may then be neglected. Again, when φ is very small, the factor $c^{-111.\sin.^2\frac{1}{2}\varphi}$ is nearly equal to 1; but as φ increases, it becomes quite small. For, if we put successively [10397d] $\sin.\frac{1}{2}\varphi$ equal to 0,1; 0,2; 0,3; 0,4, &c.; corresponding respectively to φ equal to 13°, 26°, 39°, 52°, &c., nearly; the factor $c^{-111.\sin.^2\frac{1}{2}\varphi}$ will become $c^{-1.11}$, $c^{-4.44}$, [10397e] $c^{-9.99}$, $c^{-17.76}$, and as $c=2.71$, nearly; these terms will be nearly represented by $\frac{1}{3}$, $\frac{1}{85}$, $\frac{1}{20000}$, $\frac{1}{50000000}$, &c., respectively. In consequence of the smallness of this factor, [10397f] when φ is of a rather large magnitude, the elements of the integral in the second member of [10396] are nearly insensible, and may be neglected in comparison with those depending on [10397g]

[10398] $$2u\sqrt{2\alpha}.\sin.^2\tfrac{1}{2}\varphi = t^2,$$

or

[10398'] $$\sin.\tfrac{1}{2}\varphi = \frac{t}{\sqrt{2u\sqrt{2\alpha}}},$$

we shall have *

[10399] $$z = \frac{c^{u\sqrt{2\alpha}}}{\alpha b\pi\sqrt{2u\sqrt{2\alpha}}}.\int 2dt.c^{-tt}.\left\{1+\frac{tt}{4u\sqrt{2\alpha}}\right\} - \frac{1}{\alpha b}.$$

The integral relative to t must be taken from $t^2=0$ to $t^2=2u\sqrt{2\alpha}$ [10399c];

[10400] but, $c^{-2u\sqrt{2\alpha}}$ being by supposition an insensible quantity, we may take this integral from $t=0$ to $t=$ infinite; and then we shall have, as in [8331],

[10401] $$2\int_0^\infty dt.c^{-tt} = \sqrt{\pi};$$

therefore,†

[10402] $$z = \frac{c^{u\sqrt{2\alpha}}}{\alpha b\sqrt{2\pi.u\sqrt{2\alpha}}}.\left\{1+\frac{1}{8u\sqrt{2\alpha}}\right\} - \frac{1}{\alpha b}.$$ [Value of z on the upper surface of the drop, from the vertex towards the borders.]

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the smaller values of φ; and as we have just shown that, for these smaller values of φ, the terms in [10396, line 2] are nearly insensible, we may neglect it wholly, and retain only the upper line, or that in [10396, line 1]; and by substituting it in [10393], we get

[10397h] $$z = \frac{c^{u\sqrt{2\alpha}}}{\alpha b\pi}.\int_0^\pi d\varphi.\cos.\tfrac{1}{2}\varphi.\{1+\tfrac{1}{2}\sin.^2\tfrac{1}{2}\varphi\}.c^{-2u\sqrt{2\alpha}.\sin.^2\frac{1}{2}\varphi} - \frac{1}{\alpha b}.$$

* (4402) The differential of [10398'] gives, by considering φ, t, as the variable quantities,

[10399a] $d\varphi.\cos.\frac{1}{2}\varphi = \frac{2dt}{\sqrt{2u\sqrt{2\alpha}}}$; substituting this and [10398'] in [10397h], we get

[10399b] $$z = \frac{c^{u\sqrt{2\alpha}}}{\alpha b\pi}.\int\frac{2dt}{\sqrt{2u\sqrt{2\alpha}}}.\left\{1+\frac{\frac{1}{2}t^2}{2u\sqrt{2\alpha}}\right\}c^{-tt} - \frac{1}{\alpha b},$$

which is easily reduced to the form [10399]. The limits of this integral are easily obtained

[10399c] from those in [10385]; since $\varphi=0$, gives, in [10398], $t^2=0$, for the first limit; and $\varphi=\pi$ [10385] gives, in [10398], $t^2=2u\sqrt{2\alpha}$, for the second limit, as in [10400]; and as this last quantity, in the experiment mentioned in the last note, is nearly equal to 111

[10399d] [10397b], it will give $c^{-2u\sqrt{2\alpha}} = c^{-111}$, an insensible quantity; whence it is evident that we may extend this last limit of t, to $t=\infty$, as in [10400].

[10402a] † (4403) From [1534o, p], we get $\int_0^\infty t^2dt.c^{-tt} = \frac{1}{2}\int_0^\infty dt.c^{-tt}$, observing that the elements of these integrals have the same values for $+t$ as for $-t$; substituting in the

[10402b] preceding integral the value [10401], we get $\int_0^\infty t^2dt.c^{-tt} = \frac{1}{4}\sqrt{\pi}$. Introducing both these integrals into [10399], we obtain [10402].

We shall now resume the differential equation [10380], which, by using the values of z', q [10378', 10377'], becomes * [10403]

$$\frac{\frac{ddz'}{du^2}}{\left(1+\frac{dz'^2}{du^2}\right)^{\frac{3}{2}}}+\frac{\frac{1}{u}\cdot\frac{dz'}{du}}{\left(1+\frac{dz'^2}{du^2}\right)^{\frac{1}{2}}}+2\alpha q-2\alpha z'=-\frac{2}{b}.$$ [General differential equation of the surface.] [10404]

We shall now put, as in [10265, or 10379'], ϖ *for the angle which the tangent of the curve makes with the radius* u; and we shall have † ϖ [10405]

$$\frac{dz'}{du}=-\text{tang.}\varpi, \quad \text{or} \quad dz'=-du.\text{tang.}\varpi;$$ [10406]

thus the preceding equation becomes

$$\frac{d\varpi}{du}.\cos.\varpi+\frac{1}{u}.\sin.\varpi=2\alpha q+\frac{2}{b}-2\alpha z';$$ [10406']

multiplying the first term of this equation by $-du.\text{tang.}\varpi$, and the others by the equal quantity dz' [10406], we obtain

$$-d\varpi.\sin.\varpi+\frac{dz'}{u}.\sin.\varpi=2\alpha q.dz'+\frac{2}{b}.dz'-2\alpha z'dz';$$ [10407]

hence, by integration, we get

$$\cos.\varpi+\int\frac{dz'.\sin.\varpi}{u}=\left(2\alpha q+\frac{2}{b}\right).z'-\alpha z'^2+\text{constant}.$$ [10408]

To determine the constant quantity, we shall put ϖ' [10379''], *for the value of* ϖ [10379'], *when* z' *is nothing;* ϖ' *will be the obtuse angle formed by the surface of the drop with the plane.* Commencing the integral of the preceding [10408'] [10409]

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* (4404) Substituting $z=q-z'$ [10378'], and its differentials $dz=-dz'$, $ddz=-ddz'$, in [10380], and then changing the signs of all the terms, we get [10404]. [10404a]

† (4405) The angle GBb, fig. 172, page 971, being represented by ϖ [10379'], we evidently have $\text{tang.}\varpi=\frac{dz}{du}=-\frac{dz'}{du}$ [10404a], as in [10406]; its differential gives [10406a] $\frac{ddz'}{du^2}=-\frac{d\varpi}{du}\cdot\frac{1}{\cos.^2\varpi}$; moreover $\left(1+\frac{dz'^2}{du^2}\right)^{\frac{1}{2}}=(1+\text{tang.}^2\varpi)^{\frac{1}{2}}=\frac{1}{\cos.\varpi}$; substituting these in [10406b] [10404], we get [10406'], by changing the signs of all the terms. The expression $\text{tang.}\varpi=\frac{dz}{du}$ gives

$$\sin.\varpi=\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}},$$ [10406c]

which will be of use hereafter.

ϖ [10409′] equation with z', *we shall have constant* $=\cos.\varpi'$. We shall at first neglect

[10410] the part under the sign of integration, also the term* $\frac{2}{b}.z'$, and then we shall have nearly

[10411] $$\cos.\varpi=2\alpha qz'-\alpha z'^2+\cos.\varpi'.$$

[10412] We may, for a first approximation, suppose $z'=q$, when the tangent is horizontal, or when $\cos.\varpi=1$; thus we shall have†

[10413] $$1-\cos.\varpi'=\alpha q^2,$$

or

[10414] $$q=\sqrt{\frac{2}{\alpha}}.\sin.\tfrac{1}{2}\varpi',$$ [Approximate value of the height of the drop q.]

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* (4406) The term $\frac{2}{b}.z'$ is much smaller than the term $2\alpha qz'$, with which it is

[10411a] connected. For the ratio of the first of these terms to the second is $\frac{1}{\alpha qb}$; and by [10414], $\alpha q=\sqrt{2\alpha}.\sin.\frac{1}{2}\varpi'$ nearly, may be considered as of the order $\sqrt{2\alpha}=\sqrt{\frac{1}{13}}=\frac{1}{1^{mi.},8}$ [10351′];

[10411b] hence this ratio is of the order $\frac{1^{mi.},8}{b}$; and as b [10377] represents the radius of curvature of the surface of the drop at the vertex A, fig. 172, page 971, where the curve is nearly

[10411c] horizontal, it must evidently be much greater than $1^{mi.},8$, when the radius $l=ED=50^{mi.}$ nearly [10397a′]; therefore the term in question $\frac{2}{b}.z'$ may be neglected in the first

[10411d] approximation. Again, the integral $\int\frac{dz'.\sin.\varpi}{u}$, whose approximate value is computed in [10420], is to the first term of the expression [10408], namely, $\cos.\varpi$, as

[10411e] $$\frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\left(\frac{\cos.^3\frac{1}{2}\varpi-\cos.^3\frac{1}{2}\varpi'}{\cos.\varpi}\right)\text{ to }1;$$

[10411f] and we may consider the factor $\left(\frac{\cos.^3\frac{1}{2}\varpi-\cos.^3\frac{1}{2}\varpi'}{\cos.\varpi}\right)$ as being in general of the order 1; so

[10411g] that the integral expression [10411d] may be considered as of the order $\frac{2}{3l}.\sqrt{\frac{2}{\alpha}}$. Now,

[10411h] putting $l=50^{mi.}$ [10411c], and $\frac{2}{\alpha}=13^{mi.mi.}$ [10351′], it becomes $\frac{1}{75}\sqrt{13}=\frac{1}{21}$ nearly; and as this is so small, we may also neglect it, in a first approximation; and then the expression [10408] becomes as in [10411], using the constant quantity [10409′].

† (4407) As the point B, fig. 172, page 971, passes from C towards A, the angle ϖ

[10413a] [10379′] decreases, and becomes $\varpi=0$, at the vertex A, where $z'=q$ [10377′]. Substituting these values in [10411], we get at the vertex $1=\alpha q^2+\cos.\varpi'$, as in [10413];

[10413b] from which we get $\alpha q^2=1-\cos.\varpi'=2\sin.^2\frac{1}{2}\varpi'$ [1], Int. Dividing this by α, and extracting the square root, we get the approximate value of q [10414].

therefore *

$$q-z'=\sqrt{\frac{2}{\alpha}}.\sin.\tfrac{1}{2}\varpi=z, \qquad [10415]$$

which gives

$$dz'=-\tfrac{1}{2}\sqrt{\frac{2}{\alpha}}.d\varpi.\cos.\tfrac{1}{2}\varpi; \qquad [10416]$$

hence we get the following integral,

$$\int\frac{dz'.\sin.\varpi}{u}=-\sqrt{\frac{2}{\alpha}}.\int\frac{d\varpi.\sin.\frac{1}{2}\varpi.\cos.^2\frac{1}{2}\varpi}{u}. \qquad [10417]$$

The preceding integral is insensible when the angle ϖ is very small; for, although the denominator of the differential $\frac{dz'.\sin.\varpi}{u}$ may be then very small, [10418] and even nothing, yet the expression itself is very small, and much less than when the angle ϖ is larger, as we may easily prove.† In this last case, *the*

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* (4408) Substituting $\cos.\varpi'=1-\alpha q^2$ [10413] in [10411], we get

$$\cos.\varpi=2\alpha qz'-\alpha'z'^2+1-\alpha q^2; \qquad [10417a]$$

hence, by transposition,

$$\alpha(q^2-2qz'+z'^2)=1-\cos.\varpi=2\sin.^2\tfrac{1}{2}\varpi \text{ [1], Int.} \qquad [10417b]$$

Dividing by α, and extracting the square root, we get [10415], which is the same as the value of z [10378']. The differential of [10415] gives dz' [10416]; and by substituting it in the first member of [10417], it becomes as in its second member, observing that $\sin.\varpi=2\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi$ [31], Int. [10417c]

† (4409) We shall suppose the point B, fig. 172, page 971, to be very near to the vertex A; then, drawing the tangent Bb and the ordinate BG, we shall have the angle $GBb=\varpi$ [10379'], and $BG=u$ [10379]. Now the arc AB, for a very small distance [10418a] about the vertex A, being supposed to be circular, with the radius b; we shall have ϖ equal to the angle formed at the centre of this circle, by the radii drawn through the points A, B; and from the nature of the circle, we shall have $u=BG=b.\sin.\varpi=2b.\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi$ [10418b] [31], Int. Substituting this in the second member of [10417], we get, when ϖ, u, are very small,

$$\int\frac{dz'.\sin.\varpi}{u}=-\sqrt{\frac{2}{\alpha}}.\int\frac{d\varpi.\cos.\frac{1}{2}\varpi}{2b}=-\sqrt{\frac{2}{\alpha}}.\int\frac{d\varpi}{2b}=-\sqrt{\frac{2}{\alpha}}.\frac{\varpi}{2b}. \qquad [10418c]$$

Now, the upper part of a large drop being very little curved near the vertex, we shall have b quite large, and ϖ small, even when u is of a considerable magnitude; hence it is evident [10418d] that the integral expression [10117] must be very small when the angle ϖ is small; agreeing with the remarks in [10418, &c.]. From what has been said, it is evident that we may [10418e] neglect all the parts of the integral [10417], except where u is very large, and nearly equal to $DE=l$; and if we put $u=l$, and substitute also for $d\varpi.\sin.\frac{1}{2}\varpi$ its value $-2d.\cos.\frac{1}{2}\varpi$,

value of u *is very nearly equal to the semi-diameter of the circular section of the contact of the mercury with the plane. We shall put this semi-diameter equal*
l *to* l; and we may, without any sensible error, suppose $u=l$ in the preceding
[10419] integral; and then, by supposing the integral to commence with $\varpi=\varpi'$, we shall obtain, as in [10418g],

[10420] $$\int_0^{z'} \frac{dz'.\sin.\varpi}{u} = \frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\{\cos.^3\tfrac{1}{2}\varpi - \cos.^3\tfrac{1}{2}\varpi'\};$$

therefore we shall have*

[10421] $$\cos.\varpi + \frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\{\cos.^3\tfrac{1}{2}\varpi - \cos.^3\tfrac{1}{2}\varpi'\} = \left(2\alpha q + \frac{2}{b}\right).z' - \alpha z'^2 + \cos.\varpi';$$

substituting $z'=q-z$ [10378'], it becomes

[10422] $$\cos.\varpi + \frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\{\cos.^3\tfrac{1}{2}\varpi - \cos.^3\tfrac{1}{2}\varpi'\} = \alpha q^2 + \frac{2}{b}.(q-z) - \alpha z^2 + \cos.\varpi'.$$

[10423] Now, z being nothing when† $\varpi=0$, we shall have ϖ^2 equal to a series ascending according to the powers of z; by substituting it in the preceding

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[10418f] the expression of the second member of [10417] will become $\sqrt{\frac{2}{\alpha}}.\frac{2}{l}.\int\cos.^2\tfrac{1}{2}\varpi.d.\cos.\tfrac{1}{2}\varpi$, whose integral is $\frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\{\cos.^3\tfrac{1}{2}\varpi + \text{constant}\}$; and if we commence the integral at the
[10418g] point D, where $\varpi=\varpi'$, the constant quantity will be $-\cos.^3\tfrac{1}{2}\varpi'$, and the integral [10417] will become as in [10120].

* (4410) Substituting the constant quantity $\cos.\varpi'$ [10409'], and the value of the
[10422a] integral [10420], in [10408], it becomes as in [10421]; and by using $z'=q-z$ [10378'], we get [10422].

† (4411) At the vertex A, fig. 172, page 971, where the tangent to the surface is
[10424a] horizontal, we have $z=0$ [10378], and $\varpi=0$ [10413a]; substituting these values in [10422], we get

[10424b] $$1 + \frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\{1 - \cos.^3\tfrac{1}{2}\varpi'\} = \alpha q^2 + \frac{2}{b}.q + \cos.\varpi';$$

transposing $\cos.\varpi'$, we get [10424]. Moreover, if we substitute, in the first member of
[10424c] [10422], the expression $\cos.\varpi = 1 - \tfrac{1}{2}\varpi^2 + \tfrac{1}{24}\varpi^4 - \&c.$ [44], Int., and then subtract from the resulting equation that in [10121], it will become

[10424d] $$-\tfrac{1}{2}\varpi^2 + \tfrac{1}{24}\varpi^4 - \&c. + \frac{2}{3l}.\sqrt{\frac{2}{\alpha}}.\{-\tfrac{3}{2}(\tfrac{1}{8}\varpi^2) + \&c.\} = -\frac{2}{b}.z - \alpha z^2;$$

whence we can deduce ϖ^2, in a series arranged according to the powers of z, as is stated in [10423].

equation, and comparing together the coefficients of these powers, the coefficient independent of z will give, as in [10424b],

$$1-\cos.\varpi'+\frac{2}{3l}\cdot\sqrt{\frac{2}{\alpha}}\cdot(1-\cos.^3\tfrac{1}{2}\varpi')=\alpha q^2+\frac{2}{b}\cdot q;$$ [10424]

$\frac{1}{b}$ is a very small fraction, whose square may be neglected when the drop has a great width; and in this case, the preceding equation gives very nearly * [10425]

$$q+\frac{1}{\alpha b}=\sqrt{\frac{2}{\alpha}}\cdot\sin.\tfrac{1}{2}\varpi'+\frac{(1-\cos.^3\frac{1}{2}\varpi')}{3\alpha l.\sin.\frac{1}{2}\varpi'}.$$ [10426]

We shall now determine the constant quantity $\frac{1}{b}$. For this purpose we shall resume the equation † [10406, 10416],

$$dz'=-du.\operatorname{tang}.\varpi=-\tfrac{1}{2}\sqrt{\frac{2}{\alpha}}.d\varpi.\cos.\tfrac{1}{2}\varpi;$$ [10427]

whence we deduce

$$du=\frac{1}{2\sqrt{2\alpha}}.d\varpi.\left\{\frac{1}{\sin.\frac{1}{2}\varpi}-2\sin.\tfrac{1}{2}\varpi\right\},$$ [10428]

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* (4412) Dividing [10424] by α, then substituting $1-\cos.\varpi'=2\sin.^2\frac{1}{2}\varpi'$, also $\left(q+\frac{1}{\alpha b}\right)^2$, for $q^2+\frac{2q}{\alpha b}$, which may be done by neglecting the very small quantity $\left(\frac{1}{\alpha b}\right)^2$, we get

$$\frac{2}{\alpha}.\sin.^2\tfrac{1}{2}\varpi'+2\sqrt{\frac{2}{\alpha}}.\frac{(1-\cos.^3\frac{1}{2}\varpi')}{3\alpha l}=\left(q+\frac{1}{\alpha b}\right)^2.$$ [10426a]

Extracting the square root of this expression, we get

$$q+\frac{1}{\alpha b}=\sqrt{\frac{2}{\alpha}}.\sin.\tfrac{1}{2}\varpi'.\left\{1+\frac{2(1-\cos.^3\frac{1}{2}\varpi')}{3l\sqrt{2\alpha}.\sin.^2\frac{1}{2}\varpi'}\right\}^{\frac{1}{2}}.$$ [10426b]

Developing the second member in a series, and retaining only the two first terms, it becomes as in [10426]. The neglected terms must be small, because they depend on the square and higher powers of the factor $\frac{2}{3l\sqrt{2\alpha}}$ [10426b], which, in the example [10411g], is of the [10426c] order of the square and higher powers of $\frac{1}{42}$, as is evident from the values of $\sqrt{2\alpha}=\frac{1}{1^{\text{mi.}},8}$ [10426d] and $l=50^{\text{mi.}}$ [10411b, c].

† (4413) Equating the two expressions of dz' [10406, 10416], we get [10427]; now we have identically, as in [10281n], $\operatorname{tang}.\varpi=\frac{2\sin.\frac{1}{2}\varpi.\cos.\frac{1}{2}\varpi}{1-2\sin.^2\frac{1}{2}\varpi}$; dividing the two last [10427a] expressions of [10427] by these expressions of $\operatorname{tang}.\varpi$, and then changing the signs, we get $du=\frac{1}{2}\sqrt{\frac{2}{\alpha}}.d\varpi.\frac{(1-2\sin.^2\frac{1}{2}\varpi)}{2\sin.\frac{1}{2}\varpi}$, which is easily reduced to the form [10428], or [10427b] $du\sqrt{2\alpha}=\frac{\frac{1}{2}d\varpi}{\sin.\frac{1}{2}\varpi}-d\varpi.\sin.\frac{1}{2}\varpi$, whose integral is as in [10429], as is evident from the [10427c] formula [10281q].

which gives, by integration [10427c],

[10429] $$u\sqrt{2\alpha} = \log.\text{tang.}\tfrac{1}{4}\varpi + 2\cos.\tfrac{1}{2}\varpi + \text{constant}.$$

[10430] To determine the constant quantity, we shall observe that, when $\varpi = \varpi'$ [10408′], we have $u = l$ [10419]; and then [10429] gives

[10431] $$\text{constant} = l\sqrt{2\alpha} - \log.\text{tang.}\tfrac{1}{4}\varpi' - 2\cos.\tfrac{1}{2}\varpi';$$

therefore we shall have *

[10432] $$\text{tang.}\tfrac{1}{4}\varpi = \text{tang.}\tfrac{1}{4}\varpi'.c^{(u-l).\sqrt{2\alpha} - 2\cos.\frac{1}{2}\varpi + 2\cos.\frac{1}{2}\varpi'},$$

[10433] c being the number whose hyperbolic logarithm is 1. This equation gives very nearly, when the angle ϖ is very small,

[10434] $$\text{tang.}\varpi = 4\,\text{tang.}\tfrac{1}{4}\varpi'.c^{(u-l).\sqrt{2\alpha} - 4\sin.^2\frac{1}{4}\varpi'}.$$

[10435] Now, if we take the differential of the expression of z [10402], in which ϖ is supposed to be small, we shall have †

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* (4414) Substituting the value of the constant quantity [10431] in [10429], we
[10432a] obtain $u\sqrt{2\alpha} = \log.\text{tang.}\frac{1}{4}\varpi + 2\cos.\frac{1}{2}\varpi + l\sqrt{2\alpha} - \log.\text{tang.}\frac{1}{4}\varpi' - 2\cos.\frac{1}{2}\varpi'$, whence we
[10432b] deduce $\log.\text{tang.}\frac{1}{4}\varpi = \log.\text{tang.}\frac{1}{4}\varpi' + \{(u-l)\sqrt{2\alpha} - 2\cos.\frac{1}{2}\varpi + 2\cos.\frac{1}{2}\varpi'\}$. Multiplying the terms between the braces in the second member by $\log.c = 1$, and then reducing to natural numbers, we get $\text{tang.}\frac{1}{4}\varpi$ [10432]. When ϖ is very small, we shall have very nearly
[10432c] $\text{tang.}\frac{1}{4}\varpi = \frac{1}{4}\text{tang.}\varpi$, also $2\cos.\frac{1}{2}\varpi = 2$ nearly; and then we have
[10432d] $-2\cos.\frac{1}{2}\varpi + 2\cos.\frac{1}{2}\varpi' = -2 + 2\cos.\frac{1}{2}\varpi' = -2(1 - \cos.\frac{1}{2}\varpi') = -4\sin.^2\frac{1}{4}\varpi'$ [1], Int. Substituting these in [10432], and then multiplying by 4, we get [10434].

† (4415) We have supposed, in [10381, &c.], that the top part of the drop, for a
[10436a] considerable distance from the vertex A, fig. 172, page 971, is nearly horizontal; so that u may become quite large, and almost equal to l [10419], while ϖ [10379′] is quite small;
[10436b] and it is for such small values of ϖ, and large values of u, that the expression of z [10402] is supposed to correspond. Now, taking the differential of this expression of z, relative to u,
[10436c] and dividing it by du, we get [10436]; in which we may neglect the two last terms of the second member containing the divisor $8u\sqrt{2\alpha}$, and its square, because these divisors are quite large; since we have seen, in [10397b], that, in the experiment there mentioned,
[10436d] $2u\sqrt{2\alpha} = 111$, near the border of the drop. In this case, the expression [10436] will be
[10436e] reduced to its first term, $\frac{dz}{du} = \frac{\sqrt{2\alpha}.c^{u\sqrt{2\alpha}}}{\alpha b\sqrt{2\pi.u\sqrt{2\alpha}}}$; and since $\frac{dz}{du} = -\frac{dz'}{du} = \text{tang.}\varpi$ [10406a], we
[10436f] get $\text{tang.}\varpi = \frac{\sqrt{2\alpha}.c^{u\sqrt{2\alpha}}}{\alpha b\sqrt{2\pi.u\sqrt{2\alpha}}}$, for the value of tang.$\varpi$, near the borders of the drop; and if we then suppose, as in [10419, 10430], $u = l$, in the denominator of this expression of tang.ϖ, it becomes as in [10439].

$$\frac{dz}{du}=\frac{\sqrt{2\alpha}.c^{u\sqrt{2\alpha}}}{\alpha b\sqrt{2\pi.u\sqrt{2\alpha}}}.\left\{1-\frac{3}{8u\sqrt{2\alpha}}-\frac{12}{(8u\sqrt{2\alpha})^2}\right\}; \qquad [10436]$$

when l is very large, we may neglect, in this expression, the terms $-\frac{3}{8u\sqrt{2\alpha}}$ and $-\frac{12}{(8u\sqrt{2\alpha})^2}$, in comparison with unity, and suppose, in the denominator, [10437]
$u=l$, which is the same thing as to neglect the powers of $\frac{l-u}{l}$, as we have done in the preceding expression of tang.ϖ, or $\frac{dz}{du}$; and then we shall have [10438]

$$\text{tang.}\varpi=\frac{\sqrt{\alpha}.c^{u\sqrt{2\alpha}}}{\alpha b\sqrt{\pi l.\sqrt{2\alpha}}}. \qquad [10439]$$

If we compare this expression of tang.ϖ with that in [10434], we shall obtain *

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* (4416) Putting the two expressions of tang.ϖ [10434, 10439] equal to each other, and multiplying by $\frac{\sqrt{\pi l.\sqrt{2\alpha}}}{\sqrt{\alpha}.c^{u\sqrt{2\alpha}}}$, we get [10440]; and if we substitute the values $\varpi'=152^{\circ}$ [10450], $\alpha=\frac{2}{13}$ [10351′], it becomes [10439a]

$$\frac{1}{\alpha b}=2.5854.l^{\frac{1}{2}}.c^{-l\times 0,5547}=2.5854.l^{\frac{1}{2}}.10^{-l\times 0,2409}, \qquad [10439b]$$

the second of these expressions being deduced from the first, by substituting $c=10^{0,43429}$. [10439c]
Multiplying [10439b] by $\alpha=\frac{2}{13}$, we obtain

$$\frac{1}{b}=0,3977.l^{\frac{1}{2}}.10^{-l\times 0,2409}, \qquad [10439d]$$

or

$$b=2,514.l^{-\frac{1}{2}}.10^{l\times 0,2409}. \qquad [10439d']$$

If l is large, as for example $25^{\text{mi.}}$, or more, the factor $10^{-l\times 0,2409}$ becomes less than 10^{-6}, and this renders the expression [10439b] so small, that it may be neglected, as in [10450′]. [10439e]
If we substitute the same values of ϖ', α [10439a], in [10426], we shall get

$$q+\frac{1}{\alpha b}=3^{\text{mi.}},3524+\frac{2^{\text{mi.}},2141}{l}. \qquad [10439f]$$

Subtracting from this the expression [10439b], we obtain

$$q=3^{\text{mi.}},3524+\frac{2^{\text{mi.}},2141}{l}-2,5854.l^{\frac{1}{2}}.10^{-l\times 0,2409}. \qquad [10439g]$$

When l is infinite, this expression of q becomes

$$q=3^{\text{mi.}},3524, \qquad [10439h]$$

which may be considered as the limit of the height of a drop when its diameter is very large.
If $l=50^{\text{mi.}}$, the last term of [10439g] is insensible, and the expression of q becomes [10439i]

[10440] $$\frac{1}{\alpha b}=\frac{4}{\sqrt{\alpha}}\cdot\sqrt{\pi l\sqrt{2\alpha}}\cdot\text{tang.}\tfrac{1}{4}\varpi'.c^{-l\sqrt{2\alpha}-4\sin.^2\frac{1}{4}\varpi'}.$$ [Depression of the mercury in a large tube.]

$q=3^{\text{mi.}},3524+0^{\text{mi.}},0442=3^{\text{mi.}},3966$, as in [10451]; hence it appears that q increases while l decreases, always supposing l to be a large quantity. There is, however, a limit to this increase, in the height of the drop, because, when the quantity of mercury decreases so as
[10439k] to make the drop become very small, all its dimensions, and therefore its height, must be very small.

In the preceding formulas, l is supposed to be given to find q; but, instead of this, we may
[10439l] suppose the volume V of the fluid to be given, which is easily obtained by dividing the mass of the fluid by its density D, ϖ' and α being considered as known quantities. In finding q from V, it is necessary to obtain an analytical expression of V from the integral of the
[10439m] equation [10380]; this is obtained in the following manner. Multiplying the equation [10380] by udu, and transposing the last term, we get

[10439n] $$\frac{u.\frac{ddz}{du}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}+\frac{dz}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{1}{2}}}-\frac{2udu}{b}-2\alpha zudu=0,$$

whose integral is

[10439o] $$\frac{u.\frac{dz}{du}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{1}{2}}}-\frac{u^2}{b}-\alpha zu^2+\alpha.\textstyle\int_0^z u^2dz=0,$$

as is easily proved by differentiation; no constant quantity being added, because the integral vanishes when $u=0$, and $z=0$. Multiplying this by $\frac{\pi}{\alpha}$, and transposing the three first terms, we get

[10439p] $$\pi.\textstyle\int_0^z u^2dz=\pi.\left(zu^2+\frac{u^2}{\alpha b}-\frac{u}{\alpha}.\frac{\frac{dz}{du}}{\sqrt{1+\frac{dz^2}{du^2}}}\right).$$

Now the first member of this expression evidently represents the volume V of the mercury in the upper part of the drop, and limited in the lower part by the horizontal circle passing
[10439q] through the ordinate y, at the distance z from the vertex. Substituting this symbol V in the first member, also $\sin.\varpi$ [10406c] in the second member, it becomes

[10439r] $$V=\pi u^2.\left(z+\frac{1}{\alpha b}\right)-\frac{\pi u}{\alpha}.\sin.\varpi.$$

To obtain the whole volume V of the drop, we must substitute, in this expression, the values
[10439s] of z, u, ϖ, corresponding to the point where the extreme limit of the drop touches the horizontal plane upon which it rests; hence we get for the whole volume of the drop the following expression;

[10439t] $$\mathrm{V}=\pi l^2.\left(q+\frac{1}{\alpha b}\right)-\frac{\pi l}{\alpha}.\sin.\varpi'.$$

This value of $\frac{1}{\alpha b}$ gives, as in [9750a], the depression depending upon the capillary action, in a barometer tube of a large diameter. For it is evident that [10440′]

Substituting the value of $q+\frac{1}{\alpha b}$ [10426], it becomes

$$V=\pi l^2.\sqrt{\tfrac{2}{\alpha}}.\sin.\tfrac{1}{2}\varpi'-\tfrac{\pi l}{\alpha}.\left\{\sin.\varpi'-\frac{(1-3\cos.^3\tfrac{1}{2}\varpi')}{3\sin.\tfrac{1}{2}\varpi'}\right\}; \qquad [10439u]$$

which is the same as the expression (p) in page 218 of the *Nouvelle Théorie, &c.*, of M. Poisson, changing V, l, α, ϖ', into v, r', a^{-2}, $\pi-w'$, respectively, to conform to his notation. [10439v]

If we substitute, in [10439u], the values of ϖ', α [10439a], it will become

$$V=10{,}5317\,.\,l^2-7{,}0230\,.\,l. \qquad [10439w]$$

As an example of the use of this formula, we shall take an observation of Gay-Lussac, given by M. Poisson, in page 219 of his *Nouvelle Théorie, &c.;* where the weight of the drop of mercury was $6^{\text{gram.}},013$, its height $q=3^{\text{mi.}},34$, at a mean temperature when the density was [10439x] $\frac{10435}{770}$, that of water being unity. Dividing the weight in milligrammes 6013, by this density, we get the volume of the drop in cubic millimetres equal to $\frac{770}{10435}\times 6013=443{,}7=V$. Substituting this in [10439w], we get $443{,}7=10{,}5317\,.\,l^2-7{,}0230\,.\,l$; and from this [10439$y$] quadratic equation we easily deduce

$$l=6^{\text{mi.}},833. \qquad [10439z]$$

M. Poisson makes $l=6^{\text{mi.}},8687$, in page 220 of his work, where he uses the symbol r' instead of l; the difference of the two results arises from his using the values of ϖ', α [10440a] [10355c], which differ a little from those which are used by La Place [10439a]. The value of l [10439z] being substituted in [10439g], gives for the height q of the drop the value $q=3^{\text{mi.}},52$, instead of $3^{\text{mi.}},34$, as found by observation [10439x]; the difference arises chiefly [10440b] from the terms omitted in the approximate formula [10439g], which is designed only for larger values of l than that in the present example. M. Poisson, by a somewhat different, but [10440c] equally imperfect approximation, makes $q=3^{\text{mi.}},1996$, in page 221 of his work, where it is denoted by the symbol k.

The points C, H, fig. 172, page 971, where $\varpi=\frac{1}{2}\pi$, correspond to *the greatest value of u, which we shall represent by* u; substituting these values of ϖ, u, in [10432], and then [10440d] putting $2\cos.\frac{1}{2}\varpi=\sqrt{2}$, $2\cos.\frac{1}{2}\varpi'=2-4\sin.^2\frac{1}{4}\varpi'$, we get [10440$e$]

$$\text{tang.}\tfrac{1}{8}\pi=\text{tang.}\tfrac{1}{4}\varpi'.c^{(\mathrm{u}-l)\sqrt{2\alpha}-\sqrt{2}+2-4\sin.^2\frac{1}{4}\varpi'}. \qquad [10440f]$$

Multiplying this by $c^{-\mathrm{u}\sqrt{2\alpha}-2+\sqrt{2}}.(\text{tang.}\frac{1}{4}\varpi')^{-1}$, we get

$$c^{-l\sqrt{2\alpha}-4\sin.^2\frac{1}{4}\varpi'}=\frac{\text{tang.}\frac{1}{8}\pi}{\text{tang.}\frac{1}{4}\varpi'}.c^{-\mathrm{u}\sqrt{2\alpha}-2+\sqrt{2}}. \qquad [10440g]$$

Substituting this in [10440], and multiplying by α, we obtain

$$\frac{1}{b}=4\sqrt{\alpha}.\sqrt{\pi l.\sqrt{2\alpha}}.\text{tang.}\tfrac{1}{8}\pi.c^{-\mathrm{u}\sqrt{2\alpha}-2+\sqrt{2}}. \qquad [10440h]$$

[10441] the surface of the mercury in the tube is the same as that of the drop which we have just considered; but at the point where this surface is terminated, it
[10441'] makes with the sides of the tube, an angle whose complement is* ϖ'.

[10440i] Now from [10281t], we have $\tang.\frac{1}{8}\pi=\frac{1}{\sqrt{2}+1}$; and if we put, for brevity,

[10440i'] $$u'=\mathrm{u}+\frac{\sqrt{2}-1}{\sqrt{\mathrm{a}}},$$

we shall get, from [10440h], the value of b [10440k], which is easily reduced to the form [10440l], by putting, as in [9323p], $\mathrm{a}=\alpha^{-2}$,

[10440k] $$b=\frac{1}{4\sqrt{\mathrm{a}}\cdot\sqrt{\pi l.\sqrt{2\mathrm{a}}}}\cdot(\sqrt{2}+1)\cdot c^{\mathrm{u}'\sqrt{2\mathrm{a}}}$$

[10440l] $$=\frac{\alpha.(\sqrt{2}+1)}{4\sqrt{\pi.\sqrt{2}}}\cdot\sqrt{\frac{\alpha}{l}}\cdot c^{\frac{u'\sqrt{2}}{\alpha}}.$$

When l is very large, the factor $\sqrt{\frac{\alpha}{l}}$ differs but very little from $\sqrt{\frac{\alpha}{u'}}$; and if we make this
[10440m] substitution in [10440l], and then change b, α, u', into μ, a, l', respectively, to conform to M. Poisson's notation, we shall obtain the value of μ, in page 216 of his work; observing that he, like La Place, neglects terms of the order $\frac{\mathrm{u}-l}{l}$, in these calculations.

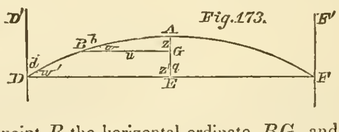
If we resubstitute, in [10440f], the value $2\cos.\frac{1}{2}\varpi'$ [10440e], and then reduce it to
[10440n] logarithms, we get, by a slight reduction, the formula [10440o]; and by putting $\mathrm{a}=\alpha^{-2}$, it becomes as in [10440p], using $\tang.\frac{1}{8}\pi$ [10440i],

[10440o] $$\mathrm{u}=l+\frac{1}{\sqrt{\mathrm{a}}}-\frac{\sqrt{2}}{\sqrt{\mathrm{a}}}.\cos.\tfrac{1}{2}\varpi'-\frac{1}{\sqrt{2\mathrm{a}}}.\text{hyp.log.}\frac{\tang.\frac{1}{4}\varpi'}{\tang.\frac{1}{8}\pi}$$

[10440p] $$=l+\alpha-\sqrt{2}.\alpha.\cos.\tfrac{1}{2}\varpi'-\frac{\alpha}{\sqrt{2}}.\text{hyp.log.}\{(1+\sqrt{2}).\tang.\tfrac{1}{4}\varpi'\}.$$

This expression is the same as that given by M. Poisson, in page 218 of his work, for finding
[10440q] the value of r', changing u, l, α, ϖ', into l, r', a, $\pi-w'$, to conform to his notation.

* (4417) In the annexed figure 173, $FAbBdD$ represents the convex surface of the mercury in a large barometer or cylindrical tube $DD'F'F$. This surface touches the sides of the
[10442a] tube in the points D, F, situated on the horizontal line DEF, which is bisected, in the point E,
[10442b] by the vertical axis EGA, passing through the vertex of the surface.



Then, drawing through any point B the horizontal ordinate BG, and
[10442c] using a notation similar to that in [10377—10379'']; observing also that the letters of reference in fig. 173 are exactly the same as those in fig. 172, page 971, corresponding to a large drop; we shall have

With respect to a fluid, which, like water and alcohol, completely [10442]

$$AE=q, \qquad ED=l, \qquad AG=z, \qquad EG=z', \qquad GB=u,$$
$$\text{angle } GBb=\varpi, \qquad \text{angle } EDd=\varpi', \text{ \&c.},$$ [10442d]

b being equal to the radius of curvature of the surface at the point A. Then by a slight attention [10442e]
to the method of calculation in the article [10377, 10440], we shall readily perceive, that what has been proved relative to the surface of the drop, is also true relative to the surface of the [10442f]
fluid in the tube; remarking, however, that the mercury forms the acute angle $D'Dd=48^{\circ}$ [10442g]
[10353], with the horizontal plane $D'D$, fig. 172, p. 971, or with the vertical side of the tube DD', fig. 173; so that, in the case of the drop, we shall have, in fig. 172, $\varpi'=EDd$ [10409], or $\varpi'=\pi-D'Dd=200^{\circ}-48^{\circ}=152^{\circ}$, as in [10450]; but in the case of the tube, [10442h]
fig. 173, $\varpi'=EDd$ becomes $\varpi=100^{\circ}-48^{\circ}=52^{\circ}$, as in [10441', or 10453]. The [10442i]
remarks we have here made relative to the convex surface of mercury, fig. 173, apply with hardly any variation to the concave surface of water, alcohol, &c. in a large tube, as in [10442k]
fig. 174.

If we substitute $\frac{H}{g}=\frac{1}{\alpha}$ [9328] in [9355], we shall get

$$q=\frac{\sin.\theta'}{\alpha l}-\frac{l}{\sin.\theta'}.\left\{1-\tfrac{2}{3}.\frac{(1-\cos.^3\theta')}{\sin.^2\theta'}\right\},$$ [10442l]

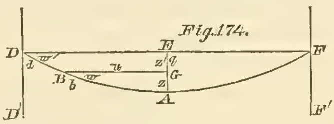


Fig. 174.

the angle θ' [9346] being the same as ϖ' [10442i]; so that if we substitute, in [10442l], the values $\theta'=\varpi'=52^{\circ}$ [10442i], $\frac{1}{\alpha}=6^{\text{mi.}},5$ [10351'], we shall get [10442m]

$$q=\frac{4,7383}{l}-0,2029.l,$$ [10442n]

for the depression q arising from the capillary action in a barometrical tube whose radius is l, *the unit of measure being a millimetre;* observing, however, that, in computing this formula, it is supposed that $l\sqrt{\alpha}$ is a small fraction, whose higher powers are neglected in the [10442o]
computation of the formula [9355]. M. Poisson, in page 287 of his *Nouvelle Théorie, &c.*, supposes $\varpi'=45^{\text{d.}}\,30^{\text{m.}}=50^{\circ}\,33'\tfrac{1}{3}$, and $\frac{1}{\alpha}=a^2=6^{\text{mi.}},5262$; and by substituting these [10442p]
values in [10442l], it becomes

$$q=\frac{4,5743}{l}-0,1931.l,$$ [10442q]

which agrees very nearly with the formula (1) in page 287 of his work, namely,

$$q=\frac{4,5746}{l}-0,1932.l+0,0559.l^3;$$ [10442r]

observing that his symbols δ, α, correspond respectively to q, l, in La Place's notation. [10442s]
Moreover, M. Poisson has used a somewhat different method in developing the series [9355], and has retained the small term depending on l^3 which is neglected by La Place, on account

[10442′] wets the sides of a glass tube, $\frac{1}{\alpha b}$ will express, in such a tube,

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[10442s′] of its smallness. The formula [10442q, or 10442r] *must not be used except for very small values of l*, and we shall hereafter give some examples of its use. If we suppose the
[10442t] diameter of the tube $2l$ to be equal to a ten thousandth part of a millimetre, or $l=0^{mi.},0005$, the expression of q [10442n] will give $q=94766^{mi.}$, being the same as is used by La Place in page 316 of the *Connaissance des Tems* for the year 1812, in a paper entitled *Sur la Dépression du Mercure dans un Tube de Baromètre, due à sa Capillarité.* When
[10442u] the radius of the tube is *very great*, we can determine the depression by means of the
[10442v] formula [10440], as is shown in [10455, &c., 10456a—f]. *For intermediate values of l*, we may use a process of quadratures like that proposed by La Place in the paper just mentioned, for computing the table of depressions [10443z] of the upper point of the surface
[10442w] of the mercury arising from the capillary action. This table was first computed by M. Bouvard, by the formulas [10443g—q], and was afterwards corrected and augmented by him,
[10442x] in a paper inserted in the *Connaissance des Tems* for the year 1829, as in [10443z]. We shall now explain this method of quadratures as it is given by La Place in the above-mentioned paper.

[10442y] If we substitute the value of $\frac{1}{R}$ [9326], and that of $\sin.\varpi$ [10406c], in [10380], we shall get the following equation of the surface of the fluid,

[10442z] $$\frac{1}{R}+\frac{1}{u}.\sin.\varpi-2\alpha z=\frac{2}{b}, \quad \text{or} \quad \frac{1}{R}=\frac{2}{b}+2\alpha z-\frac{1}{u}.\sin.\varpi,$$

which gives a very simple expression of the radius of curvature R of the generating curve,
[10443a] or curve formed by the section of the surface by a vertical plane passing through the axis of the tube. Therefore, if we consider this curve as being formed by a series of small arcs of a circle, described with different radii, and touching each other at their extremities, we can
[10443b] obtain the coordinates of the curve with so much greater degree of accuracy as the number
[10443c] of divisions of the amplitude shall be increased. The whole amplitude $\varpi=52^{\circ}$ [10453] is divided by La Place into 12 equal parts; and the depression of the mercury in the barometer is
[10443d] assumed successively, by Bouvard, at $5^{mi.},0$, $4^{mi.},5$, $4^{mi.},0$, $3^{mi.},5$, $3^{mi.},0$, $2^{mi.},5$, $2^{mi.},0$, $1^{mi.},5$, $1^{mi.},2$, $1^{mi.}0$. Below $1^{mi.},0$, the variations of the assumed values are $0^{mi.}$, 1 to $0^{mi.},4$; below $0^{mi.},4$, the
[10443e] variations are $0^{mi.},05$; finally, in the two last expressions, the depressions are supposed to be $0^{mi.},06$, $0^{mi.},03$. Taking successively these depressions for the values of q, and using $\alpha=\frac{1}{6,5}$
[10443f] [10442m], and $q=\frac{1}{\alpha b}$ [9354], we get the following expression of the radius of curvature b at the summit of the surface;

[10443g] $$b=\frac{1}{\alpha q}.$$

We shall now put

the elevation of the lowest point of the surface above the level [10442″]

V_r = the inclination of the side of the curve to the horizon, or value of ϖ [10379′], at the lower extremity of the division r of the arc; [10443h]

z_r = the vertical ordinate, or value of z [10378], corresponding to that extremity; [10443i]

u_r = the horizontal ordinate, or value of u [10379], corresponding to that extremity; [10443k]

b_r = the radius of curvature R [10442y], at the same point; b being its value at the vertex [10442e]; [10443l]

With this notation, the second of the equations [10442z] becomes

$$\frac{1}{b_r}=\frac{2}{b}+2\alpha z_r-\frac{1}{u_r}.\sin.V_r. \qquad [10443m]$$

The limits of the division r of the arc being V_r, V_{r+1}, and the corresponding radius of the circle being b_r, the difference of their sines to that radius will give the difference of the values of u_{r+1} and u_r; and the difference of their cosines will give the difference of the values of z_{r+1} and z_r; hence we easily deduce the following expressions, by using the formulas [19, 17], Int.; [10443n]

$$u_{r+1}=u_r+2b_r.\sin.\tfrac{1}{2}(V_{r+1}-V_r).\cos.\tfrac{1}{2}(V_{r+1}+V_r); \qquad [10443o]$$

$$z_{r+1}=z_r+2b_r.\sin.\tfrac{1}{2}(V_{r+1}-V_r).\sin.\tfrac{1}{2}(V_{r+1}+V_r). \qquad [10443p]$$

At the first division, where $r=0$, $V_0=0$, these expressions become, by a slight reduction,

$$u_1=b.\sin.V_1, \qquad z_1=2b.\sin.^2V_1. \qquad [10443q]$$

Instead of dividing the amplitude $\varpi'=52^{\text{d.}}=46^{\text{d.}}\,48^{\text{m.}}$, into twelve equal parts, according to the directions of La Place, M. Bouvard takes ten arcs, of $4^{\text{d.}}$ each, from $4^{\text{d.}}$ to $40^{\text{d.}}$; and the remaining part of the amplitude he divides into two equal parts, of $3^{\text{d.}}\,21^{\text{m.}}$ each. These values are used when the depression of the mercury *exceeds* $1^{\text{mi.}}$. When the depressions are less than $1^{\text{mi.}}$, he uses the two formulas [10385e, f], selecting in the first place for u a value which will give, in the formula [10385f], the angle ϖ or V equal to $4^{\text{d.}}$. Then this angle is successively augmented by $2^{\text{d.}}$, till $V=12^{\text{d.}}$; from $12^{\text{d.}}$ to $40^{\text{d.}}$, the intervals were $4^{\text{d.}}$; from $40^{\text{d.}}$ to $46^{\text{d.}}\,48^{\text{m.}}$, the intervals were $3^{\text{d.}}\,24^{\text{m.}}$. [10443r] [10443s] [10443t] [10443u]

The values of u corresponding to the greatest amplitude $46^{\text{d.}}\,48^{\text{m.}}$, being found for all the depressions [10443d, e], will represent the radii of the tubes, in which these depressions respectively occur. Upon examining this table, La Place found that the difference of the logarithms of the depressions, divided by the difference of the diameters of the tubes, forms a series of quotients which vary with extreme slowness; and he makes use of this property to simplify the table by adjusting it to equal decrements of the diameters of the tubes. Thus, if a, a', are the depressions corresponding to the diameters d, d', we shall have [10443v] [10443w]

$$\frac{\log.a'-\log.a}{d-d'}=C=\text{constant}; \qquad \text{or} \qquad \log.a'-\log.a=C.(d-d'). \qquad [10443x]$$

Then, if x be the depression which is required to be found for the diameter d_1, which occurs in the table, we shall have, in like manner, $\log.x-\log.a=C.(d-d_1)$; whence we get

[10443] [10443f], and we shall have $\varpi' = \frac{1}{2}\pi$;* which gives for this elevation,

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[10443y] $$\log.x = \log.a + C.(d - d_{1});$$

and, by means of this formula, the diameters of the tubes were arranged, as in the following table, so as to have a regular decrement of $0^{mi.},5$, from $21^{mi.}$ to $2^{mi.}$, in the first column; the second column contains the corresponding depressions, and the third column the differences of these quantities.

[10443z]

Internal diameters of the tubes.	Depressions.	Differences.	Internal diameters of the tubes.	Depressions.	Differences.
$21^{mi.},0$	$0^{mi.},028$	$0^{mi.},004$	$11^{mi.},5$	$0^{mi.},293$	$0^{mi.},037$
20 ,5	0 ,032	0 ,004	11 ,0	0 ,330	0 ,042
20 ,0	0 ,036	0 ,005	10 ,5	0 ,372	0 ,047
19 ,5	0 ,041	0 ,006	10 ,0	0 ,419	0 ,054
19 ,0	0 ,047	0 ,006	9 ,5	0 ,473	0 ,061
18 ,5	0 ,053	0 ,007	9 ,0	0 ,534	0 ,070
18 ,0	0 ,060	0 ,008	8 ,5	0 ,604	0 ,080
17 ,5	0 ,068	0 ,009	8 ,0	0 ,684	0 ,091
17 ,0	0 ,077	0 ,010	7 ,5	0 ,775	0 ,102
16 ,5	0 ,087	0 ,012	7 ,0	0 ,877	0 ,118
16 ,0	0 ,099	0 ,013	6 ,5	0 ,995	0 ,141
15 ,5	0 ,112	0 ,015	6 ,0	1 ,136	0 ,170
15 ,0	0 ,127	0 ,016	5 ,5	1 ,306	0 ,201
14 ,5	0 ,143	0 ,018	5 ,0	1 ,507	0 ,245
14 ,0	0 ,161	0 ,020	4 ,5	1 ,752	0 ,301
13 ,5	0 ,181	0 ,023	4 ,0	2 ,053	0 ,362
13 ,0	0 ,201	0 ,026	3 ,5	2 ,415	0 ,487
12 ,5	0 ,230	0 ,030	3 ,0	2 ,902	0 ,692
12 ,0	0 ,260	0 ,033	2 ,5	3 ,594	0 ,985
11 ,5	0 ,293		2 ,0	4 ,579	

This table is calculated upon the supposition that the angle $\varpi' = 52^{\circ} = 46^{d.}\,48^{m.}$ [10442i], and that, if the diameter of the tube be $0^{mi.},001$, the depression δ of the mercury will be
[10444a] $\delta = 94776^{mi.}$ [10442t]; some small variations will be found if we use the values adopted y M. Poisson, namely, $\varpi' = 45^{d.}\,30^{m.}$ [10442p], and $\delta = \frac{4,5746}{0,0005} = 91492^{mi.}$ [10442r].

[10444a'] * (4418) With water or alcohol, we have the angle $D'Dd = 0$, fig. 174, page 989; and then $\varpi' = 100^{\circ} - D'Dd$ [10442h] becomes $\varpi' = 100^{\circ} = \frac{1}{2}\pi$ [10443]. Hence we have
[10444b] $\text{tang.}\frac{1}{4}\varpi' = \frac{1}{\sqrt{2}+1}$ [10440i], and $-4\sin.^{2}\frac{1}{4}\varpi' = -2 + 2\cos.\frac{1}{2}\varpi' = -2 + \sqrt{2}$ [10440e]; substituting these in [10440], it becomes as in [10444], or, as it may be written,

$$\frac{1}{\alpha b}=\frac{4}{(1+\sqrt{2})\cdot\sqrt{\alpha}}\cdot\sqrt{\pi l\sqrt{2\alpha}}\cdot c^{-l\sqrt{2\alpha}-2+\sqrt{2}} \qquad [10444]$$

[Elevation of water, &c., in a large tube.]

$$=\frac{1,63476}{\sqrt{\alpha}}\cdot\sqrt{l\sqrt{2\alpha}}\cdot c^{-l\sqrt{2\alpha}}. \qquad [10445]$$

We shall now compare these results with experiments.

M. Gay-Lussac has observed, at the temperature of 12°,8, the thickness of a large drop of mercury, which was circular, and a decimetre, or $100^{\text{mi.}}$, in [10446] diameter, resting upon a plane of white glass, perfectly horizontal. He found, by means of a very accurate micrometer, that the thickness was $3^{\text{mi.}},378$. This [10447] value differs but very little from that which Segner found by a similar method, and which, being reduced to millimetres, is equal to $3^{\text{mi.}},40674$. This thickness [10448] may be calculated from the preceding expression of* $q+\frac{1}{\alpha b}$ [10426], putting, [10449]

$$\frac{1}{\alpha b}=\left\{\frac{4}{1+\sqrt{2}}\cdot\sqrt{\pi}\cdot c^{-2+\sqrt{2}}\right\}\cdot\frac{1}{\sqrt{\alpha}}\cdot\sqrt{l\sqrt{2\alpha}}\cdot c^{-l\sqrt{2\alpha}}. \qquad [10444c]$$

Now, substituting the values of $c=2,71828$, $\pi=3,141592$, we find that the factor between [10444d] the braces becomes equal to 1,63476, and then the expression [10444] becomes as in [10445]. We may remark that, in the original work, by an error of the press, this factor 1,63476, is [10444e] printed 3,63476.

* (4419) When $l=50^{\text{mi.}}$, the expression of $\frac{1}{\alpha b}$ [10439b] becomes insensible; and by [10450a] neglecting it in [10439f], we get $q=3^{\text{mi.}},3966$, as in [10439i, or 10451]. This observation is discussed by M. Poisson, in page 218 of his *Nouvelle Théorie, &c.*, by a method which is [10450b] the inverse of that in [10451, &c.]. For, instead of finding q from the assumed values of α, ϖ', or α, w', he determines the angle w' or ϖ', by the condition that it shall give, in the [10450c] formula [10426], a value of q which is equal to $3^{\text{mi.}},378$, as in the observation of M. [10450d] Gay-Lussac [10447]. With this, and an experiment on the depression of mercury in a glass tube, which gives $\alpha^2.\cos.w'=4^{\text{mi.mi.}},5746$, M. Poisson finds $\alpha^2=6^{\text{mi.mi.}},5262$, instead of [10450e] $\alpha^2=6^{\text{mi.mi.}},5$ [10351'], and $w'=45^{\text{d.}}\,30^{\text{m.}}$, instead of $\pi-\varpi'=48°=43^{\text{d.}}\,12^{\text{m.}}$, which are [10450f] used by La Place, in [10450, &c.]. If we neglect the term $\frac{1}{\alpha b}$, and the last term of [10426], which are quite small, in a large barometer, we get very nearly $q=\sqrt{\frac{2}{\alpha}}.\sin.\frac{1}{2}\varpi'$; [10450f'] substituting $\frac{2}{\alpha}=13^{\text{mi.mi.}}$ [10449'], and $\varpi'=52°$ [10453], it becomes

$$q=\sqrt{13^{\text{mi.mi.}}}.\sin.26°=1^{\text{mi.}},432, \qquad [10450g]$$

for the difference of elevations of the mercury at the sides of the tube and at its vertex, as in [10453'].

In all the calculations of this article, the drop of mercury is supposed to rest on a horizontal

[10449'] as in [10351'], $\frac{2}{\alpha} = 13^{\mathrm{mi.mi.}}$, and supposing the *acute* angle, formed by the

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[10450g] plane of glass; and we may apply the same principles to other similar problems; as, for example, where the glass, upon which the drop rests, is in the form of a concave spherical
[10450h] surface of a great radius. We may also suppose the drop to rest on another fluid, as in a case treated of by M. Poisson, in page 241 of his *Nouvelle Théorie, &c.*, where a large drop of water or other fluid is supposed to be placed upon a surface of mercury contained in a
[10450i] vessel of great extent. We shall now give the methods of investigating the figures of the surfaces of the two fluids, supposing them to be mercury and water; it being very evident that the formulas which are used in this case may be applied to any other fluids, by
[10450k] merely varying the constant quantities, so as to adapt them to the case actually under consideration.

[10450l] When a drop of water rests upon a surface of mercury, there are three distinct parts of these surfaces which are subjected to different conditions, and of course require different equations to express them. This is manifest in the annexed figure 175, in which Oo is a vertical axis passing through the centre of the drop;
[10450m] $On_{,}Na'A'AA''a''CnoHb''B''BB'b'$, a section of the drop by a plane passing through this axis; and
[10450n] $WvGN'a_{,}A_{,}AA'a'Nn_{,}Ob'B'BB_{,}b_{,}FV$, a section of the surface of the mercury by the same plane. Then the three surfaces we have alluded to are, *First*, the *upper* surface of

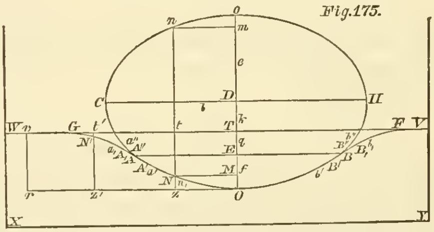

Fig.175.

[10450o] the drop $A''oB''$, which we shall, for brevity, denote by S'; this part is not in contact with the mercury, and its greatest horizontal diameter is the line CDH, the *highest* part of the drop being at the point o, in the axis Oo; *Second*, the *lower* part of the drop $A'OB'$, in
[10450p] contact with the mercury, and which we shall denote by $S_{,}$, O being its lowest point, and AEB the line of division between the upper and lower part of the section of the drop;
[10450p'] *Third*, the section of the surface of the mercury $WvGN'a_{,}A_{,}A$, or $VFb_{,}B_{,}B$, wholly free from the drop, which we shall denote by S. The distances of the points $A_{,}$, A', A'', from
[10450q] A, are supposed to be insensible, but greater than the quantity λ [9240a], which expresses the limit of the corpuscular action; moreover, the distances of the points $B_{,}$, B', B'', from
[10450r] B are supposed to be insensible, but greater than λ; and all the arcs $A_{,}a_{,}$, $A'a'$, $A''a''$, $B_{,}b_{,}$, $B'b'$, $B''b''$, $Nn_{,}$, are supposed to be infinitely small. The points F, G, are taken so far from the drop, that the parts FV, GW, may be considered as in the horizontal plane or
[10450s] level of the mercury $WvGt'tTFV$. Finally, we shall suppose that a canal $ntNzrv$ is drawn, as in [10020c], with two vertical branches rv, zn, meeting the surfaces in v, n, $N_{,}$;

surface of the mercury and by that of the glass, at the contact to be equal to [10449′]

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the lower points r, z, being connected by a horizontal branch rz; also another canal $vrz'N'$, whose vertical branches rv, $z'N'$, are terminated at the surface of the mercury in v, N', and their lower points r, z', are connected by the horizontal branch rz'. [10450t]

We shall take O for the origin of the rectangular coordinates u, z; the axis of u being the horizontal line Or, and the axis of z the vertical line Oo; these coordinates being supposed to correspond to the surface S; we shall also suppose that the rectangular coordinates u, z′, correspond to the *upper* surface S', and the coordinates u, $z_{,}$, to the *lower* surface $S_{,}$; hence [10450u] [10450v]

$Oz'=u$, $z'N'=$ z, are the coordinates of the point N' of the surface of the mercury S; [10450w]
$Oz=u$, $zn=$ z′, are the coordinates of the point n of the *upper* surface of the drop S'; [10450x]
$Oz=u$, $zN=\mathrm{z}_{,}$, are the coordinates of the point N of the *lower* surface of the drop $S_{,}$. [10450y]

We shall also use the following symbols;

$$OE=f, \quad ET=q, \quad TD=k, \quad Do=e, \quad TO=q+f, \quad To=k+e,$$
$$Oo=f+q+k+e=\varepsilon, \quad AE=r, \quad CD=l; \qquad [10450z]$$

$D=$ the density of the mercury; [10451a]
$D'=$ the density of the water; [10451b]
$g=$ the force of gravity. [10451c]

The principle of the equilibrium of the fluid in the canal $vrz'N'$ will give the differential equation of the surface S, in like manner as in [9315]. For if we put B for the radius of curvature at the point N' of the section $GN'A$, in the plane of the figure, and B' for the radius of curvature at the same point N', in a plane perpendicular to the plane of the figure, the corpuscular action at N', in the canal $N'z'$, in the direction of gravity, will be $K+\frac{1}{2}H.\left(\frac{1}{B}-\frac{1}{B'}\right)$ [9301]; the sign of B' being made negative, as in [9294′], because it corresponds to a concave arc whose centre is situated on the axis of revolution of the drop Oo. Adding this expression to the action of gravity in the canal $z'N'$, which is represented by $g.D\times z'N'$, we get the whole pressure at the point z', at the bottom of the canal $K+\frac{1}{2}H.\left(\frac{1}{B}-\frac{1}{B'}\right)+g.D\times z'N'$. In like manner, the corpuscular action at v being equal to K, the pressure at r will be $K+g.D\times rv$. Putting these two expressions equal to each, and neglecting the quantities which destroy each other, we get, by observing that $rv=TO=q+f$, and $z'N'=$ z, [10451d] [10451e] [10451f] [10451g] [10451h]

$$\tfrac{1}{2}H.\left(\frac{1}{B}-\frac{1}{B'}\right)=g.D.(q+f-\mathrm{z}). \qquad [10451i]$$

The value of $\frac{1}{B'}$ is the same as that of $\frac{1}{R'}$ [9326′], changing z into z, the radical $\left(1+\frac{d\mathrm{z}^2}{du^2}\right)^{\frac{1}{2}}$ being supposed positive; and as z increases with u, dz has the same sign as the constant differential du; hence we have

[10450] 48° [10353], which gives, in the expression just mentioned, $\varpi'=152°$

[10451k]
$$\frac{1}{B'}=\frac{\frac{1}{u}\cdot\frac{dz}{du}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{1}{2}}}.$$

In like manner, the value of $\frac{1}{B}$ is the same as that of $\frac{1}{R}$ [9326], changing the sign of ddz, because the surface $A_{,}N'G$ is convex, so that ddz must be negative; and we shall therefore have

[10451l]
$$\frac{1}{B}=-\frac{\frac{ddz}{du^2}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}}.$$

Substituting these values [10451k, l], in [10451i], and then multiplying by $-\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$, we get, for *the equation of the surface of the mercury S, not in contact with the drop,* the following differential expression;

Surface S. [10451m]
$$\tfrac{1}{2}H.\left\{\frac{ddz}{du^2}+\frac{1}{u}\cdot\frac{dz}{du}\cdot\left(1+\frac{dz^2}{du^2}\right)\right\}=g.D.(z-q-f)\cdot\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}.$$

The differential equations of the surfaces $S_{,}$, S', are found by means of the equilibrium of the canal $vrzNn$, in the same manner as we have already explained in treating of two fluids

[10451n] in a tube [10027a—z]. If we substitute the values of $\frac{1}{R}$, $\frac{1}{R'}$ [9326, 9326′], in the equation [10027x], and then multiply by $\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}$, changing z into $z_{,}$, to conform to the present notation, we shall get *the following equation of the lower surface of the drop $S_{,}$, in contact with the mercury;*

Surface $S_{,}$. [10451o]
$$\tfrac{1}{2}G.\left\{\frac{ddz_{,}}{du^2}+\frac{1}{u}\cdot\frac{dz_{,}}{du}\cdot\left(1+\frac{dz_{,}^2}{du^2}\right)\right\}=\{(D-D').gz_{,}-(D-D').gn+c\}\left(1+\frac{dz_{,}^2}{du^2}\right)^{\frac{3}{2}};$$

G being a constant quantity depending on the capillary action of the two fluids near

[10451p] the lower surface $S_{,}$ [10027k], and c a constant quantity [10027o″] to be determined by observation.

In like manner, the equation of the upper convex surface of the drop S' [10027z] is of the following form, H being changed into H', to conform to the notation of M. Poisson;

[10451q]
$$-\tfrac{1}{2}H'.\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)=(D'gz'-D'gn-c).$$

The values of $\frac{1}{\mu}$, $\frac{1}{\mu'}$, may be deduced from [9326, 9326′], by changing R into μ, and R' into μ'; whence we get

[10451r]
$$\frac{1}{\mu}=\frac{\frac{ddz}{du^2}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}},\qquad\qquad \frac{1}{\mu'}=\frac{\frac{1}{u}\cdot\frac{dz}{du}}{\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}}};$$

[10439a]; and if we neglect the term $\frac{1}{\alpha b}$, which becomes insensible relative [10450′]

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observing that, in fig. 117, page 719, the *lowest point O of the concave surface ONA* is the origin of the rectangular coordinates $OM=z$, $MN=u$; the ordinate MN being horizontal, [10451s] and the ordinate OM vertical, or *in a direction from the origin O contrary to that of gravity;* but in fig. 118, page 725, the *highest point O of the convex surface ONA* is the origin of the rectangular coordinates $OM=z$, $MN=u$; the ordinate OM being in the direction of gravity from the origin O. [10451t] Hence it follows that, if we refer the formulas [10151r] *to the upper part of the convex surface of the drop onC*, we must take o for the origin of the rectangular coordinates $om=z$, $mn=u$, of any point n. [10451u] This value of $om=z$, being subtracted from $Oo=\varepsilon$ [10450z], gives $Om=zn=\varepsilon-z$; and as this is represented by z' [10450x], we get $\varepsilon-z=\mathrm{z}'$, whose [10451$v$] differential gives $dz=-d\mathrm{z}'$, $ddz=-dd\mathrm{z}'$. Substituting these in [10451r], we get the values of $\frac{1}{\mu}$, $\frac{1}{\mu'}$, to be used in [10451q], which, by this means, becomes

$$\tfrac{1}{2}H'.\left\{\frac{dd\mathrm{z}'}{du^2}+\frac{1}{u}.\frac{d\mathrm{z}'}{du}.\left(1+\frac{d\mathrm{z}'^2}{du^2}\right)\right\}=\{D'g\mathrm{z}'-D'gn-c\}.\left(1+\frac{d\mathrm{z}'^2}{du^2}\right)^{\frac{3}{2}}.$$ Surface S'. [10451w]

This represents the equation of the upper or convex surface onC of the drop; H' being a constant quantity, depending on the capillary action of the fluid which forms the drop, and c being of the same value, as in [10151p].

Having thus obtained the differential equations of the three surfaces S, $S_{\prime}$, S' [10451m, o, w], we shall now reduce them to more symmetrical forms, by putting the arbitrary constant quantity n [10027v'] equal to TO or $q+f$, which is the same as to [10451x] change the origin of the coordinates z, $\mathrm{z}_{\prime}$, z', from the point O to T, using also the following additional symbols;

$$H=g.D.\alpha^2, \qquad G=(D-D').g.\alpha_{\prime}^2, \qquad H'=D'.g.\alpha'^2,$$ [10451y]

$$n=q+f, \qquad \mathrm{z}=-z+n, \qquad \mathrm{z}_{\prime}=z_{\prime}+n-\frac{c}{(D-D').g}, \qquad \mathrm{z}'=-z'+n+\frac{c}{D'g}.$$ [10451z]

Now, substituting the values of H, z, in [10151m], multiplied by $-\frac{2}{H}$, we get [10452b]. The values of G, $\mathrm{z}_{\prime}$, being substituted in [10451o], multiplied by $\frac{2}{G}$, give [10452c]; [10452a] and the values of H', z', being substituted in [10451w], multiplied by $-\frac{2}{H}$, give [10452d]; by this means the differential equations of the surfaces S, $S_{\prime}$, S', become, respectively,

$$\frac{ddz}{du^2}+\frac{1}{u}.\frac{dz}{du}.\left(1+\frac{dz^2}{du^2}\right)=\frac{2z}{\alpha^2}.\left(1+\frac{dz^2}{du^2}\right)^{\frac{3}{2}},$$ [Surface S of the mercury not touching the drop.] [10452b]

$$\frac{ddz_{\prime}}{du^2}+\frac{1}{u}.\frac{dz_{\prime}}{du}.\left(1+\frac{dz_{\prime}^2}{du^2}\right)=\frac{2z_{\prime}}{\alpha_{\prime}^2}.\left(1+\frac{dz_{\prime}^2}{du^2}\right)^{\frac{3}{2}},$$ [Surface $S_{\prime}$ where the drop is in contact with the mercury.] [10452c]

$$\frac{ddz'}{du^2}+\frac{1}{u}.\frac{dz'}{du}.\left(1+\frac{dz'^2}{du^2}\right)=\frac{2z'}{\alpha'^2}.\left(1+\frac{dz'^2}{du^2}\right)^{\frac{3}{2}}.$$ [Surface S' of the drop not in contact with the mercury.] [10452d]

[10450''] to a drop whose diameter is a decimetre or $100^{mi.}$ [10450a], the

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The equations we have here computed are equivalent to those given by M. Poisson in his
[10452e] *Nouvelle Théorie, &c.* For the equations [10451m, o, w] are the same as his system of equations (a), page 242, changing the origin of z from the point O to the point T, or in
[10452f] other words by decreasing the values of z, $z_{,}$, z', by the quantity $q+f$, to conform to his notation, and putting $u=t$, $\alpha=a$, $D=\rho$, $D'=\rho'$. Moreover, the first of his equations (a), page 242, is easily reduced to the form [10452b]; the first of his equations (f),
[10452g] page 246, is reduced to the form [10452c]; lastly, the second of these equations (f) is reduced to the form [10452d]. We may here remark, that the expressions of $z_{,}$, z'
[10452h] [10451z], may be put under other forms, by the substitution of the value of c. For, the
[10452i] figure of the drop being that of a surface of revolution, if we put λ for its radius of curvature at its lowest point O, where $z_{,}=0$, we shall have $R=R'=\lambda$; substituting this in [10027x], after changing z into $z_{,}$, to conform to the notation of the surface $S_{,}$, we get
[10452k] $c=\frac{G}{\lambda}+(D-D').gn$; and by substituting it in the general expression of $z_{,}$ [10451z],
[10452l] it becomes $z_{,}=z_{,}-\frac{G}{(D-D').g\lambda}$. In like manner, at the upper point o of the surface of the
[10452m] drop, where $z'=\varepsilon$ [10450z], and $\mu=\mu'$, we get, from [10027z], $c=\frac{H'}{\mu}+D'g.(\varepsilon-n)$;
[10452n] and since $\varepsilon-n=k+e$ [10450z, 10451z], we finally get $c=\frac{H'}{\mu}+D'g.(k+e)$; hence the general expression of z' [10451z] becomes

[10452o]
$$z'=-z'+n+k+e+\frac{H'}{D'g\mu}=-z'+\varepsilon+\frac{H'}{D'g\mu} \quad [10450z,\ 10451z].$$

Substituting in $z_{,}$, z' [10452l, o], the values of G, H, n [10451y, z], we get

[10452p]
$$z_{,}=z_{,}-\frac{\alpha_{,}^{2}}{\lambda}, \qquad \text{or} \qquad z_{,}=z_{,}+\frac{\alpha_{,}^{2}}{\lambda};$$

[10452q]
$$z'=-z'+\varepsilon+\frac{\alpha'^{2}}{\mu}, \quad \text{or} \quad z'=\varepsilon+\frac{\alpha'^{2}}{\mu}-z';$$

which are equivalent to the two equations given by M. Poisson in the lower part of page 245 of his *Nouvelle Théorie, &c.* The origin of z, $z_{,}$, z', is at the same point O, fig. 175, page 991 [10450u, &c.], and as $z=-z+n$ [10451z], also $n=TO$ [10451x],
[10452r] it follows that the origin of z must be at the point T. Moreover, as $z_{,}=z_{,}+\frac{\alpha_{,}^{2}}{\lambda}$ [10452p], the origin of $z_{,}$ must be on the continuation of the axis oO *below* O, and at a distance from O which is equal to $\frac{\alpha_{,}^{2}}{\lambda}$; the positive values of z being counted from their origin T in the direction of gravity TO, and the positive values of $z_{,}$ in the opposite or vertical direction
[10452r'] from their origin *below* O. Lastly, it follows from the expression of z' [10452q], that the origin of z' is on the continuation of the line Oo, at the distance $\varepsilon+\frac{\alpha'^{2}}{\mu}$ from O, or at the distance $\frac{\alpha'^{2}}{\mu}$ *above* o, and that the positive values of z' are in the downward direction oO, from its origin situated *above* o.

expression [10426, or 10439g], will give, for the thickness of the [10450‴]

If we put the expressions of c [10452k, n] equal to each other, we shall get

$$G.\frac{1}{\lambda}-H'.\frac{1}{\mu}=D'g.(k+c)-(D-D').gn=D'g.(k+\varepsilon+n)-Dgn$$
$$=D'g.\varepsilon-Dg.(q+f) \qquad [10451z], \qquad [10452r'']$$

which is the same as the equation (c) in page 245 of the *Nouvelle Théorie*, &c., of M. Poisson.

The equations [10452b, c, d], are integrated by approximation, in M. Poisson's *Nouvelle Théorie*, &c., by methods which are substantially the same as those which are used [10452s] by La Place in calculating the figure of a large drop of mercury resting upon a plane surface [10380—10445]. It is not therefore absolutely necessary to go into a minute detail of these integrations, and the determination of the constant quantities; because the reader can easily [10452t] apply the methods of La Place to these objects, in case the dimensions and figure of a drop, in any experiment, should be proposed for examination. There are, however, some objects which require particular attention, and we shall therefore point out the general methods of [10452u] solution, with such additional remarks as may be thought necessary for the elucidation of the subject.

We have seen, in [10450, &c.], that when a drop of mercury, fig. 172, page 971, rests on a glass plane DE, the curvature of the surface suddenly varies near the point D, so that [10452v] the tangent changes from the direction ED, to another which is inclined to it by the angle 48° [10450], while passing over a distance Dd which is insensible, but greater than the [10452w] limits to which the corpuscular action is sensible. The same thing occurs on the surface of the mercury in passing over the space $A'AA_{,}$, fig. 175, page 994, the drop of water acting [10452x] in the same manner as a wet pane of glass or wet tube; so that the angle of inclination of [10452y] the two infinitely small arcs $A'a'$, $A_{,}a_{,}$, or of the tangents drawn through them, may be represented by $w=48°$. In like manner there is a rapid change of the curvature of the [10452z] drop in passing over the space $A_{,}AA''$; so that the angle of inclination of the two infinitely small arcs $A'a'$, $A''a''$, may be represented by w'; this angle being the same as that which [10453a] would be observed if the drop were placed on a solid horizontal plane of mercury, supposing its action not to be in any manner affected by its becoming solid. If we now suppose that [10453b] the lines $A_{,}a_{,}$, $A'a'$, $A''a''$, form with the horizon, angles which are represented by ϖ, $\varpi_{,}$, ϖ', respectively, we shall evidently have [10453c]

$$w=\varpi_{,}-\varpi, \qquad w'=\varpi'-\varpi_{,}; \qquad [10453d]$$

and from [22, 34′], Int., we have

$$\sin.w=\sin.(\varpi_{,}-\varpi)=\sin.\varpi_{,}.\cos.\varpi-\cos.\varpi_{,}.\sin.\varpi=\cos.\varpi.\cos.\varpi_{,}.(\text{tang}.\varpi_{,}-\text{tang}.\varpi). \qquad [10453e]$$

We find also, in like manner as in [10406b, a],

$$\cos.\varpi=\left(1+\frac{dz^2}{du^2}\right)^{-\frac{1}{2}}, \qquad \cos.\varpi_{,}=\left(1+\frac{dz_{,}^2}{du^2}\right)^{-\frac{1}{2}}, \qquad \text{tang}.\varpi=\frac{dz}{du}, \qquad \text{tang}.\varpi_{,}=\frac{dz_{,}}{du}. \qquad [10453f]$$

Dividing the last expression of $\sin.w$ [10453e] by $\cos.\varpi.\cos.\varpi_{,}$, and then substituting the

[10450''''] drop q, the following value,

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values [10453f], we get [10453h]. In the same manner, we may find [10453i], or more simply by derivation, by changing $\varpi_{,}$ into ϖ', and ϖ into $\varpi_{,}$, which requires that we should

[10453g] change $z_{,}$ into z', and z into $z_{,}$; consequently, also, w into w' [10453a];

[10453h] $$\frac{dz_{,}}{du}-\frac{dz}{du}=\sin.w.\left(1+\frac{dz^2}{du^2}\right)^{\frac{1}{2}}.\left(1+\frac{dz_{,}^2}{du^2}\right)^{\frac{1}{2}};$$

[10453i] $$\frac{dz'}{du}-\frac{dz_{,}}{du}=\sin.w'.\left(1+\frac{dz_{,}^2}{du^2}\right)^{\frac{1}{2}}.\left(1+\frac{dz'^2}{du^2}\right)^{\frac{1}{2}};$$

and it is evident, from [10451z], that we may substitute in these equations $dz=-dz$,

[10453k] $dz_{,}=dz$, $dz'=-dz'$, also $u=r$ [10450z]. These equations will serve to determine two

[10453l] of the constant quantities which occur in [10450z], after we have determined $\frac{dz}{du}$, $\frac{dz_{,}}{du}$, $\frac{dz'}{du}$, by means of the integrals of the equations of the surfaces [10452b, c, d]; and we shall now make some remarks relative to these integrals.

Multiplying the equation [10452b] by $-dz.\left(1+\frac{dz^2}{du^2}\right)^{-\frac{3}{2}}$, and taking the integral so that it may vanish when $u=\infty$ and $z=0$, we get

[10453m] $$\frac{1}{\sqrt{1+\frac{dz^2}{du^2}}}=1-\frac{z^2}{\alpha^2}+\int_0^u\frac{\frac{dz^2}{du^2}.du}{u\sqrt{1+\frac{dz^2}{du^2}}},$$

as is easily proved by taking the differentials, and making some slight reductions. In this equation of the surface $GN'A_{,}$, fig. 175, page 994, the ordinate u is never less than AE or r [10450z]; we may therefore neglect the term under the sign $\int$ [10453m], on account of its smallness, and then we shall have

[10453n] $$\frac{1}{\sqrt{1+\frac{dz^2}{du^2}}}=1-\frac{z^2}{\alpha^2}.$$

Now, squaring and reducing this equation, we obtain

[10453o] $$du=\frac{(z^2-\alpha^2).dz}{z\sqrt{2\alpha^2-z^2}}.$$

When $z=0$, u becomes infinite, the origin of z and u being at the point T, and the positive values of z being taken in the direction of gravity TO [10452r]. As z increases, u decreases, so that dz, du, must have contrary signs; and the radical in the denominator of [10453o] must therefore be considered as positive. The integral of this equation is found as

[10453o'] in [10181u, v], by taking the constant quantity so that $u=r$, when $z=q$ or $z=f$ [10451z, 10450z]; hence we get

Surface S. [10453p] $$u=r+\sqrt{2\alpha^2-q^2}-\sqrt{2\alpha^2-z^2}+\frac{\alpha}{\sqrt{2}}.\log.\frac{(\alpha\sqrt{2}-\sqrt{2\alpha^2-q^2}).z}{(\alpha\sqrt{2}-\sqrt{2\alpha^2-z^2}).q};$$

remarking that we have here changed the sign of the arbitrary constant factor $\sqrt{2\alpha^2-q^2}-\alpha\sqrt{2}$ introduced into the logarithmic part of the expression [10181v], so as to

$$q = 3^{\text{mi.}},39664 \quad [10439i],$$ [10451]

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make the logarithm vanish when $z = q$ [10453p], instead of making it vanish when $z = -q$, as in [10181s, v]; it being evident that this does not alter the form of du deduced from [10453q]; so that it will correspond to [10453o], in the same manner as y [10181v] corresponds to dy [10183u]. [10453p']

If we put $u = -\mathrm{u}$, $r = -\mathrm{r}$, and $q = \mathrm{q}$, in [10453o], and then change the signs of all the terms, we shall get, by some slight reductions, the following values of $d\mathrm{u}$, and of its integral u, r being the value of u when $z = \mathrm{q}$; [10453q]

$$d\mathrm{u} = \frac{(\alpha^2 - z^2) \cdot dz}{z\sqrt{2\alpha^2 - z^2}}; \qquad [10453q']$$

$$\mathrm{u} = \mathrm{r} - \sqrt{2\alpha^2 - \mathrm{q}^2} + \sqrt{2\alpha^2 - z^2} + \frac{\alpha}{\sqrt{2}} \cdot \log. \frac{(\alpha\sqrt{2} - \sqrt{2\alpha^2 - z^2}) \cdot \mathrm{q}}{(\alpha\sqrt{2} - \sqrt{2\alpha^2 - \mathrm{q}^2}) \cdot z}; \qquad [10453r]$$

which will be of use hereafter. If we wish to obtain a more correct value of the integral of the equation [10452b], we may substitute this value of u in the part of [10453m] which is under the sign $\int$, and neglected in the preceding calculation; thus, by successive operations, we may obtain corrected expressions of u, arranged according to the powers of r, its first term being the value of the second member of the equation [10453p]. This term is, however, sufficiently accurate for practical purposes, and it is the same as that marked (d) in page 245 of the *Nouvelle Théorie, &c.*, of M. Poisson; representing very nearly the equation of the surface of the mercury at a sensible distance from the drop. [10453r'] [10453s] [10453t]

In those parts of the surface of the drop which are but little inclined to the horizon, we may reduce the differential equations [10452c, d], to linear forms, as in [10382], and their integrals may be found as in [10384], or rather as in [10385m, n]. For, by putting, in [10385m], $\mathrm{z} = z_{,}$, $\alpha^2 = \alpha_{,}^2$, $b = \lambda$, it becomes as in [10452c], and its integral [10385n] gives the value of $z_{,}$ [10453w]. In like manner, by putting $\mathrm{z} = z'$, $\alpha^2 = \alpha'^2$, $b = \mu$, the equation [10385m] changes into [10452d], and its integral [10385n] gives the value of z' [10453x], [10453u] [10453v]

$$z_{,} = \frac{\alpha_{,}^2}{\lambda\pi} \cdot \int_0^{\pi} d\varphi \cdot c^{\frac{u\sqrt{2}}{\alpha_{,}} \cdot \cos.\varphi}; \qquad [10453w]$$

$$z' = \frac{\alpha'^2}{\mu\pi} \cdot \int_0^{\pi} d\varphi \cdot c^{\frac{u\sqrt{2}}{\alpha'} \cdot \cos.\varphi}; \qquad [10453x]$$

λ being the radius of curvature of the drop at its lowest point O, and μ the radius of curvature at the upper point o, as in [10452i, k]. [10453y]

If we use the approximate value of the integral [10385p], these expressions of $z_{,}$, z', will become as in [10454a, b], which can be used when u is supposed to be considerably greater than $\alpha_{,}$ or α'; [10453z]

$$z_{,} = \frac{\alpha_{,}^2}{\lambda\sqrt{2\pi\sqrt{2}}} \cdot \sqrt{\frac{\alpha_{,}}{u}} \cdot c^{\frac{u\sqrt{2}}{\alpha_{,}}}; \qquad [10454a]$$

$$z' = \frac{\alpha'^2}{\mu\sqrt{2\pi\sqrt{2}}} \cdot \sqrt{\frac{\alpha'}{u}} \cdot c^{\frac{u\sqrt{2}}{\alpha'}}. \qquad [10454b]$$

[10451'] which differs but little from the experiments [10447, 10448].

We may also find the integrals of the equations [10452c, d], by a process like that in
[10454c] [10404, &c.]; or in the following manner. Multiplying [10452c] by $-dz_{,}.\left(1+\frac{dz_{,}^2}{du^2}\right)^{-\frac{3}{2}}$, integrating and transposing the second and third terms, we get [10454e]. In like manner,
[10454d] multiplying [10452d] by $-dz'.\left(1+\frac{dz'^2}{du^2}\right)^{-\frac{3}{2}}$, integrating and transposing, we get [10454f]; the integrals being supposed to vanish when $u=0$,

[10454e] $$\frac{1}{\sqrt{1+\frac{dz_{,}^2}{du^2}}}-1+\frac{1}{\alpha_{,}^2}.\left(z_{,}^2-\frac{\alpha_{,}^4}{\lambda^2}\right)=\int_0^u\frac{\frac{dz_{,}^2}{du^2}.du}{u\sqrt{1+\frac{dz_{,}^2}{du^2}}};$$

[10454f] $$\frac{1}{\sqrt{1+\frac{dz'^2}{du^2}}}-1+\frac{1}{\alpha'^2}.\left(z'^2-\frac{\alpha'^4}{\mu^2}\right)=\int_0^u\frac{\frac{dz'^2}{du^2}.du}{u\sqrt{1+\frac{dz'^2}{du^2}}};$$

[10454g] observing that, when $u=0$, we have $z_{,}=0$, $z'=\varepsilon$ [10450x, y, z], also $\frac{dz_{,}}{du}=\frac{dz_{,}}{du}=0$, $\frac{dz'}{du}=-\frac{dz'}{du}=0$, because the tangents at the points o, O, are horizontal; and if we substitute these values of $z_{,}$, z', in the expressions of $z_{,}$, z' [10452p, q], we shall get, when
[10454h] $u=0$, $z_{,}=\frac{\alpha_{,}^2}{\lambda}$, $z'=\frac{\alpha'^2}{\mu}$. If the arc $ON=s$ be very small, we may consider it as a circular arc whose radius is λ; and then we shall have, from the usual rules for the differentials of circular arcs, &c. [9328c, d], $dz_{,} : ds :: u : \lambda$; and as ds is, in this case,
[10454i] very nearly equal to du, we shall have very nearly $\frac{dz_{,}^2}{du^2}=\frac{u^2}{\lambda^2}$, which is a very small quantity when the point N falls near the point O, where λ is much greater than u. In like manner, $\frac{dz'^2}{du^2}=\frac{u^2}{\mu^2}$ is a very small quantity when the point n falls near to o. We may therefore neglect the second members of the integrals [10454e, f], when u is very small. These quantities may also be neglected when we increase the values of u so as to correspond
[10454k] to the approximate values of $z_{,}$, z' [10453w, x], because these values produce in $\frac{dz_{,}^2}{du^2}$, $\frac{dz'^2}{du^2}$, terms which have the great divisor λ^2, or μ^2. Finally, as we approach towards the borders of the drop, the value of u is so much increased, that we may even then neglect the
[10454l] second members of [10454e, f], on account of the divisor u; so that in all cases we may neglect these integral expressions. If we also neglect the terms of the first members divided by λ^2 or μ^2, we shall get, by some slight reductions,

[10454m] $$\frac{\alpha_{,}^2}{\sqrt{1+\frac{dz_{,}^2}{du^2}}}=\alpha_{,}^2-z_{,}^2,\qquad \frac{\alpha'^2}{\sqrt{1+\frac{dz'^2}{du^2}}}=\alpha'^2-z'^2;$$

or, by squaring and reducing,

M. Gay-Lussac has also observed, in a very large glass vessel whose sides were vertical, the distance from the point of contact of the surface of the [10452] mercury with the sides, to the highest point of that surface; and he found it to be $1^{\text{mi.}},455$. This distance is, by what has been said, equal to $\sqrt{\frac{2}{\alpha}}.\sin.\frac{1}{2}\varpi'$ [10453] [10450f']; and in this case we have $\varpi'=52^{\circ}$ [10442i]; hence we find that

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$$du=\frac{(\alpha_{,}^2-z_{,}^2).dz_{,}}{z_{,}\sqrt{2\alpha_{,}^2-z_{,}^2}}, \qquad [10454n]$$

$$du=\frac{(\alpha'^2-z'^2).dz'}{z'\sqrt{2\alpha'^2-z'^2}}; \qquad [10454o]$$

observing that when u, z, are small, du, $dz_{,}$, have the same signs; and the same remark may be made relative to du, dz'; so that we may consider the radicals as positive in the denominators of the expressions of du [10454n, o].

The differential equations [10454n, o], are of the same form as [10453q'], and their integrals may be derived from that in [10453r]. Thus, if we change u, α, r, q [10453q', r], into u, $\alpha_{,}$, r, $f_{,}$, respectively, the expression of du [10453q'] will become like that of du [10454p] [10454n]; and its integral u [10453r] will become as in [10454s], which represents the [10454q] integral of [10454n]. In like manner, by changing u, α, r, q [10453q', r], into u, α', l, f', respectively, the expression of du [10453q'] will become like that of du [10454o], and the value of u [10453r] will become as in [10454t], which represents the integral [10454r] of [10454o];

$$u=r-\sqrt{2\alpha_{,}^2-f_{,}^2}+\sqrt{2\alpha_{,}^2-z_{,}^2}+\frac{\alpha_{,}}{\sqrt{2}}.\log.\frac{(\alpha_{,}\sqrt{2}-\sqrt{2\alpha_{,}^2-z_{,}^2}).f_{,}}{(\alpha_{,}\sqrt{2}-\sqrt{2\alpha_{,}^2-f_{,}^2}).z_{,}}, \qquad [10454s]$$

Surfaces $S_{,}$ S'.

$$u=l-\sqrt{2\alpha'^2-f'^2}+\sqrt{2\alpha'^2-z'^2}+\frac{\alpha'}{\sqrt{2}}.\log.\frac{(\alpha'\sqrt{2}-\sqrt{2\alpha'^2-z'^2}).f'}{(\alpha'\sqrt{2}-\sqrt{2\alpha'^2-f'^2}).z'}; \qquad [10454t]$$

r being the value of u [10454s] when $z_{,}=f_{,}$, and l the value of u [10454t] when $z'=f'$. [10454u] If we suppose $f_{,}$ to correspond to the point A, or the greatest value of $z_{,}$, we shall have $z_{,}=OE=f$ [10450z]; substituting this in the value of $z_{,}$ [10452p], we get [10454v] $z_{,}=f+\frac{\alpha_{,}^2}{\lambda}=f_{,}$, and with this value of $z_{,}$, the expression of u [10454s] becomes $u=r$. [10454w] Again, if we suppose f' to correspond to the point C, or the greatest value of the ordinate u [10454t], we shall have $z'=OD=f+q+k=\varepsilon-e$ [10450z]; substituting this in the value of z' [10452q], we get $z'=-(\varepsilon-e)+\varepsilon+\frac{\alpha'^2}{\mu}=e+\frac{\alpha'^2}{\mu}=f'$. [10454$x$]

When accurate experiments on the figure of a drop of water, or of any other fluid, resting upon a surface of mercury, shall be made, we may compare the results of the observations with the integral expressions given in this note, and, after determining the constant quantities [10454y] which enter into the integrals, we shall have the formulas which give the figure and dimensions of the drop, and the figure of the surface of mercury; but it will not be necessary to enter into any discussion relative to this subject, because the whole calculation is similar to that [10454z] which has been used in other cases of capillary attraction treated of in this work.

[10453′] this distance is equal to $1^{mi.},432$, which differs but little from the result of the experiment.

[10454] *To compare the analysis of the depression of mercury in very large tubes with the result of observations,* we have selected the experiments made by Mr. Charles Cavendish, reported in the Philosophical Transactions for the year 1776. They give, in English inches, this depression equal to $0^{in.},005\,[=0^{mi.},1270]$, in a glass tube whose diameter is $0^{in.},6\,[=15^{mi.},24]$;
[10455] and a depression of $0^{in.},007\,[=0^{mi.},1778]$, in a tube whose diameter is $0^{in.},5\,[=12^{mi.},70]$; also a depression of $0^{in.},015\,[=0^{mi.},3810]$, in a tube whose diameter is $0^{in.},4\,[=10^{mi.},16]$. The preceding expression of $\frac{1}{\alpha b}$ [10440], by
[10455′] substituting $\varpi'=52^{\circ}$ [10453], and reducing the results into English inches,* makes the depression 0,0038 in the first tube; 0,0069 in the second tube;
[10456] and 0,0126 in the third tube; which agree with the experiments as well as can be expected in these observations, where such small quantities are estimated.

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* (4420) An English inch is equal to $25^{mi.},3918$ [9678]; multiplying this by the diameters $\frac{6}{10}$,
[10456a] $\frac{5}{10}$, $\frac{4}{10}$ [10455], and taking half the corresponding products, we shall get the corresponding values
[10456b] of the radii of the tube in millimetres, $7^{mi.},62$, $6^{mi.},35$, $5^{mi.},08$, nearly. Substituting these successively for l, in [10440], and putting $\varpi'=52^{\circ}$ [10455′], $\frac{2}{\alpha}=13^{mi.mi.}$ [10449′], we obtain the corresponding values of $\frac{1}{\alpha b}$ in millimetres, $0^{mi.},0954$, $0^{mi.},1762$, $0^{mi.},3187$, which, being divided
[10456c] by 25,3918, give the values of $\frac{1}{\alpha b}$ in English inches, namely, $0^{in.},0038$, $0^{in.},0069$, $0^{in.},0126$, respectively, as in [10456]. These differ but very little from the results of the experiments in
[10456d] [10455], namely, $0^{in.},005$, $0^{in.},007$, $0^{in.},015$, respectively. M. Poisson computes the values of $\frac{1}{\alpha b}$, or δ in his notation, and finds, in the three preceding examples, $\delta=0^{mi.},0945$,
[10456e] $\delta=0^{mi.},1747$, $\delta=0^{mi.},3175$, in page 290 of his *Nouvelle Théorie, &c.*, by a method which is very nearly the same as that in [10440], though varying a little in its form, and including
[10456f] some very small quantities which were neglected by La Place, as in [10442s, t]. These three examples include the three greatest diameters in the table of Mr. Cavendish; his two smallest diameters are $0^{in.},10=2^{mi.},54$, and $0^{in.},15=3^{mi.},81$, and the observed
[10456g] depressions $\delta=0^{in.},140=3^{mi.},556$, and $\delta=0^{in.},092=2^{mi.},336$, respectively. These depressions are computed by M. Poisson by the rule given by him in [10442r], and found to be
[10456h] $\delta=3^{mi.},4712$, and $\delta=2^{mi.},4199$. La Place's rule [10442q] gives $\delta=3^{mi.},4733$, $\delta=2^{mi.},1008$, respectively; which agree very nearly with the observations; the result of M. Poisson's
[10456i] calculation for the larger tube being rather more accurate than the other, on account of his carrying on the series to terms of the order l^3 in [10442r]. Similar results would be obtained
[10456k] by comparing the depressions computed by the formulas [10440, 10442q], with those given in the table [10443z], for the greatest and least diameters.

M. Gay-Lussac found, by the mean of five experiments, the elevation of the lowest point of the surface of alcohol, in a glass tube whose diameter was $10^{\text{mi.}},508$, equal to $0^{\text{mi.}},3835$. The temperature was 16° during these [10457]
experiments, and the specific gravity of the alcohol was 0,813467, at this temperature. The lowest point of the same fluid rose, at the same temperature, to the height of* $9^{\text{mi.}},07850$, in a glass tube whose diameter was $1^{\text{mi.}},2944$; [10458]
whence we deduce

$$\frac{2}{\alpha} = 12^{\text{mi.}},0306. \qquad [10459]$$

The formula [10445] gives $0^{\text{mi.}},3374$ for the elevation of the alcohol in a large tube, in which, by experiment, this elevation was $0^{\text{mi.}},3835$. The difference [10460]
$0^{\text{mi.}},0461$ is within the limits of the errors of the experiment and of the [10460′]
formula, which is only an approximation.

GENERAL OBSERVATIONS.

From what has been said, we perceive the agreement which is found between the capillary phenomena and the results of the law of attraction of the particles of the bodies, decreasing with extreme rapidity, so as to become insensible at the least distances perceptible to our senses. This law of nature is the source [10461]
of chemical affinities; like gravity, it is not arrested at the surfaces of bodies, but penetrates them, *acting beyond the point of contact, but at imperceptible distances.* Upon this depends the influence of the masses in chemical phenomena, or the capacity for saturation, whose effects have been so beautifully developed by M. Berthollet. Thus two acids, acting upon the same base, are divided in proportion to their affinities with it; which would not take place, if this affinity acted only when in contact; for then the most powerful acid

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* (4421) We have changed the value $7^{\text{mi.}},07850$, given in the original work, into [10460a]
$9^{\text{mi.}},07850$, to correct a supposed typographical mistake, and to make the observation agree with the calculated result in [10459]; and we may remark that this corrected value agrees very nearly with an observation of M. Gay-Lussac given by M. Poisson, in page 225 of his *Nouvelle Théorie, &c.* To find $\frac{2}{\alpha}$, from the observation [10458], we must proceed as in [10460b]
[10296h], and add one sixth part of the diameter of the tube $1^{\text{mi.}},2944$ [10458], or $0^{\text{mi.}},2159$ to the height $9^{\text{mi.}},0785$, and then multiply the sum $9^{\text{mi.}},2944$ by the diameter [10460c]
$1^{\text{mi.}},2964$; the product $12^{\text{mi.}},0306$ gives the value of $\frac{2}{\alpha}$, as in [10459]. Using this value of [10460d]
$\frac{2}{\alpha}$, and that of $2l = 10^{\text{mi.}},508$ [10457], we find that the expression of $\frac{1}{\alpha b}$ [10445], becomes $0^{\text{mi.}},3374$, as in [10460]. [10460e]

would retain the whole base. The figure of the elementary particles, the heat, and other causes, being combined with this law, modify the effects of it. The discussion of these causes, and of the circumstances which develop them, is the most delicate part of chemistry, and constitutes the philosophy of that science, making known to us, as much as possible, the intimate nature of the bodies, the law of the attractions of their particles, and that of the foreign attractions which operate upon them.

The particles of a solid body have that position in which their resistance to a change of situation is the greatest possible. Each particle, when it is removed from that situation an infinitely small space, tends to return to it by means of the forces which act upon it. It is this which constitutes the elasticity with which we may suppose all bodies to be endowed, when the figure of the bodies is but very little changed. But when the relative situations of the particles
[10461′] suffer considerable changes, these particles will form new states of stable equilibrium, as happens with hardened metals, and generally with bodies which, by their softness, are susceptible of retaining all the forms which may be given to them by pressure. The hardness of bodies, and their viscosity, appear to me to be only the resistance of the particles to these changes in the state of equilibrium. The expansive force of heat being opposed to the attractive force
[10462] of the particles, it decreases little by little their tenacity or mutual adhesion by successive increments; and when the particles of a body oppose only a very slight resistance to a change of relative situation, within its surface, it becomes a liquid. But its viscosity, although very much weakened, still exists, until, by an increase of temperature, it becomes nothing, or insensible. Then
[10463] each particle, in all positions, suffers the same attractive force and the same repulsive force of heat; it yields to the slightest pressure, and the liquid enjoys a perfect fluidity. We may with probability conjecture that this takes place with those liquids which, like alcohol, have a temperature far above that in which they begin to congeal. This influence of the figure of the particles is
[10464] very sensible in the phenomena of congelation and crystallization, which become much more rapid by immersing in the liquid a piece of ice, or a crystal formed from the same liquid; the particles of the surface of this solid, are presented to the liquid particles which touch them, in the situation the most favorable to their union with them. We can easily conceive that the influence
[10465] of the figure, when the distance increases, must decrease much more rapidly than the attraction itself. It is so with the astronomical phenomena depending upon the figure of the planets, as the precession of the equinoxes; for this influence decreases as the cube of the distance, whilst the attraction decreases only in the ratio of the square of the distance.

Therefore it appears that the state of solidity depends upon the attraction of the particles, combined with their figure; so that an acid, although it may [10466] exert upon a base a less attraction at a distance than upon another base, yet it will combine and will crystallize in preference with the former, if the figure of its particles is such as to render its contact with that base more intimate. The influence of the figure, although it may continue sensible in viscous fluids, is nothing in those which enjoy a perfect fluidity. Lastly, every thing leads us to believe that, in a gaseous state, not only the influence of the figure of the [10467] particles, but also that of their attractive forces, is insensible in comparison with the repulsive force of heat. These particles appear to be only an obstacle to the expansion of that force; for we may, without changing the tension of any given magnitude of gas, substitute for several of the particles disseminated through this mass, a like number of particles of another gas. This is the reason why several gases placed in contact are finally mixed together in a [10468] uniform manner; for it is only in this case that they are in a stable state of equilibrium. If one of these gases is vapor, the equilibrium is not stable [10469] except when the vapor is equal or less in quantity than that of the same vapor which would be diffused at the same temperature in a void space equal to that occupied by the mixture. If the vapor is in greater quantity, the excess must, for the stability of the equilibrium, be condensed under a liquid form.*

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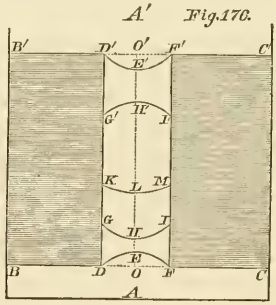
* (4422) We may, in this connexion, mention the subject of a newly-discovered capillary phenomenon, first observed by M. Dutrochet, and named *endosmose*, which is treated of by [10469*a*] M. Poisson in page 296, &c., of his *Nouvelle Théorie, &c.*, and explained in nearly the following manner.

When two different liquids A and A' are separated by a membrane $BCC'B'$, which, for [10469*b*] greater simplicity, we shall suppose to be limited by the horizontal surfaces BC, $B'C'$, fig. 176, one of these liquids, as, for example, the lower one A, will penetrate the membrane, and, if it be very thin, will pass [10469*c*] through it, and mix with the upper liquid A', till it is raised up considerably above its natural level, and this is what constitutes the phenomenon of *endosmose*. M. Dutrochet found that it occurs when certain inorganic [10469*d*] substances are used instead of a membrane, as for example when a thin lamina of slate is employed.

A' *Fig.176.*



In considering the nature of this phenomenon, we shall suppose a canal OO', of a very small diameter, to pass through the whole thickness of the membrane, and we [10469*e*] shall suppose at first that neither of the liquids has penetrated into it. Then, if the two liquids are not of such a nature as to rise up in a tube of

[10469'] The consideration of the stability of the equilibrium of a system of particles

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[10469f] the same matter as this canal, they will not penetrate into the interior of the canal, but will form the convex surfaces DEF, $D'E'F'$, at the two extremities, like mercury at the side of a portable barometer, in which a very small capillary aperture has been made. If, on the
[10469g] contrary, the two liquids are susceptible of wetting the membrane, or of rising in a tube of the same matter as this membrane, they will penetrate into the interior of the canal OO'. At the beginning, both surfaces will be concave, as GHI, $G'H'I'$; but when they meet each other,
[10469h] they will have a common surface KLM, so that one of them will be more or less *concave*, and the other *convex*. Then the motion will continue from O towards O', if the surface of
[10469i] the fluid H be *concave*, as is the case in the present figure; but from O' towards O, if the surface of the fluid A' be *concave*; so that in the first case, for example, some of the fluid A
[10469k] will pass wholly through the membrane, and begin to mingle with the fluid A', and thus produce a mixed fluid whose density will depend on the portions of the two liquids of which it
[10469l] is composed; and with this change in the density, there very naturally follows a corresponding change in the capillary action between the mixed fluid and the membrane.

[10469m] From what has been said, it is manifest that, if the membrane is traversed by an extremely great number of canals similar to OO', we may extend to all of them the remarks we have
[10469n] applied to one, and the liquid A will ascend in all the canals which satisfy the conditions mentioned in [10469k]. Although the membrane is supposed to be homogeneous, it will not,
[10469o] however, be impossible that, on account of the different disposition of the particles along the different canals, or from some other cause, there may be a disposition to descend along some of the canals, instead of ascending as is supposed in [10469i, &c.]; then there will be
[10469p] simultaneously, through these canals, a chief motion of the fluid A towards A', and a much less motion of the fluid A' towards A, as appears to be indicated by experiments. From the law of the ascent of fluids in capillary tubes [9361, &c.], it will follow, that, when other
[10469q] circumstances are alike, the difference of the level of the mixed fluid on both sides of the membrane, will be inversely proportional to the diameters of these canals; so that the intensity
[10469r] of the phenomenon will decrease with the increased diameter of these canals, and the endosmose will be insensible when the tissue of the membrane becomes quite open. The principal and essential condition of the phenomenon is the difference of the two liquids; it will
[10469s] never be produced if they are both of the same nature, or have the same properties relative to capillary action. In order that the membrane may not be unfit for the purpose of the production of the phenomenon of endosmose, it is necessary that its contexture should be
[10469t] such that it can be traversed by very slender canals through its whole thickness, which is most commonly the case with any organic membrane. On the other hand, the liquids suffer a considerable friction along the sides of these canals, on account of the smallness of their
[10469u] diameters; this friction may balance the capillary action, and prevent the two liquids from penetrating sufficiently deep into the membrane to meet each other and form a common
[10469v] surface, so as to change the concavity of the surface of one of them into a convexity, as in [10469h]; on this account, a too great thickness of the membrane may be an obstacle to the production of endosmose.

reacting upon each other, is very useful in the explanation of many phenomena. In like manner as in a system of solid and fluid bodies, affected by gravity, the [10470] principles of mechanics make known several states of stable equilibrium; and chemistry affords, by a combination of the same principles, several permanent states of equilibrium. Sometimes two principles unite together, and the particles formed by their union unite to those of a third principle. Such, in all probability, is the combination of the constituent principles of an acid with a [10471] base. In other cases, the principles of a substance, without being united together, as they are in the substance itself, are united to other principles, and form with them some triple or quadruple combination; so that this substance, extracted by a chemical process, is then a product of this operation. The component particles may also be united by different faces, and thus produce [10472] some crystals, differing in form, hardness, specific gravity, and in their action upon light. Lastly, the condition of a stable equilibrium appears to me to be what determines the fixed proportions according to which different principles are combined in a great number of circumstances. All these phenomena depend upon the form of the elementary particles, upon the laws of their attractive [10473] force, upon the repulsive force of heat, and perhaps upon other forces yet unknown. Our ignorance of these data, and their extreme complication, do not permit us to reduce the results to mathematical analysis. To compensate for this, we can compare together those facts which have been well ascertained, and deduce from their comparison certain general relations, which, by connecting together a great number of phenomena, form the basis of the [10474] chemical theories, whose applications to the arts are thus rendered more extensive and perfect.

At the surface of a fluid, the attraction of the particles, modified by the curvature of the surface and of the sides of the vessel which contains it, produces the capillary attraction. Therefore these phenomena, and all those which chemistry presents, correspond to one and the same law, of which now [10475] there can be no doubt. Some philosophers have attributed the capillary phenomena to the adhesion of the fluid particles, either to each other, or to the sides of the vessels which contain them; but this cause is not sufficient to produce the effect. For, if we suppose the surface of water, contained in a glass tube, to be horizontal, and upon a level with that of the water in the [10476] vessel into which the tube is dipped by its lower end; the tenacity of the fluid, and its adhesion to the tube, would not curve this surface and render it concave. To produce that, it is necessary to suppose that there is an attraction in the [10477] upper part of the tube, which is not immediately in contact with the fluid.

Besides, the surface of the fluid contained in the tube, if it were concave, would be drawn vertically downwards by the vertical columns of the fluid adhering to
[10478] it; and when the surface is convex, as is the case with mercury in a glass tube, or with water in a tube having a convex drop at its lowest end [10016], this convex surface will be pressed perpendicularly in each of its points by the weight of the superior columns of the fluid. This surface would not, therefore, be the same in these two cases, and the capillary phenomena would not follow the same laws; which is contrary to experience. We must, therefore, admit
[10479] that the phenomena do not depend wholly upon the action at the point of contact, but upon an attraction which extends beyond it; decreasing with extreme rapidity.

The viscosity of fluids is so far from being the cause of the capillary phenomena, that it is, on the contrary, a cause of disturbance. The phenomena
[10480] do not rigorously conform to the theory, except when the fluid enjoys a perfect mobility; for the forces upon which the phenomena depend are so small, that the slightest obstacle may modify the effects of them in a sensible manner. It is to the tenacity of water that we must attribute the considerable differences which have been observed by philosophers, in the elevations of this fluid in glass capillary tubes of the same diameter. The second manner in which we
[10481] have considered the capillary action, shows us that the interior surface of a tube raises at first a lamina of water; this raises a second, which draws up a third; and so on towards the axes of the tube. The existence of these laminæ may be rendered sensible, by means of some grains of dust adhering to the
[10482] sides of the glass; we see these little bodies agitated by the impulsion of these laminæ, before the surface of the liquor has risen up to them. The attraction of these laminæ upon each other is in an oblique direction to the surface of the sides, and tends to make the particles of the second lamina penetrate within the first; which cannot be done without raising it up or breaking it. When
[10483] the tube is but very little moistened, this first lamina, which is then very small, resists these efforts, by its adhesion to the glass and the viscosity of its particles. And, if I am not mistaken, this is the reason why Newton and M. Haüy have found the ascent of water in a glass tube of a millimetre in diameter to be only
[10484] about thirteen millimetres [9670], whilst, in a similar, well-moistened tube, the water has risen to the height of above thirty millimetres [10322].

At the top of a glass tube, the first laminæ of water cannot ascend any higher without altering the figure of their upper surface; and at the moment when this
[10485] surface becomes convex, it tends to depress the fluid below it, and thus presents an obstacle to its ascent. This cause, together with the viscosity of the fluid,

and its adhesion to the glass, accounts for the small resistance which the water meets in its elevation, when it comes near the extremity of the tube; a resistance which ought to be nothing, and is so in fact, in fluids which, like [10486]
alcohol, have a perfect mobility.

The friction of the fluid against the surface of the sides, and the adhesion of the air at the surface of the bodies, are also causes of anomaly in capillary attractions. It is necessary to notice them in the comparison of any experiment [10487]
with the theory, and the agreement will be found more complete as these causes have less influence.

It is almost impossible to determine, by experiment, the intensity of the attractive force of the particles of bodies; we only know that it is incomparably [10488]
greater than the capillary action. We have seen already, that water supports itself above the level in the axis of a capillary tube, by the difference of the actions of the fluid upon its own particles, at the surface of the fluid in the [10488′]
vessel in which the tube is dipped, and at the surface of the fluid within the tube. This difference is the action of the liquid meniscus cut off by a horizontal plane drawn through the lowest point of this last surface; and this action is measured by the height of the elevated column. To obtain the action of the whole mass of the fluid, we shall suppose, in an indefinite mass of stagnant water, *an infinitely narrow vertical canal* to be placed, having its sides [10489]
infinitely thin, so as not to prevent the action of the particles without the canal upon the column of water contained within it; and we shall determine the pressure of this column upon a base perpendicular to the sides of the canal, and placed at a sensible distance below the surface of the fluid; this base being taken for unity. It is easy to show that, if we have several similar canals, of the same width, but of different lengths, in each of which the water may be urged [10490]
by different forces, varying according to any laws whatever, *the pressures of this fluid upon the bases of the canals, will be to each other as the squares of the velocities acquired by bodies setting out from a state of rest, and moving in a* [10491]
*vacuum through the whole length of these canals, supposing that they are acted upon at each point by the same forces as those which act upon the corresponding particles of water, included in the canal.** If the action of water upon its

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* (4423) The calculations we have made in [8158a—p] may be used in illustrating the remarks of the author on this subject, by varying a little his hypothesis, and supposing the infinitely narrow canal [10489] to be *external* relative to the attracting fluid, instead of [10491a]
internal. In this case, by referring to fig. 107, page 448, we may suppose that the upper part of the attracting fluid is limited by the horizontal surface AE, and that its lowest surface BD [10491b]

own particles be equal to its action upon light, it would follow, from [8174, &c.], that the square of the velocity acquired in the canal of which we have spoken,
[10491′] would be equal to* $2K$, *the density of water being taken for unity.* In a canal whose height is s, and in which the constant force is equal to gravity,
[10492] the square of the acquired velocity is $2gs$, g being the force of gravity, or twice the space which gravity causes the body to describe in the first *unit* of

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is at a great distance from AB. The attracted canal is represented by TD, drawn
[10491c] perpendicular to the surface AE, and continued upwards far beyond the limits of the capillary attraction of the fluid $AEDB$. Then, using the same notation as in [8158—8162″], namely,
[10491d] $TD = s'$ [8159], and ρ equal to the density of the fluid, supposing it to be constant [8158h],
[10491e] we shall have $\rho . \Pi_{,}(s')$ [8158h], for the attraction of the fluid $AEDB$ upon a mass of the
[10491f] fluid represented by unity, situated at D. Multiplying this by ds', we get $\rho . ds' . \Pi_{,}(s')$, for the attraction of the mass $AEDB$ upon the portion of the canal ds'. Integrating this from $s'=0$ to $s'=\infty$, we get the whole action or pressure p at the base T of the canal; so that
[10491g] we shall have $p = \rho . \int_0^\infty ds' . \Pi_{,}(s')$; and if we use the abridged symbol $K = \int_0^\infty ds' . \Pi_{,}(s')$ [8162], we shall have

[10491h]
$$p = K\rho;$$

and if we put $\rho = 1$, as in [10491′], it becomes $p = K$, agreeing with the calculation in [9254], where the radius b of the external surface of the fluid AE is infinite, as in the case
[10491i] treated of in fig. 107, page 448. We have, in [8162′, 8162″], n for the velocity of a
[10491k] ray of light when at a sensible distance from the attracting mass $AEDB$, and n for its velocity at any point D after the attraction has had a sensible operation; then we have generally, as in [8164k],

[10491l]
$$\mathrm{n}^2 - n^2 = 2K\rho - 2\int_0^{s'} \rho . ds' . \Pi_{,}(s').$$

At the point T of the surface of the fluid where $s' = 0$, the integral part of the expression
[10491m] [10491l] vanishes; and if we suppose n to be the velocity of the light at that point T, we shall have

[10491n]
$$\mathrm{n}^2 - n^2 = 2K\rho,$$

or, by substituting [10491h],

[10491o]
$$\mathrm{n}^2 - n^2 = 2p.$$

[10491p] If we suppose the initial velocity of the ray to be nothing, or $n = 0$, the expression [10491o] will become

[10491q]
$$\mathrm{n}^2 = 2p;$$

[10491r] and, in this case, the pressure p will be as the square of the acquired velocity n; as in [10491].

[10492a] * (4424) Putting $\rho = 1$, as in [10491′], we find that the increment of the square of the velocity of the ray $2K\rho$ [8174] becomes $2K$, as in [10491′]. Moreover we have, as in
[10492b] [67c], $2gs$ for the square of the velocity acquired by falling from rest, by the force of gravity, through the space s; as in [10492].

time, which we shall suppose to be a decimal second. The pressure of the columns of water upon the bases of the two canals will therefore be to each other as $2K$ is to $2gs$; therefore, if they are equal, we shall have [10493]

$$s=\frac{K}{g}. \qquad [10494]$$

In the present hypothesis, this value of s represents the height of a second canal, in which the water is supposed to be acted upon only by gravity, and always with the same force as it has at the surface of the earth, so as to produce a pressure of this fluid on the base of this second canal, which is equal to the whole action of the indefinite mass of water upon the water in the first canal. [10495]

We have, by means of the formula [8192],*

$$R^2-1=\frac{4K}{n^2}, \qquad [10496]$$

where n is the space described by light in a unit of time [8162′], or in one second [10493], and R equal to the ratio of the sine of incidence to the sine of refraction, in the passage of a luminous ray from vacuum into water; therefore we shall have [10497]

$$s=\frac{(R^2-1).n^2}{4g}. \qquad [10498]$$

Using the most exact values of the sun's parallax and the velocity of light, we find that s *exceeds ten thousand*† *times the sun's distance from the earth.* [10498′]

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* (4425) Putting $\rho=1$ [10492a], in [8192], and changing i [8191′] into R [10497], in order to conform to the present notation, we get $\frac{4K}{n^2}=R^2-1$, as in [10496]; which gives $K=\frac{(R^2-1).n^2}{4}$; hence s [10494] becomes as in [10498]. [10495a]

† (4426) We have, in [8523—8524], $R=\frac{529}{396}$, or $R=\frac{4}{3}$ nearly; hence $R^2-1=\frac{7}{9}$; and, by substitution in [10498], we get $s=\frac{7}{36}\cdot\frac{n^2}{g}$. Now, by taking a sexagesimal second for [10496a]
the unit of time, and using English measures, we have $g=32$ feet $=\frac{1}{165}$ mile nearly; hence [10496b]
$s=\frac{7}{36}\times 165n^2=32n^2$ nearly; and as the velocity of light per second is $n=195000$ miles, [10496c]
it becomes $s=32.(195000)^2=12.10^{11}$ miles nearly; and as the sun's distance from the [10496d]
earth is 95.10^6 miles, the value of s, expressed in parts of the sun's distance, will be [10496e]
$s=\frac{12.10^{11}}{95.10^6}=12000$ nearly, which exceeds 10000, as is observed in [10498′]. Therefore, if we [10496f]

Such a prodigious value of the action of water upon its own particles, cannot
[10499] be admitted with any probability; therefore it appears that this action is much
[10500] less than the action of water upon light, *but it is extremely great relative to the capillary action*, and from this arises a very great compression in the strata of liquids. For if, within an indefinite mass of stagnant water, we suppose an infinitely narrow canal to be placed, whose sides are infinitely thin, and whose ends terminate at the surface of the water; the fluid strata of the canal, placed at a sensible distance below that surface, will suffer by the action of the water
[10501] near one of its extremities a pressure K [9259, 9262], which will be balanced by an equal and opposite pressure, produced by the action of the water towards the other extremity; each stratum of the comprised liquid is therefore compressed by these two opposite forces. *At the surface of the fluid, this*
[10502] *pressure is evidently nothing; it increases with extreme rapidity from this surface, and becomes constant at the least sensible distance below it.*

[10502'] *These great variations in the compression may sensibly vary the densities of the strata of a fluid very near to its surface; and in the mixture of two fluids,*
[10503] *as alcohol and water, they may produce variations not only in the densities of the fluid strata extremely near to the surface, but also in the proportion of the two fluids, included by these strata and the fluid lamina adhering to the sides of the tube.* These variations have no influence upon the refraction which, when the
[10504] luminous ray has arrived at a sensible distance below the surface, is the same as if the nature and density of the liquid did not suffer any change [8231, 8235]; but they can have upon the capillary phenomena a very sensible
[10505] influence, which seems to be indicated by several experiments of M. Gay-Lussac,

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admit the Newtonian theory of the emission of the particles of light, it will follow, as in [10499], that the action of water upon its own particles is much less than its action upon
[10496g] light; but we cannot make this inference if we adopt the wave theory [8137e, &c.]. Upon the same principles, we may find the ratio of the capillary action $\frac{H}{b}$ to the corpuscular action
[10496h] K. This ratio is expressed by e in [9253g]; so that we have $e=\frac{1}{K}\cdot\frac{H}{b}$; and, by substituting
[10496i] $K=sg$ [10494], it becomes $e=\frac{1}{s}\cdot\frac{H}{gb}$; or, since $q=\frac{H}{gb}$ [9354], we finally get $e=\frac{q}{s}$, q being, as in [9353], the elevation of water in a capillary tube whose radius is b, which is only
[10496k] a few millimetres, and s being the immense distance spoken of in [10498']; so that, upon these principles, e may be considered as an excessively small quantity, as we have already stated in [9253k].

upon the elevation of different mixtures of alcohol and water in capillary tubes.

A lamina of insulated water, whose thickness is less than the radius of the sensible sphere of activity of the particles, suffers, then, a much less compression than [10506] *a similar lamina placed within a considerable mass of that fluid; it is natural from thence to conclude that its density is much less than the density of that mass. Is it unreasonable to suppose that this is the case with the aqueous envelop of* [10507] *vesicular vapor, which by this means becomes lighter, and in an intermediate state between that of liquid and vapor?*

We have not noticed, in this theory, either the pressure of the atmosphere or the repulsive force of heat. The consideration of these forces is unnecessary, [10508] because they are the same upon the whole surface of the fluid,* and they are independent of its curvature. Therefore heat has no influence upon capillary phenomena, except by decreasing the density of fluids; and experience shows [10508′] that, in fluids having a perfect mobility, the variations of these phenomena, produced by the increment of temperature, are exactly such as are given by [10509] the theory.

The effects of the capillary action being reduced to a mathematical theory, there is only wanting, in this interesting branch of physical science, a series of very accurate experiments, by means of which we may compare the results of the theory with nature. The necessity of such experiments is felt in proportion as physical science becomes more perfect, and falls within the domain of [10510] analysis. We may then obtain with great precision the results of the theories; and, by comparison with very accurate experiments, we may elevate these theories to the greatest degree of certainty that natural science is susceptible of. Fortunately, the experiments which Rumford and Gay-Lussac have made upon the capillary phenomena, have left but little to be done relative to this subject; and we have seen the agreement of this theory with the results of [10511] Gay-Lussac, who has introduced into this kind of experiments the accuracy of astronomical observations.

When we have at length ascertained the true cause of any phenomenon, it is

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* (4427) If there be any difference of the heat, in consequence of the great difference of the pressures near the surface of the fluid, or from any other cause, it has no other effect [10508a] than to alter the function $\varphi(f)$ [9173e], which represents the capillary action; but this does [10508b] not alter the form of the resulting force [9254], but merely the values of the constant quantities K, H, which occur in it; as in [9173f, &c.].

an object of curiosity to look back, and see how near the hypotheses that have been framed to explain it approach towards the truth. One of the oldest and most commonly received opinions relative to the capillary action is that of Jurin. This author attributes the elevation of the water in a glass capillary tube, to the attraction of the annular part of the tube, to which the surface of the water
[10512] is contiguous and adheres; "for," he says, "this is the only part of the tube from which the water must recede on its subsiding, and consequently the only one which, by the force of its cohesion or attraction, opposes the descent of the water. This is also a cause proportional to the effect which it produces, since that periphery and the column suspended are both in the same proportion as the diameter of the tube." (Philosophical Transactions, No. 355, 1718; Hutton's, &c., Abridgment, vol. 6, page 333.) Clairaut, in his Treatise on the
[10512'] Figure of the Earth, objects to this method of using the principle, that the effects are proportional to the cause, except we ascend to a first cause; and that we ought not to do it when we examine an effect arising from the combination of several particular causes, unless we estimate each one separately. Therefore, although we should admit that the glass ring alone, which adheres to the surface of the water, is the cause of the elevation of this
[10513] uid, we must not thence conclude that the weight of the elevated fluid is proportional to its diameter; because we cannot ascertain the force of that ring, except by summing up the attractions of all its parts. Clairaut, therefore, substituted, instead of Jurin's hypothesis, an accurate analysis of all the forces which keep a column of water suspended in equilibrium in an infinitely narrow canal passing through the axis of the tube. But this great mathematician has not explained the most noted capillary phenomenon, namely, that of the ascent
[10514] or descent of a fluid, in a very slender tube, in the inverse ratio of the diameter of the tube; he merely observes, without any proof, that an infinite number of laws of attraction would produce this phenomenon. The supposition which he makes, that the action of the glass is sensible, even upon the particles situated in the axis of the tube, must have led him astray from the true explanation of the phenomenon; but it is remarkable that, if he had used the hypothesis of an attraction insensible at sensible distances, and had applied to the particles situated within the sphere of activity of the sides of the tube, the analysis of the forces which he used for the particles situated in the axis, he would have been led not only to Jurin's result, but also to those which we have obtained by the second method we have used in computing the capillary action.
[10515] We see by this method, that, if the fluid perfectly wets the sides of the tube,

we may imagine that the part of the tube which is immediately above the surface of the fluid, and at an insensible distance, is the only part which tends to elevate it, and keep it suspended in equilibrium, when the weight of the elevated column exactly balances the attraction of this annulus of the tube. This approaches very near to Jurin's theory, and leads to his result, namely, that the weight of the column is proportional to the circumference of the interior base of the tube; and this result may be extended generally to any prismatic tube, whatever be its interior form [9986′, 9988b], and the ratio of [10516]
the attraction of its particles upon the fluid to the attraction of the particles of the fluid upon each other.

The resemblance between the surface of a fluid drop, or the surface of a fluid contained within a capillary space, and the surface with which geometricians were occupied at the time of the origin of the differential calculus, under the name of the catenarian curve, or the elastic curve, &c., naturally led several philosophers to consider fluids as being enveloped by similar surfaces, which, by their tension and elasticity, impressed upon the fluids the forms indicated by experiment. Segner, who was one of the first that formed this [10517]
idea, was very sensible that it was only a fiction, which might be useful to represent the phenomena, but that it ought not to be admitted except it corresponded to the law of an attraction insensible at sensible distances. (Tom. 1. of the early Memoirs of the Royal Society of Gottingen.) Therefore he endeavored to establish this dependence; but, by the examination of his reasoning, it is easy to discover the inaccuracy of it, and the results which he has obtained are a sufficient proof of it. For example, he finds, that it is only necessary to notice the curvature of the vertical section of a drop, and not that of the horizontal section, which is not correct. Besides, he has not perceived that the tension of the surface is the same, whatever be the [10518]
magnitude of the drop; which a correct reasoning would have made known to him. Moreover, we see, by the note which terminates his researches, that he was not well satisfied with them himself. When I was occupied upon this subject, Mr. Thomas Young was likewise making upon it the ingenious researches which he has inserted in the Philosophical Transactions for the year 1805. In comparing, as Segner had done, the capillary action to the tension of a surface which would envelop the fluids, and applying to this force the known results of the tension of surfaces, he has discovered that it is [10519]
necessary to notice the curvature of the fluid surfaces in two directions perpendicular to each other; he has also supposed that these surfaces, with the

same fluid, form with the sides of the tube the same angle, when they are made of the same substance, whatever may be their figure; which, as we have seen, is not correct at the extremities of these sides.* But he has not attempted, like Segner, to derive this hypothesis from the law of the attraction of the particles, decreasing with extreme rapidity, which is indispensable for confirming it. This cannot be done but by a rigorous demonstration, like that
[10520] we have given in the first method, to which the explanations of Segner and Mr. Thomas Young correspond, as that of Jurin does to the second method.

[10519a] * (4428) This remark of La Place is not correct, as we have already stated in [10012a, &c.].

NOTE.—The equation [10384o], of page 974, appears not to have received the final revision of the author, and to have been left in an imperfect state by the omission, at the end, of z'', which has been subsequently inserted, together with the parentheses contained in the succeeding line, in which the signification of z'' is defined.

FINIS.